THE SEMI-INFINITE INTERSECTION COHOMOLOGY SHEAF-II: THE RAN SPACE VERSION

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Abstract. This paper is a sequel to [Ga1]. We study the semi-infinite category on the Ran version of the affine Grassmannian, and study a particular object in it, denoted $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$, which we call the *semi-infinite intersection cohomology sheaf.*

Unlike the situation of [Ga1], this $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ is defined as the middle of extension of the constant (more precisely, dualizing) sheaf on the basic stratum, in a certain t-structure. We give several explicit description and characterizations of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$: we describe its !- and *- stalks; we present it explicitly as a colimit; we relate it to the IC sheaf of Drinfeld's relative compactification $\overline{\mathrm{Bun}}_N$; we describe $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ via the Drinfeld-Plucker formalism.

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Introduction

0.1. **What are trying to do?**

0.1.1. This paper is a sequel of [Ga1]. In *loc. cit.* an attempt was made to construct a certain object, denoted $\mathrm{IC}^{\frac{\infty}{2}}$, in the (derived) category $\mathrm{Shv}(\mathrm{Gr}_G)$ of sheaves on the affine Grassmannian, whose existence had been predicted by G. Lusztig.

Notionally, $\mathrm{IC}^{\frac{\infty}{2}}$ is supposed to be the intersection cohomology complex on the closure $\overline{S}^0$ of the unit $N(\!(t)\!)$-orbit $S^0 \subset \mathrm{Gr}_G$. Its stalks are supposed to be given by periodic Kazhdan-Lusztig polynomials. Ideally, one would want the construction of $\mathrm{IC}^{\frac{\infty}{2}}$ to have the following properties:

- It should be local, i.e., only depend on the formal disc, where we are thinking of Gr_G as $G(\!(t)\!)/G[\![t]\!]$;
- When our formal disc is the formal neighborhood of a point x in a global curve X, then $\mathrm{IC}^{\frac{\infty}{2}}$ should be the pullback along the map $\overline{S}^0 \to \overline{\mathrm{Bun}}_N$ of the intersection cohomology sheaf of $\overline{\mathrm{Bun}}_N$, where the latter is Drinfeld's relative compactification of the stack of G-bundles equipped with a reduction to N (which is an algebraic stack locally of finite type, so $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$ is well defined).

The construction in [Ga1] indeed produced such an object, but with the following substantial drawback: in *loc. cit.*, $\mathrm{IC}^{\frac{\infty}{2}}$ was given by an *ad hoc* procedure; namely, it was written as a certain explicit direct limit. In particular, $\mathrm{IC}^{\frac{\infty}{2}}$ was *not* the middle extension of the constant[1] sheaf on S^0 with respect to the natural t-structure on $\mathrm{Shv}(\mathrm{Gr}_G)$ (however, $\mathrm{IC}^{\frac{\infty}{2}}$ does belong to the heart of this t-structure).

0.1.2. In the present paper we will construct a variant of $\mathrm{IC}^{\frac{\infty}{2}}$, denoted $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$, closely related to $\mathrm{IC}^{\frac{\infty}{2}}$, that is actually given by the procedure of middle extension in a certain t-structure.

Namely, instead of the single copy of the affine Grassmannian Gr_G, we will consider its Ran space version, denoted $\mathrm{Gr}_{G,\mathrm{Ran}}$. We will equip the corresponding category $\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})$ with a t-structure, and we will define $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ as the middle extension of the dualizing sheaf on $S^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$.

[1] Technically, not constant but rather dualizing.

Remark 0.1.3. Technically, the Ran space is attached to a smooth (but not necessarily complete) curve X, and one may think that this compromises the locality property of the construction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$. However, if one day a formalism becomes available for working with the Ran space of a formal disc, the construction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ will become purely local.

0.1.4. For a fixed point $x \in X$, we have the embedding

$$\mathrm{Gr}_G \simeq \{x\} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}},$$

and we will show that the restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ along this map recovers $\mathrm{IC}^{\frac{\infty}{2}}$ from [Ga1].

0.1.5. Our $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ retains the relation to $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$. Namely, we have a natural map

$$\overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_N$$

and we will prove that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ identifies with the pullback of $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$ along this map.

In particular, this implies the isomorphism

$$\mathrm{IC}^{\frac{\infty}{2}} \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_N}|_{\overline{S}^0},$$

which had been established in [Ga1] by a different method.

0.1.6. To summarize, we can say that we still do not know how to intrinsically characterize $\mathrm{IC}^{\frac{\infty}{2}}$ on an individual Gr_G as an intersection cohomology sheaf, but we can do it, once we allow the point of the curve to move along the Ran space.

But *ce n'est pas grave*: as was argued in [Ga1, Sect. 0.4], our $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$, equipped with its factorization structure, is perhaps a more fundamental object than the original $\mathrm{IC}^{\frac{\infty}{2}}$.

0.2. **What is done in this paper?** The main constructions and results of this paper can be summarized as follows:

0.2.1. We define the *semi-infinite* category on the Ran version of the affine Grassmannian, denoted $\mathrm{SI}_{\mathrm{Ran}}$, and equip it with a t-structure. This is largely parallel to [Ga1].

We define $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}} \in \mathrm{SI}_{\mathrm{Ran}}$ as the middle extension of the dualizing sheaf on stratum $S^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$. (We will also show that the corresponding !- and *- extensions both belong to the heart of the t-structure, see Proposition 2.1.7; this contrasts with the situation for $\mathrm{IC}^{\frac{\infty}{2}}$, see [Ga1, Theorem 1.5.5]).

We describe explicitly the !- and *-restrictions of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ to the strata $S^\lambda_{\mathrm{Ran}} \subset \overline{S}^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$ (here λ is an element of Λ^{neg}, the negative span of positive simple coroots), see Theorem 2.4.5. These descriptions are given in terms of the combinatorics of the Langlands dual Lie algebra; more precisely, in terms, of the *factorization algebras* attached to $\mathcal{O}(\check{N})$ and $U(\check{\mathfrak{n}}^-)$.

We give an explicit presentation of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ as a colimit (parallel to the *definition* of $\mathrm{IC}^{\frac{\infty}{2}}$ in [Ga1]), see Theorem 2.7.2. This implies the identification $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}|_{\mathrm{Gr}_G} \simeq \mathrm{IC}^{\frac{\infty}{2}}$, where $\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathrm{Gr}_G)$ is the object constructed in [Ga1].

0.2.2. We show that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ identifies canonically (up to a cohomological shift by $[d]$, $d = \dim(\mathrm{Bun}_N)$) with the pullback of $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$ along the map

$$\overline{S}^0 \to \overline{\mathrm{Bun}}_N, \tag{0.1}$$

see Theorem 3.3.3.

In fact, we show that the above pullback functor is t-exact (up to the shift by $[d]$), when restricted to the subcategory $\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$ that consists of objects equivariant with respect to the action of the adelic N, see Corollary 3.5.7.

The proof of this t-exactness property is based on applying Braden's theorem to $\mathrm{Gr}_{G,\mathrm{Ran}}$ and the *Zastava space*.

We note that, unlike [Ga1], the resulting proof of the isomorphism

$$\mathrm{IC}_{\overline{\mathrm{Bun}}_N}|_{\overline{S}^0_{\mathrm{Ran}}}[d] \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}} \tag{0.2}$$

does not use the computation of $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$ from [BFGM], but rather reproves it.

As an aside we prove an important geometric fact that the map (0.1) is *universally homologically contractible* (=the pullback functor along any base change of this map is fully faithful), see Theorem 3.4.4.

0.2.3. We show that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ has an eigen-property with respect to the action of the Hecke functors for G and T, see Theorem 4.5.7.

In the course of the proof of this theorem, we give yet another characterization of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ (which works for $\mathrm{IC}^{\frac{\infty}{2}}$ as well):

We show that the δ-function $\delta_{1_{\mathrm{Gr},\mathrm{Ran}}} \in \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})$ on the unit section $\mathrm{Ran} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ possesses a natural *Drinfeld-Plucker* structure with respect to the Hecke actions of G and T (see Sect. 4.4 for what this means), and that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ can be obtained from $\delta_{1_{\mathrm{Gr},\mathrm{Ran}}}$ by applying the functor from the Drinfeld-Plücker category to the graded Hecke category, left adjoint to the tautological forgetful functor (see Sect. 4.5).

Finally, we establish the compatibility of the isomorphism (0.2) with the Hecke eigen-structures on $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ and $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$, respectively (see Theorem 5.3.5).

0.3. **Organization.**

0.3.1. In Sect. 1 we recall the definition of the Ran space $\mathrm{Ran}(X)$, the Ran version of the affine Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$, and the stratification of the closure $\overline{S}^0_{\mathrm{Ran}}$ of the adelic N-orbit S^0_{Ran} by locally closed substacks S^λ_{Ran}.

We define the semi-infinite category $\mathrm{SI}_{\mathrm{Ran}}$ and study the standard functors that link it to the corresponding categories on the strata.

0.3.2. In Sect. 2 we define the t-structure on $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$ and our main object of study, $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$.

We state Theorem 2.4.5 that describes the *- and !- restrictions of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ to the strata S^λ_{Ran}. The proof of the statement concerning *-restrictions will be given in this same section (it will result from Theorem 2.7.2 mentioned below). This proof of the statement concerning !-restrictions will be given in Sect. 3.

We state and prove Theorem 2.7.2 that gives a presentation of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ as a colimit.

0.3.3. In Sect. 3, we recall the definition of Drinfeld's relative compactification $\overline{\mathrm{Bun}}_N$.

We define the global semi-infinite category $\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$. We prove that the pullback functor along (0.1), viewed as a functor

$$\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \to \mathrm{SI}^{\leq 0}_{\mathrm{Ran}},$$

is t-exact (up to the shift by $[d]$). From here we deduce the identification (0.2), which is Theorem 3.3.3.

We also state Theorem 3.4.4, whose proof is given in Sect. A.

0.3.4. In Sect. 4 we establish the Hecke eigen-property of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$. In the process of doing so we discuss the formalism of *lax central objects* and *Drinfeld-Plücker* structures, and their relation to the Hecke eigen-structures.

In Sect. 5 we prove the compatibility between the eigen-property of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ and that of $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$.

0.4. **Background, conventions and notation.** The notations and conventions in this follow closely those of [Ga1]. Here is a summary:

0.4.1. This paper uses higher category theory. It appears already in the definition of our basic object of study, the *category of sheaves* on the Ran Grassmannian, $\mathrm{Gr}_{G,\mathrm{Ran}}$.

Thus, we will assume that the reader is familiar with the basics of higher categories and higher algebra. The fundamental reference is [Lu1, Lu2], but shorter expositions (or user guides) exist as well, for example, the first chapter of [GR].

0.4.2. Our algebraic geometry happens over an arbitrary algebraically closed ground field k. Our geometric objects are classical (i.e., this paper *does not* need derived algebraic geometry).

We let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ denote the category of (classical) affine schemes of finite type over k.

By a prestack (locally of finite type) we mean an arbitrary functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Groupoids} \tag{0.3}$$

(we de not need to consider higher groupoids).

We let $\mathrm{PreSk}_{\mathrm{lft}}$ denote the category of such prestacks. It contains $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ via the Yoneda embedding. All other types of geometric objects (schemes, algebraic stacks, ind-schemes) are prestacks with some specific *properties* (but *not additional pieces of structure*).

0.4.3. We let G be a connected reductive group over k. We fix a Borel subgroup $B \subset G$ and the opposite Borel $B^- \subset G$. Let $N \subset B$ and $N^- \subset B^-$ denote their respective unipotent radicals.

Set $T = B \cap B^-$; this is a Cartan subgroup of G. We use it to identify the quotients

$$B/N \simeq T \simeq B^-/N^-.$$

We let Λ denote the coweight lattice of G, i.e., the lattice of cocharacters of T. We let $\Lambda^{\mathrm{pos}} \subset \Lambda$ denote the sub-monoid consisting of linear combinations of positive simple roots with non-negative integral coefficients. We let $\Lambda^+ \subset \Lambda$ denote the sub-monoid of *dominant coweights*.

0.4.4. While our geometry happens over a field k, the representation-theoretic categories that we study are *DG categories* over another field, denoted e (assumed algebraically closed and of characteristic 0). For a crash course on DG categories, the reader is referred to [GR, Chapter 1, Sect. 10].

All our DG categories are assumed presentable. When considering functors, we will only consider functors that preserve colimits. We denote the ∞-category of DG categories by DGCat. It carries a symmetric monoidal structure (i.e., one can consider tensor products of DG categories). The unit object is the DG category of complexes of e-vector spaces, denoted Vect.

We will use the notion of t-structure on a DG category. Given a t-structure on $\mathcal{C}$, we will denote by $\mathcal{C}^{\leq 0}$ the corresponding subcategory of cohomologically connective objects, and by $\mathcal{C}^{>0}$ its right orthogonal. We let $\mathcal{C}^{\heartsuit}$ denote the heart $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$.

0.4.5. The source of DG categories will be a *sheaf theory*, which is a functor

$$\mathrm{Shv} : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}, \quad S \mapsto \mathrm{Shv}(S).$$

For a morphism of affine schemes $f : S_0 \to S_1$, the corresponding functor

$$\mathrm{Shv}(S_1) \to \mathrm{Shv}(S_0)$$

is the !-pullback $f^!$.

The main examples of sheaf theories are:

(i) We take $\mathsf{e} = \overline{\mathbb{Q}}_\ell$ and we take $\mathrm{Shv}(S)$ to be the ind-completion of the (small) DG category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

(ii) When $k = \mathbb{C}$ and e arbitrary, we take $\mathrm{Shv}(S)$ to be the ind-completion of the (small) DG category of constructible e-sheaves on $S(\mathbb{C})$ in the analytic topology.

(iii) If k has characteristic 0, we take $\mathsf{e} = k$ and we take $\mathrm{Shv}(S)$ to be the DG category of D-modules on S.

In examples (i) and (ii), the functor $f^!$ always has a left adjoint, denoted $f_!$. In example (iii) this is not the case. However, the partially defined left adjoint $f_!$ is defined on holonomic objects. It is defined on the entire category if f is proper.

0.4.6. *Sheaves on prestacks.* We apply the procedure of right Kan extension along the embedding

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}}$$

to the functor Shv, and thus obtain a functor (denoted by the same symbol)

$$\mathrm{Shv} : (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}} \to \mathrm{DGCat}\,.$$

By definition, for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ we have

$$\mathrm{Shv}(\mathcal{Y}) = \lim_{S\in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\, y:S\to\mathcal{Y}} \mathrm{Shv}(S), \tag{0.4}$$

where the transition functors in the formation of the limit are the !-pullbacks[2].

For a map of prestacks $f : \mathcal{Y}_0 \to \mathcal{Y}_1$ we thus have a well-defined pullback functor

$$f^! : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_0).$$

We denote by $\omega_{\mathcal{Y}}$ the dualizing sheaf on $\mathcal{Y}$, i.e., the pullback of

$$\mathsf{e} \in \mathrm{Vect} \simeq \mathrm{Shv}(\mathrm{pt})$$

along the tautological map $\mathcal{Y} \to \mathrm{pt}$.

0.4.7. This paper is closely related to the geometric Langlands theory, and the geometry of the Langlands dual group $\check{G}$ makes it appearance.

By definition, $\check{G}$ is a reductive group over e and geometric objects constructed out of $\check{G}$ give rise to e-linear DG categories by considering quasi-coherent (resp., ind-coherent) sheaves on them.

The most basic example of such a category is

$$\mathrm{QCoh}(\mathrm{pt}\,/\check{G}) =: \mathrm{Rep}(\check{G}).$$

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1. The Ran version of the semi-infinite category

In this section we extend the definition of the semi-infinite category given in [Ga1] from the affine Grassmannian $\mathrm{Gr}_{G,x}$ corresponding to a fixed point $x \in X$ to the Ran version, denoted $\mathrm{Gr}_{G,\mathrm{Ran}}$.

1.1. **The Ran Grassmannian.**

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[2]Note that even though the index category (i.e., $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}$) is ordinary, the above limit is formed in the ∞-category DGCat. This is how ∞-categories appear in this paper.

1.1.1. We recall that the Ran space of X, denoted $\mathrm{Ran}(X)$, is the prestack that assigns to an affine test scheme Y the set of finite non-empty subsets

$$\mathcal{I} \subset \mathrm{Hom}(Y, X).$$

One can explicitly exhibit $\mathrm{Ran}(X)$ as a colimit (in PreStk) of schemes:

$$\mathrm{Ran}(X) \simeq \underset{I}{\mathrm{colim}}\, X^I,$$

where the colimit is taken over the category opposite to the category $\mathrm{Fin}^{\mathrm{surj}}$ of finite non-empty sets and surjective maps, where to a map $\phi : I \twoheadrightarrow J$ we assign the corresponding diagonal embedding

$$X^J \overset{\Delta_\phi}{\hookrightarrow} X^I.$$

1.1.2. We can consider the Ran version of the affine Grassmannian, denoted $\mathrm{Gr}_{G,\mathrm{Ran}}$, defined as follows.

It assigns to an affine test scheme Y, the set of triples $(\mathcal{I}, \mathcal{P}_G, \alpha)$, where $\mathcal{I}$ is an Y-point of $\mathrm{Ran}(X)$, $\mathcal{P}_G$ is a G-bundle on $Y \times X$, and α is a trivialization of $\mathcal{P}_G$ on the open subset of $Y \times X$ equal to the complement of the union $\Gamma_\mathcal{I}$ of the graphs of the maps $Y \to X$ that comprise $\mathcal{I}$.

We also consider the Ran versions of the loop and arc groups (ind)-schemes, denoted

$$\mathfrak{L}^+(G)_{\mathrm{Ran}} \subset \mathfrak{L}(G)_{\mathrm{Ran}}.$$

The Ran Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$ identifies with the étale (equivalently, fppf) sheafification of the prestack quotient $\mathfrak{L}(G)_{\mathrm{Ran}}/\mathfrak{L}^+(G)_{\mathrm{Ran}}$.

1.1.3. For a fixed finite non-empty set I, we denote

$$\mathrm{Gr}_{G,I} := X^I \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}, \quad \mathfrak{L}(G)_I := X^I \underset{\mathrm{Ran}(X)}{\times} \mathfrak{L}(G)_{\mathrm{Ran}}, \quad \mathfrak{L}^+(G)_I := X^I \underset{\mathrm{Ran}(X)}{\times} \mathfrak{L}^+(G)_{\mathrm{Ran}}.$$

For a map of finite sets $\phi : I \twoheadrightarrow J$, we will denote by Δ_ϕ the corresponding map $\mathrm{Gr}_{G,J} \to \mathrm{Gr}_{G,I}$, so that we have the Cartesian square:

$$\begin{array}{ccc} \mathrm{Gr}_{G,J} & \xrightarrow{\Delta_\phi} & \mathrm{Gr}_{G,I} \\ \downarrow & & \downarrow \\ X^J & \xrightarrow{\Delta_\phi} & X^I. \end{array}$$

1.1.4. We introduce also the following closed (resp., locally closed) subfunctors

$$S^0_{\mathrm{Ran}} \subset \overline{S}^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}.$$

Namely, a Y-point $(\mathcal{I}, \mathcal{P}_G, \alpha)$ belongs to $\overline{S}^0_{\mathrm{Ran}}$ if for every dominant weight $\check{\lambda}$, the composite meromorphic map of vector bundles on $Y \times X$

$$\mathcal{O} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}^0_G} \xrightarrow{\alpha} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \tag{1.1}$$

is regular. In the above formula the notations are as follows:

- $\mathcal{V}^{\check{\lambda}}$ denotes the Weyl module over G with highest weight $\check{\lambda}$;
- $\mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$ (resp., $\mathcal{V}^{\check{\lambda}}_{\mathcal{P}^0_G}$) denotes the vector bundles associated with $\mathcal{V}^{\check{\lambda}}$ and the G-bundle $\mathcal{P}_G$ (resp., the trivial G-bundle $\mathcal{P}^0_G$);
- $\mathcal{O} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}^0_G}$ is the map coming from the highest weight vector in $\mathcal{V}^{\check{\lambda}}$.

A point as above belongs to S^0_{Ran} if the above composite map is a bundle map (i.e., has no zeroes).

1.2. **The semi-infinite category.**

1.2.1. Since $\mathrm{Gr}_{G,\mathrm{Ran}}$ a prestack locally of finite type, we have a well-defined category

$$\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

We have

$$\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}}) := \lim_{I} \mathrm{Shv}(\mathrm{Gr}_{G,I}),$$

where the limit is formed using the !-pullback functors.

1.2.2. Although the group ind-scheme $\mathfrak{L}(N)_{\mathrm{Ran}}$ is *not* locally of finite type, we have a well-defined full subcategory

$$\mathrm{SI}_{\mathrm{Ran}} := \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})^{\mathfrak{L}(N)_{\mathrm{Ran}}} \subset \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

Namely, for every fixed finite non-empty set I, we consider the full subcategory

$$\mathrm{SI}_I := \mathrm{Shv}(\mathrm{Gr}_{G,I})^{\mathfrak{L}(N)_I} \subset \mathrm{Shv}(\mathrm{Gr}_{G,I}),$$

defined as in [Ga1, Sect. 1.2].

We say that the object of $\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})$ belongs to $\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})^{\mathfrak{L}(N)_{\mathrm{Ran}}}$ if its restriction to any $\mathrm{Gr}_{G,I}$ yields an object of $\mathrm{Shv}(\mathrm{Gr}_{G,I})^{\mathfrak{L}(N)_I}$. By construction, we have an equivalence

$$\mathrm{SI}_{\mathrm{Ran}} := \lim_{I} \mathrm{SI}_I .$$

1.2.3. Let $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{SI}_{\mathrm{Ran}}$ be the full subcategory consisting of objects supported on $\overline{S}^0_{\mathrm{Ran}}$. I.e.,

$$\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} = \mathrm{SI}_{\mathrm{Ran}} \cap \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}}),$$

and

$$\mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}}) \simeq \lim_{I} \mathrm{Shv}(\overline{S}^0_I).$$

1.2.4. We are going to introduce a t-structure on $\overline{S}^0_{\mathrm{Ran}}$ and describe a certain object in its heart, called the *semi-infinite intersection cohomology sheaf.*

In order to do so we will first need to describe the corresponding stratification of $\overline{S}^0_{\mathrm{Ran}}$, whose open stratum will be S^0_{Ran}.

1.3. **Stratification.**

1.3.1. For $\lambda \in \Lambda^{\mathrm{neg}}$, let X^λ denote the corresponding partially symmetrized power of X. I.e., if

$$\lambda = \sum_i (-n_i) \cdot \alpha_i, \quad n_i \geq 0$$

where α_i are the simple positive coroots, then

$$X^\lambda = \prod_i X^{(n_i)}.$$

In other words, S-points of X^λ are effective Λ^{neg}-valued divisors on X.

