THE SEMI-INFINITE INTERSECTION COHOMOLOGY SHEAF

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To David Kazhdan, with gratitude and affection

ABSTRACT. We introduce the semi-infinite category of sheaves on the affine Grassmannian, and construct a particular object in it, which we call the the semi-infinite intersection cohomology sheaf. We relate it to several other entities naturally appearing in the geometric Langlands theory.

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Introduction

0.1. **Our goals.**

0.1.1. This paper deals with sheaves on infinite-dimensional algebro-geometric objects. Specifically, we consider the *affine Grassmannian* of a reductive group G,

$$\mathrm{Gr}_G := G(\!(t)\!)/G[\![t]\!],$$

and we consider sheaves that are equivariant with respect to the action of the group $N(\!(t)\!)$. We denote this category by $\mathrm{SI}(\mathrm{Gr}_G)$. Since $N(\!(t)\!)$-orbits on Gr_G are all infinite-dimensional, objects from $\mathrm{SI}(\mathrm{Gr}_G)$ necessarily have infinite-dimensional support.

We refer the reader to Sect. 0.5.7, where we explain what we mean by sheaves on infinite-dimensional objects such as Gr_G, so that the category of $N(\!(t)\!)$-equivariant sheaves makes sense. Let us mention, however, that in order to set up such a theory we need to work from the start with the derived category of sheaves (or, more precisely, its DG model). There is no hope to develop a rich enough theory while staying within an abelian category (of either sheaves or perverse sheaves).

The word "semi-infinite" in the title of the paper refers to the fact that $N(\!(t)\!)$-orbits on Gr_G have also an infinite codimension. We recall that they are parameterized by coweights of G:

$$S^\lambda = N(\!(t)\!) \cdot t^\lambda.$$

The orbit S^μ lies in the closure $\overline{S}^\lambda$ of S^λ if and only if $\lambda - \mu$ lies in the "wide cone" Λ^{pos}, i.e., is the sum of simple positive roots with non-negative integral coefficients.

0.1.2. The goal of this paper is to construct and describe a particular object in $\mathrm{SI}(\mathrm{Gr}_G)$ which we call the *semi-infinite intersection cohomology sheaf*, and denote by $\mathrm{IC}^{\frac{\infty}{2}}$. Notionally, $\mathrm{IC}^{\frac{\infty}{2}}$ is the intersection cohomology sheaf of $\overline{S}^0$, the closure of the $N[\![t]\!]$-orbit S^0. In what precise sense our $\mathrm{IC}^{\frac{\infty}{2}}$ really is an intersection cohomology sheaf will be discussed in this introduction.

It is obviously not the case the we have randomly decided to construct one particular object in $\mathrm{SI}(\mathrm{Gr}_G)$ and call it $\mathrm{IC}^{\frac{\infty}{2}}$. Our construction is motivated (or, rather, necessitated) by the role that $\mathrm{IC}^{\frac{\infty}{2}}$ is supposed to perform in and around the geometric Langlands theory. We will list a certain number of such roles in Sect. 0.4.

0.1.3. The initial cue for what $\mathrm{IC}^{\frac{\infty}{2}}$ should be was provided by [Lus]. Namely, we want our semi-infinite intersection cohomology sheaf to be such that its fibers are given by the *periodic* (or *stable*) affine Kazhdan-Lusztig polynomials.

These Kazhdan-Lusztig polynomials are defined combinatorially, and the challenge was to find a geometry (i.e., a space and a sheaf on it) that realizes it. In fact, what the space should be is dictated by the initial setting of Lusztig's: it is the double quotient

$$N((t))\backslash G((t))/G[[t]].$$

So, the sought-for sheaf should be an object of $\mathrm{SI}(\mathrm{Gr}_G)$, and our $\mathrm{IC}^{\frac{\infty}{2}}$ is designed so that it has the desired fibers. Namely, its (!)-restriction to the orbit S^μ equals the constant (more precisely, dualizing) sheaf tensored with the cohomologically graded vector space

$$\mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu), \tag{0.1}$$

where $\check{\mathfrak{n}}^-$ is the unipotent radical of the (negative) Borel in the Langlands dual Lie algebra $\check{\mathfrak{g}}$.

0.1.4. However, !-fibers do not determine a sheaf, and one may want to seek further confirmation as to why our $\mathrm{IC}^{\frac{\infty}{2}}$ is the "right thing". Such further confirmation comes from the paper [FFKM].

Namely, the authors of *loc. cit.* considered a certain *finite-dimensional* algebro-geometric object, namely, the algebraic stack $\overline{\mathrm{Bun}}_B$ (the definition of an object, equivalent to $\overline{\mathrm{Bun}}_B$ from the point of view of singularities, denoted $\overline{\mathrm{Bun}}_N$, is recalled in Sect. 3.1).

We choose a projective curve X, and consider the stacks Bun_B and Bun_G that classify B-bundles and G-bundles on X, respectively. The stack $\overline{\mathrm{Bun}}_B$ is Drinfeld's relative compactification of Bun_B along the fibers of the projection $\mathrm{Bun}_B \to \mathrm{Bun}_G$.