For $\lambda = 0$ we by definition have $X^0 = \mathrm{pt}$.

1.3.2. Let
$$(\mathrm{Ran}(X) \times X^\lambda)^\supset \subset \mathrm{Ran}(X) \times X^\lambda$$
be the closed subfunctor, whose S-points are pairs $(\mathcal{I}, D)$ for which the support of the divisor D is *set-theoretically* supported on the union of the graphs of the maps $S \to X$ that comprise $\mathcal{I}$.

In other words,
$$(\mathrm{Ran}(X) \times X^\lambda)^\supset = \underset{I}{\mathrm{colim}}\, (X^I \times X^\lambda)^\supset,$$
where
$$(X^I \times X^\lambda)^\supset \subset X^I \times X^\lambda$$
is the correspondence incidence subvariety.

For future use we note:

Lemma 1.3.3. *The map*
$$\mathrm{pr}^\lambda_{\mathrm{Ran}} : (\mathrm{Ran}(X) \times X^\lambda)^\supset \to X^\lambda$$
is universally homologically contractible.

The proof will be given in Sect. A.2.8 (we refer the reader to [Ga2, Sect. 2.5] for the notion of universal local contractibility).

Corollary 1.3.4. *The pullback functor*
$$(\mathrm{pr}^\lambda_{\mathrm{Ran}})^! : \mathrm{Shv}(X^\lambda) \to \mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset)$$
is fully faithful.

1.3.5. We let
$$\overline{S}^\lambda_{\mathrm{Ran}} \subset (\mathrm{Ran}(X) \times X^\lambda)^\supset \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}$$
be the closed subfunctor defined by the following condition:

An S-point $(\mathcal{I}, D, \mathcal{P}_G, \alpha)$ of the fiber product $(\mathrm{Ran}(X) \times X^\lambda)^\supset \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}$ belongs to $\overline{S}^\lambda_{\mathrm{Ran}}$ if for every dominant weight $\check{\lambda}$ the map (1.1) extends to a regular map

$$\mathcal{O}(\check{\lambda}(D)) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}. \tag{1.2}$$

We denote by $\bar{\mathbf{i}}^\lambda$ the composite map
$$\overline{S}^\lambda_{\mathrm{Ran}} \to (\mathrm{Ran}(X) \times X^\lambda)^\supset \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}.$$

This map is proper, and its image is contained in $\overline{S}^0_{\mathrm{Ran}}$.

Note that for $\lambda = 0$, the map $\bar{\mathbf{i}}^0$ is the identity map on $\overline{S}^0_{\mathrm{Ran}}$.

Let $\overline{p}^\lambda_{\mathrm{Ran}}$ denote the projection
$$\overline{S}^\lambda_{\mathrm{Ran}} \to (\mathrm{Ran}(X) \times X^\lambda)^\supset.$$

1.3.6. We define the open subfunctor
$$S^\lambda_{\mathrm{Ran}} \subset \overline{S}^\lambda_{\mathrm{Ran}}$$
to correspond to those quadruples $(\mathcal{I}, D, \mathcal{P}_G, \alpha)$ for which the map (1.2) is a bundle map.

The projection
$$p^\lambda_{\mathrm{Ran}} := \overline{p}^\lambda_{\mathrm{Ran}}|_{S^\lambda_{\mathrm{Ran}}} : S^\lambda_{\mathrm{Ran}} \to (\mathrm{Ran}(X) \times X^\lambda)^\supset \tag{1.3}$$
admits a canonically defined section
$$s^\lambda_{\mathrm{Ran}} : (\mathrm{Ran}(X) \times X^\lambda)^\supset \to S^\lambda_{\mathrm{Ran}}. \tag{1.4}$$

Namely, it sends $(\mathcal{I}, D)$ to the quadruple $(\mathcal{I}, D, \mathcal{P}_G, \alpha)$, where $\mathcal{P}_G$ is the G-bundle induced from the T-bundle $\mathcal{P}_T := \mathcal{P}^0_T(D)$, and α is the trivialization of $\mathcal{P}_G$ induced by the tautological trivialization of $\mathcal{P}_T$ away from the support of D.

1.3.7. We let

$$\mathbf{j}^\lambda : S^\lambda_{\mathrm{Ran}} \hookrightarrow \overline{S}^\lambda_{\mathrm{Ran}}, \quad \mathbf{i}^\lambda = \bar{\mathbf{i}}^\lambda \circ \mathbf{j}^\lambda : S^\lambda_{\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$$

denote the resulting maps.

For a fixed finite non-empty set I, we obtain the corresponding subfunctors

$$\overline{S}^\lambda_I \subset (X^I \times X^\lambda)^\supset \underset{X^I}{\times} \mathrm{Gr}_{G,I}$$

and

$$S^\lambda_I \subset \overline{S}^\lambda_I,$$

and maps, denoted by the same symbols $\mathbf{j}^\lambda$, $\bar{\mathbf{i}}^\lambda$, $\mathbf{i}^\lambda$. Let p^λ_I (resp., $\overline{p}^\lambda_I$) denote the corresponding map from S^λ_I (resp., $\overline{S}^\lambda_I$) to $(X^I \times X^\lambda)^\supset$.

1.3.8. The following results easily from the definitions:

Lemma 1.3.9. *The maps*

$$\mathbf{i}^\lambda : S^\lambda_{\mathrm{Ran}} \to \overline{S}^0_{\mathrm{Ran}} \text{ and } S^\lambda_I \to \overline{S}^0_I$$

are locally closed embeddings. Every field-valued point of $\overline{S}^0_{\mathrm{Ran}}$ (resp., $\overline{S}^0_I$) belongs to the image of exactly one such map.

1.4. **Structure of the semi-infinite category.**

1.4.1. As in the case of $\mathrm{Gr}_{G,\mathrm{Ran}}$, we can consider the full subcategories

$$\mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^\lambda_{\mathrm{Ran}}) \text{ and } \mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \subset \mathrm{Shv}(S^\lambda_{\mathrm{Ran}}),$$

obtained by imposing the equivariance condition with respect to the corresponding group ind-scheme.

The maps $\mathbf{j}^\lambda$, $\bar{\mathbf{i}}^\lambda$ and $\mathbf{i}^\lambda$ induce functors

$$(\bar{\mathbf{i}}^\lambda)_! := (\bar{\mathbf{i}}^\lambda)_* : \mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}} \to \mathrm{SI}^{\leq 0}_{\mathrm{Ran}},$$
$$(\bar{\mathbf{i}}^\lambda)^! : \mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \to \mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}};$$
$$(\mathbf{j}^\lambda)^* := (\mathbf{j}^\lambda)^! : \mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}} \to \mathrm{SI}^{=\lambda}_{\mathrm{Ran}};$$
$$(\mathbf{j}^\lambda)_* : \mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \to \mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}};$$
$$(\mathbf{i}^\lambda)^! \simeq (\mathbf{j}^\lambda)^! \circ (\bar{\mathbf{i}}^\lambda)^! : \mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \to \mathrm{SI}^{=\lambda}_{\mathrm{Ran}};$$
$$(\mathbf{i}^\lambda)_* \simeq (\bar{\mathbf{i}}^\lambda)_* \circ (\mathbf{j}^\lambda)_* : \mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \to \mathrm{SI}^{\leq 0}_{\mathrm{Ran}}.$$

The above discussion applies to Ran replaced by X^I for a fixed finite non-empty set I.

1.4.2. In Sect. 1.6 we will prove:

Proposition 1.4.3.
(a) *For a fixed finite set I, the partially defined left adjoint $(\mathbf{i}^\lambda)^*$ of*

$$(\mathbf{i}^\lambda)_* : \mathrm{Shv}(S^\lambda_I) \to \mathrm{Shv}(\overline{S}^0_I)$$

is defined on the full subcategory $\mathrm{SI}^{\leq 0}_I \subset \mathrm{Shv}(\overline{S}^0_I)$, and takes values in $\mathrm{SI}^{=\lambda}_I \subset \mathrm{Shv}(S^\lambda_I)$.
(b) *For $\mathcal{F} \in \mathrm{Shv}(\overline{S}^0_I)$ and $\mathcal{F}' \in \mathrm{Shv}(X^I)$, the map*[3]

$$(\mathbf{i}^\lambda)^*((\overline{p}^0_I)^!(\mathcal{F}') \overset{!}{\otimes} \mathcal{F}) \to (p^\lambda_I)^!(\mathcal{F}'|_{(X^I \times X^\lambda)^\supset}) \overset{!}{\otimes} (\mathbf{i}^\lambda)^*(\mathcal{F})$$

is an isomorphism, provided that $\mathcal{F} \in \mathrm{SI}^{\leq 0}_I$.
(c) *For a map of finite sets $\phi : I \twoheadrightarrow J$, the natural transformation*

$$(\mathbf{i}^\lambda)^* \circ (\Delta_\phi)^! \to (\Delta_\phi)^! \circ (\mathbf{i}^\lambda)^*, \quad \mathrm{Shv}(\overline{S}^0_I) \rightrightarrows \mathrm{Shv}(S^\lambda_J)$$

is an isomorphism, when evaluated on objects of $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$.

From here, by a formal argument, we obtain:

[3] In the formula below $-|_{(X^I \times X^\lambda)^\supset}$ denotes the !-restriction along the projection $(X^I \times X^\lambda)^\supset \to X^I$.

Corollary 1.4.4.
(a) *For a fixed finite set I, the partially defined left adjoint $(\mathbf{i}^\lambda)_!$ of*

$$(\mathbf{i}^\lambda)^! : \mathrm{Shv}(\overline{S}^0_I) \to \mathrm{Shv}(S^\lambda_I)$$

is defined on the full subcategory $\mathrm{SI}^{=\lambda}_I \subset \mathrm{Shv}(S^\lambda_I)$, and takes values in $\mathrm{SI}^{\leq 0}_I \subset \mathrm{Shv}(\overline{S}^0_I)$.
(b) *For $\mathcal{F} \in \mathrm{Shv}(S^\lambda_I)$ and $\mathcal{F}' \in \mathrm{Shv}(X^I)$, the map*

$$(\mathbf{i}^\lambda)_!((p^\lambda_I)^!(\mathcal{F}'|_{(X^I \times X^\lambda)^\supset}) \overset{!}{\otimes} \mathcal{F}) \to (\overline{p}^0_I)^!(\mathcal{F}') \overset{!}{\otimes} (\mathbf{i}^\lambda)_!(\mathcal{F})$$

is an isomorphism.
(c) *For a map of finite sets $\phi : I \twoheadrightarrow J$, the natural transformation*

$$(\mathbf{i}^\lambda)_! \circ (\Delta_\phi)^! \to (\Delta_\phi)^! \circ (\mathbf{i}^\lambda)_!, \quad \mathrm{Shv}(S^\lambda_J) \rightrightarrows \mathrm{Shv}(\overline{S}^0_I)$$

is an isomorphism, when evaluated on objects of $\mathrm{SI}^{=\lambda}_I \subset \mathrm{Shv}(S^\lambda_I)$.

As a corollary we obtain:

Corollary 1.4.5.
(a) *The partially defined left adjoint $(\mathbf{i}^\lambda)^*$ of*

$$(\mathbf{i}^\lambda)_* : \mathrm{Shv}(S^\lambda_{\mathrm{Ran}}) \to \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$$

is defined on the full subcategory $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$, and takes values in $\mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \subset \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$.
(a') *The partially defined left adjoint $(\mathbf{i}^\lambda)_!$ of*

$$(\mathbf{i}^\lambda)^! : \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}}) \to \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$$

is defined on the full subcategory $\mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \subset \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$, and takes values in $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$.
(b) *For $\mathcal{F} \in \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$ and $\mathcal{F}' \in \mathrm{Shv}(\mathrm{Ran}(X))$, the map*

$$(\mathbf{i}^\lambda)^*((\overline{p}^0_{\mathrm{Ran}})^!(\mathcal{F}') \overset{!}{\otimes} \mathcal{F}) \to (p^\lambda_{\mathrm{Ran}})^!(\mathcal{F}'|_{(\mathrm{Ran}(X) \times X^\lambda)^\supset}) \overset{!}{\otimes} (\mathbf{i}^\lambda)^*(\mathcal{F})$$

is an isomorphism, provided that $\mathcal{F} \in \mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$.
(b') *For $\mathcal{F} \in \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$ and $\mathcal{F}' \in \mathrm{Shv}(\mathrm{Ran}(X))$, the map*

$$(\mathbf{i}^\lambda)_!((p^\lambda_{\mathrm{Ran}})^!(\mathcal{F}'|_{(\mathrm{Ran}(X) \times X^\lambda)^\supset}) \overset{!}{\otimes} \mathcal{F}) \to (\overline{p}^0_{\mathrm{Ran}})^!(\mathcal{F}') \overset{!}{\otimes} (\mathbf{i}^\lambda)_!(\mathcal{F})$$

is an isomorphism.

Remark 1.4.6. A slight variation of the proof of Proposition 1.4.3 shows that the assertions remain valid for $\mathbf{i}^\lambda$ replaced by $\bar{\mathbf{i}}^\lambda$. Similarly, the assertion of Corollary 1.4.4 remains valid for $\mathbf{i}_\lambda$ replaced by $\mathbf{j}_\lambda$, and the same is true for their Ran variants.

1.4.7. We have the following explicit description of the category on each stratum separately:

Lemma 1.4.8. *Pullback along the map p^λ_{Ran} of* (1.3) *defines an equivalence*

$$\mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset) \to \mathrm{SI}^{=\lambda}_{\mathrm{Ran}}.$$

The inverse equivalence is given by restriction to the section s^λ_{Ran} of (1.4), *and similarly for* $\mathrm{Ran}(X)$ *replaced by X^I for an individual I.*

Proof. Follows from the fact that the action of the group ind-scheme

$$(\mathrm{Ran}(X) \times X^\lambda)^\supset \underset{\mathrm{Ran}(X)}{\times} \mathfrak{L}(G)_{\mathrm{Ran}}$$

on S^λ_{Ran} transitive along the fibers of the map (1.3), with the stabilizer of the section s^λ_{Ran} being a pro-unipotent group-scheme over $(\mathrm{Ran}(X) \times X^\lambda)^\supset$. □

1.5. **An aside: the ULA property.** Consider the object

$$(\mathbf{j}^0)_!(\omega_{S_I^0}) \in \mathrm{Shv}(\overline{S}_I^0).$$

We will now formulate a certain strong acyclicity property that it enjoys with respect to the projection

$$\overline{p}_I^0 : \overline{S}_I^0 \to X^I.$$

1.5.1. Let Y be a scheme, and let $\mathcal{C}$ be a DG category equipped with an action of the $\mathrm{Shv}(Y)$, viewed as a monoidal category with respect to $\overset{!}{\otimes}$.

We shall say that an object $c \in \mathcal{C}$ is ULA with respect to Y if for any compact $\mathcal{F} \in \mathrm{Shv}(Y)^c$, and any $c' \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F} \overset{!}{\otimes} c, c') \to \mathrm{Hom}(\mathbb{D}(\mathcal{F}) \overset{!}{\otimes} \mathcal{F} \overset{!}{\otimes} c, \mathbb{D}(\mathcal{F}) \overset{!}{\otimes} c') \to \mathrm{Hom}(\mathsf{e}_Y \overset{!}{\otimes} c, \mathbb{D}(\mathcal{F}) \overset{!}{\otimes} c')$$

is an isomorphism.

In the above formula, $\mathbb{D}(-)$ denotes the Verdier duality anti-equivalence of $\mathrm{Shv}(Y)^c$, and e_Y is the "constant sheaf" on Y, i.e., $\mathsf{e}_Y := \mathbb{D}(\omega_Y)$. Note that when Y is smooth of dimension d, we have $\mathsf{e}_Y \simeq \omega_Y[-2d]$.

1.5.2. We regard $\mathrm{Shv}(\overline{S}_I^0)$ as tensored over $\mathrm{Shv}(X^I)$ via

$$\mathcal{F} \in \mathrm{Shv}(X^I), \mathcal{F}' \in \mathrm{Shv}(\overline{S}_I^0) \mapsto (\overline{p}_I^0)^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}'.$$

We claim:

Proposition 1.5.3. *The object* $(\mathbf{j}^0)_!(\omega_{S_I^0}) \in \mathrm{Shv}(\overline{S}_I^0)$ *is ULA with respect to* X^I*.*

Proof. For $\mathcal{F} \in \mathrm{Shv}(X^I)$ and $\mathcal{F}' \in \mathrm{Shv}(\overline{S}_I^0)$, we have a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}((p_I^0)^!(\mathcal{F}), (\mathbf{j}^0)^!(\mathcal{F}')) & \longrightarrow & \mathrm{Hom}((p_I^0)^!(\mathsf{e}_{X^I}), (p_I^0)^!(\mathbb{D}(\mathcal{F})) \overset{!}{\otimes} (\mathbf{j}^0)^!(\mathcal{F}')) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Hom}((\mathbf{j}^0)_! \circ (p_I^0)^!(\mathcal{F}), \mathcal{F}') & & \mathrm{Hom}((\mathbf{j}^0)_! \circ (p_I^0)^!(\mathsf{e}_{X^I}), (\overline{p}_I^0)^!(\mathbb{D}(\mathcal{F})) \overset{!}{\otimes} \mathcal{F}') \\
\uparrow & & \uparrow \\
\mathrm{Hom}((\overline{p}_I^0)^!(\mathcal{F}) \overset{!}{\otimes} (\mathbf{j}^0)_!(\omega_{S_I^0}), \mathcal{F}') & \longrightarrow & \mathrm{Hom}((\overline{p}_I^0)^!(\mathsf{e}_{X^I}) \overset{!}{\otimes} (\mathbf{j}^0)_!(\omega_{S_I^0}), (\overline{p}_I^0)^!(\mathbb{D}(\mathcal{F})) \overset{!}{\otimes} \mathcal{F}').
\end{array}$$

In this diagram the lower vertical arrows are isomorphisms by Corollary 1.4.4(b). The top horizontal arrow is an isomorphism because S_I^0 can be exhibited as a union of closed subschemes, each being smooth over X^I.

Hence, the bottom horozontal arrow is an isomorphism, as required.

□

1.6. **An application of Braden's theorem.** In this subsection we will prove Proposition 1.4.3.

1.6.1. Let

$$S_I^{-,\lambda} \overset{\mathbf{j}^{-,\lambda}}{\hookrightarrow} \overline{S}_I^{-,\lambda} \overset{\bar{\mathbf{i}}^{-,\lambda}}{\to} \mathrm{Gr}_{G,I}$$

be the objects defined in the same way as their counterparts

$$S_I^\lambda \overset{\mathbf{j}^\lambda}{\hookrightarrow} \overline{S}_I^\lambda \overset{\bar{\mathbf{i}}^\lambda}{\to} \mathrm{Gr}_{G,I},$$

but where we replace N by N^-.

Choose a regular dominant coweight $\mathbb{G}_m \to T$. It gives rise to an action of $\mathbb{G}_m$ on $\overline{S}_I^0$ along the fibers of the projection $\overline{p}_I^0$. We have:

Lemma 1.6.2. *The attracting (resp., repelling) locus of the above $\mathbb{G}_m$ action identifies with*

$$\underset{\lambda\in\Lambda^{\rm neg}}{\sqcup} S^\lambda_I \text{ and } \underset{\lambda\in\Lambda^{\rm neg}}{\sqcup} S^{-,\lambda}_I,$$

respectively. The fixed point locus identifies with

$$\underset{\lambda\in\Lambda^{\rm neg}}{\sqcup} s^\lambda_I(X^I).$$

1.6.3. Let us prove point (a) of Proposition 1.4.3. A priori, the functor $(\mathbf{i}^\lambda)^*$ applied to objects from $\mathrm{SI}^{\leq 0}_I$ maps to the pro-completion of $\mathrm{SI}^{=\lambda}_I$, viewed as a full subcategory in the pro-completion of $\mathrm{Shv}(S^\lambda_I)$. Now, using Lemma 1.4.8, it suffices to show that for $\mathcal{F}\in\mathrm{SI}(G)^{\leq 0}_I$, the object

$$(s^\lambda_I)^!\circ(\mathbf{i}^\lambda)^*\in\mathrm{Pro}(\mathrm{Shv}(X^I))$$

actually belongs to $\mathrm{Shv}(X^I)$.

It is easy to see that every object in $\mathrm{Shv}(\overline{S}^0_I)$ is $\mathbb{G}_m$-monodromic. Using Braden's theorem (see [DrGa]), combined with Lemma 1.6.2, we obtain that $(s^\lambda_I)^!\circ(\mathbf{i}^\lambda)^*$ indeed belongs to $\mathrm{Shv}(X^I)$, as required.

1.6.4. Point (b) of Proposition 1.4.3 also follows from Braden's theorem: we identify

$$(s^\lambda_I)^!\circ(\mathbf{i}^\lambda)^*\simeq(p^{-,\lambda})_*\circ(\mathbf{i}^{-,\lambda})^!,$$

and the assertion follows by base change.

Point (c) is a formal corollary of point (b).

2. The t-structure and the semi-infinite IC sheaf

In this section we define a t-structure on $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$, and define the main object of study in this paper– the Ran version of the semi-infinite intersection cohomology sheaf, denoted $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$.

We will also give a presentation of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ as a colimit, and describe explicitly its *- and !-restrictions to the strata S^λ_{Ran}.

2.1. The t-structure on the semi-infinite category.

2.1.1. We introduce a t-structure on the category $\mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset)$ as follows.

Note that the projection

$$\mathrm{pr}^\lambda_{\mathrm{Ran}}:(\mathrm{Ran}(X)\times X^\lambda)^\supset\to X^\lambda \tag{2.1}$$

is pseudo-proper (see [Ga2, Sect. 1.5] for what this means). In particular, the functor

$$(\mathrm{pr}^\lambda_{\mathrm{Ran}})_!:\mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset\to\mathrm{Shv}(X^\lambda),$$

left adjoint to $(\mathrm{pr}^\lambda_{\mathrm{Ran}})^!$, is well-defined.

We declare an object $\mathcal{F}\in\mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset)$ to be *connective* if

$$\mathcal{H}om(\mathcal{F},(\mathrm{pr}^\lambda_{\mathrm{Ran}})^!(\mathcal{F}'))=0$$

for all $\mathcal{F}'\in\mathrm{Shv}(X^\lambda)$ that are *strictly coconnective* (in the perverse t-structure).

By construction, the functor

$$(\mathrm{pr}^\lambda_{\mathrm{Ran}})^!:\mathrm{Shv}(X^\lambda)\to\mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset)$$

is left t-exact.

However, we claim:

Proposition 2.1.2. *The functor* $(\mathrm{pr}^\lambda_{\mathrm{Ran}})^!:\mathrm{Shv}(X^\lambda)\to\mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset)$ *is t-exact.*

Proof. We need to show that the composition

$$(\mathrm{pr}^\lambda_{\mathrm{Ran}})_! \circ (\mathrm{pr}^\lambda_{\mathrm{Ran}})^!$$

is right t-exact. However, we claim that the above composition is actually isomorphic to the identity functor.

Indeed, the latter assertion is equivalent to the fact that the functor $(\mathrm{pr}^\lambda_{\mathrm{Ran}})^!$ is fully faithful, which is the assertion of Corollary 1.3.4.

□

2.1.3. We define a t-structure on $\mathrm{SI}^{=\lambda}_{\mathrm{Ran}}$ as follows. We declare an object $\mathcal{F} \in \mathrm{SI}^{=\lambda}_{\mathrm{Ran}}$ to be connective/coconnective if

$$(s^\lambda_{\mathrm{Ran}})^!(\mathcal{F})[\langle\lambda, 2\check{\rho}\rangle] \in \mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset)$$

is connective/coconnective.

In other words, this t-structure is transferred from $\mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset)$ via the equivalences

$$(s^\lambda_{\mathrm{Ran}})^! : \mathrm{SI}^{=\lambda}_{\mathrm{Ran}} \to \mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset) : (p^\lambda_{\mathrm{Ran}})^!$$

of Lemma 1.4.8, up to the cohomological shift $[\langle\lambda, 2\check{\rho}\rangle]$.

2.1.4. We define a t-structure on $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$ by declaring that an object $\mathcal{F}$ is coconnective if

$$(\mathbf{i}^\lambda)^!(\mathcal{F}) \in \mathrm{SI}^{=\lambda}_{\mathrm{Ran}}$$

is coconnective in the above t-structure.

By construction, the subcategory of connective objects in $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$ is generated under colimits by objects of the form

$$(\mathbf{i}^\lambda)_! \circ (p^\lambda_{\mathrm{Ran}})^!(\mathcal{F})[-\langle\lambda, 2\check{\rho}\rangle], \quad \lambda \in \Lambda^{\mathrm{neg}} \tag{2.2}$$

where $\mathcal{F}$ is a connective object of $\mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset)$.

We claim:

Lemma 2.1.5. *An object $\mathcal{F} \in \mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$ is connective if and only if $(\mathbf{i}^\lambda)^*(\mathcal{F}) \in \mathrm{SI}^{\leq\lambda}_{\mathrm{Ran}}$ is connective for every $\lambda \in \Lambda^{\mathrm{neg}}$.*

Proof. It is clear that for objects of the form (2.2), their *-pullback to any S^λ_{Ran} is connective. Hence, the same is true for any connective object of $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$.

Vice versa, let $0 \neq \mathcal{F}$ be a strictly coconnective object of $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$. We need to show that if all $(\mathbf{i}^\lambda)^*(\mathcal{F})$ are connective, then $\mathcal{F} = 0$. Let λ be the largest element such that $(\mathbf{i}^\lambda)^!(\mathcal{F}) \neq 0$. On the one hand, since $\mathcal{F}$ is strictly coconnective, $(\mathbf{i}^\lambda)^!(\mathcal{F})$ is strictly coconnective. On the other hand, by the maximality of λ, we have

$$(\mathbf{i}^\lambda)^!(\mathcal{F}) \simeq (\mathbf{i}^\lambda)^*(\mathcal{F}),$$

and the assertion follows.

□

2.1.6. By construction, the object $(\mathbf{i}^\lambda)_!(\omega_{S^\lambda_{\mathrm{Ran}}})[-\langle\lambda, 2\check{\rho}\rangle]$ (resp., $(\mathbf{i}^\lambda)_*(\omega_{S^\lambda_{\mathrm{Ran}}})[-\langle\lambda, 2\check{\rho}\rangle]$) of $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}}$ is connective (resp., coconnective).

However, in Sect. 3.5.10 we will prove:

Proposition 2.1.7. *The objects*

$$(\mathbf{i}^\lambda)_!(\omega_{S^\lambda_{\mathrm{Ran}}})[-\langle\lambda, 2\check{\rho}\rangle] \text{ and } (\mathbf{i}^\lambda)_*(\omega_{S^\lambda_{\mathrm{Ran}}})[-\langle\lambda, 2\check{\rho}\rangle]$$

belong to $(\mathrm{SI}^{\leq 0}_{\mathrm{Ran}})^\heartsuit$.

2.2. Definition of the semi-infinite IC sheaf.

2.2.1. Consider the canonical map

$$(\mathbf{i}^\lambda)_!(\omega_{S^\lambda_{\text{Ran}}})[-\langle\lambda,2\check{\rho}\rangle] \to (\mathbf{i}^\lambda)_*(\omega_{S^\lambda_{\text{Ran}}})[-\langle\lambda,2\check{\rho}\rangle].$$

According to Proposition 2.1.7 both sides belong to $(\text{SI}^{\leq 0}_{\text{Ran}})^\heartsuit$. We let

$$\text{IC}^{\frac{\infty}{2}}_{\text{Ran}} \in (\text{SI}^{\leq 0}_{\text{Ran}})^\heartsuit$$

denote the image of this map.

The above object is the main character of this paper.

2.2.2. Our goal is to describe $\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$ as explicitly as possible. Specifically, we will do the following:

- We will describe the !- and *- restrictions of $\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$ to the strata S^λ_{Ran};
- We will exhibit the values of $\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$ in $\text{Shv}(\text{Gr}_{G,I})$ explicitly as colimits;
- We will relate $\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$ to the intersection cohomology sheaf of Drinfeld's compactification $\overline{\text{Bun}}_N$.

2.3. **Digression: from commutative algebras to factorization algebras.** Let A be a commutative algebra, graded by Λ^{neg} with $A(0) \simeq \mathsf{e}$. Then to A we can attach an object

$$\text{Fact}^{\text{alg}}(A)_{X^\lambda} \in \text{Shv}(X^\lambda),$$

characterized by the property that its !-fiber at a divisor

$$\underset{k}{\Sigma}\, \lambda_k \cdot x_k \in X^\lambda, \quad 0 \neq \lambda_k \in \Lambda^{\text{neg}}, \quad \underset{k}{\Sigma}\, \lambda_k = \lambda, \quad k' \neq k'' \Rightarrow x_{k'} \neq x_{k''}$$

equals $\underset{k}{\bigotimes} A(\lambda_k)$.

In the present subsection we recall this construction.

2.3.1. Consider the category TwArr_λ whose objects are diagrams

$$\Lambda^{\text{neg}} - 0 \overset{\underline{\lambda}}{\longleftarrow} I \overset{\phi}{\twoheadrightarrow} J, \quad \underset{i\in I}{\Sigma}\, \underline{\lambda}(i) = \lambda, \tag{2.3}$$

where I and J are finite non-empty sets. A morphism between two such objects is a diagram

$$\begin{array}{ccccc}
\Lambda^{\text{neg}} - 0 & \overset{\underline{\lambda}_1}{\longleftarrow} & J_1 & \overset{\phi_1}{\longrightarrow} & K_1 \\
{\scriptstyle \text{id}}\downarrow & & {\scriptstyle \psi_J}\downarrow & & \uparrow{\scriptstyle \psi_K} \\
\Lambda^{\text{neg}} - 0 & \overset{\underline{\lambda}_2}{\longleftarrow} & J_2 & \overset{\phi_2}{\longrightarrow} & K_2,
\end{array}$$

where:

- The right square commutes;
- The maps ψ_J and ψ_K are surjective;
- $\underline{\lambda}_2(j_2) = \underset{j_1\in\psi_J^{-1}(j_2)}{\Sigma} \underline{\lambda}_1(j_1)$.

2.3.2. The algebra A defines a functor

$$\text{TwArr}(A) : \text{TwArr}_\lambda \to \text{Shv}(X^\lambda),$$

constructed as follows.

For an object (2.3), let $\Delta_{K,\lambda}$ denote the map $X^K \to X^\lambda$ that sends a point $\{x_k, k \in K\} \in X^K$ to the divisor

$$\underset{k\in K}{\Sigma} \Big(\underset{j\in\phi^{-1}(k)}{\Sigma} \underline{\lambda}(j) \Big) \cdot x_k \in X^\lambda.$$

Then the value of $\text{TwArr}(A)$ on (2.3) is

$$(\Delta_{K,\lambda})_*(\omega_{X^K}) \bigotimes \left(\underset{j\in J}{\otimes} A(\lambda_j) \right),$$

where $\lambda_j = \underline{\lambda}(j)$.

The structure of functor on $\mathrm{TwArr}(A)$ is provided by the commutative algebra structure on A.

2.3.3. We set

$$\mathrm{Fact}^{\mathrm{alg}}(A)_{X^\lambda} := \underset{\mathrm{TwArr}_\lambda}{\mathrm{colim}}\ \mathrm{TwArr}(A) \in \mathrm{Shv}(X^\lambda).$$

2.3.4. *An example.* Let V be a Λ^{neg}-graded vector space with $V(0)=0$. Let us take $A=\mathrm{Sym}(V)$. In this case $\mathrm{Fact}^{\mathrm{alg}}(A)_{X^\lambda}$, can be explicitly described as follows:

It is the direct sum over all ways to write λ as a sum

$$\lambda = \sum_k n_k\cdot\lambda_k,\quad n_k>0,\ \lambda_k\in\Lambda^{\mathrm{neg}}-0$$

of the direct images of

$$\omega_{\prod_k X^{(n_k)}} \otimes \bigotimes_k \mathrm{Sym}^{n_k}(V(\lambda_k))$$

along the corresponding maps

$$\prod_k X^{(n_k)} \to X^\lambda,\quad \{D_k\in X^{(n_k)}\}\mapsto \sum_k \lambda_k\cdot D_k.$$

2.3.5. Dually, if A is a co-commutative co-algebra, graded by Λ^{neg} with $A(0)\simeq \mathsf{e}$, then to A we attach an object $\mathrm{Fact}^{\mathrm{coalg}}(A)_{X^\lambda}\in\mathrm{Shv}(X^\lambda)$ characterized by the property that its *-fiber at a divisor

$$\sum_k \lambda_k\cdot x_k\in X^\lambda,\quad 0\neq\lambda_k\in\Lambda^{\mathrm{neg}},\quad \sum_k\lambda_k=\lambda,\quad k'\neq k''\ \Rightarrow\ x_{k'}\neq x_{k''}$$

equals $\bigotimes\limits_k A(\lambda_k)$.

If all the graded components of A are finite-dimensional, we can view the dual $A^\vee$ of A as a Λ^{neg}-graded algebra, and we have

$$\mathbb{D}(\mathrm{Fact}^{\mathrm{coalg}}(A)_{X^\lambda})\simeq \mathrm{Fact}^{\mathrm{alg}}(A^\vee)_{X^\lambda}.$$

2.4. Restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ to strata.

2.4.1. We apply the above construction to A being the (classical) algebra $\mathcal{O}(\check{N})$ (resp., co-algebra $U(\check{\mathfrak{n}}^-)$).

Thus, we obtain the objects

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda} \text{ and } \mathrm{Fact}^{\mathrm{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda}$$

in $\mathrm{Shv}(X^\lambda)$.

Note also that $U(\check{\mathfrak{n}}^-)$ is the graded dual of $\mathcal{O}(\check{N})$, and so the objects $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}$ and $\mathrm{Fact}^{\mathrm{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda}$ are Verdier dual to each other.

2.4.2. From the construction it follows that for $\lambda\neq 0$,

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}\in\mathrm{Shv}(X^\lambda)^{<0},$$

and hence

$$\mathrm{Fact}^{\mathrm{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda}\in\mathrm{Shv}(X^\lambda)^{>0}.$$

Remark 2.4.3. Note that by the PBW theorem, when viewed as a co-commuatative co-algebra, $U(\check{\mathfrak{n}}^-)$ is canonically identified with $\mathrm{Sym}(\check{\mathfrak{n}}^-)$; in this paper we will not use the algebra structure on $U(\check{\mathfrak{n}}^-)$ that allows to distinguish it from $\mathrm{Sym}(\check{\mathfrak{n}}^-)$. So $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}$ falls into the paradigm of Example 2.3.4.

Similarly, when viewed just as a commutative algebra (i.e., ignoring the Hopf algebra structure), $\mathcal{O}(\check{N})$ is canonically identified with $\mathrm{Sym}(\check{\mathfrak{n}}^*)$.

2.4.4. In Sect. 3.6 we will prove:

Theorem 2.4.5. *The objects*

$$(\mathbf{i}^\lambda)^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}) \text{ and } (\mathbf{i}^\lambda)^*(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}})$$

of $\mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$ *identify with the !-pullback along*

$$S^\lambda_{\mathrm{Ran}} \overset{p^\lambda_{\mathrm{Ran}}}{\longrightarrow} (\mathrm{Ran}(X) \times X^\lambda)^{\supset} \overset{\mathrm{pr}^\lambda_{\mathrm{Ran}}}{\longrightarrow} X^\lambda$$

of $\mathrm{Fact}^{\mathrm{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda}[-\langle\lambda, 2\check{\rho}\rangle]$ *and* $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}[-\langle\lambda, 2\check{\rho}\rangle]$*, respectively,*

2.5. **Digression: categories over the Ran space.** We will now discuss a variant of the construction in Sect. 2.3 that attaches to a symmetric monoidal category $\mathcal{A}$ a category spread over the Ran space, denoted $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$.

2.5.1. Consider the category TwArr whose objects are

$$I \overset{\phi}{\twoheadrightarrow} J, \tag{2.4}$$

where I and J are finite non-empty sets. A morphism between two such objects is is a commutative diagram

$$\begin{array}{ccc} J_1 & \xrightarrow{\phi_1} & K_1 \\ {\scriptstyle\psi_J}\downarrow & & \uparrow{\scriptstyle\psi_K} \\ J_2 & \xrightarrow{\phi_2} & K_2, \end{array} \tag{2.5}$$

where the maps ψ_J and ψ_K are surjective.

2.5.2. To $\mathcal{A}$ we attach a functor

$$\mathrm{TwArr}(\mathcal{A}) : \mathrm{TwArr} \to \mathrm{DGCat}$$

by sending an object (2.4) to $\mathrm{Shv}(X^K) \otimes \mathcal{A}^{\otimes J}$, and a morphism (2.5) to a functor comprised of

$$\mathrm{Shv}(X^{K_1}) \overset{(\Delta_{\psi_J})_*}{\longrightarrow} \mathrm{Shv}(X^{K_2})$$

and the functor

$$\mathcal{A}^{\otimes J_1} \to \mathcal{A}^{\otimes J_2},$$

given by the symmetric monoidal structure on $\mathcal{A}$.

2.5.3. We set

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} := \underset{\mathrm{TwArr}}{\mathrm{colim}}\, \mathrm{TwArr}(\mathcal{A}) \in \mathrm{DGCat}\,. \tag{2.6}$$

2.5.4. Let us consider some examples.

(i) Let $\mathcal{A} = \mathrm{Vect}$. Then $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} \simeq \mathrm{Shv}(\mathrm{Ran})$.

(ii) Let $\mathcal{A}$ be the (non-unital) symmetric monoidal category consisting of vector spaces graded by the semi-group $\Lambda^{\mathrm{neg}} - 0$. We have a canonical functor

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} \to \mathrm{Shv}(\underset{\lambda\in\Lambda^{\mathrm{neg}}-0}{\sqcup} X^\lambda), \tag{2.7}$$

and it follows from [Ga2, Lemma 7.4.11(d)] that this functor is an equivalence.

2.5.5. Similarly, if $\mathcal{A}$ is a symmetric co-monoidal category, we can form the limit of the corresponding functor

$$\mathrm{TwArr}(\mathcal{A}) : \mathrm{TwArr}^{\mathrm{op}} \to \mathrm{DGCat},$$

and obtain a category that we denote by $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})_{\mathrm{Ran}}$.

2.5.6. Recall that whenever we have a diagram of categories

$$t \mapsto \mathcal{C}_t$$

indexed by some category T, then

$$\underset{t \in T}{\operatorname{colim}}\, \mathcal{C}_t$$

is canonically equivalent to

$$\underset{t \in T^{\mathrm{op}}}{\lim}\, \mathcal{C}_t,$$

where the transition functors are given by the right adjoints of those in the original family.

2.5.7. Let $\mathcal{A}$ be again a symmetric monoidal category. Applying the observation of Sect. 2.5.6 to the colimit (2.6), and obtain that $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$ can also be written as a limit.

Assume now that $\mathcal{A}$ is such that the functor

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

right adjoint to the tensor product operation, is continuous. In this case, the above tensor co-product operation makes $\mathcal{A}$ into a symmetric co-monoidal category, and we can form $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})_{\mathrm{Ran}}$.

We note however, that the *limit* presentation of $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$ tautologically coincides with the limit defining $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})_{\mathrm{Ran}}$. I.e., we have a canonical equivalence:

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} \simeq \mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})_{\mathrm{Ran}}.$$

Hence, in what follows we will sometimes write simply $\mathrm{Fact}(\mathcal{A})_{\mathrm{Ran}}$, having both of the above realizations in mind.

2.5.8. Let I be a fixed finite non-empty set. The above constructions have a variant, where instead of TwArr we use its variant, denoted $\mathrm{TwArr}_{I/}$, whose objects are commutative diagrams

$$I \twoheadrightarrow J \overset{\phi}{\twoheadrightarrow} K,$$

and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} I & \longrightarrow & J_1 & \xrightarrow{\phi_1} & K_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \psi_J}\downarrow & & \uparrow{\scriptstyle \psi_K} \\ I & \longrightarrow & J_2 & \xrightarrow{\phi_2} & K_2, \end{array}$$

We denote the resulting category by $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_I$.

2.5.9. For a surjective morphism $I_1 \twoheadrightarrow I_2$, we have the corresponding functor

$$\mathrm{TwArr}_{I_2/} \to \mathrm{TwArr}_{I_1/},$$

which induces a functor

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{I_2} \to \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{I_1}.$$

An easy cofinality argument shows that the resulting functor

$$\underset{I}{\operatorname{colim}}\, \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_I \to \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} \tag{2.8}$$

is an equivalence.

2.5.10. When we think of $\mathrm{Fact}(\mathcal{A})_I$ as a *limit*, the functor $\mathrm{TwArr}_{I_2/} \to \mathrm{TwArr}_{I_1/}$ defines a functor

$$\mathrm{Fact}(\mathcal{A})_{I_1} \to \mathrm{Fact}(\mathcal{A})_{I_2}.$$

The corresponding functor

$$\mathrm{Fact}(\mathcal{A})_{\mathrm{Ran}} \to \underset{I}{\lim}\, \mathrm{Fact}(\mathcal{A})_I$$

is an equivalence obtained from (2.8) by the procedure of Sect. 2.5.6.

2.6. Presentation of $\mathrm{IC}^{\frac{\infty}{2}}$ as a colimit.

2.6.1. Denote

$$\mathrm{Sph}_{G,I} := \mathrm{Shv}(\mathfrak{L}^+(G)_I\backslash \mathrm{Gr}_{G,I}) \text{ and } \mathrm{Sph}_{G,\mathrm{Ran}} := \mathrm{Shv}(\mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$

Consider the symmetric monoial category $\mathrm{Rep}(\check{G})$. Geometric Satake defines functors

$$\mathrm{Sat}_{G,I} : \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I \to \mathrm{Sph}_{G,I}$$

that glue to a functor

$$\mathrm{Sat}_{G,\mathrm{Ran}} : \mathrm{Fact}(\mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \to \mathrm{Sph}_{G,\mathrm{Ran}}.$$

2.6.2. For a fixed finite non-empty set I and a map $\underline{\lambda} : I \to \Lambda^+$, we consider the following object of $\mathrm{Fact}(\mathrm{Rep}(\check{G}))_I$, denoted $V^{\underline{\lambda}}$.

Informally, $V^{\underline{\lambda}}$ is designed so its !-fiber at a point

$$I \to X, \quad I = \underset{k}{\sqcup} I_k,\ I_k \mapsto x_k, \quad x_{k'} \neq x_{k''}$$

is

$$\bigotimes_k V^{\lambda_k} \in \mathrm{Rep}(\check{G})^{\otimes k}.$$

2.6.3. It terms of the presentation of $\mathrm{Fact}(\mathrm{Rep}(\check{G}))_I$ as a colimit

$$\mathrm{Fact}(\mathrm{Rep}(\check{G}))_I = \underset{\mathrm{TwArr}_{I/}}{\mathrm{colim}}\ \mathrm{TwArr}(\mathrm{Rep}(\check{G})),$$

the object $V^{\underline{\lambda}}$ corresponds to the colimit over $\mathrm{TwArr}_{I/}$ of the functor

$$\mathrm{TwArr}_{I/} \to \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I$$

that sends

$$(I \twoheadrightarrow J \twoheadrightarrow K) \in \mathrm{TwArr}_{I/} \rightsquigarrow V^{\underline{\lambda}}_{I\twoheadrightarrow J\twoheadrightarrow K} \in \mathrm{Shv}(X^K) \otimes \mathrm{Rep}(\check{G})^{\otimes J} \to \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I, \tag{2.9}$$

where

$$V^{\underline{\lambda}}_{I\twoheadrightarrow J\twoheadrightarrow K} = \omega_{X^K} \bigotimes \left(\underset{j\in J}{\otimes} V^{\lambda_j}\right), \quad \lambda_j = \sum_{i\in I, i\mapsto j} \underline{\lambda}(i).$$

The structure of a functor $\mathrm{TwArr}_{I/} \to \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I$ on (2.9) is given by the Plücker maps

$$\underset{i}{\otimes} V^{\lambda_i} \to V^{\lambda}, \quad \lambda = \sum_i \lambda_i.$$

2.6.4. Denote

$$\mathrm{Sph}_{G,I} := \mathrm{Shv}(\mathfrak{L}^+(G)_I\backslash \mathrm{Gr}_{G,I}) \text{ and } \mathrm{Sph}_{G,\mathrm{Ran}} := \mathrm{Shv}(\mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$

Consider the symmetric monoial category $\mathrm{Rep}(\check{G})$. Geometric Satake defines functors

$$\mathrm{Sat}_{G,I} : \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I \to \mathrm{Sph}_{G,I}$$

that glue to a functor

$$\mathrm{Sat}_{G,\mathrm{Ran}} : \mathrm{Fact}(\mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \to \mathrm{Sph}_{G,\mathrm{Ran}}.$$

2.6.5. Consider the object

$$\mathrm{Sat}_{G,I}(V^{\underline{\lambda}}) \in \mathrm{Sph}_{G,I}.$$

The element $\underline{\lambda}$ gives rise to a section

$$s^{\underline{\lambda}}_I : X^I \to \mathrm{Gr}_{G,I}.$$

Denote

$$\delta_{\underline{\lambda}} := (s^{\underline{\lambda}}_I)_!(\omega_{X^I}) \in \mathrm{Shv}(\mathrm{Gr}_{G,I}).$$

Consider the object

$$\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}})[\langle \lambda, 2\check{\rho}\rangle] \in \mathrm{Shv}(\mathrm{Gr}_{G,I}).$$

In the above formula, $\lambda = \sum_{i\in I} \underline{\lambda}(i)$, and $\star$ denotes the (right) convolution action of $\mathrm{Sph}_{G,I}$ on $\mathrm{Shv}(\mathrm{Gr}_{G,I})$.