The stack $\overline{\mathrm{Bun}}_B$ also has strata numbered by elements of the negative wide cone $-\Lambda^{\mathrm{pos}}$, and the stratum with index μ, denoted $(\overline{\mathrm{Bun}}_B)_{=\mu}$, is isomorphic to

$$\mathrm{Bun}_B \times X^\mu,$$

where X^μ is the spaces of $(-\Lambda^{\mathrm{pos}})$-valued divisors on X of total degree μ.

Let $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}}$ denote the intersection cohomology sheaf of $\overline{\mathrm{Bun}}_B$. The subscript "glob" meant to signify that $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}}$ is global in nature, i.e., its definition involves a global curve X. Pick a point $x \in X$, and let us identify the completed local ring of X at x with the formal power series ring $k[[t]]$ that we used in the definition of $G[[t]], G((t))$, etc.

The main result of [FFKM] can be interpreted as saying that if we take the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}}$ to the stratum $(\overline{\mathrm{Bun}}_B)_{=\mu}$ *and the further* !-restriction to

$$\mathrm{Bun}_B \simeq \mathrm{Bun}_B \times \{\mu \cdot x\} \subset \mathrm{Bun}_B \times X^\mu = (\overline{\mathrm{Bun}}_B)_{=\mu},$$

the result is isomorphic to the intersection cohomology sheaf of Bun_B (which is constant, because Bun_B is smooth) tensored with (0.1).

0.1.5. Thus, the geometry of $\overline{\mathrm{Bun}}_B$ does reproduce Lusztig's answer, but with the following caveats:

(i) $\overline{\mathrm{Bun}}_B$ is not the same as $N((t))\backslash\overline{S}^0$;

(ii) The strata in $\overline{\mathrm{Bun}}_B$ are "bigger" than just copies of Bun_B: we have all those floating points of X that we need to assemble to x in order to get a copy of Bun_B.

That said, we do have a map

$$\pi : \overline{S}^0 \to \overline{\mathrm{Bun}}_B.$$

We prove that our $\mathrm{IC}^{\frac{\infty}{2}}$ is isomorphic to the !-pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}}$. This provides another piece of evidence that $\mathrm{IC}^{\frac{\infty}{2}}$ that we define is a sensible object.

Remark 0.1.6. Of course, one could have simply defined $\mathrm{IC}^{\frac{\infty}{2}}$ as the !-pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{glob}}$ along the map π. However, the drawback of this approach is that such a definition is not intrinsically local: for the applications we have in mind we wish to define $\mathrm{IC}^{\frac{\infty}{2}}$ purely in terms of Gr_G and the $N(\!(t)\!)$-action on it.

0.1.7. Being of infinite dimension and infinite codimension, $N(\!(t)\!)$-orbits on Gr_G do not have an intrinsically defined dimension function. However, one can talk about their relative dimension. Namely, we set the relative dimension of S^μ and S^λ to be (the negative of) the relative dimension of the stabilizer subgroups of points on these orbits. The latter works out to be

$$\dim.\mathrm{rel.}(\mathrm{Ad}_{t^{-\lambda}}(N[\![t]\!]), \mathrm{Ad}_{t^{-\mu}}(N[\![t]\!])) = \langle \lambda - \mu, 2\check{\rho}\rangle.$$

This notion of relative dimension leads to a t-structure on $\mathrm{SI}(\mathrm{Gr}_G)$. The inevitable question that one asks is the following: is our $\mathrm{IC}^{\frac{\infty}{2}}$ *the* minimal extension of the constant sheaf on S^0? The answer is "no, but..."

It is true that $\mathrm{IC}^{\frac{\infty}{2}}$ lies in the heart of this t-structure. However, as a curious feature of our situation (which reflects the fact that it is substantially infinite-dimensional) is that the minimal extension of the constant sheaf on S^0 is *the !-extension*.

So, the "no" aspect of the answer is that our $\mathrm{IC}^{\frac{\infty}{2}}$ is *not* the minimal extension. The "but" aspect is the following:

Instead of the affine Grassmannian Gr_G (considered as attached to a point of a curve X) we can consider its version over the Ran space $\mathrm{Ran}(X)$ of X. Denote the corresponding objects by $\mathrm{Gr}_{G,\mathrm{Ran}}, S^0_{\mathrm{Ran}}$, etc. In this situation one can also introduce a t-structure and consider the minimal extension of the constant sheaf on S^0_{Ran}; denote it by $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$. Then it will be true that our $\mathrm{IC}^{\frac{\infty}{2}}$ is the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ to

$$\mathrm{Gr}_G = \{x\} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}.$$

This will be performed in the forthcoming paper [Ga1].

This, we can say that our $\mathrm{IC}^{\frac{\infty}{2}}$ is not the intersection cohomology sheaf in any t-structure, as long as we do not allow the point $x \in X$ to move. However, it is the restriction of the IC sheaf of a more global object in at least two different contexts ($\overline{\mathrm{Bun}}_B$ or $\mathrm{Ran}(X)$), in both of which x moves along X. A related fact is that in both of these models, the codimension of the μ-stratum is $\langle \mu, \check{\rho}\rangle$, while on Gr_G itself, this codimension is twice that amount.

0.2. **Relation to the work of Bouthier-Kazhdan.** In the paper [BK] another approach to the construction of the semi-infinite IC sheaf is taken.