2.6.6. Consider now the set $\mathrm{Maps}(I, \Lambda^+)$ of maps

$$\underline{\lambda} : I \to \Lambda^+.$$

We equip it with a partial order by declaring

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \Leftrightarrow \underline{\lambda}_2(i) - \underline{\lambda}_1(i) \in \Lambda^+, \forall i \in I.$$

The assignment

$$\underline{\lambda} \mapsto \delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}})[\langle \lambda, 2\check{\rho} \rangle] \in \mathrm{Shv}(\mathrm{Gr}_{G,I})$$

has a structure of a functor

$$\mathrm{Maps}(I, \Lambda^+) \to \mathrm{Shv}(\mathrm{Gr}_{G,I}),$$

see Sects. 4.4.6 and 4.5.2.

Set

$$\mathrm{IC}_I^{\frac{\infty}{2}} := \underset{\underline{\lambda} \in \mathrm{Maps}(I, \Lambda^+)}{\mathrm{colim}} \delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}})[\langle \lambda, 2\check{\rho} \rangle] \in \mathrm{Shv}(\mathrm{Gr}_{G,I}).$$

As in [Ga1, Proposition 2.3.7(a,b,c)] one shows:

Lemma 2.6.7. *The object* $\mathrm{IC}_I^{\frac{\infty}{2}}$ *has the following properties:*

(a) *It is supported on* $\overline{S}_I^0$*;*

(b) *It belongs to* $\mathrm{SI}_I^{\leq 0} = \mathrm{Shv}(\overline{S}_I^0)^{\mathfrak{L}(N)_I} \subset \mathrm{Shv}(\overline{S}_I^0)$*;*

(c) *Its restriction to* S_I^0 *is identified with* $\omega_{S_I^0}$.

2.6.8. For a surjective map

$$\phi : I_2 \twoheadrightarrow I_1$$

and the corresponding map

$$\Delta_\phi : \mathrm{Gr}_{G,I_1} \to \mathrm{Gr}_{G,I_2},$$

we have a canonical identification

$$(\Delta_\phi)^!(\mathrm{IC}_{I_2}^{\frac{\infty}{2}}) \simeq \mathrm{IC}_{I_1}^{\frac{\infty}{2}}.$$

One endows this system of isomorphisms with a homotopy-coherent system of compatibilities, thus making the assignment

$$I \mapsto \mathrm{IC}_I^{\frac{\infty}{2}}$$

into an object of $\mathrm{SI}_{\mathrm{Ran}}^{\leq 0}$, see Sect. 4.4.8.

We denote this object by $'\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$. By Lemma 2.6.7(c), the restriction of $'\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$ to S_{Ran}^0 identifies canonically with $\omega_{S_{\mathrm{Ran}}^0}$.

2.6.9. Fix a point $x \in X$, and consider the restriction of $'\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$ along the map

$$\mathrm{Gr}_{G,x} \simeq \{x\} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}.$$

It follows from the construction, that this restriction identifies canonically with the object

$$\mathrm{IC}_x^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathrm{Gr}_{G,x}),$$

constructed in [Ga1, Sect. 2.3].

2.7. Presentation of $\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$ as a colimit.

2.7.1. We will prove:

Theorem 2.7.2. *There exists a canonical isomorphism* $'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}} \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$.

The rest of this section is devoted to the proof of Theorem 2.7.2. It amounts to the combination of the following two assertions:

Proposition 2.7.3. *For* $\mu \in \Lambda^{\mathrm{neg}}$*, the object*

$$(\mathbf{i}^\mu)^*('\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}) \in \mathrm{SI}^{=\mu}_{\mathrm{Ran}}$$

identifies canonically with the !-pullback along

$$S^\mu_{\mathrm{Ran}} \overset{p^\mu_{\mathrm{Ran}}}{\longrightarrow} (\mathrm{Ran}(X) \times X^\mu)^{\supset} \overset{\mathrm{pr}^\mu_{\mathrm{Ran}}}{\longrightarrow} X^\mu$$

of $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}[-\langle \mu, 2\check{\rho}\rangle]$.

Proposition 2.7.4. *For* $0 \neq \mu \in \Lambda^{\mathrm{neg}}$*, the object*

$$(\mathbf{i}^\mu)^!('\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}})[\langle \mu, 2\check{\rho}\rangle] \in \mathrm{SI}^{=\mu}_{\mathrm{Ran}}$$

is a pullback along $\mathrm{pr}^\mu_{\mathrm{Ran}} \circ p^\mu_{\mathrm{Ran}}$ *of an object of* $\mathrm{Shv}(X^\mu)$ *that is strictly coconnective.*

Remark 2.7.5. Note that Theorem 2.7.2 and Proposition 2.7.3 imply the assertion of Theorem 2.4.5 about the *-fibers.

2.7.6. As a corollary of Theorem 2.7.2 and Sect. 2.6.9 we obtain:

Corollary 2.7.7. *The restriction of* $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ *along the map*

$$\mathrm{Gr}_{G,x} \simeq \{x\} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$$

identifies canonically with the object $\mathrm{IC}^{\frac{\infty}{2}}_x \in \mathrm{Shv}(\mathrm{Gr}_{G,x})$ *of* [Ga1, Sect. 2.3].

2.7.8. Note that by construction, $'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ has the following *factorization property* with respect to Ran:

Let $(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}$ denote the *disjoint locus*. I.e., for an affine test scheme Y,

$$\mathrm{Hom}(Y, (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}) \subset \mathrm{Hom}(Y, \mathrm{Ran}(X)) \times \mathrm{Hom}(Y, \mathrm{Ran}(X))$$

consists of those pairs $\mathcal{I}_1, \mathcal{I}_2 \in \mathrm{Hom}(Y, X)$, for which for every $i_1 \in I_1$ and $i_2 \in I_2$, the corresponding two maps $Y \rightrightarrows X$ have non-intersecting images.

It is well-known that we have a canonical isimorphism

$$(\mathrm{Gr}_{G,\mathrm{Ran}} \times \mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathrm{Ran}(X)\times\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \simeq \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}, \tag{2.10}$$

where

$$(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \to \mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X)$$

is the map

$$\mathcal{I}_1, \mathcal{I}_2 \mapsto \mathcal{I}_1 \cup \mathcal{I}_2.$$

Then, in terms of the identification (2.10), we have a canonical isomorphism

$$('\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}} \boxtimes {'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}})|_{(\mathrm{Gr}_{G,\mathrm{Ran}} \times \mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathrm{Ran}(X)\times\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X)\times\mathrm{Ran}(X))_{\mathrm{disj}}} \simeq \tag{2.11}$$

$$\simeq {'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}}|\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}.$$

2.7.9. The rest of this section is devoted to the proof of Theorem 2.7.2.

2.8. **Description of the *-restriction to strata.** The goal of this subsection is to prove Proposition 2.7.3.

2.8.1. We will compute

$$(\mathbf{i}^\mu)^*('\mathrm{IC}_I^{\frac{\infty}{2}}) \in \mathrm{SI}_I^{\leq 0}$$

for each individual finite non-empty set I, and obtain the !-pullback of $(\mathrm{pr}^\mu_{\mathrm{Ran}} \circ p^\mu_{\mathrm{Ran}})^!(\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda})$ along $S^\mu_I \to S^\mu_{\mathrm{Ran}}$.

By Lemma 1.4.8, we need to construct an identification

$$(p^\mu_I)_! \circ (\mathbf{i}^\mu)^*('\mathrm{IC}_I^{\frac{\infty}{2}}) \simeq (\mathrm{pr}^\mu_I)^!(\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}), \tag{2.12}$$

where pr^μ_I denotes the map

$$(X^I \times X^\mu)^\supset \to X^\mu.$$

2.8.2. We will compute

$$(p^\mu_I)_! \circ (\mathbf{i}^\mu)^*(\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}}))[\langle \lambda, 2\check{\rho}\rangle] \in \mathrm{Shv}((X^I \times X^\mu)^\supset) \tag{2.13}$$

for each individual $\underline{\lambda} : I \to \Lambda^+$.

Namely, we will show that (2.12) identifies with the following object, denoted

$$V^{\underline{\lambda}}(\underline{\lambda} + \mu) \in \mathrm{Shv}((X^I \times X^\mu)^\supset),$$

described below.

2.8.3. Before we give the definition of $V^{\underline{\lambda}}(\underline{\lambda} + \mu)$, let us describe what its !-fibers are. Fix a point of $(X^I \times X^\mu)^\supset$. By definition, a datum of such a point consists of:

- A partition $I = \underset{k}{\sqcup} I_k$;
- A collection of distinct points x_k of x;
- As assignment $x_k \mapsto \mu_k \in \Lambda^{\mathrm{neg}}$, so that $\underset{k}{\Sigma}\, \mu_k = \mu$.

Then the !-fiber of $V^{\underline{\lambda}}(\underline{\lambda} + \mu)$ at a such a point is

$$\underset{k}{\otimes}\, V^{\lambda_k}(\lambda_k + \mu_k),$$

where $\lambda_k = \underset{i \in I_k}{\Sigma}\, \underline{\lambda}(i)$, and where $V(\nu)$ denotes the ν-weight space in a $\check{G}$-representation V.

2.8.4. Consider the category, denoted $\mathrm{TwArr}_{\mu, I/}$, whose objects are commutative diagrams

$$\begin{array}{ccccc}
I & \xrightarrow{\upsilon} & J & \xrightarrow{\psi} & K \\
 & & \phi_J \big\downarrow & & \big\downarrow \phi_K \\
 & & \widetilde{J} & \xrightarrow{\widetilde{\psi}} & \widetilde{K} \\
 & & \phi'_J \big\uparrow & & \big\uparrow \phi'_K \\
\Lambda^{\mathrm{neg}} & \xleftarrow{\underline{\mu}} & \widetilde{J}' & \xrightarrow{\widetilde{\psi}'} & \widetilde{K}',
\end{array}$$

where the maps $\upsilon, \psi, \widetilde{\psi}, \widetilde{\psi}', \phi_J, \phi_K$ are surjective (but ϕ'_J and ϕ'_K are not necessarily so), and

$$\underset{\widetilde{j}' \in \widetilde{J}'}{\Sigma}\, \underline{\mu}(\widetilde{j}') = \mu.$$

Morphisms in this category are defined by the same principle as in TwArr_μ and $\mathrm{TwArr}_{I/}$ introduced earlier, i.e., the sets J, $\widetilde{J}$, $\widetilde{J}'$ map forward and the sets K, $\widetilde{K}$, $\widetilde{K}'$ map backwards.

Let $\Delta_{\widetilde{K}, I, \lambda}$ denote the map

$$X^{\widetilde{K}} \to X^I \times X^\mu,$$

comprised of

$$\Delta_{\phi_K \circ \psi \circ \upsilon} : X^{\widetilde{K}} \to X^I$$

and

$$X^{\widetilde{K}} \overset{\Delta_{\phi'_K}}{\longrightarrow} X^{\widetilde{K}'} \overset{\Delta_{\widetilde{K}',\mu}}{\longrightarrow} X^{\mu}.$$

We let $V^{\underline{\lambda}}(\underline{\lambda}+\mu)$ be the colimit over $\mathrm{TwArr}_{\mu,I/}$ of the objects

$$(\Delta_{\widetilde{K},I,\lambda})_*(\omega_{X^{\widetilde{K}}}) \bigotimes \left(\underset{\widetilde{j}\in\widetilde{J}}{\otimes} V^{\lambda_{\widetilde{j}}}(\lambda_{\widetilde{j}}+\mu_{\widetilde{j}}) \right),$$

where

$$\lambda_{\widetilde{j}} = \underset{i\in I, i\mapsto \widetilde{j}}{\Sigma} \underline{\lambda}(i) \text{ and } \mu_{\widetilde{j}} = \underset{\widetilde{j}'\in\widetilde{J}', \widetilde{j}'\mapsto\widetilde{j}}{\Sigma} \underline{\mu}(\widetilde{j}').$$

2.8.5. Applying Sect. 1.6, we obtain a canonical isomorphism

$$(p^{\mu}_I)_! \circ (\mathbf{i}^{\mu})^*(\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}})) \simeq (p^{-,\mu}_I)_* \circ (\mathbf{i}^{-,\mu})^!(\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}})).$$

Now, the properties of the geometric Satake functor $\mathrm{Sat}_{G,I}$ imply a canonical isomorphism

$$(p^{-,\mu}_I)_* \circ (\mathbf{i}^{-,\mu})^!(\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}}))[\langle\lambda, 2\check{\rho}\rangle] \simeq V^{\underline{\lambda}}(\underline{\lambda}+\mu),$$

giving rise to the desired expression for (2.13).

2.8.6. Finally, it is not difficult to see that

$$\underset{\underline{\lambda}\in\mathrm{Maps}(I,\Lambda^+)}{\mathrm{colim}} V^{\underline{\lambda}}(\underline{\lambda}+\mu)$$

identifies canonically with $(\mathrm{pr}^{\mu}_I)^!(\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}))_{X^{\lambda}})$.

Indeed, this follows from the fact that we have a canonical identification

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, V^{\lambda}(\lambda+\mu) \simeq \mathcal{O}(\check{N})(\mu),$$

where Λ^+ is endowed with the order relation

$$\lambda_1 \leq \lambda_2 \;\Leftrightarrow\; \lambda_2 - \lambda_1 \in \Lambda^+.$$

2.9. **Proof of coconnectivity.** In this subsection we will prove Proposition 2.7.4, thereby completing the proof of Theorem 2.7.2.

2.9.1. Consider the diagonal stratification of X^{μ}. For each parameter β of the stratification, let X^{μ}_{β} let denote the corresponding stratum, and denote by

$$(\mathrm{Ran}(X)\times X^{\mu}_{\beta})^{\supset} := X^{\mu}_{\beta} \underset{X^{\mu}}{\times} (\mathrm{Ran}(X)\times X^{\mu})^{\supset}) \overset{\iota_\beta}{\hookrightarrow} (\mathrm{Ran}(X)\times X^{\mu})^{\supset}$$

and

$$(\mathrm{Ran}(X)\times X^{\mu}_{\beta})^{\supset} \overset{\mathrm{pr}^{\mu}_{\mathrm{Ran},\beta}}{\longrightarrow} X^{\mu}_{\beta}$$

the resulting maps.

Let $\mathcal{F}^{\mu} \in \mathrm{Shv}((\mathrm{Ran}(X)\times X^{\mu})^{\supset})$ be such that

$$(\mathbf{i}^{\mu})^!({}'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}) \simeq (p^{\mu}_{\mathrm{Ran}})^!(\mathcal{F}^{\mu}).$$

To prove Proposition 2.7.4, its suffices to show that each

$$(\iota_\beta)^! \circ \mathcal{F}^{\mu} \in \mathrm{Shv}((\mathrm{Ran}(X)\times X^{\mu}_{\beta})^{\supset})$$

is of the form

$$(\mathrm{pr}^{\mu}_{\mathrm{Ran},\beta})^!(\mathcal{F}^{\mu}_{\beta}),$$

where $\mathcal{F}^{\mu}_{\beta} \in \mathrm{Shv}(X^{\mu}_{\beta})$ is such that $\mathcal{F}^{\mu}_{\beta}[\langle\mu, 2\check{\rho}\rangle]$ is strictly coconnective.

2.9.2. By the factorization property of $'\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ (see (2.11)), it suffices to prove the above assertion for β corresponding to the main diagonal $X \to X^\mu$. Denote the corresponding stratum in $(\mathrm{Ran}(X) \times X^\mu)^\supset$ by

$$(\mathrm{Ran}(X) \times X)^\supset.$$

Denote the corresponding map $\mathrm{pr}^\mu_{\mathrm{Ran},\beta}$ by

$$\mathrm{pr}^\mu_{(\mathrm{Ran}(X)\times X)^\supset} : (\mathrm{Ran}(X) \times X)^\supset \to X.$$

Denote the restriction of the section

$$s^\mu_{\mathrm{Ran}} : (\mathrm{Ran}(X) \times X^\mu)^\supset \to S^\mu_{\mathrm{Ran}}$$

to this stratum by $s^\mu_{(\mathrm{Ran}(X)\times X)^\supset}$.

We claim that

$$(s^\mu_{(\mathrm{Ran}(X)\times X)^\supset})^!('\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}) \simeq (\mathrm{pr}^\mu_{(\mathrm{Ran}(X)\times X)^\supset})^!(\omega_X) \otimes \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu)[-\langle \mu, 2\check{\rho}\rangle].$$

This would prove the required estimate (the important thing is that $\mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu)$ lives in cohomological degrees ≥ 2).

2.9.3. In fact, we claim that for every I, we have:

$$(s^\mu_{(X^I\times X)^\supset})^!(\mathrm{IC}^{\frac{\infty}{2}}_I) \simeq (\mathrm{pr}^\mu_{(X^I\times X)^\supset})^!(\omega_X) \otimes \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu)[-\langle \mu, 2\check{\rho}\rangle],$$

where

$$\mathrm{pr}^\mu_{(X^I\times X)^\supset} := \mathrm{pr}^\mu_{(\mathrm{Ran}(X)\times X)^\supset}|_{(X^I\times X)^\supset}.$$

2.9.4. Indeed, it follows from the definitions that for any $\underline{\lambda} : I \to \Lambda^+$,

$$(s^\mu_{(X^I\times X)^\supset})^!(\delta_{\underline{\lambda}} \star \mathrm{Sat}_{G,I}(V^{\underline{\lambda}}))[\langle \lambda, 2\check{\rho}\rangle] \simeq (\mathrm{pr}^\mu_{(X^I\times X)^\supset})^!(\omega_X) \otimes W_{\lambda,\mu}[-\langle \mu, 2\check{\rho}\rangle],$$

where $W_{\lambda,\mu}$ is the cohomogically graded vector space such that

$$\mathrm{Sat}(V^\lambda)|_{\mathrm{Gr}^{\lambda+\mu}_G} \simeq \mathrm{IC}_{\mathrm{Gr}^{\lambda+\mu}_G} \otimes W_{\lambda,\mu},$$

where $-|_-$ means !-restriction (again, the important thing is that $W_{\lambda,\mu} \in \mathrm{Vect}$ is cohomologically ≥ 2).

Now, it follows (see [Ga1, Sect. 4.2.4]) that

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, W_{\lambda,\mu} \simeq \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu).$$

3. The semi-infinite IC sheaf and Drinfeld's compactification

In this section we will express $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ in terms of an *actual intersection cohomology sheaf*, i.e., one arising in finite-dimensional algebraic geometry (technically, on an algebraic stack locally of finite type).

3.1. Drinfeld's compactification.

3.1.1. Let $\overline{\mathrm{Bun}}_B$ Drinfeld's relative compactification of the stack Bun_B along the fibers of the map $\mathrm{Bun}_B \to \mathrm{Bun}_G$.

I.e., $\overline{\mathrm{Bun}}_B$ is the algebraic stack that classifies triples $(\mathcal{P}_G, \mathcal{P}_T, \kappa)$, where:

(i) $\mathcal{P}_G$ is a G-bundle on X;

(ii) $\mathcal{P}_T$ is a T-bundle on X;

(iii) κ is a *Plücker* data, i.e., a system of non-zero maps

$$\kappa^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_T) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G},$$

(here $\mathcal{V}^{\check{\lambda}}$ denotes the Weyl module with highest weight $\check{\lambda} \in \check{\Lambda}^+$) that satisfy Plücker relations, i.e., for $\check{\lambda}_1$ and $\check{\lambda}_2$ the diagram

$$\begin{array}{ccc} \check{\lambda}_1(\mathcal{P}_T) \otimes \check{\lambda}_2(\mathcal{P}_T) & \xrightarrow{\kappa^{\check{\lambda}_1} \otimes \kappa^{\check{\lambda}_2}} & \mathcal{V}^{\check{\lambda}_1}_{\mathcal{P}_G} \otimes \mathcal{V}^{\check{\lambda}_2}_{\mathcal{P}_G} \\ \sim \uparrow & & \uparrow \\ (\check{\lambda}_1 + \check{\lambda}_2)(\mathcal{P}_T) & \xrightarrow{\kappa^{\check{\lambda}_1 + \check{\lambda}_2}} & \mathcal{V}^{\check{\lambda}_1 + \check{\lambda}_2}_{\mathcal{P}_G} \end{array}$$

must commute.

3.1.2. For $\lambda \in \Lambda^{\text{neg}}$ we let $\bar{\mathbf{i}}^{\lambda}_{\text{glob}}$ denote the map

$$\overline{\text{Bun}}_B^{\leq \lambda} := \overline{\text{Bun}}_B \times X^\lambda \to \overline{\text{Bun}}_B,$$

given by

$$(\mathcal{P}_G, \mathcal{P}_T, \kappa, D) \mapsto (\mathcal{P}'_G, \mathcal{P}'_T, \kappa')$$

with $\mathcal{P}'_G = \mathcal{P}_G$, $\mathcal{P}'_T = \mathcal{P}_T(D)$ and κ' given by precomposing κ with the natural maps

$$\check{\lambda}(\mathcal{P}'_T) = \check{\lambda}(\mathcal{P}_T)(\check{\lambda}(D)) \hookrightarrow \check{\lambda}(\mathcal{P}_T).$$

It is known that $\bar{\mathbf{i}}^{\lambda}_{\text{glob}}$ is a finite morphism.

3.1.3. Let $\mathbf{j}^{\lambda}_{\text{glob}}$ denote the open embedding

$$\overline{\text{Bun}}_N^{=\lambda} =: \text{Bun}_B \times X^\lambda \hookrightarrow \overline{\text{Bun}}_B \times X^\lambda =: \overline{\text{Bun}}_N^{\leq \lambda}.$$

Denote

$$\mathbf{i}^{\lambda}_{\text{glob}} = \bar{\mathbf{i}}^{\lambda}_{\text{glob}} \circ \mathbf{j}^{\lambda}_{\text{glob}}.$$

Note that by definition $\mathbf{i}^{0}_{\text{glob}} = \mathbf{j}^{0}_{\text{glob}}$; this is the embedding of the open substack

$$\text{Bun}_B \hookrightarrow \overline{\text{Bun}}_B.$$

The following is known:

Lemma 3.1.4. *The maps $\mathbf{i}^{\lambda}_{\text{glob}}$ are locally closed embeddings. Every field-valued point of $\overline{\text{Bun}}_B$ belongs to the image of exactly one such map.*

3.2. **The global semi-infinite category.**

3.2.1. Denote

$$\overline{\text{Bun}}_N := \overline{\text{Bun}}_B \underset{\text{Bun}_T}{\times} \text{pt}, \ \overline{\text{Bun}}_N^{\leq \lambda} := \overline{\text{Bun}}_B^{\leq \lambda} \underset{\text{Bun}_T}{\times} \text{pt}, \ \overline{\text{Bun}}_N^{=\lambda} := \overline{\text{Bun}}_B^{=\lambda} \underset{\text{Bun}_T}{\times} \text{pt},$$

where $\text{pt} \to \text{Bun}_T$ corresponds to the trivial bundle.