0.2.1. In [BK], the authors consider the affine closure $\overline{G/N}$ of G/N and the scheme of arcs

$$\overline{G/N}[\![t]\!].$$

Inside $\overline{G/N}[\![t]\!]$ one considers the "good" open part $\overset{\circ}{\overline{G/N}}[\![t]\!]$, whose field valued-points are those maps

$$\mathrm{Spec}(k[\![t]\!]) \to \overline{G/N},$$

for which the composite

$$\mathrm{Spec}(k(\!(t)\!)) \to \mathrm{Spec}(k[\![t]\!]) \to \overline{G/N}$$

lands in G/N.

0.2.2. Note that at the set-theoretic level, the quotients $N(\!(t)\!)\backslash\overline{S}^0$ and $G[\![t]\!]\backslash\overset{\circ}{\overline{G/N}}[\![t]\!]$ are isomorphic.

The main difference of the two approaches is that our basic geometric object is $\overline{S}^0$ (which is the quotient of an appropriate closed ind-subscheme of $G(\!(t)\!)$ by $G[\![t]\!]$); it is an ind-scheme of *ind-finite type* (i.e., a rising union of schemes of finite type under closed embeddings), and we consider the category of sheaves on it, which is built out of categories of sheaves on schemes of finite type that comprise $\overline{S}^0$.

The authors of [BK] consider $\overset{\circ}{\overline{G/N}}[\![t]\!]$, which is a *scheme*, but *of infinite type.* There are several ways to define the category of sheaves on arbitrary schemes, but in general this category will be quite ill-behaved; in particular, one would not be able to define the intersection cohomology sheaf of a subscheme.

However, the authors of [BK] show, that in the case of $\overset{\circ}{\overline{G/N}}[\![t]\!]$ (or more generally, for $\overline{G/N}$ replaced by an affine scheme Z with a smooth open $\overset{\circ}{Z} \subset Z$), it can be approximated by finite-dimensional schemes in the pro-smooth topology so that one obtains a reasonably behaved category of sheaves, equipped with a Verdier auto-duality functor and a t-structure. In particular, one can consider the intersection cohomology sheaf of the open subscheme

$$G/N[\![t]\!] \subset \overset{\circ}{\overline{G/N}}[\![t]\!].$$

It follows from the results of [BK] that the !-fibers of their intersection cohomology sheaf identify with those of our $\mathrm{IC}^{\frac{\infty}{2}}$.

0.2.3. At the moment, the author does not know what is the relationship between the $G[\![t]\!]$-equivariant part of the Bouthier-Kazhdan category and the portion of our $\mathrm{SI}(\mathrm{Gr}_G)$ supported on $\overline{S}^0$.

Most probably, they are *not* equivalent; further, it is possible that the Bouthier-Kazhdan category is equivalent to the category $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$ mentioned in Sect. 0.4.2 below.

0.2.4. Thus, the Bouthier-Kazhdan approach produces the right intersection cohomology sheaf, and probably the entire category. However, for the needs of the geometric Langlands theory (see Sect. 0.4) we still the realization of these objects via the affine Grassmannian, as developed in the present paper.

0.3. **What is done in this paper?**

0.3.1. We want to define $\mathrm{IC}^{\frac{\infty}{2}}$ so it its !-fibers are given by the stable affine Kashdan-Lusztig polynomials. By definition, the value of the latter on $\mu \in -\Lambda^{\mathrm{pos}}$ is the direct limit over λ (ranging over the poset of dominant coweights of G, see Sect. 2.3.1) of the !-fibers of $\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}$ at $t^{\lambda+\mu}$, with the cohomological shift $[\langle \lambda+\mu, 2\check{\rho}\rangle]$.

With this in mind, we let $\mathrm{IC}^{\frac{\infty}{2}}$ be the direct limit over λ of the objects

$$t^{-\lambda}\cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}[\langle \lambda, 2\check{\rho}\rangle],$$

where $g\cdot -$ means translation by an element $g \in G(\!(t)\!)$.

It is almost immediate to see that $\mathrm{IC}^{\frac{\infty}{2}}$ defined in this way is indeed $N(\!(t)\!)$-equivariant, i.e., is an object of $\mathrm{SI}(\mathrm{Gr}_G)$.

One advantage of the above definition of $\mathrm{IC}^{\frac{\infty}{2}}$ is that it is easy to see that it *factorizes* over the Ran space, see Sect. 0.4.1.

0.3.2. Next, we consider the map

$$\pi : \overline{S}^0 \to \overline{\mathrm{Bun}}_N$$

and show that $\mathrm{IC}^{\frac{\infty}{2}}$ is canonically isomorphic (up to a cohomological shift) to the !-pullback of the intersection cohomology sheaf on $\overline{\mathrm{Bun}}_N$.

Among the rest, this isomorphism provides a geometric way to construct an isomorphism between the stable fibers of the perverse sheaves $\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}$ and the fibers of the intersection cohomology sheaf on $\overline{\mathrm{Bun}}_N$, something that the authors of [BFGM] (including the author of the present paper) did not see how to do previously.

0.3.3. Third, we reproduce an argument from [Ras] showing that the category $\mathrm{SI}(\mathrm{Gr}_G)$ is canonically equivalent to a category of finite-dimensional nature, namely, $\mathrm{Shv}(\mathrm{Gr})^{I^0}$, where I^0 is the unipotent radical of the Iwahori subgroup.

In terms of this equivalence, our $\mathrm{IC}^{\frac{\infty}{2}}$ corresponds to a remarkable object of $\mathrm{Shv}(\mathrm{Gr})^{I^0}$, known as the (dual) *baby Verma object*. This object appeared among the rest in [ABBGM] and [FG] in the geometric descriptions of the (dual) baby Verma module over the small quantum group and the Wakimoto module at the critical level, respectively.

0.4. **What do we need $\mathrm{IC}^{\frac{\infty}{2}}$ for?** This subsection plays a motivational role only, and many of the objects mentioned here do not have proper references.

0.4.1. We should say right away that the object of crucial importance is not so much $\mathrm{IC}^{\frac{\infty}{2}}$ itself, but rather is *factorizable* version. Namely, the ind-scheme Gr_G (resp. the group ind-scheme $N(\!(t)\!)$), when viewed as attached to a point $x \in X$, admits a natural factorization structure, and so the category $\mathrm{SI}(\mathrm{Gr}_G)$ has a natural structure of factorization category over the Ran space $\mathrm{Ran}(X)$.

It is a key feature of $\mathrm{IC}^{\frac{\infty}{2}}$ that it carries a canonical structure of *factorization algebra* in $\mathrm{SI}(\mathrm{Gr}_G)_{\mathrm{Ran}}$; denote it by $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$.

Here are some of the roles that $\mathrm{IC}^{\frac{\infty}{2}}$ and $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ are supposed to perform.

0.4.2. It was a dream since the late 1980's, put forth by B. Feigin and E. Frenkel, that there should exist a *category of sheaves/D-modules on the semi-infinite flag space*

$$\mathrm{Fl}^{\frac{\infty}{2}} := G(\!(t)\!)/N(\!(t)\!).$$

In the context of D-modules, it was expected that there should exist a functor of global sections from $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$ to the category of modules over the Kac-Moody algebra at the critical level.

The problem is that $\mathrm{Fl}^{\frac{\infty}{2}}$ is "badly" infinite-dimensional and cannot be approximated by finite-dimensional algebro-geometric objects. So at the time, it was not clear how to even define the category $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$.

However, nowadays one can give *a* definition: namely, the category $\mathrm{Shv}(G(\!(t)\!))$ is well-defined, and one can formally take the corresponding equivariant category

$$\mathrm{Shv}(G(\!(t)\!))^{N(\!(t)\!)},$$

where we consider the $N(\!(t)\!)$-action on $G(\!(t)\!)$ by right translations.

For example, the category of spherical (i.e., $G[\![t]\!]$-equivariant) objects in $\mathrm{Shv}(G(\!(t)\!))^{N(\!(t)\!)}$ identifies by definition with our category $\mathrm{SI}(\mathrm{Gr}_G)$.

The problem is that the resulting category is *not* the desired category, i.e., it has different Hom spaces for some basic objects, than what is expected from *the* category $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$.

For example, the category of Iwahori-equivariant objects in $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$ is supposed to be equivalent to the category of ind-coherent sheaves on

$$(\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \{0\})/\check{T},$$

while the category of Iwahori-equivariant objects in $\mathrm{Shv}(G((t)))^{N((t))}$ is that on

$$(\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G},$$

where $\widetilde{\widetilde{\mathcal{N}}} \to \check{\mathfrak{g}}$ and $\widetilde{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}$ are the Spring and Grothendieck-Springer resolutions, respectively, for the Langlands dual Lie algebra.

Here is how our object $\mathrm{IC}^{\frac{\infty}{2}}$ (or rather, its close relative) is supposed to provide a remedy. Instead of just $\mathrm{Shv}(G((t)))^{N((t))}$, which is a factorization module category over $\mathrm{Shv}(\mathrm{Gr}_G)^{N((t))} = \mathrm{SI}(\mathrm{Gr}_G)$, we consider the category of factorization $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$-modules in $\mathrm{Shv}(\mathrm{Gr}_G)^{N((t))}$.

Calculations performed by the author indicate that the above category of factorization modules may well be the desired category $\mathrm{Shv}(\mathrm{Fl}^{\frac{\infty}{2}})$, and that in the case of D-modules, it does admit the desired functor of global sections to Kac-Moody modules at the critical level.

0.4.3. Another place where the object $\mathrm{IC}^{\frac{\infty}{2}}$ appears is the paper [Ga]. There one considers the Kazhdan-Lusztig category $\mathrm{KL}(G)_\kappa$. This is the category of $G[\![t]\!]$-integrable representations of the Kac-Moody Lie algebra attached to $\mathfrak{g}$ at a (negative integral) level κ. According to Kazhdan and Lusztig, there is a canonical equivalence between $\mathrm{KL}(G)_\kappa$ and the category of modules over the quantum group $U_q(G)$, where q is determined by κ.

We wish to describe the functor

$$\mathrm{KL}(G)_\kappa \to \mathrm{KL}(T)_\kappa \tag{0.2}$$

that makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{KL}(G)_\kappa & \xrightarrow{\sim} & U_q(G)\text{-mod} \\ \downarrow & & \downarrow \\ \mathrm{KL}(T)_\kappa & \xrightarrow{\sim} & U_q(T)\text{-mod}, \end{array} \tag{0.3}$$

where the right vertical arrow is the functor of cohomology with respect to $\mathfrak{u}_q(N^+)$, the positive part of the *small* quantum group.