In particular,

$$\overline{\text{Bun}}_N^{=\lambda} \simeq \text{Bun}_B \underset{\text{Bun}_T}{\times} X^\lambda,$$

where $X^\lambda \to \text{Bun}_T$ is the Abel-Jacobi map.

We will denote by the same symbols the corresponding maps

$$\bar{\mathbf{i}}^{\lambda}_{\text{glob}} : \overline{\text{Bun}}_N^{\leq \lambda} \to \overline{\text{Bun}}_N, \ \mathbf{j}^{\lambda}_{\text{glob}} : \overline{\text{Bun}}_N^{=\lambda} \to \overline{\text{Bun}}_N^{\leq \lambda}, \ \mathbf{i}^{\lambda}_{\text{glob}} : \overline{\text{Bun}}_N^{=\lambda} \to \overline{\text{Bun}}_N.$$

Denote by $p^{\lambda}_{\text{glob}}$ the projection

$$\overline{\text{Bun}}_N^{=\lambda} \to X^\lambda.$$

3.2.2. We define

$$\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N) \tag{3.1}$$

to be the full subcategory defined by the following condition:

An object $\mathcal{F} \in \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$ belongs to $\mathrm{SI}^{\leq 0}_{\mathrm{glob}}$ if and only if for every $\lambda \in \Lambda^{\mathrm{neg}}$, the object

$$(\mathbf{i}^\lambda_{\mathrm{glob}})^!(\mathcal{F}) \in \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda})$$

belongs to the full subcategory

$$\mathrm{SI}^{=\lambda}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda}),$$

equal by definition to the essential image of the pullback functor

$$(p^\lambda_{\mathrm{glob}})^! : \mathrm{Shv}(X^\lambda) \to \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda}).$$

We note that the above pullback functor is fully faithful, since the map $p^\lambda_{\mathrm{glob}}$, being a base change of $\mathrm{Bun}_B \to \mathrm{Bun}_T$, is smooth with homologically contractible fibers.

3.2.3. We note that the full subcategory (3.1) can also be defined by an equivariance condition with respect to a certain pro-unipotent groupoid, see [Ga3, Sect. 6.1-6.2]. In particular, this fully faithful embedding admits a *right* adjoint[4].

We have the naturally defined functors

$$(\mathbf{i}^\lambda_{\mathrm{glob}})^! : \mathrm{SI}^{\leq 0}_{\mathrm{glob}} \to \mathrm{SI}^{=\lambda}_{\mathrm{glob}}$$

and

$$(\mathbf{i}^\lambda_{\mathrm{glob}})_* : \mathrm{SI}^{=\lambda}_{\mathrm{glob}} \to \mathrm{SI}^{\leq 0}_{\mathrm{glob}}.$$

In addition, one shows that the partially defined functor $(\mathbf{i}^\lambda_{\mathrm{glob}})_!$ left adjoint to $(\mathbf{i}^\lambda_{\mathrm{glob}})^!$, is defined on $\mathrm{SI}^{=\lambda}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda})$ and takes values in $\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$.

It follows formally that the partially defined functor $(\mathbf{i}^\lambda_{\mathrm{glob}})^*$, left adjoint to $(\mathbf{i}^\lambda_{\mathrm{glob}})_*$, is defined on $\mathrm{SI}^{\leq 0}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$ and takes values in $\mathrm{SI}^{=\lambda}_{\mathrm{glob}} \subset \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda})$.

3.2.4. The embeddings

$$\mathrm{SI}^{=\lambda}_{\mathrm{glob}} \hookrightarrow \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda}) \text{ and } \mathrm{SI}^{\leq 0}_{\mathrm{glob}} \hookrightarrow \mathrm{Shv}(\overline{\mathrm{Bun}}_N)$$

are compatible with the t-structure on the target categories. This follows from the fact that the right adjoints to these embeddings are right t-exact.

Hence, the categories $\mathrm{SI}^{=\lambda}_{\mathrm{glob}}$ and $\mathrm{SI}^{\leq 0}_{\mathrm{glob}}$ acquire t-structures. By construction, an object $\mathcal{F} \in \mathrm{SI}^{\leq 0}_{\mathrm{glob}}$ is connective (resp., coconnective) if and only if $(\mathbf{i}^\lambda_{\mathrm{glob}})^*(\mathcal{F})$ (resp., $(\mathbf{i}^\lambda_{\mathrm{glob}})^!(\mathcal{F})$ is connective (resp., coconnective) for every $\lambda \in \Lambda^{\mathrm{neg}}$.

3.2.5. We will denote by

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}} \in (\mathrm{SI}^{\leq 0}_{\mathrm{glob}})^\heartsuit$$

the minimal extension of $\mathrm{IC}_{\mathrm{Bun}_N} \in (\mathrm{SI}^{=0}_{\mathrm{glob}})^\heartsuit$ along $\mathbf{j}^0_{\mathrm{glob}}$.

3.3. **Local vs global compatibility for the semi-infinite IC sheaf.**

3.3.1. For every finite set I we have a canonically defined map

$$\pi_I : \overline{S}^0_I \to \overline{\mathrm{Bun}}_N.$$

Together these maps combine to a map

$$\pi_{\mathrm{Ran}} : \overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_N.$$

[4]The corresponding assertion would be *false* for the corresponding embedding $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$.

3.3.2. Let $d = \dim(\mathrm{Bun}_N) = (g-1)\cdot \dim(N)$. The main result of this section is:

Theorem 3.3.3. *There exists a (unique) isomorphism*

$$(\pi_{\mathrm{Ran}})^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}})[d] = \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}.$$

3.3.4. The rest of this section is devoted to the proof of this theorem. Modulo auxiliary assertions, the proof will be given in Sect. 3.5.8.

3.4. **The local vs global compatibility for the semi-infinite category.** This subsection contains some preparatory material for the proof of Theorem 3.3.3.

3.4.1. First, we observe:

Lemma 3.4.2. *For every λ, we have a commutative diagram*

$$\begin{array}{ccc} \overline{S}^{\leq\lambda}_{\mathrm{Ran}} & \xrightarrow{\bar{\mathbf{i}}^\lambda} & \overline{S}^0_{\mathrm{Ran}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathrm{Ran}}} \\ \overline{\mathrm{Bun}}_N^{\leq\lambda} & \xrightarrow{\bar{\mathbf{i}}^\lambda_{\mathrm{glob}}} & \overline{\mathrm{Bun}}_B. \end{array}$$

The corresponding diagram

$$\begin{array}{ccc} S^{=\lambda}_{\mathrm{Ran}} & \xrightarrow{\mathbf{i}^\lambda} & \overline{S}^0_{\mathrm{Ran}} \\ {\scriptstyle \pi^\lambda_{\mathrm{Ran}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathrm{Ran}}} \\ \overline{\mathrm{Bun}}_N^{=\lambda} & \xrightarrow{\mathbf{i}^\lambda_{\mathrm{glob}}} & \overline{\mathrm{Bun}}_B. \end{array} \tag{3.2}$$

is Cartesian, and we have a commutative diagram

$$\begin{array}{ccc} S^{=\lambda}_{\mathrm{Ran}} & \xrightarrow{p^\lambda_{\mathrm{Ran}}} & (\mathrm{Ran}(X)\times X^\lambda)^{\supset} \\ {\scriptstyle \pi^\lambda_{\mathrm{Ran}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}^\lambda_{\mathrm{Ran}}} \\ \overline{\mathrm{Bun}}_N^{=\lambda} & \xrightarrow{p^\lambda_{\mathrm{glob}}} & X^\lambda. \end{array}$$

The assertions parallel to those in the above lemma hold for $\mathrm{Ran}(X)$ replaced by X^I for an individual finite set I.

3.4.3. The following assertion is not necessary for the needs of this paper, but we will prove it for the sake of completeness (see Sect. A.1.9):

Theorem 3.4.4. *The functor*

$$(\pi_{\mathrm{Ran}})^! : \mathrm{Shv}(\overline{\mathrm{Bun}}_N) \to \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$$

is fully faithful.

When working with an individual stratum, a stronger assertion is true (to be proved in Sect. 3.8): Consider the map

$$(p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}}) : S^\lambda_{\mathrm{Ran}} \to (\mathrm{Ran}(X)\times X^\lambda)^{\supset} \underset{X^\lambda}{\times} \overline{\mathrm{Bun}}_N^{=\lambda}.$$

Proposition 3.4.5. *The functor*

$$(p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})^! : \mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^{\supset} \underset{X^\lambda}{\times} \overline{\mathrm{Bun}}_N^{=\lambda}) \to \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$$

is fully faithful.

Combining with Lemma 1.3.3, we obtain:

Corollary 3.4.6. *The functor*

$$(\pi^\lambda_{\mathrm{Ran}})^! : \mathrm{Shv}(\overline{\mathrm{Bun}}_N^{=\lambda}) \to \mathrm{Shv}(S^\lambda_{\mathrm{Ran}})$$

is fully faithful.

3.4.7. Next we claim:

Proposition 3.4.8. *For every finite set I, the functor*

$$(\pi_I)^! : \mathrm{Shv}(\overline{\mathrm{Bun}}_N) \to \mathrm{Shv}(\overline{S}^0_I)$$

sends $\mathrm{SI}^{\leq 0}_{\mathrm{glob}}$ *to* $\mathrm{SI}^{\leq 0}_I$.

Proof. Note that an object $\mathcal{F} \in \mathrm{Shv}(\overline{S}^0_I)$ belongs to $\mathrm{SI}^{\leq 0}_I$ if and only if $(\mathbf{i}^\lambda)^!(\mathcal{F})$ belongs to $\mathrm{SI}^{=\lambda}_I$ for every λ. Now the result follows from the identification

$$\mathrm{pr}^\lambda_I \circ p^\lambda_I = p^\lambda_{\mathrm{glob}} \circ \pi^\lambda_I.$$

□

We will now deduce:

Corollary 3.4.9. *An object of* $\mathrm{Shv}(\overline{\mathrm{Bun}}_N)$ *belongs to* $\mathrm{SI}^{\leq 0}_{\mathrm{glob}}$ if and only if *its pullback under* $(\pi_{\mathrm{Ran}})^!$ *belongs to* $\mathrm{SI}^{\leq 0}_{\mathrm{Ran}} \subset \mathrm{Shv}(\overline{S}^0_{\mathrm{Ran}})$.

Proof. The "only if" direction is the content of Proposition 3.4.8.

For the "if" direction, we need to show that if an object $\mathcal{F} \in \mathrm{Shv}(\overline{\mathrm{Bun}}^{=\lambda}_N)$ is such that

$$(\pi^\lambda_{\mathrm{Ran}})^!(\mathcal{F}) \simeq (p^\lambda_{\mathrm{Ran}})^!(\mathcal{F}')$$

for some $\mathcal{F}' \in \mathrm{Shv}((\mathrm{Ran}(X) \times X^\lambda)^\supset)$, then $\mathcal{F}$ is the pullback of an object in $\mathrm{Shv}(X^\lambda)$ along $p^\lambda_{\mathrm{glob}}$.

By Proposition 3.4.5, in the diagram

$$\begin{array}{ccc}
S^\lambda & & \\
\Big\downarrow{\scriptstyle p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}}} & & \\
(\mathrm{Ran}(X) \times X^\lambda)^\supset \underset{X^\lambda}{\times} \overline{\mathrm{Bun}}^{=\lambda}_N & \xrightarrow{\mathrm{id}_{(\mathrm{Ran}(X)\times X^\lambda)^\supset} \times p^\lambda_{\mathrm{glob}}} & (\mathrm{Ran}(X) \times X^\lambda)^\supset \\
\Big\downarrow{\scriptstyle \mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N}} & & \Big\downarrow{\scriptstyle \mathrm{pr}^\lambda_{\mathrm{Ran}}} \\
\overline{\mathrm{Bun}}^{=\lambda}_N & \xrightarrow{p^\lambda_{\mathrm{glob}}} & X^\lambda
\end{array}$$

we have

$$\begin{aligned}
\mathcal{F} &\simeq (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})_! \circ (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})^!(\mathcal{F}) \simeq \\
&\simeq (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})_! \circ (p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})_! \circ (p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})^! \circ (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})^!(\mathcal{F}) \simeq \\
&\simeq (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})_! \circ (p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})_! \circ (p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})^! \circ (\mathrm{id}_{(\mathrm{Ran}(X)\times X^\lambda)^\supset} \times p^\lambda_{\mathrm{glob}})^!(\mathcal{F}') \simeq \\
&\simeq (\mathrm{pr}^\lambda_{\mathrm{Ran}} \times \mathrm{id}_{\overline{\mathrm{Bun}}^{=\lambda}_N})_! \circ (\mathrm{id}_{(\mathrm{Ran}(X)\times X^\lambda)^\supset} \times p^\lambda_{\mathrm{glob}})^!(\mathcal{F}') \simeq (p^\lambda_{\mathrm{glob}})^! \circ (\mathrm{pr}^\lambda_{\mathrm{Ran}})_!(\mathcal{F}'),
\end{aligned}$$

as required (the last isomorphism is base change, which holds due to the fact that the map $\mathrm{pr}^\lambda_{\mathrm{Ran}}$ is pseudo-proper).

□

3.5. **The key isomorphism.**

3.5.1. The base change isomorphism

$$(\pi_I)^! \circ (\mathbf{i}^\lambda_{\text{glob}})_* \simeq (\mathbf{i}^\lambda)_* \circ (\pi_I)^!$$

in the diagram (3.2) gives rise to a natural transformation

$$(\mathbf{i}^\lambda)^* \circ (\pi_I)^! \to (\pi_I^\lambda)^! \circ (\mathbf{i}^\lambda_{\text{glob}})^* \tag{3.3}$$

as functors

$$\text{Shv}(\overline{\text{Bun}}_N) \rightrightarrows \text{Pro}(\text{Shv}(S_I^\lambda)).$$

Note that both sides in (3.3) belong to

$$\text{SI}_I^{=\lambda} \subset \text{Shv}(S_I^\lambda) \subset \text{Pro}(\text{Shv}(S_I^\lambda)),$$

when evaluated on objects of $\text{SI}^{\leq 0}_{\text{glob}}$.

3.5.2. In Sect. 3.7, we will prove:

Proposition 3.5.3. *The natural transformation* (3.3) *is an isomorphism when evaluated on objects of* $\text{SI}^{\leq 0}_{\text{glob}}$.

We will now deduce some corollaries of Proposition 3.5.3; these will easily imply Theorem 3.3.3, see Sect. 3.5.8.

First, combining Proposition 3.5.3 with Proposition 1.4.3(c), we obtain:

Corollary 3.5.4. *The natural transformation*

$$(\mathbf{i}^\lambda)^* \circ (\pi_{\text{Ran}})^! \to (\pi^\lambda_{\text{Ran}})^! \circ (\mathbf{i}^\lambda_{\text{glob}})^*$$

is an isomorphism when evaluated on objects of $\text{SI}^{\leq 0}_{\text{glob}}$.

Next, from Proposition 3.5.3 we formally obtain:

Corollary 3.5.5. *The natural transformation*

$$(\mathbf{i}^\lambda)_! \circ (\pi_I^\lambda)^! \to (\pi_I)^! \circ (\mathbf{i}^\lambda_{\text{glob}})_!,$$

arising by adjunction from

$$(\pi_I^\lambda)^! \circ (\mathbf{i}^\lambda_{\text{glob}})^! \simeq (\mathbf{i}^\lambda)^! \circ (\pi_I)^!,$$

is an isomorphism when evaluated on objects of $\text{SI}^{=\lambda}_{\text{glob}}$.

Combining Corollary 3.5.5 with Corollary 1.4.4(c), we obtain:

Corollary 3.5.6. *The natural transformation*

$$(\mathbf{i}^\lambda)_! \circ (\pi^\lambda_{\text{Ran}})^! \to (\pi_{\text{Ran}})^! \circ (\mathbf{i}_{\text{glob}})_!$$

is an isomorphism when evaluated on objects of $\text{SI}^{=\lambda}_{\text{glob}}$.

Finally, we claim:

Corollary 3.5.7. *The functor*

$$\pi^![d] : \text{SI}^{\leq 0}_{\text{glob}} \to \text{SI}^{\leq 0}_{\text{Ran}}$$

is t-exact.

Proof. This follows from Corollary 3.5.4, combined with the (tautological) isomorphism

$$(\mathbf{i}^\lambda)^! \circ (\pi_{\text{Ran}})^! \simeq (\pi^\lambda_{\text{Ran}})^! \circ (\mathbf{i}^\lambda_{\text{glob}})^!.$$

□

3.5.8. Note that Corollary 3.5.7 immediately implies Theorem 3.3.3.

Remark 3.5.9. In Sect. 5.4 we will present another construction of the map in one direction

$$\text{IC}^{\frac{\infty}{2}}_{\text{Ran}} \to \pi^!(\text{IC}^{\frac{\infty}{2}}_{\text{glob}})[d],$$

where we will realize $\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$ as $'\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}$.

3.5.10. Let us now prove Proposition 2.1.7.

Proof. By Corollary 3.5.7, it suffices to show that the objects

$$(\mathbf{i}^\lambda_{\text{glob}})_!(\text{IC}_{\overline{\text{Bun}}_N^{=\lambda}}) \text{ and } (\mathbf{i}^\lambda_{\text{glob}})_*(\text{IC}_{\overline{\text{Bun}}_N^{=\lambda}})$$

belong to the heart of the t-structure (i.e., are perverse sheaves on $\overline{\text{Bun}}_B$).

We claim that the morphism $\mathbf{i}^\lambda_{\text{glob}}$ is affine, which would imply that the functors $(\mathbf{i}^\lambda_{\text{glob}})_!$ and $(\mathbf{i}^\lambda_{\text{glob}})_*$ are t-exact.

Indeed, $\mathbf{i}^\lambda_{\text{glob}}$ is the base change of the morphism

$$\mathbf{i}^\lambda_{\text{glob}} : \text{Bun}_B \times X^\lambda \to \overline{\text{Bun}}_B,$$

which we claim to be affine.

Indeed, $\mathbf{i}^\lambda_{\text{glob}} = \bar{\mathbf{i}}^\lambda_{\text{glob}} \circ \mathbf{j}^\lambda_{\text{glob}}$, where $\bar{\mathbf{i}}^\lambda_{\text{glob}}$ is a finite morphism, and $\mathbf{j}^\lambda_{\text{glob}}$ is known to be an affine open embedding (see [FGV, Proposition 3.3.1]). □

3.6. **Computation of fibers.** In this subsection we will prove Theorem 2.4.5. One proof follows from the description of the objects

$$(\mathbf{i}^\lambda_{\text{glob}})^!(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \text{ and } (\mathbf{i}^\lambda_{\text{glob}})^*(\text{IC}^{\frac{\infty}{2}}_{\text{glob}})$$

in [BG2, Poroposition 4.4], combined with Corollary 3.5.7. But in fact one can reprove the description from [BG2, Proposition 4.4], see Theorem 3.6.2 below.

3.6.1. Thus, we first prove:

Theorem 3.6.2.
(a) $(\mathbf{i}^\lambda_{\text{glob}})^*(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \simeq (p^\lambda_{\text{glob}})^!(\text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda})[-d - \langle \lambda, 2\check{\rho}\rangle]$.
(b) $(\mathbf{i}^\lambda_{\text{glob}})^!(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \simeq (p^\lambda_{\text{glob}})^!(\text{Fact}^{\text{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda})[-d - \langle \lambda, 2\check{\rho}\rangle]$.

Proof. Let $\mathcal{F}^\lambda \in \text{Shv}(X^\lambda)$ we such that

$$(\mathbf{i}^\lambda_{\text{glob}})^*(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \simeq (p^\lambda_{\text{glob}})^!(\mathcal{F}^\lambda)[-d - \langle \lambda, 2\check{\rho}\rangle].$$

We will show that

$$\mathcal{F}^\lambda \simeq \text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}.$$

Indeed, by Corollary 3.5.7 and Theorem 3.3.3, we have:

$$(\pi^\lambda_{\text{Ran}})^! \circ (\mathbf{i}^\lambda_{\text{glob}})^*(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \simeq (\mathbf{i}^\lambda)^*(\text{IC}^{\frac{\infty}{2}}_{\text{Ran}})[-d]. \tag{3.4}$$

We have

$$(\pi^\lambda_{\text{Ran}})^! \circ (\mathbf{i}^\lambda_{\text{glob}})^*(\text{IC}^{\frac{\infty}{2}}_{\text{glob}}) \simeq (\pi^\lambda_{\text{Ran}})^! \circ (p^\lambda_{\text{glob}})^!(\mathcal{F}^\lambda)[-d - \langle \lambda, 2\check{\rho}\rangle] \simeq (p^\lambda_{\text{Ran}})^! \circ (\text{pr}^\lambda_{\text{Ran}})^!(\mathcal{F}^\lambda)[-d - \langle \lambda, 2\check{\rho}\rangle] \tag{3.5}$$

and by Remark 2.7.5, we have:

$$(\mathbf{i}^\lambda)^*(\text{IC}^{\frac{\infty}{2}}_{\text{Ran}}) \simeq (p^\lambda_{\text{Ran}})^! \circ (\text{pr}^\lambda_{\text{Ran}})^!(\text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda})[-\langle \lambda, 2\check{\rho}\rangle]. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain

$$(p^\lambda_{\text{Ran}})^! \circ (\text{pr}^\lambda_{\text{Ran}})^!(\mathcal{F}^\lambda) \simeq (p^\lambda_{\text{Ran}})^! \circ (\text{pr}^\lambda_{\text{Ran}})^!(\text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda}).$$

Since the functor $(p^\lambda_{\text{Ran}})^! \circ (\text{pr}^\lambda_{\text{Ran}})^!$ is fully faithful, we obtain the desired

$$\mathcal{F}^\lambda \simeq \text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda},$$

proving point (a).

Since $\text{IC}^{\frac{\infty}{2}}_{\text{glob}}$ is Verdier self-dual, and using the fact that

$$\mathbb{D}(\text{Fact}^{\text{coalg}}(U(\check{\mathfrak{n}}^-))_{X^\lambda}) \simeq \text{Fact}^{\text{alg}}(\mathcal{O}(\check{N}))_{X^\lambda},$$

we obtain

$$(\mathbf{i}^\lambda_{\rm glob})^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm glob}) \simeq (p^\lambda_{\rm glob})^*(\mathrm{Fact}^{\rm coalg}(U(\check{\mathfrak{n}}^-))_{X^\lambda})[d+\langle\lambda,2\check\rho\rangle] \simeq$$
$$\simeq (p^\lambda_{\rm glob})^!(\mathrm{Fact}^{\rm coalg}(U(\check{\mathfrak{n}}^-))_{X^\lambda})[-d-\langle\lambda,2\check\rho\rangle],$$

the latter isomorphism because $p^\lambda_{\rm glob}$ is smooth of relative dimension $d+\langle\lambda,2\check\rho\rangle$. This proves point (b).