Now, a general formalism says that objects of

$$\mathrm{Shv}(\mathrm{Gr}_G)^{N((t))\cdot T[\![t]\!]}$$

give rise to functors (0.2) (namely, convolve and take semi-infinite cohomology with respect to $\mathfrak{n}((t))$). It turns out that our object $\mathrm{IC}^{\frac{\infty}{2}}$ gives rise to the functor that we need for the commutativity of the diagram (0.3).

0.4.4. There is an analogous situation to the one above, where instead of $\mathrm{KL}(G)_\kappa$ we consider the *metaplectic Whittaker category* for the dual group, denoted $\mathrm{Whit}_q(\mathrm{Gr}_{\check{G}})$. According to the conjecture of J. Lurie and the author, we still have an equivalence

$$\mathrm{Whit}_q(\mathrm{Gr}_{\check{G}}) \simeq U_q(G)\text{-mod}.$$

We wish to have a functor

$$\mathrm{Whit}_q(\mathrm{Gr}_{\check{G}}) \to \mathrm{Whit}_q(\mathrm{Gr}_{\check{T}}) \tag{0.4}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{Whit}_q(\mathrm{Gr}_{\check{G}}) & \xrightarrow{\sim} & U_q(G)\text{-mod} \\ \downarrow & & \downarrow \\ \mathrm{Whit}_q(\mathrm{Gr}_{\check{T}}) & \xrightarrow{\sim} & U_q(T)\text{-mod}, \end{array}$$

commutative, where the right vertical arrow is the same as in (0.3).

Again, any object in the metaplectic $\mathrm{SI}_q(\mathrm{Gr}_{\check{G}})$ version of $\mathrm{SI}(\mathrm{Gr}_{\check{G}})$ gives to a functor in (0.4), and the sought-for functor is given by the appropriate metaplectic version $\mathrm{IC}_q^{\frac{\infty}{2}} \in \mathrm{SI}_q(\mathrm{Gr}_{\check{G}})$ of $\mathrm{IC}^{\frac{\infty}{2}}$. This is the subject of the forthcoming paper [GLys].

0.5. **Background and conventions.**

0.5.1. This paper unavoidably uses higher category theory. It appears in the very definition of our basic object of study, the semi-infinite category on the affine Grassmannian.

Thus, we will assume that the reader is familiar with the basics tenets of the theory. The fundamental reference is [Lu], but shorter expositions (or user guides) exist as well, for example, the first chapter of [GR].

0.5.2. Our algebraic geometry happens over an arbitrary algebraically closed ground field k. Our geometric objects are classical, i.e., non-derived (see, however, Sect. 0.5.8).

By a prestack we mean an arbitrary (accessible) functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Groupoids} \tag{0.5}$$

(we de not need to consider higher groupoids).

We shall say that a prestack $\mathcal{Y}$ is locally of finite type if the corresponding functor (0.5) takes filtered limits of affine schemes to colimits of groupoids. Equivalently, $\mathcal{Y}$ is locally of finite type if it is the left Kan extension of a functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Groupoids}$$

along the fully faithful embedding

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

Still equivalently, $\mathcal{Y}$ is locally of finite type if for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $y : S \to \mathcal{Y}$, the category of factorizations of y as

$$S \to S' \to \mathcal{Y},$$

where $S' \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, is *contractible*.

We let $\mathrm{PreStk}_{\mathrm{lft}}$ denote the category of prestacks locally of finite type. We can identify it with the category of all functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Groupoids}\,. \tag{0.6}$$

0.5.3. We let G be a connected reductive group over k. We fix a Borel subgroup $B \subset G$ and the opposite Borel $B^- \subset G$. Let $N \subset B$ and $N^- \subset B^-$ denote their respective unipotent radicals.

Set $T = B \cap B^-$; this is a Cartan subgroup of G. We use it to identify the quotients

$$B/N \simeq T \simeq B^-/N^-.$$

We let Λ denote the coweight lattice of G, i.e., the lattice of cocharacters of T. We let $\Lambda^{\mathrm{pos}} \subset \Lambda$ denote the sub-monoid consisting of linear combinations of positive simple roots with non-negative integral coefficients. We let $\Lambda^+ \subset \Lambda$ denote the sub-monoid of *dominant coweights*.

0.5.4. For an affine scheme Z, we let $\mathfrak{L}(Z)$ (resp., $\mathfrak{L}^+(Z)$) denote the scheme (resp., ind-scheme), whose S-points for $S = \mathrm{Spec}(A)$ are given by $\mathrm{Hom}(\mathrm{Spec}(A(\!(t)\!)), Z)$ (resp., $\mathrm{Hom}(\mathrm{Spec}(A[\![t]\!]), Z)$). Thus, we replace the notations

$$Z(\!(t)\!) \rightsquigarrow \mathfrak{L}(Z),\ \ Z[\![t]\!] \rightsquigarrow \mathfrak{L}^+(Z)$$

from earlier in the introduction.

For $Z = G$, we consider the étale (equivalently, fppf) quotient $\mathrm{Gr}_G := \mathfrak{L}(G)/\mathfrak{L}^+(G)$. This is the affine Grassmannian of G. It is known to be an ind-scheme of ind-finite type.

0.5.5. While our geometry happens over a field k, the representation-theoretic categories that we study are DG categories over another field, denoted e (assumed algebraically closed and of characteristic 0).

All our DG categories are assumed presentable. When considering functors, we will only consider functors that preserve colimits. We denote the ∞-category of DG categories by DGCat. It carries a symmetric monoidal structure (i.e., one can consider tensor products of DG categories). The unit object is the DG category of complexes of e-vector spaces, denoted Vect.

For a pair of objects c_0, c_1 in a DG category $\mathcal{C}$ we will denote by $\mathcal{H}om_{\mathcal{C}}(c_0, c_1)$ the corresponding object of Vect.

We will use the notion of t-structure on a DG category. Given a t-structure on $\mathcal{C}$, we will denote by $\mathcal{C}^{\leq 0}$ the corresponding subcategory of cohomologically connective objects, and by $\mathcal{C}^{>0}$ its right orthogonal. We let $\mathcal{C}^{\heartsuit}$ denote the heart $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$.

0.5.6. The source of DG categories will be a *sheaf theory*, which is a functor

$$\mathrm{Shv} : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}, \quad S \mapsto \mathrm{Shv}(S).$$

For a morphism of affine schemes $f : S_0 \to S_1$, the corresponding functor

$$\mathrm{Shv}(S_1) \to \mathrm{Shv}(S_0)$$

is the !-pullback $f^!$.

The main examples of sheaf theories are:

(i) We take $\mathsf{e} = \overline{\mathbb{Q}}_\ell$ and we take $\mathrm{Shv}(S)$ to be the ind-completion of the (small) DG category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

(ii) When $k = \mathbb{C}$ and e arbitrary, we take $\mathrm{Shv}(S)$ to be the ind-completion of the (small) DG category of constructible e-sheaves on $S(\mathbb{C})$ in the analytic topology.

(iii) If k has characteristic 0, we take $\mathsf{e} = k$ and we take $\mathrm{Shv}(S)$ to be the DG category of D-modules on S.

In examples (i) and (ii), the functor $f^!$ always has a left adjoint, denoted $f_!$. In example (iii) this is not the case. However, the partially defined left adjoint $f_!$ is defined on holonomic objects. It is defined on the entire category if f is proper.

0.5.7. *Sheaves on prestacks.* We apply the procedure of right Kan extension along the embedding

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}}$$

to the functor Shv, and thus obtain a functor (denoted by the same symbol)

$$\mathrm{Shv} : (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}} \to \mathrm{DGCat}\,.$$

By definition, for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ we have

$$\mathrm{Shv}(\mathcal{Y}) = \lim_{S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, y : S \to \mathcal{Y}} \mathrm{Shv}(S), \tag{0.7}$$

where the transition functors in the formation of the limit are the !-pullbacks[1].

For a map of prestacks $f : \mathcal{Y}_0 \to \mathcal{Y}_1$ we thus have a well-defined pullback functor

$$f^! : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_0).$$

We denote by $\omega_{\mathcal{Y}}$ the dualizing sheaf on $\mathcal{Y}$, i.e., the pullback of

$$\mathsf{e} \in \mathrm{Vect} \simeq \mathrm{Shv}(\mathrm{pt})$$

along the tautological map $\mathcal{Y} \to \mathrm{pt}$.

[1] Note that even though the index category (i.e., $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}$) is ordinary, the above limit is formed in the ∞-category DGCat. This is how ∞-categories appear in this paper.

0.5.8. This paper is closely related to the geometric Langlands theory, and the geometry of the Langlands dual group $\check{G}$ makes it appearance.

By definition, $\check{G}$ is a reductive group over e and geometric objects constructed out of $\check{G}$ give rise to e-linear DG categories by considering quasi-coherent (resp., ind-coherent) sheaves on them.

However, unlike our geometric objects over k, the objects of this dual geometry are often *derived.* A typical example is the derived enhancement of the flag variety:

$$\widetilde{\check{\mathfrak{g}}} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt}\,.$$

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1. The semi-infinite category on the affine Grassmannian

1.1. The category of sheaves on the affine Grassmannian.

1.1.1. Recall that for any prestack $\mathcal{Y}$ locally of finite type, we define the category $\mathrm{Shv}(\mathcal{Y})$ by (0.7).

Let now $\mathcal{Y}$ be an ind-scheme of ind-finite type, i.e., a prestack that can be written as

$$\underset{i\in I}{\mathrm{colim}}\, Y_i, \tag{1.1}$$

where I is a filtered index category, Y_i's are schemes of finite type, and for every $(i \to i') \in I$, the corresponding map

$$Y_i \overset{f_{i,i'}}{\longrightarrow} Y_{i'}$$

is a closed embedding[2].

In this case we have

$$\mathrm{Shv}(\mathcal{Y}) \simeq \underset{i\in I}{\lim}\, \mathrm{Shv}(Y_i),$$

where for $(i \to i') \in I$, the corresponding functor $\mathrm{Shv}(Y_{i'}) \to \mathrm{Shv}(Y_i)$ is $f^!_{i,i'}$.

1.1.2. If $\mathcal{Y}$ is an ind-scheme, the category $\mathrm{Shv}(\mathcal{Y})$ has a t-structure. It is uniquely characterized by the following (equivalent) conditions:

(i) An object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ belongs to $(\mathrm{Shv}(\mathcal{Y}))^{\geq 0}$ if and only if its (!)-pullback to any Y_i belongs to $(\mathrm{Shv}(Y_i))^{\geq 0}$.