□

3.6.3. Let us now prove Theorem 2.4.5.

Proof. By Remark 2.7.5, it remains to prove the assertion regarding $(\mathbf{i}^\lambda)^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm Ran})$.

Let $\mathcal{G}^\lambda\in \mathrm{Shv}((\mathrm{Ran}(X)\times X^\lambda)^\supset)$ be such that

$$(\mathbf{i}^\lambda)^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm Ran}) \simeq (p^\lambda_{\rm Ran})^!(\mathcal{G}^\lambda)[-\langle\lambda,2\check\rho\rangle].$$

Let us show that

$$\mathcal{G}^\lambda \simeq (\mathrm{pr}^\lambda_{\rm Ran})^!(\mathrm{Fact}^{\rm coalg}(U(\check{\mathfrak{n}}^-))_{X^\lambda}).$$

Indeed, by Theorem 3.3.3 and Theorem 3.6.2(b), we have:

$$(p^\lambda_{\rm Ran})^!(\mathcal{G}^\lambda)[-\langle\lambda,2\check\rho\rangle] = (\mathbf{i}^\lambda)^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm Ran}) \simeq (\mathbf{i}^\lambda)^!\circ(\pi_{\rm Ran})^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm glob})[d] \simeq$$
$$\simeq (\pi^\lambda_{\rm Ran})^!\circ(\mathbf{i}^\lambda_{\rm glob})^!(\mathrm{IC}^{\frac{\infty}{2}}_{\rm glob})[d] \simeq (\pi^\lambda_{\rm Ran})^!\circ(p^\lambda_{\rm glob})^!(\mathrm{Fact}^{\rm coalg}(U(\check{\mathfrak{n}}^-))_{X^\lambda})[-\langle\lambda,2\check\rho\rangle] \simeq$$
$$\simeq (p^\lambda_{\rm Ran})^!\circ(\mathrm{pr}^\lambda_{\rm Ran})^!(\mathrm{Fact}^{\rm coalg}(U(\check{\mathfrak{n}}^-))_{X^\lambda})[-\langle\lambda,2\check\rho\rangle].$$

Since $(p^\lambda_{\rm Ran})^!$ is fully faithful, this gives the desired isomorphism.

□

3.7. **Proof of Proposition 3.5.3.**

3.7.1. Let $\mathcal{F}$ be an object of $\mathrm{SI}^{\leq 0}_{\rm glob}$. We need to establish the isomorphism

$$(s^\lambda_I)^!\circ(\mathbf{i}^\lambda)^*\circ(\pi_I)^!(\mathcal{F}) \simeq (s^\lambda_I)^!\circ(\pi^\lambda_I)^!\circ(\mathbf{i}^\lambda_{\rm glob})^*(\mathcal{F}). \tag{3.7}$$

Note, however, that by Sect. 1.6, we have

$$(s^\lambda_I)^!\circ(\mathbf{i}^\lambda)^*\circ(\pi_I)^!(\mathcal{F}) \simeq (p^{-,\lambda}_I)_*\circ(\mathbf{i}^{-,\lambda})^!\circ(\pi_I)^!(\mathcal{F}). \tag{3.8}$$

3.7.2. For $\lambda\in\Lambda^{\rm neg}$, let $\mathcal{Z}^\lambda$ be the Zastava space, i.e., this is the open substack of

$$\overline{\mathrm{Bun}}_N \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{-\lambda}_{B^-},$$

corresponding to the condition that the B^--reduction and the generalized N-reduction of a given G-bundle are generically transversal.

Let $\mathfrak{q}$ denote the forgetful map $\mathcal{Z}^\lambda\to\overline{\mathrm{Bun}}_N$. Let $\mathfrak{p}$ denote the projection

$$\mathcal{Z}^\lambda\to X^\lambda,$$

and let $\mathfrak{s}$ denote its section

$$X^\lambda\to\mathcal{Z}^\lambda.$$

3.7.3. Note that we have a canonical identification

$$(X^I\times X^\lambda)^\supset \underset{X^\lambda}{\times}\mathcal{Z}^\lambda \simeq \overline{S}^0_I\cap S^{-,\lambda}_I, \tag{3.9}$$

so that the projection

$$(\mathrm{id}_{(X^I\times X^\lambda)^\supset}\times\mathfrak{p}) : (X^I\times X^\lambda)^\supset \underset{X^\lambda}{\times}\mathcal{Z}^\lambda \to (X^I\times X^\lambda)^\supset$$

identifies with

$$\overline{S}^0_I\cap S^{-,\lambda}_I \to S^{-,\lambda}_I \overset{p^{-,\lambda}_I}{\longrightarrow} (X^I\times X^\lambda)^\supset,$$

3.7.4. Hence, the right-hand side in (3.8) can be rewritten as

$$(\mathrm{id}_{(X^I\times X^\lambda)^\supset}\times \mathfrak{p})_*\circ (\mathrm{pr}_I^\lambda\times \mathrm{id}_{\mathcal{Z}^\lambda})^!\circ \mathfrak{q}^!(\mathcal{F}). \tag{3.10}$$

where the maps are as shown in the diagram

$$\begin{array}{ccccc}
(X^I\times X^\lambda)^\supset \underset{X^\lambda}{\times} \mathcal{Z}^\lambda & \xrightarrow{\mathrm{pr}_I^\lambda\times \mathrm{id}_{\mathcal{Z}^\lambda}} & \mathcal{Z}^\lambda & \xrightarrow{\mathfrak{q}} & \overline{\mathrm{Bun}}_N \\
\downarrow{\scriptstyle \mathrm{id}_{(X^I\times X^\lambda)^\supset}\times\mathfrak{p}} & & \downarrow{\scriptstyle\mathfrak{p}} & & \\
(X^I\times X^\lambda)^\supset & \xrightarrow{\mathrm{pr}_I^\lambda} & X^\lambda. & &
\end{array}$$

By base change, we rewrite (3.10) as

$$(\mathrm{pr}_I^\lambda)^!\circ \mathfrak{p}_*\circ \mathfrak{q}^!(\mathcal{F}). \tag{3.11}$$

3.7.5. Applying the contraction principle for the action of $\mathbb{G}_m$ along the fibers of $\mathfrak{p}$ (see [DrGa, Proposition 3.2.2]), we rewrite (3.11) as

$$(\mathrm{pr}_I^\lambda)^!\circ \mathfrak{s}^*\circ \mathfrak{q}^!(\mathcal{F}).$$

3.7.6. Note that we have a Cartesian diagram

$$\begin{array}{ccc}
X^\lambda & \xrightarrow{\mathfrak{s}} & \mathcal{Z}^\lambda \\
\downarrow{\scriptstyle\mathfrak{q}^\lambda} & & \downarrow{\scriptstyle\mathfrak{q}} \\
\overline{\mathrm{Bun}}_N^{=\lambda} & \xrightarrow{\mathbf{i}^\lambda_{\mathrm{glob}}} & \overline{\mathrm{Bun}}_N,
\end{array} \tag{3.12}$$

where the map $\mathfrak{q}^\lambda$ is given by

$$X^\lambda \simeq X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_T \to X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \simeq \overline{\mathrm{Bun}}_N^{=\lambda}.$$

Note also that the map

$$(X^I\times X^\lambda)^\supset \xrightarrow{s_I^\lambda} S_I^\lambda \xrightarrow{\pi_I^\lambda} \overline{\mathrm{Bun}}_N^{=\lambda}$$

identifies with

$$(X^I\times X^\lambda)^\supset \xrightarrow{\mathrm{pr}_I^\lambda} X^\lambda \xrightarrow{\mathfrak{q}^\lambda} \overline{\mathrm{Bun}}_N^{=\lambda}.$$

Hence, the right-hand side in (3.7) identifies with

$$(\mathrm{pr}_I^\lambda)^!\circ (\mathfrak{q}^\lambda)^!\circ (\mathbf{i}^\lambda_{\mathrm{glob}})^*(\mathcal{F}).$$

3.7.7. Hence, in order to establish the isomorphism (3.7), it suffices to show that the natural transformation

$$\mathfrak{s}^*\circ \mathfrak{q}^! \to (\mathfrak{q}^\lambda)^!\circ (\mathbf{i}^\lambda_{\mathrm{glob}})^*, \tag{3.13}$$

coming from the Cartesian square (3.12), is an isomorphism, when evaluated on objects from $\mathrm{SI}^{\leq 0}_{\mathrm{glob}}$.

However, the latter is done by repeating the argument of [Ga1, Sect. 3.9]:

We first consider the case when $-\lambda$ is *sufficiently dominant*, in which case the morphism $\mathfrak{q}$ is smooth, being the base change of $\mathrm{Bun}_{B^-}^{-\lambda}\to \mathrm{Bun}_G$. In this case, the fact that (3.13) is an isomorphism follows by smoothness.

Then we reduce the case of a general λ to one above using the factorization property of $\mathcal{Z}^\lambda$.

□

3.8. **Proof of Proposition 3.4.5.**

3.8.1. Consider the morphism

$$(p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}}) : S^\lambda_{\mathrm{Ran}} \to (\mathrm{Ran}(X) \times X^\lambda)^{\supset} \underset{X^\lambda}{\times} \overline{\mathrm{Bun}}_N^{=\lambda}.$$

A point of S^λ_{Ran} is the following data:

(i) A B-bundle $\mathcal{P}_B$ on X (denote by $\mathcal{P}_T$ the induced T-bundle);

(ii) A Λ^{neg}-valued divisor D on X (we denote by $\mathcal{O}(D)$ the corresponding T-bundle);

(iii) An identification $\mathcal{P}_T \simeq \mathcal{O}(D)$;

(iv) A finite non-empty set $\mathcal{I}$ of points of X that contains the support of D;

(v) A trivialization α of $\mathcal{P}_B$ away from $\mathcal{I}$, such that the induced trivialization of $\mathcal{P}_T|_{X-\mathcal{I}}$ agrees with the tautological trivialization of $\mathcal{O}(D)|_{X-\mathcal{I}}$.

3.8.2. The map $(p^\lambda_{\mathrm{Ran}} \times \pi^\lambda_{\mathrm{Ran}})$ amounts to forgetting the data of (v) above. It is clear that for an affine test-scheme Y and a Y-point of

$$(\mathrm{Ran}(X) \times X^\lambda)^{\supset} \underset{X^\lambda}{\times} \overline{\mathrm{Bun}}_N^{=\lambda},$$

the set of its lifts to a Y-point of S^λ_{Ran} is non-empty and is a torsor for the group

$$\mathrm{Maps}(Y \times X - \Gamma_{\mathcal{I}}, N).$$

For a given Y and $\mathcal{I} \subset \mathrm{Maps}(Y, X)$, let $\mathbf{Maps}_Y(X - \mathcal{I}, N)$ be the prestack over Y that assigns to $Y' \to Y$ the set of maps

$$\mathrm{Maps}(Y' \times X - (Y' \underset{Y}{\times} \Gamma_{\mathcal{I}}), N).$$

Thus, it suffices to show that the projection $\mathbf{Maps}_Y(X - \mathcal{I}, N) \to Y$ is *universally homologically contractible*, see [Ga2, Sect. 2.5] for what this means.

3.8.3. Since N is unipotent, it is isomorphic to $\mathbb{A}^m$, where $m = \dim(N)$. Hence, it suffices to show that the map

$$\mathbf{Maps}_Y(X - \mathcal{I}, \mathbb{A}^1) \to Y$$

is universally homologically contractible.

However, the latter is clear: the prestack $\mathbf{Maps}_Y(X - \mathcal{I}, \mathbb{A}^1)$ is isomorphic to the ind-scheme $\mathbb{A}^\infty \times Y$, where

$$\mathbb{A}^\infty \simeq \underset{n}{\mathrm{colim}}\, \mathbb{A}^n.$$

□

4. The Hecke property of the semi-infinite IC sheaf

The goal of this section is to show that the object $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ that we have constructed satisfies the (appropriately formulate) Hecke eigen-property.

4.1. **Pointwise Hecke property.**

4.1.1. Consider the category $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathrm{Gr}_{G,\mathrm{Ran}})$, i.e., we impose the structure of equivariance with respect to group-scheme of arcs into T over the relevant base prestack.

The action of $\mathfrak{L}(T)_{\mathrm{Ran}}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ by left multiplication defines an action of $\mathrm{Sph}_{T,\mathrm{Ran}}$ on $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathrm{Gr}_{G,\mathrm{Ran}})$.

4.1.2. We consider $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathrm{Gr}_{G,\mathrm{Ran}})$ as acted on by the monoidal category $\mathrm{Sph}_{G,\mathrm{Ran}}$ on the right by convolutions.

This action commutes with the left action of $\mathrm{Sph}_{T,\mathrm{Ran}}$.

4.1.3. Since $\mathfrak{L}(T)_{\mathrm{Ran}}$ normalizes $\mathfrak{L}(N)_{\mathrm{Ran}}$, the category

$$(\mathrm{SI}_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} := \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}\cdot \mathfrak{L}(N)_{\mathrm{Ran}}}$$

inherits an action of $\mathrm{Sph}_{T,\mathrm{Ran}}$ and a commuting $\mathrm{Sph}_{G,\mathrm{Ran}}$-action.

Working with this version of the semi-infinite category, we can define a t-structure on it in the same way as for

$$\mathrm{SI}_{\mathrm{Ran}} := \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})^{\mathfrak{L}(N)_{\mathrm{Ran}}},$$

so that the forgetful functor

$$(\mathrm{SI}_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \to \mathrm{SI}_{\mathrm{Ran}}$$

is t-exact.

Thus, we obtain that the object $\mathrm{IC}^{\infty}_{\mathrm{Ran}} \in \mathrm{SI}_{\mathrm{Ran}} \subset \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})$ naturally lifts to an object of

$$(\mathrm{SI}_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}});$$

by a slight abuse of notation we denote it by the same character $\mathrm{IC}^{\infty}_{\mathrm{Ran}}$.

4.1.4. We fix a point x, and consider the geometric Satake functor

$$\mathrm{Sat}_{G,x} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}_{G,x},$$

which we compose with the monoidal (but non-unital) functor

$$\mathrm{Sph}_{G,x} \to \mathrm{Sph}_{G,\mathrm{Ran}}.$$

We modify the geometric Satake functor for T by applying the cohomological shift by $[-\langle \lambda, 2\check{\rho}\rangle]$ on $\mathsf{e}^{\lambda} \in \mathrm{Rep}(\check{T})$. Denote the resulting functor by

$$\mathrm{Sat}'_{T,x} : \mathrm{Rep}(\check{T}) \to \mathrm{Sph}_{T,x},$$

which we also compose with the monoidal (but non-unital) functor

$$\mathrm{Sph}_{T,x} \to \mathrm{Sph}_{T,\mathrm{Ran}}.$$

4.1.5. Thus, we obtain that $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$ is a bimodule category for $(\mathrm{Rep}(\check{T}), \mathrm{Rep}(\check{G}))$. In this case, we can talk about the category of *graded Hecke objects* in $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$, denoted

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})),$$

see [Ga1, Sect. 4.3.5], and also Sect. 4.4.1 below.

These are objects $\mathcal{F} \in \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$, equipped with a system of isomorphisms

$$\mathcal{F} \star \mathrm{Sat}_{G,x}(V) \overset{\phi(V,\mathcal{F})}{\longrightarrow} \mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V)) \star \mathcal{F}, \quad V \in \mathrm{Rep}(\check{G})$$

that are compatible with the monoidal structure on $\mathrm{Rep}(\check{G})$ in the sense that the diagrams

$$\begin{array}{ccc}
\mathcal{F} \star \mathrm{Sat}_{G,x}(V_1) \star \mathrm{Sat}_{G,x}(V_2) & \xrightarrow{\phi(V_1,\mathcal{F})} & \mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V_1)) \star \mathcal{F} \star \mathrm{Sat}_{G,x}(V_2) \\
\sim\downarrow & & \downarrow \phi(V_2,\mathcal{F}) \\
\mathcal{F} \star \mathrm{Sat}_{G,x}(V_1 \otimes V_2) & \longrightarrow & \mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V_1)) \star \mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V_2)) \star \mathcal{F} \\
\phi(V_1\otimes V_2,\mathcal{F})\downarrow & & \downarrow\sim \\
\mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V_1 \otimes V_2)) \star \mathcal{F} & \xrightarrow{\sim} & \mathrm{Sat}'_{T,x}(\mathrm{Res}^{\check{G}}_{\check{T}}(V_1) \otimes \mathrm{Res}^{\check{G}}_{\check{T}}(V_2)) \star \mathcal{F},
\end{array}$$

along with a coherent system of higher compatibilities.

4.1.6. We will prove:

Theorem-Construction 4.1.7. *The object* $\mathrm{IC}^{\infty}_{\mathrm{Ran}} \in \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$ *naturally lifts to an object of* $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}))$.

Several remarks are in order.

Remark 4.1.8. In the proof of Theorem 4.1.7, the $\mathrm{IC}^{\infty}_{\mathrm{Ran}}$ will come as its incarnation as $'\mathrm{IC}^{\infty}_{\mathrm{Ran}}$, constructed in Sect. 2.6.

Remark 4.1.9. Consider the restriction

$$\mathrm{IC}^{\infty}_{x} := \mathrm{IC}^{\infty}_{\mathrm{Ran}}|_{\mathrm{Gr}_{G,x}}.$$

The Hecke structure on $\mathrm{IC}^{\infty}_{\mathrm{Ran}}$ induces one on IC^{∞}_{x}. It will follow from the construction and [Ga1, Sect. 6.2.5] that the resulting Hecke structure on IC^{∞}_{x} coincides with one constructed in [Ga1, Sect. 5.1].

Remark 4.1.10. In order to prove Theorem 4.1.7 we will need to consider the Hecke action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$ over the entire Ran space. The next few subsections are devoted to setting up the corresponding formalism.

4.2. **Categories over the Ran space, continued.**

4.2.1. Recall the construction

$$\mathcal{A} \rightsquigarrow \mathrm{Fact}(\mathcal{A})_I \tag{4.1}$$

of Sect. 2.5, viewed as a functor $\mathrm{DGCat}^{\mathrm{SymMon}} \to \mathrm{Shv}(X^I)$-**mod**.

Note that the functor (4.1) has a natural right-lax symmetric monoidal structure, i.e., we have the natural transformation

$$\mathrm{Fact}(\mathcal{A}')_I \underset{\mathrm{Shv}(X^I)}{\otimes} \mathrm{Fact}(\mathcal{A}'')_I \to \mathrm{Fact}(\mathcal{A}' \otimes \mathcal{A}'')_I.$$

In particular, since any $\mathcal{A} \in \mathrm{DGCat}^{\mathrm{SymMon}}$ can be viewed as an object in $\mathrm{ComAlg}(\mathrm{DGCat}^{\mathrm{SymMon}})$, we obtain that $\mathrm{Fact}(\mathcal{A})_I$ itself acquires a structure of symmetric monoidal category.

4.2.2. For a surjection of finite sets $I_1 \twoheadrightarrow I_2$, the corresponding functor

$$\mathrm{Fact}(\mathcal{A})_{I_1} \to \mathrm{Fact}(\mathcal{A})_{I_2} \tag{4.2}$$

is naturally symmetric monoidal. In particular, we obtain that

$$\mathrm{Fact}(\mathcal{A})_{\mathrm{Ran}} \simeq \lim_{I} \mathrm{Fact}(\mathcal{A})_I$$

acquires a natural symmetric monoidal structure, and a homomorphism

$$\mathrm{Shv}(\mathrm{Ran}(X)) \to \mathrm{Fact}(\mathcal{A})_{\mathrm{Ran}}.$$

4.2.3. Let $\mathcal{A}' \to \mathcal{A}''$ be a right-lax symmetric monoidal functor. The functor (4.1) gives rise to a right-lax symmetric monoidal functor

$$\mathrm{Fact}(\mathcal{A}')_I \to \mathrm{Fact}(\mathcal{A}'')_I,$$

compatible with the restriction functors (4.2). Varying I, we obtain a right-lax symmetric monoidal functor

$$\mathrm{Fact}(\mathcal{A}')_{\mathrm{Ran}} \to \mathrm{Fact}(\mathcal{A}'')_{\mathrm{Ran}}.$$

In particular, a commutative algebra object A in $\mathcal{A}$, viewed as a right-lax symmetric monoidal functor $\mathrm{Vect} \to \mathcal{A}$, gives rise to a commutative algebra

$$\mathrm{Fact}^{\mathrm{alg}}(A)_I \in \mathrm{Fact}(\mathcal{A})_I.$$

These algebra objects are compatible under the restriction functors (4.2). Varying I, we obtain a commutative algebra object

$$\mathrm{Fact}^{\mathrm{alg}}(A)_{\mathrm{Ran}} \in \mathrm{Fact}(\mathcal{A})_{\mathrm{Ran}}.$$

4.2.4. *Examples.* Let us consider the two examples of $\mathcal{A}$ from Sect. 2.5.4.

(i) Let $\mathcal{A} = \text{Vect}$. We obtain that to $A \in \text{ComAlg}(\text{Vect})$ we can canonically assign an object $\text{Fact}^{\text{alg}}(A)_{\text{Ran}} \in \text{Shv}(\text{Ran})$.

(ii) Let $\mathcal{A}$ be the category of $\Lambda^{\text{neg}} - 0$ graded vector spaces. Note that a commutative algebra A in $\mathcal{A}$ is the same as a commutative Λ^{neg}-algebra with $A(0) = k$. On the one hand, the construction of Sect. 2.3 assigns to such an A a collection of objects

$$\text{Fact}^{\text{alg}}(A)_{X^\lambda} \in \text{Shv}(X^\lambda), \quad \lambda \in \Lambda^{\text{neg}} - 0.$$

On the other hand, we have the above object

$$\text{Fact}^{\text{alg}}(A)_{\text{Ran}} \in \text{Fact}(\mathcal{A})_{\text{Ran}}.$$

By unwinding the constructions we obtain that these two objects match up under the equivalence (2.7).

4.3. **Digression: right-lax central structures.**

4.3.1. Let $\mathcal{A}$ and $\mathcal{A}'$ be symmetric monoidal categories, and let $\mathcal{C}$ be a $(\mathcal{A}', \mathcal{A})$-bimodule category. Let $F : \mathcal{A} \to \mathcal{A}'$ be a right-lax symmetric monoidal functor.