(ii) The subcategory $(\mathrm{Shv}(\mathcal{Y}))^{\leq 0}$ is generated under filtered colimits by direct images of objects $\mathcal{F}_i \in (\mathrm{Shv}(Y_i))^{\leq 0}$.

1.1.3. We apply the above discussion to $\mathcal{Y} = \mathrm{Gr}_G$, and thus obtain a well-defined category $\mathrm{Shv}(\mathrm{Gr}_G)$, equipped with a t-structure.

1.2. **Definition of the semi-infinite subcategory.** We note that the ind-scheme Gr_G is acted on by the group ind-scheme $\mathfrak{L}(N)$. We define the category

$$\mathrm{SI}(\mathrm{Gr}_G) := \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$$

as the full subcategory of objects $\mathrm{Shv}(\mathrm{Gr}_G)$ that are $\mathfrak{L}(N)$-invariant.

By definition, this means the following.

[2] In this case each Y_i is a closed subfunctor of $\mathcal{Y}$, and $\mathcal{Y}$ admits a universal presentation as in (1.1), where the index category is taken to be that of *all* closed subfunctors of $\mathcal{Y}$ that are representable by schemes of finite type. This category is automatically filtered, and the initial category I is cofinal in the universal one.

1.2.1. First off, since N is unipotent, $\mathfrak{L}(N)$ naturally comes from an ind-object in the category of group-schemes

$$\mathfrak{L}(N) \simeq \underset{\alpha\in A}{\operatorname{colim}}\, N_\alpha, \tag{1.2}$$

where A is a filtered category.

We set

$$\operatorname{Shv}(\operatorname{Gr}_G)^{\mathfrak{L}(N)} := \lim_{\alpha} \operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha}, \tag{1.3}$$

where each $\operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha}$ is a full subcategory of $\operatorname{Shv}(\operatorname{Gr}_G)$ and for $(\alpha \to \alpha') \in A$, we have

$$\operatorname{Shv}(\operatorname{Gr}_G)^{N_{\alpha'}} \subset \operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha}$$

as full subcategories in $\operatorname{Shv}(\operatorname{Gr}_G)$. Note that the limit in (1.3) amounts to the intersection

$$\bigcap_{\alpha} \operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha}$$

as full subcategories in $\operatorname{Shv}(\operatorname{Gr}_G)$.

Let us now explain what the subcategories

$$\operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha} \subset \operatorname{Shv}(\operatorname{Gr}_G)$$

are.

1.2.2. For a fixed α, the prestack Gr_G, when viewed as equipped with an action of N_α, is naturally an ind-object in the category of schemes equipped with an action of N_α.

I.e., we can represent Gr_G as

$$\underset{i\in I}{\operatorname{colim}}\, Y_i$$

where each Y_i is a closed subscheme of Gr_G, stable under the N_α-action.

We set

$$\operatorname{Shv}(\operatorname{Gr}_G)^{N_\alpha} := \lim_{i\in I} \operatorname{Shv}(Y_i)^{N_\alpha},$$

viewed as a full subcategory of

$$\operatorname{Shv}(\operatorname{Gr}_G) \simeq \lim_{i\in I} \operatorname{Shv}(Y_i).$$

Thus, it remains to explain what we mean by

$$\operatorname{Shv}(Y_i)^{N_\alpha} \subset \operatorname{Shv}(Y_i)$$

for each α and i, and why for $(i \to i')$, the corresponding functor

$$\operatorname{Shv}(Y_{i'}) \xrightarrow{f^!_{i,i'}} \operatorname{Shv}(Y_i)$$

sends $\operatorname{Shv}(Y_{i'})^{N_\alpha}$ to $\operatorname{Shv}(Y_i)^{N_\alpha}$.

1.2.3. The group-scheme N_α, can be naturally written as

$$\lim_{\beta\in B_{\alpha,i}} N_{\alpha,\beta},$$

where:

(i) $B_{\alpha,i}$ is a filtered category;

(ii) Each $N_{\alpha,\beta}$ is a unipotent algebraic group (of finite type);

(iii) For every $(\beta \to \beta') \in B_{\alpha,i}$ the corresponding map $N_{\alpha,\beta'} \to N_{\alpha,\beta}$ is surjective;

(iv) The action of N_α on Y_i comes from a compatible family of actions of $N_{\alpha,\beta}$'s on Y_i.

For any $\beta \in B_{\alpha,i}$, we can consider the corresponding equivariant category $\operatorname{Shv}(Y_i)^{N_{\alpha,\beta}}$. Since $N_{\alpha,\beta}$ is unipotent, the forgetful functor

$$\operatorname{Shv}(Y_i)^{N_{\alpha,\beta}} \to \operatorname{Shv}(Y_i)$$

is fully faithful, and for every $(\beta \to \beta') \in B_{\alpha,i}$, we have the inclusion of subcategories

$$\mathrm{Shv}(Y_i)^{N_{\alpha,\beta}} = \mathrm{Shv}(Y_i)^{N_{\alpha,\beta'}}$$

as subcategories of $\mathrm{Shv}(Y_i)$.