A right-lax central structure on an object $c \in \mathcal{C}$ with respect to F is a system of maps

$$F(a) \otimes c \overset{\phi(a,c)}{\longrightarrow} c \otimes a, \quad a \in \mathcal{A}$$

that make the diagrams

$$\begin{array}{ccc}
F(a_1) \otimes (F(a_2) \otimes c) & \xrightarrow{\phi(a_2,c)} & F(a_1) \otimes (c \otimes a_2) \\
\sim\downarrow & & \downarrow\sim \\
(F(a_1) \otimes F(a_2)) \otimes c & & (F(a_1) \otimes c) \otimes a_2 \\
\downarrow & & \downarrow{\scriptstyle \phi(a_1,c)} \\
F(a_1 \otimes a_2) \otimes c & & (c \otimes a_1) \otimes a_2 \\
{\scriptstyle \phi(a_1 \otimes a_2, c)}\downarrow & & \downarrow\sim \\
c \otimes (a_1 \otimes a_2) & \xrightarrow{\text{id}} & c \otimes (a_1 \otimes a_2),
\end{array}$$

commute, along with a coherent system of higher compatibilities.

Denote the category of objects of $\mathcal{C}$ equipped with a right-lax central structure on an object with respect to F by $Z_F(\mathcal{C})$.

4.3.2. From now on we will assume that $\mathcal{A}$ is rigid (see [GR, Chapter 1, Sect. 9.1] for what this means).

If $\mathcal{A}$ is compactly generated, this condition is equivalent to requiring that the class of compact objects in $\mathcal{A}$ coincides with the class of objects that are dualizable with respect to the symmetric monoidal structure on $\mathcal{A}$.

4.3.3. Assume for a moment that F is strict (i.e., is a genuine symmetric monoidal functor). We have:

Lemma 4.3.4. *If $c \in Z_F(\mathcal{C})$, then the morphisms $\phi(a, c)$ are isomorphisms.*

In other words, this lemma says that if F is genuine, then any right-lax central structure is a genuine central structure (under the assumption that $\mathcal{A}$ is rigid).

4.3.5. Let $R_{\mathcal{A}} \in \mathcal{A} \otimes \mathcal{A}$ be the (commutative) algebra object, obtained by applying the right adjoint

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

of the monoidal operation $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, to the unit object $\mathbf{1}_{\mathcal{A}} \in \mathcal{A}$.

Consider the (commutative) algebra object

$$R^F_{\mathcal{A}} := (F \otimes \mathrm{id})(R_{\mathcal{A}}) \in \mathcal{A}' \otimes \mathcal{A}.$$

We have:

Lemma 4.3.6. *A datum of right-lax central structure on an object $c \in \mathcal{C}$ is equivalent to upgrading c to an object of $R^F_{\mathcal{A}}\text{-mod}(\mathcal{C})$.*

4.3.7. Let F' be another right-lax symmetric monoidal functor, and let $F \to F'$ be a right-lax symmetric monoidal natural transformation. Restriction defines a functor

$$Z_{F'}(\mathcal{C}) \to Z_F(\mathcal{C}). \tag{4.3}$$

In addition, we have a homomorphism of commutative algebra objects in $\mathcal{A}' \otimes \mathcal{A}$

$$R^F_{\mathcal{A}} \to R^{F'}_{\mathcal{A}}.$$

It easy to see that with respect to the equivalence of Lemma 4.3.6, the diagram

$$\begin{array}{ccc} Z_{F'}(\mathcal{C}) & \longrightarrow & Z_F(\mathcal{C}) \\ \sim\downarrow & & \downarrow\sim \\ R^{F'}_{\mathcal{A}}\text{-mod}(\mathcal{C}) & \longrightarrow & R^F_{\mathcal{A}}\text{-mod}(\mathcal{C}), \end{array}$$

commutes, where the bottom arrow is given by restriction.

In particular, we obtain that the functor (4.3) admits a left adjoint, given by

$$R^{F'}_{\mathcal{A}} \underset{R^F_{\mathcal{A}}}{\otimes} -.$$

4.3.8. We now modify our context, and we let $\mathcal{C}$ be a module category for

$$\mathrm{Fact}(\mathcal{A}' \otimes \mathcal{A})_I.$$

We have the corresponding category of right-lax central objects, denoted by the same symbol $Z_F(\mathcal{C})$, which can be identified with

$$\mathrm{Fact}(R^F_{\mathcal{A}})_I\text{-mod}(\mathcal{C}).$$

For a right-lax symmetric monoidal natural transformation $F \to F'$, the left adjoint to the restriction functor $Z_{F'}(\mathcal{C}) \to Z_F(\mathcal{C})$ is given by

$$\mathrm{Fact}(R^{F'}_{\mathcal{A}})_I \underset{\mathrm{Fact}(R^F_{\mathcal{A}})_I}{\otimes} -. \tag{4.4}$$

4.3.9. Let

$$I \rightsquigarrow \mathcal{C}_I, \quad I \in \mathrm{Fin}^{\mathrm{surj}}$$

be a compatible family of module categories over $\mathrm{Fact}(\mathcal{A}' \otimes \mathcal{A})_I$.

Set

$$\mathcal{C}_{\mathrm{Ran}} := \lim_{I \in \mathrm{Fin}^{\mathrm{surj}}} \mathcal{C}_I.$$

We can thus talk about an object $c \in \mathcal{C}_{\mathrm{Ran}}$ being equipped with a right-lax central structure with respect to F. Denote the corresponding category of right-lax central objects by $Z_F(\mathcal{C}_{\mathrm{Ran}})$.

The functors (4.4) provide a left adjoint to the forgetful functor

$$Z_{F'}(\mathcal{C}_{\mathrm{Ran}}) \to Z_F(\mathcal{C}_{\mathrm{Ran}}).$$

This follows from the fact that for a surjective map of finite sets $\phi : I_1 \twoheadrightarrow I_2$, the natural transformation in the diagram

$$\begin{array}{ccc} Z_F(\mathcal{C}_{I_1}) & \xrightarrow{\Delta^!_\phi} & Z_F(\mathcal{C}_{I_2}) \\ {\scriptstyle \mathrm{Fact}(R^{F'}_{\mathcal{A}})_{I_1} \underset{\mathrm{Fact}(R^F_{\mathcal{A}})_{I_1}}{\otimes} -}\downarrow & & \downarrow{\scriptstyle \mathrm{Fact}(R^{F'}_{\mathcal{A}})_{I_2} \underset{\mathrm{Fact}(R^F_{\mathcal{A}})_{I_2}}{\otimes} -} \\ Z_{F'}(\mathcal{C}_{I_1}) & \xrightarrow{\Delta^!_\phi} & Z_{F'}(\mathcal{C}_{I_2}) \end{array}$$

is an isomorphism.

4.4. **Hecke and Drinfeld-Plücker structures.** We will be interested in the following particular cases of the above situation[5].

4.4.1. Take $\mathcal{A} = \mathrm{Rep}(\check{G})$ and $\mathcal{A}' = \mathrm{Rep}(\check{T})$ with F' being given by restriction along $\check{T} \to \check{G}$. We denote the corresponding category $Z_{F'}(\mathcal{C})$ by

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}).$$

By Lemma 4.3.4, its objects are $c \in \mathcal{C}$, equipped with a system of *iso*morphisms

$$\mathrm{Res}^{\check{G}}_{\check{T}}(V) \otimes c \simeq c \otimes V, \quad V \in \mathrm{Rep}(\check{G}),$$

compatible with tensor products of the V's.

For this reason, we call a (right-lax) central structure on an object of $\mathcal{C}$ in this case a *graded Hecke structure*.

Equivalently, these are objects of $\mathcal{C}$ equipped with an action of the algebra

$$R^{F'}_{\mathcal{A}} := (\mathrm{Res}(\check{G}_{\check{T}}) \otimes \mathrm{id})(R_{\check{G}}),$$

where $R_{\check{G}} \in \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G})$ is the regular representation.

4.4.2. Let us now take $\mathcal{A} = \mathrm{Rep}(\check{G})$ and $\mathcal{A}' = \mathrm{Rep}(\check{T})$, but the functor F is given by the *non-derived* functor of $\check{N}$-invariants

$$V^\lambda \mapsto V^\lambda(\lambda) = \mathsf{e}^\lambda.$$

The corresponding algebra object

$$R^F_{\mathcal{A}} \in \mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$$

is $\mathcal{O}(\overline{\check{N}\backslash\check{G}})$, where $\overline{\check{N}\backslash\check{G}}$ is the base affine space of $\check{G}$, viewed as acted on on the left by $\check{T}$ and on the right by $\check{G}$.

We denote the corresponding category $Z_F(\mathcal{C})$ by

$$\mathrm{DrPl}(\mathcal{C}).$$

By defintiion, its objects are $c \in \mathcal{C}$, equipped with a collection of maps

$$\mathsf{e}^\lambda \otimes c \overset{\phi(\lambda,c)}{\longrightarrow} c \otimes V^\lambda$$

[5]The formalism described in this subsection (as well as the term was suggested by S. Raskin.

that make the diagrams

$$\begin{CD}
\mathsf{e}^{\lambda}\otimes(\mathsf{e}^{\mu}\otimes c) @>{\phi(\mu,c)}>> \mathsf{e}^{\lambda}\otimes(c\otimes V^{\mu}) \\
@V{\sim}VV @VV{\sim}V \\
(\mathsf{e}^{\lambda}\otimes\mathsf{e}^{\mu})\otimes c @. (\mathsf{e}^{\lambda}\otimes c)\otimes V^{\mu} \\
@V{\sim}VV @VV{\phi(\lambda,c)}V \\
\mathsf{e}^{\lambda+\mu}\otimes c @. (c\otimes V^{\lambda})\otimes V^{\mu} \\
@V{\phi(\lambda+\mu,c)}VV @VV{\sim}V \\
c\otimes V^{\lambda+\mu} @>>> c\otimes(V^{\lambda}\otimes V^{\mu})
\end{CD}$$

commute, along with a coherent system of higher compatibilities.

We will call a right-lax central structure on an object of $\mathcal{C}$ in this case a *Drinfeld-Plücker* structure.

4.4.3. We have a right-lax symmetric monoidal natural transformation $F \to F'$,

$$\mathsf{e}^{\lambda} \to \mathrm{Res}^{\check{G}}_{\check{T}}(V^{\lambda}).$$

The corresponding morphism of commutative algebra objects in $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ is given by pull-back along the projection map

$$\check{G} \to \overline{\check{N}\backslash\check{G}}.$$

Consider the forgetful functor

$$\mathrm{Res}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}} : \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathrm{DrPl}(\mathcal{C}),$$

and its left adjoint

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}} : \mathrm{DrPl}(\mathcal{C}) \to \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}).$$

4.4.4. Let us now recall the statement of [Ga1, Proposition 6.2.4] that describes the composition

$$\mathrm{DrPl}(\mathcal{C}) \overset{\mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}}{\longrightarrow} \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathcal{C}, \tag{4.5}$$

where the second arrow is the forgetful functor.

Given an object $c \in \mathrm{DrPl}(\mathcal{C})$, the construction of [Ga1, Sect. 2.7] defines a functor $\Lambda^+ \to \mathcal{C}$, which at the level of objects sends $\lambda \in \Lambda^+$ to

$$\mathsf{e}^{-\lambda} \otimes c \otimes V^{\lambda}.$$

The assertion [Ga1, Proposition 6.2.4] says that the value of (4.5) on the above c is canonically identified with

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^{-\lambda} \otimes c \otimes V^{\lambda}.$$

4.4.5. We now place ourselves in the context of Sect. 4.3.8. Let $\mathcal{C}$ be a module category for

$$\mathrm{Fact}(\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G}))_I.$$

We denote corresponding categories $Z_{F'}(\mathcal{C})$ and $Z_F(\mathcal{C})$ by $\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C})$ and $\mathrm{DrPl}(\mathcal{C})$, respectively.

Let $c \in \mathcal{C}$ be an object of $Z_F(\mathcal{C})$. We wish to describe the value on c of the composite functor

$$\mathrm{DrPl}(\mathcal{C}) \overset{\mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}}{\longrightarrow} \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathcal{C} \tag{4.6}$$

4.4.6. For $\underline{\lambda} \in \text{Maps}(I, \Lambda^+)$, recall the object $V^{\underline{\lambda}} \in \text{Fact}(\text{Rep}(\check{G}))_I$, see Sect. 2.6.2. Similarly, we have the object

$$\mathsf{e}^{\underline{\lambda}} \in \text{Fact}(\text{Rep}(\check{T}))_I.$$

The construction of [Ga1, Sect.2.7] defines on the assignment

$$\underline{\lambda} \mapsto \mathsf{e}^{-\underline{\lambda}} \otimes c \otimes V^{\underline{\lambda}}$$

a structure of a functor

$$\text{Maps}(I, \Lambda^+) \to \mathcal{C}.$$

Generalizing [Ga1, Proposition 6.2.4] one shows:

Proposition 4.4.7. *The value of the composite functor* (4.6) *on* $c \in \text{DrPl}(\mathcal{C})$ *identifies canonically with*

$$\underset{\underline{\lambda} \in \text{Maps}(I, \Lambda^+)}{\text{colim}} \underline{\lambda} \mapsto \mathsf{e}^{-\underline{\lambda}} \otimes c \otimes V^{\underline{\lambda}}.$$

4.4.8. Let $I \rightsquigarrow \mathcal{C}_I$ be as in Sect. 4.3.9. Consider the corresponding categories $\text{DrPl}(\mathcal{C}_{\text{Ran}})$ and $\text{Hecke}_{\check{G},\check{T}}(\mathcal{C}_{\text{Ran}})$.

The compatibility of the functors $\text{Ind}_{\text{DrPl}}^{\text{Hecke}_{\check{G},\check{T}}}$ for surjections of finite sets gives to a well-defined functor

$$\text{Ind}_{\text{DrPl}}^{\text{Hecke}_{\check{G},\check{T}}} : \text{DrPl}(\mathcal{C}_{\text{Ran}}) \to \text{Hecke}_{\check{G},\check{T}}(\mathcal{C}_{\text{Ran}}),$$

left adjoint to the restriction functor.

For $c \in \text{DrPl}(\mathcal{C}_{\text{Ran}})$, the value of the composite functor

$$\text{DrPl}(\mathcal{C}) \overset{\text{Ind}_{\text{DrPl}}^{\text{Hecke}_{\check{G},\check{T}}}}{\longrightarrow} \text{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathcal{C} \to \mathcal{C}_I$$

is given by

$$\underset{\underline{\lambda} \in \text{Maps}(I, \Lambda^+)}{\text{colim}} \underline{\lambda} \mapsto \mathsf{e}^{-\underline{\lambda}} \otimes c_I \otimes V^{\underline{\lambda}},$$

where c_I is the value of c in $\mathcal{C}_I$.

4.5. **The Hecke property–enhanced statement.**

4.5.1. The key property of the geometric Satake functor

$$\text{Sat}_{G,I} : \text{Fact}(\text{Rep}(\check{G}))_I \to \text{Sph}_{G,I}$$

is that it is has a natural monoidal structure.

The same applies to the modified geometric Satake functor $\text{Sat}'_{T,I}$ for T.

Thus, we obtain that the category $\text{Shv}(\mathfrak{L}^+(T)_I\backslash\text{Gr}_{G,I})$ is as acted on by the monoidal category $\text{Fact}(\text{Rep}(\check{T}) \otimes \text{Rep}(\check{G}))_I$.

These actions are compatible under surjective maps of finite sets $I_1 \twoheadrightarrow I_2$.

4.5.2. Consider the object

$$\delta_{1_{\text{Gr}},I} := (s_I)_!(\omega_{X^I}) \in \text{Shv}(\mathfrak{L}^+(T)_I\backslash\text{Gr}_{G,I}),$$

where $s_I : X^I \to \text{Gr}_{G,I}$ is the unit section.

It follows from the construction of the functor $\text{Sat}_{G,I}$ that $\delta_{\underline{0},I}$ lifts canonically to an object of

$$\text{DrPl}(\text{Shv}(\mathfrak{L}^+(T)_I\backslash\text{Gr}_{G,I})).$$

4.5.3. Consider the corresponding object

$$\mathrm{Ind}_{\mathrm{DrPl}}^{\mathrm{Hecke}_{\check{G},\check{T}}}(\delta_{1_{\mathrm{Gr}},I}) \in \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_I\backslash \mathrm{Gr}_{G,I})).$$

It follows from Proposition 4.4.7 that its image under the forgetful functor

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_I\backslash \mathrm{Gr}_{G,I})) \to \mathrm{Shv}(\mathfrak{L}^+(T)_I\backslash \mathrm{Gr}_{G,I}) \to \mathrm{Shv}(\mathrm{Gr}_{G,I})$$

identifies canonically with the object $\mathrm{IC}_I^{\frac{\infty}{2}}$, constructed in Sect. 2.6.6.

4.5.4. Consider now the object

$$\delta_{1_{\mathrm{Gr}},\mathrm{Ran}} := (s_{\mathrm{Ran}})_!(\omega_{\mathrm{Ran}(X)}) \in \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}),$$

where $s_{\mathrm{Ran}} : \mathrm{Ran}(X) \to \mathrm{Gr}_{G,\mathrm{Ran}}$ is the unit section.

It naturally lifts to an object of

$$\mathrm{DrPl}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})).$$

Consider the corresponding object

$$\mathrm{Ind}_{\mathrm{DrPl}}^{\mathrm{Hecke}_{\check{G},\check{T}}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran}}) \in \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})).$$

By Sect. 4.4.8, the image of $\mathrm{Ind}_{\mathrm{DrPl}}^{\mathrm{Hecke}_{\check{G},\check{T}}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran}})$ under the forgetful functor

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}})) \to \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

identifies canonically with the object $'\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$, constructed in Sect. 2.6.8.

Remark 4.5.5. The latter could be used to define on the assignment

$$I \rightsquigarrow \mathrm{IC}_I^{\frac{\infty}{2}}$$

a homotopy-coherent system of compatibilities as I varies over $\mathrm{Fin}^{\mathrm{surj}}$.

4.5.6. Using the isomorphism

$$'\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}} \simeq \mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$$

of Theorem 2.7.2, we thus obtain a lift of $\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$ to an object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}))$.

Summarizing, we obtain:

Theorem 4.5.7. *The object* $\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}))$ *naturally lifts to an object of* $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash \mathrm{Gr}_{G,\mathrm{Ran}}))$.

Restricting along the symmetric monoidal functor

$$\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G}) \to \mathrm{Fact}(\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G}))_{\mathrm{Ran}},$$

corresponding to $x \in X$, we obtain the construction, whose existence is stated in Theorem 4.1.7.

5. Local vs global compatibility of the Hecke structure

In this section we will establish a compatibility between the Hecke structure on $\mathrm{IC}_{\mathrm{Ran}}^{\frac{\infty}{2}}$ constructed in the previous section and the corresponding structure on $\mathrm{IC}_{\mathrm{glob}}^{\frac{\infty}{2}}$ established in [BG1].

5.1. **The relative version of the Ran Grassmannian.**

5.1.1. We introduce a relative version of the prestack $\mathrm{Gr}_{G,\mathrm{Ran}}$ over Bun_T, denoted $\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T$, as follows.

Let $(\mathrm{Ran}(X) \times \mathrm{Bun}_T)^{\mathrm{level}}$ be the prestack that classifies the data of $(\mathcal{P}_T, \mathcal{I}, \beta)$, where:

(i) $\mathcal{I}$ is a finite non-empty collection of points on X;

(ii) $\mathcal{P}_T$ is a T-bundle on X;

(iii) β is a trivialization of $\mathcal{P}_T$ on the formal neighborhood of $\Gamma_{\mathcal{I}}$.

The prestack $(\mathrm{Ran}(X) \times \mathrm{Bun}_T)^{\mathrm{level}}$ is acted on by $\mathfrak{L}(T)_{\mathrm{Ran}}$, and the map

$$(\mathrm{Ran}(X) \times \mathrm{Bun}_T)^{\mathrm{level}} \to \mathrm{Bun}_T \times \mathrm{Ran}(X)$$

is a $\mathfrak{L}^+(T)_{\mathrm{Ran}}$-torsor, locally trivial in the étale (in fact, even Zariski, since T is a torus) topology.

We set

$$\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T := \mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \left(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Bun}_T)^{\mathrm{level}} \right).$$

We have a tautological projection

$$r : \mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T \to \mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathrm{Gr}_{G,\mathrm{Ran}}.$$

5.1.2. The right action of the groupoid

$$\mathfrak{L}^+(G)_{\mathrm{Ran}} \backslash \mathfrak{L}(G)_{\mathrm{Ran}} / \mathfrak{L}^+(G)_{\mathrm{Ran}} \tag{5.1}$$

on $\mathrm{Gr}_{G,\mathrm{Ran}}$ naturally lifts to an action on $\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T$, in a way compatible with the projection r.

In addition, by construction, we have an action of the groupoid

$$\mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathfrak{L}(T)_{\mathrm{Ran}} / \mathfrak{L}^+(T)_{\mathrm{Ran}} \tag{5.2}$$

on $\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T$, also compatible with the projection r.

In particular, we obtain that $\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T)$ is a bimodule category for $(\mathrm{Sph}_{T,\mathrm{Ran}}, \mathrm{Sph}_{G,\mathrm{Ran}})$, and hence for $(\mathrm{Fact}(\mathrm{Rep}(\check{T})_{\mathrm{Ran}}, \mathrm{Fact}(\mathrm{Rep}(\check{G}))_{\mathrm{Ran}})$, via the Geometric Satake functor, where we use the functor $\mathrm{Sat}'_{T,\mathrm{Ran}}$ to map

$$\mathrm{Fact}(\mathrm{Rep}(\check{T}))_{\mathrm{Ran}} \to \mathrm{Sph}_{T,\mathrm{Ran}}.$$

Base-changing along $X^I \to \mathrm{Ran}(X)$ we obtain a compatible family of module categories for $(\mathrm{Fact}(\mathrm{Rep}(\check{T})_I, \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I)$, for $I \in \mathrm{Fin}^{\mathrm{surj}}$.

5.1.3. Denote:

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T} := r^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}).$$

From Theorem 4.5.7, we obtain that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}$ naturally lifts to an object of

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T));$$

moreover we have:

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T} \simeq \mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T}), \tag{5.3}$$

where

$$\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T} = (s_{\mathrm{Ran},\mathrm{Bun}_T})_!(\omega_{\mathrm{Ran}(X)\times\mathrm{Bun}_T}),$$

and where $s_{\mathrm{Ran},\mathrm{Bun}_T}$ is the unit section

$$\mathrm{Ran}(X) \times \mathrm{Bun}_T \to \mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T.$$

5.2. **Hecke property in the global setting.**

5.2.1. Consider the stack $\overline{\mathrm{Bun}}_B$, and consider its version

$$(\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}$$

defined as follows:

A point of $(\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}$ is a quadruple $(\mathcal{P}_G, \mathcal{P}_T, \kappa, \mathcal{I})$, where

(i) $\mathcal{P}_G$ is a G-bundle on X;

(ii) $\mathcal{P}_T$ is a T-bundle on X;

(iii) $\mathcal{I}$ is a finite non-empty collection of points on X;

(iv) κ is a datum of maps

$$\kappa^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_T) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$$

that are allowed to have poles on $\Gamma_{\mathcal{I}}$, and that satisfy the Plücker relations.

Note that we have a closed embedding

$$\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X) \hookrightarrow (\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}},$$

corresponding to the condition that the maps $\kappa^{\check{\lambda}}$ have no poles.

5.2.2. Hecke modifications of the G-bundle (resp., T-bundle) define a right (resp., left) action of the groupoid (5.1) (resp., (5.2)) on $(\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}$.

In particular, the category $\mathrm{Shv}((\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}})$ acquires a natural structure of bimodule category for $(\mathrm{Sph}_{T,\mathrm{Ran}}, \mathrm{Sph}_{G,\mathrm{Ran}})$, and hence for $(\mathrm{Fact}(\mathrm{Rep}(\check{T}))_{\mathrm{Ran}}, \mathrm{Fact}(\mathrm{Rep}(\check{G}))_{\mathrm{Ran}})$.

Base-changing along $X^I \to \mathrm{Ran}(X)$ we obtain a compatible family of module categories for $(\mathrm{Fact}(\mathrm{Rep}(\check{T}))_I, \mathrm{Fact}(\mathrm{Rep}(\check{G}))_I)$, for $I \in \mathrm{Fin}^{\mathrm{surj}}$.

5.2.3. Denote

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T} := \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \boxtimes \omega_{\mathrm{Ran}(X)} \subset \mathrm{Shv}((\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}).$$

The following assertion is (essentially) established in [BG1, Theorem 3.1.4]:

Theorem 5.2.4. *The object* $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}$ *naturally lifts to an object of the category*

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}((\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}})).$$

5.3. **Local vs global compatibility.**

5.3.1. Note now that the map

$$\pi_{\mathrm{Ran}} : \overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_N$$

naturally extends to a map

$$\pi_{\mathrm{Ran},\mathrm{Bun}_T} : \mathrm{Gr}_{G,\mathrm{Ran}} \tilde{\times} \mathrm{Bun}_T \to (\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}.$$

We consider the functor

$$(\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^! : \mathrm{Shv}((\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}) \to \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \tilde{\times} \mathrm{Bun}_T)$$

obtained from $(\pi_{\mathrm{Ran},\mathrm{Bun}_T})^!$ by applying the shift by $[d - \langle \lambda, 2\check{\rho} \rangle]$ over the connected component Bun^{λ}_T of Bun_T.

A relative version of the calculation performed in the proof of Theorem 3.3.3 shows:

Theorem 5.3.2. *There exists a canonical isomorphism in* $\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \tilde{\times} \mathrm{Bun}_T)$

$$(\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}) \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}.$$

5.3.3. The map r is compatible with the actions of the groupoids (5.1) and (5.2). In particular, the pullback functor

$$(\pi_{\mathrm{Ran},\mathrm{Bun}_T})^! : \mathrm{Shv}((\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}) \to \mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T)$$

is a map of bimodule categories for $(\mathrm{Sph}_{T,\mathrm{Ran}}, \mathrm{Sph}_{G,\mathrm{Ran}})$.

Hence, we obtain that the functor $(\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!$ can be thought of as a map of bimodule categories for $(\mathrm{Fact}(\mathrm{Rep}(\check{T}))_{\mathrm{Ran}}, \mathrm{Fact}((\mathrm{Rep}(\check{G}))_{\mathrm{Ran}})$.

5.3.4. We are now ready to state the main result of this section:

Theorem 5.3.5. *The isomorphism* $(\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}) \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}$ *of Theorem 5.3.2 canonically lifts to an isomorphism of objects of* $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T))$.

5.4. **Proof of Theorem 5.3.5.**

5.4.1. Consider the tautological map

$$\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T} \to \mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T}). \tag{5.4}$$

Under the isomorphism

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T}) \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}$$

of (5.3), this map corresponds to the map

$$\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T} \to \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}, \tag{5.5}$$

arising, by the $((s_{\mathrm{Ran},\mathrm{Bun}_T})_!, (s_{\mathrm{Ran},\mathrm{Bun}_T})^!$ adjunction, from the isomorphism

$$\omega_{\mathrm{Ran}(X)\times\mathrm{Bun}_T} \to (s_{\mathrm{Ran},\mathrm{Bun}_T})^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T}).$$

5.4.2. Consider the composite

$$\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T} \to \mathrm{Ind}^{\mathrm{Hecke}_{\check{G},\check{T}}}_{\mathrm{DrPl}}(\delta_{1_{\mathrm{Gr}},\mathrm{Ran},\mathrm{Bun}_T}) \simeq \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T} \to (\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}). \tag{5.6}$$

We obtain that the data on the morphism

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran},\mathrm{Bun}_T} \to (\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T})$$

of a map of objects of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T))$ is equivalent to the data on (5.6) of a map of objects of $\mathrm{DrPl}(\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T))$.

5.4.3. The map (5.6) can be explicitly described as follows. By the $((s_{\mathrm{Ran},\mathrm{Bun}_T})_!, (s_{\mathrm{Ran},\mathrm{Bun}_T})^!$ adjunction, it corresponds to the (iso)mophism

$$\omega_{\mathrm{Ran}(X)\times\mathrm{Bun}_T} \to (s_{\mathrm{Ran},\mathrm{Bun}_T})^! \circ (\pi_{\mathrm{Ran},\mathrm{Bun}_T})'^!(\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}) \tag{5.7}$$

constructed as follows:

We note that the map

$$\pi_{\mathrm{Ran},\mathrm{Bun}_T} \circ s_{\mathrm{Ran},\mathrm{Bun}_T} : \mathrm{Ran}(X) \times \mathrm{Bun}_T \to (\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}$$

factors as

$$\mathrm{Ran}(X) \times \mathrm{Bun}_T \to \mathrm{Ran}(X) \times \mathrm{Bun}_B \to \mathrm{Ran}(X) \times \overline{\mathrm{Bun}}_B \to (\overline{\mathrm{Bun}}_B \times \mathrm{Ran}(X))_{\mathrm{poles}}.$$

Now, the map (5.7) is the natural isomorphism coming from the identification

$$\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}|_{\mathrm{Ran}(X)\times\mathrm{Bun}^\lambda_B}[d - \langle\lambda, 2\check{\rho}\rangle] \simeq \omega_{\mathrm{Ran}(X)\times\mathrm{Bun}^\lambda_B}.$$

5.4.4. Now, by unwinding the construction of the Hecke structure on $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob},\mathrm{Bun}_T}$ in [BG1, Theorem 3.1.4], one shows that the map (5.6) indeed canonically lifts to a map in $\mathrm{DrPl}(\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}} \widetilde{\times} \mathrm{Bun}_T))$. □

Appendix A. Proof of Theorem 3.4.4

With future applications in mind, we will prove a generalization of Theorem 3.4.4. The proof is a paraphrase of the theory developed in [Bar].

A.1. **The space of G-bundles with a generic reduction.**

A.1.1. Let Y be a test affine scheme. We shall say that an open subset of $Y \times X$ is a *domain* if it is dense in every fiber of the projection $Y \times X \to X$. Note that the intersection of two domains is again a domain.

Observe that for $\mathcal{I} \in \operatorname{Maps}(Y, \operatorname{Ran}(X))$, the subscheme $Y \times X - \Gamma_{\mathcal{I}}$ is a domain.

A.1.2. Let $\operatorname{Bun}_{G\text{-gen}}$ be the prestack that assigns to an affine test-scheme Y the groupoid, whose objects are pairs:

(i) A domain $U \subset Y \times X$;

(ii) A G-bundle $\mathcal{P}_G$ defined on U.

An (iso)morphism between two such points is by definition an isomorphism of G-bundles defined over a *subdomain* of the intersection of their respective domains of definition.

Remark A.1.3. In particular, given $(\mathcal{P}_G, U)$, if $U' \subset U$ is a sub-domain, then the points $(\mathcal{P}_G, U)$ and $(\mathcal{P}_G|_{U'}, U')$ are canonically isomorphic. Hence, in the definition of $\operatorname{Bun}_{G\text{-gen}}$ we can combine points (i) and (ii) into:

(i') A G-bundle $\mathcal{P}_G$ defined over *some* domain in $Y \times X$.

A.1.4. Let $H \to G$ be a homomorphism of algebraic groups. Consider the prestack

$$\operatorname{Bun}_{H\text{-gen}} \underset{\operatorname{Bun}_{G\text{-gen}}}{\times} \operatorname{Bun}_G .$$

By definition, for a test affine scheme Y, its groupoid of Y-points has as objects triples:

(i) A G-bundle $\mathcal{P}_G$ on $Y \times X$;

(ii) A domain $U \subset Y \times X$;

(iii) A reduction β of $\mathcal{P}_G$ to H defined over $U \subset Y \times X$;

An (iso)morphism between two such points is by definition an isomorphism of G-bundles, compatible with the reductions *over the intersection of the corresponding domains.*

Remark A.1.5. As in Remark A.1.3 above, we can combine (ii) and (iii) into:

(ii') A reduction β of $\mathcal{P}_G$ to H defined over *some* domain in $Y \times X$.

A.1.6. For $H = \{1\}$, we will use the notation

$$\operatorname{Gr}_{G,\operatorname{gen}} := \operatorname{pt} \underset{\operatorname{Bun}_{G\text{-gen}}}{\times} \operatorname{Bun}_G .$$

By definition, for an affine test scheme Y, the set $\operatorname{Maps}(Y, \operatorname{Gr}_{G,\operatorname{gen}})$ consists of pairs $(\mathcal{P}_G, \alpha)$, wgere $\mathcal{P}_G$ is a G-bundle on $Y \times X$, and α is a trivialization of $\mathcal{P}_G$ defined on *some* domain in $Y \times X$.

A.1.7. We have a canonically defined map

$$\mathrm{Gr}_{G,\mathrm{gen}} \to \mathrm{Bun}_{H\text{-gen}} \underset{\mathrm{Bun}_{G\text{-gen}}}{\times} \mathrm{Bun}_G,$$

obtained by base change along $\mathrm{Bun}_G \to \mathrm{Bun}_{G\text{-gen}}$ from the map

$$\mathrm{pt} \to \mathrm{Bun}_{H\text{-gen}}.$$

In addition, we have a canonical map

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{gen}}.$$

Composing, we obtain a map

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Bun}_{H\text{-gen}} \underset{\mathrm{Bun}_{G\text{-gen}}}{\times} \mathrm{Bun}_G. \tag{A.1}$$

The goal of this section is to prove:

Theorem A.1.8. *Assume that H is connected. Then the map* (A.1) *is universally homologically contractible.*

A.1.9. Let us show how Theorem A.1.8 implies Theorem 3.4.4. We take $H = N$. Note that there is a canonically defined map (in fact, a closed embedding)

$$\overline{\mathrm{Bun}}_N \to \mathrm{Bun}_{N\text{-gen}} \underset{\mathrm{Bun}_{G\text{-gen}}}{\times} \mathrm{Bun}_G.$$

Indeed, a Y-point of $\mathrm{Bun}_{N\text{-gen}} \underset{\mathrm{Bun}_{G\text{-gen}}}{\times} \mathrm{Bun}_G$ can be thought of as a data of $(\mathcal{P}_G, \kappa)$, where $\mathcal{P}_G$ is a G-bundle on $Y \times X$, and κ is a system of bundle maps

$$\kappa^{\check{\lambda}} : \mathcal{O}_X \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}, \quad \check{\lambda} \in \check{\Lambda}^+$$

defined over some domain $U \subset T \times X$, and satisfying the Plücker relations.

Such a point belongs to $\mathrm{Bun}_G^{N\text{-gen}}$ if and only if the maps $\kappa^{\check{\lambda}}$ extend to regular maps on all of $Y \times X$.

Finally, we note that we have a Cartesian square:

$$\begin{array}{ccc} \overline{S}^0_{\mathrm{Ran}} & \longrightarrow & \mathrm{Gr}_{G,\mathrm{Ran}} \\ \downarrow & & \downarrow \\ \overline{\mathrm{Bun}}_N & \longrightarrow & \mathrm{Bun}_{N\text{-gen}} \underset{\mathrm{Bun}_{G\text{-gen}}}{\times} \mathrm{Bun}_G. \end{array}$$

□

A.2. **Towards Theorem A.1.8.**

A.2.1. The assertion of Theorem A.1.8 is obtained as a combination of the following two statements:

Proposition A.2.2. *The map $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{gen}}$ is universally homologically contractible.*

Theorem A.2.3. *Let H be connected. Then the map $\mathrm{pt} \to \mathrm{Bun}_{H\text{-gen}}$ is universally homologically contractible.*

A.2.4. *Proof of Proposition A.2.2, Step 1.* For an affine test-scheme Y, let us be given a Y-point $(\mathcal{P}_G, \alpha)$ of $\mathrm{Gr}_{G,\mathrm{gen}}$. Consider the map

$$f : Y \underset{\mathrm{Gr}_{G,\mathrm{gen}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \to Y.$$

The map f is pseudo-proper, being the composition of the closed embedding

$$Y \underset{\mathrm{Gr}_{G,\mathrm{gen}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \hookrightarrow Y \times \mathrm{Gr}_{G,\mathrm{Ran}}$$

and the projection $Y \times \mathrm{Gr}_{G,\mathrm{Ran}} \to Y$.

In particular, $f_!$ is defined, and satisfies base change and the projection formula. Hence, it is sufficient to show that the trace map

$$f_!(\omega_{Y \underset{\mathrm{Gr}_{G,\mathrm{gen}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}}) \to \omega_Y$$

is an isomorphism.

It suffices to show that the latter map induces an isomorphism at the level of !-fibers at field-valued points of Y. By base change (and potentially extending the ground field), we are reduced to the case when $Y = \mathrm{pt}$.

A.2.5. *Interlude: the relative Rans space.* Let $\mathcal{I}_0$ be a finite subset of k-points of X. We define the relative Ran space $\mathrm{Ran}(X)^{\supset \mathcal{I}_0}$ as follows:

For an affine test-scheme Y, the set of Y-points of $\mathrm{Ran}(X)^{\supset \mathcal{I}_0}$ consists of finite non-empty subsets

$$\mathcal{I} \subset \mathrm{Hom}(Y, X),$$

such that $Y \times \mathcal{I}_0$ is set-theoretically contained in $\Gamma_{\mathcal{I}}$.

We claim:

Proposition A.2.6. *The prestack* $\mathrm{Ran}(X)^{\supset \mathcal{I}_0}$ *is homologically contractible.*

The proof repeats the proof of the homological contractibility of $\mathrm{Ran}(X)$, see [Ga4, Appendix].

A.2.7. *Proof of Proposition A.2.2, Step 2.* We continue the proof of Proposition A.2.2 with $Y = \mathrm{pt}$.

Let $U \subset X$ be the maximal open subset over which α is defined. Let $\mathcal{I}_0$ be its set-theoretic complement. Then

$$\mathrm{pt} \underset{\mathrm{Gr}_{G,\mathrm{gen}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}$$

identifies with $\mathrm{Ran}(X)^{\supset \mathcal{I}_0}$.

Now the required assertion follows from Proposition A.2.6.

□

A.2.8. *Proof of Lemma 1.3.3.* The map p^λ_{Ran} is pseudo-proper, and it suffices to show that the trace map

$$(p^\lambda_{\mathrm{Ran}})_!(\omega_{(\mathrm{Ran}(X)\times X^\lambda)^{\supset}}) \to \omega_{X^\lambda}$$

is an isomorphism at the level of !-fibers.

For a given field-valued point $D \in X^\lambda$, let $\mathcal{I}_0 \subset X$ be its support. The fiber of p^λ_{Ran} identifies with $\mathrm{Ran}(X)^{\supset \mathcal{I}_0}$.

Now the assertion follows from Proposition A.2.6.

□

A.3. **Proof of Theorem A.2.3.**

A.3.1. Let $\mathrm{Bun}_{H\text{-gen,triv}}$ be the prestack, whose value on an affine test-scheme Y is the full subgroupoid of $\mathrm{Maps}(Y, \mathrm{Bun}_{H\text{-gen}})$ consisting of objects isomorphic to the trivial one. In other words, this is the essential image of the functor

$$* = \mathrm{Maps}(Y, \mathrm{pt}) \to \mathrm{Maps}(Y, \mathrm{Bun}_{H\text{-gen}}).$$

The assertion of Theorem A.2.3 is obtained as a combination of the following two statements:

Theorem A.3.2. *For H connected, the map* $\mathrm{pt} \to \mathrm{Bun}_{H\text{-gen,triv}}$ *is universally homologically contractible.*

Theorem A.3.3. *The map* $\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H\text{-gen}}$ *is universally homologically contractible.*

A.3.4. *Proof of Theorem A.3.2.* Let $\underline{\mathrm{Maps}}(X, H)_{\mathrm{gen}}$ be the group prestack that attaches to an affine test-scheme Y the group of maps from a domain in $Y \times X$ to H. By definition

$$\mathrm{Bun}_{H\text{-gen,triv}} \simeq B(\underline{\mathrm{Maps}}(X, H)_{\mathrm{gen}}).$$

Hence, in order to prove Theorem A.3.2, it suffices to show that the prestack $\underline{\mathrm{Maps}}(X, H)_{\mathrm{gen}}$ is universally homologically contractible. However, this is essentially what is proved in [Ga2, Theorem 1.8.2].

In order to formally deduce the contractibility of $\underline{\mathrm{Maps}}(X, H)_{\mathrm{gen}}$ from [Ga2], we argue as follows:

Let $\underline{\mathrm{Maps}}(X, H)_{\mathrm{Ran}}$ be the prestack that assigns to an affine test-scheme Y the set of pairs $(\mathcal{I}, h)$, where $\mathcal{I}$ is a finite non-empty subset in $\mathrm{Hom}(Y, X)$ and h is a map

$$(Y \times X - \Gamma_{\mathcal{I}}) \to H.$$

We have a tautologically defined map

$$\underline{\mathrm{Maps}}(X, H)_{\mathrm{Ran}} \to \underline{\mathrm{Maps}}(X, H)_{\mathrm{gen}},$$

and as in Proposition A.2.2 we show that this map is universally homologically contractible.

Now, the assertion of [Ga2, Theorem 1.8.2] is precisely that for H connected, the prestack $\underline{\mathrm{Maps}}(X, H)_{\mathrm{Ran}}$ is universally homologically contractible.

□

A.3.5. The remainder of this section is devoted to the proof of Theorem A.3.3. Write

$$1 \to H_u \to H \to H_r \to 1,$$

where H_u is the unipotent radical of H and H_r is the reductive quotient.

We factor the map $\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H\text{-gen}}$ as

$$\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H_r\text{-gen,triv}} \underset{\mathrm{Bun}_{H_r\text{-gen}}}{\times} \mathrm{Bun}_{H\text{-gen}} \to \mathrm{Bun}_{H\text{-gen}}.$$

We will prove that the maps

$$\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H_r\text{-gen,triv}} \underset{\mathrm{Bun}_{H_r\text{-gen}}}{\times} \mathrm{Bun}_{H\text{-gen}} \tag{A.2}$$

and

$$\mathrm{Bun}_{H_r\text{-gen,triv}} \to \mathrm{Bun}_{H_r\text{-gen}} \tag{A.3}$$

are universally homologically contractible, which would imply the assertion of Theorem A.3.3.

Remark A.3.6. Note that in the applications for the present paper, we have $H = N$, so we do not actually need Theorem A.3.

A.3.7. In order to prove the universal homological contractibility property of (A.2), we can base change with respect to the (value-wise surjective) map $\mathrm{pt} \to \mathrm{Bun}_{H_r\text{-gen,triv}}$. We obtain a map

$$\mathrm{Bun}_{H_u\text{-gen,triv}} \to \mathrm{Bun}_{H_u\text{-gen}},$$

and the statement that (A.2) is universally homologically contractible amounts to the statement of Theorem A.3.3 for H unipotent.

However, we claim that for H unipotent, the map $\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H\text{-gen}}$ is actually an isomorphism. Indeed, every H-bundle is (non-canonically) trivial over a domain that is affine.

A.3.8. Let us observe that the statement that (A.3) is universally homologically contractible is equivalent to the statement of Theorem A.3.3 for H reductive. Hence, for the rest of the argument H will be assumed reductive.

A.4. **Proof of Theorem A.3.3 for H reductive.**

A.4.1. In order to prove that

$$\mathrm{Bun}_{H\text{-gen,triv}} \to \mathrm{Bun}_{H\text{-gen}}$$

is universally homologically contractible, it suffices to show that it becomes an isomorphism after localization in the h-topology. (We recall that h-covers include fppf covers as well as maps that are proper and surjective at the level of k-points.)

Since (A.3) is a value-wise monomorphism, it suffices to show that it is a surjection in the h-topology.

A.4.2. Consider the Cartesian square

$$\begin{array}{ccc} \mathrm{Bun}_{H\text{-gen,triv}} \underset{\mathrm{Bun}_{H\text{-gen}}}{\times} \mathrm{Bun}_H & \longrightarrow & \mathrm{Bun}_H \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{H\text{-gen,triv}} & \longrightarrow & \mathrm{Bun}_{H\text{-gen}}. \end{array}$$

It suffices to show that both maps

$$\mathrm{Bun}_{H\text{-gen,triv}} \underset{\mathrm{Bun}_{H\text{-gen}}}{\times} \mathrm{Bun}_H \to \mathrm{Bun}_H \tag{A.4}$$

and

$$\mathrm{Bun}_H \to \mathrm{Bun}_{H\text{-gen}} \tag{A.5}$$

are h-surjections.

A.4.3. The fact that map (A.4) is an h-surjection follows from [DS]; in fact the main theorem of *loc.cit.* asserts that this map is an fppf surjection.

A.4.4. Let us show that (A.5) is an h-surjection.

Fix a Y-point $(\mathcal{P}_G, U)$ of $\mathrm{Bun}_{H\text{-gen}}$ for an affine test-scheme Y. The fiber product

$$Y \underset{\mathrm{Bun}_{H\text{-gen}}}{\times} \mathrm{Bun}_H$$

is a prestack that assigns to $Y' \to Y$ the set of extensions of the G-bundle $\mathcal{P}_G|_{Y' \underset{Y}{\times} U}$ to all of $Y' \times X$.

It is easy to see that this prestack is (ind)representable by an ind-scheme, ind-proper over Y. Hence, it is enough to show that the map

$$Y \underset{\mathrm{Bun}_{H\text{-gen}}}{\times} \mathrm{Bun}_H \to Y$$

is surjective at the level of k-points.

However, the latter means that any H-bundle on open subset of X can be extended to all of X, which is well-known.

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