We set $\mathrm{Shv}(Y_i)^{N_\alpha} \subset \mathrm{Shv}(Y_i)$ to be $\mathrm{Shv}(Y_i)^{N_{\alpha,\beta}}$ for some/any $\beta \in B_{\alpha,i}$.

1.2.4. Going back, it is clear that for a map $(i \to i') \in I$, the corresponding functor

$$\mathrm{Shv}(Y_{i'}) \overset{f^!_{i,i'}}{\longrightarrow} \mathrm{Shv}(Y_i)$$

sends $\mathrm{Shv}(Y_{i'})^{N_\alpha}$ to $\mathrm{Shv}(Y_i)^{N_\alpha}$.

It is also clear that for a map $(\alpha \to \alpha') \in A$, we have

$$\mathrm{Shv}(\mathrm{Gr})^{N_{\alpha'}} \subset \mathrm{Shv}(\mathrm{Gr})^{N_\alpha}$$

as full subcategories of $\mathrm{Shv}(\mathrm{Gr})$.

1.2.5. Consider the forgetful functor

$$\mathrm{SI}(\mathrm{Gr}_G) := \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)} \to \mathrm{Shv}(\mathrm{Gr}_G).$$

As any functor, it admits a partially defined left adjoint, to be denoted $\mathrm{Av}_!^{\mathfrak{L}(N)}$.

Explicitly, for $\mathfrak{L}(N)$ written as (1.2), we have

$$\mathrm{Av}_!^{\mathfrak{L}(N)} := \underset{\alpha}{\mathrm{colim}}\ \mathrm{Av}_!^{N_\alpha},$$

where $\mathrm{Av}_!^{N_\alpha}$ is the partially defined left adjoint to the forgetful functor

$$\mathrm{Shv}(\mathrm{Gr}_G)^{N_\alpha} \to \mathrm{Shv}(\mathrm{Gr}_G).$$

1.2.6. Recall (see Sect. 1.1.2) that the category $\mathrm{Shv}(\mathrm{Gr}_G)$ is equipped with a t-structure. In Sect. 1.4.6 we will prove:

Proposition 1.2.7. *Every $\mathcal{F} \in \mathrm{Shv}(\mathrm{Gr})^{\mathfrak{L}(N)}$ is* infinitely connective*, i.e., lies in $(\mathrm{Shv}(\mathrm{Gr}_G))^{-\leq n}$ for every n.*

The above proposition says that objects of the category $\mathrm{SI}(\mathrm{Gr}_G) = \mathrm{Shv}(\mathrm{Gr})^{\mathfrak{L}(N)}$ are invisible from the point of view of the heart of $\mathrm{Shv}(\mathrm{Gr}_G)$. However, in Sect. 1.5 we will show that $\mathrm{SI}(\mathrm{Gr}_G)$ carries its own t-structure with meaningful features.

1.3. **Semi-infinite orbits and standard functors.** In this subsection we begin to investigate the structure of $\mathrm{SI}(\mathrm{Gr}_G)$.

1.3.1. For a coweight λ let S^λ denote the $\mathfrak{L}(N)$-orbit of the point $t^\lambda \in \mathrm{Gr}_G$, and let $\overline{S}^\lambda$ be its closure. We have

$$S^\mu \subset \overline{S}^\lambda \Leftrightarrow \lambda - \mu \in \Lambda^{\mathrm{pos}}.$$

The denote the corresponding maps by

$$S^\lambda \overset{\mathbf{j}_\lambda}{\hookrightarrow} \overline{S}^\lambda,$$

$$\overline{S}^\lambda \overset{\bar{\mathbf{i}}_\lambda}{\hookrightarrow} \mathrm{Gr}_G,$$

$$S^\lambda \overset{\mathbf{i}_\lambda}{\hookrightarrow} \mathrm{Gr}_G,$$

so that $\mathbf{i}_\lambda = \bar{\mathbf{i}}_\lambda \circ \mathbf{j}_\lambda$.

The map $\mathbf{j}_\lambda$ (resp., $\bar{\mathbf{i}}_\lambda$, $\mathbf{i}_\lambda$) is an open (resp., closed, locally closed) embedding of prestacks, i.e., its base change by an affine scheme yields a map of schemes with the corresponding property. In particular, S^λ and $\overline{S}^\lambda$ are ind-schemes.

1.3.2. We have the corresponding functors

$$(\bar{\mathbf{i}}_\lambda)^! : \mathrm{Shv}(\mathrm{Gr}_G) \to \mathrm{Shv}(\overline{S}^\lambda), \quad (\mathbf{j}_\lambda)^! : \mathrm{Shv}(\overline{S}^\lambda) \to \mathrm{Shv}(S^\lambda), \quad (\mathbf{i}_\lambda)^! : \mathrm{Shv}(\overline{S}^\lambda) \to \mathrm{Shv}(S^\lambda) \tag{1.4}$$

and

$$(\bar{\mathbf{i}}_\lambda)_* : \mathrm{Shv}(\overline{S}^\lambda) \to \mathrm{Shv}(\mathrm{Gr}_G), \quad (\mathbf{j}_\lambda)_* : \mathrm{Shv}(S^\lambda) \to \mathrm{Shv}(\overline{S}^\lambda), \quad (\mathbf{i}_\lambda)_* : \mathrm{Shv}(S^\lambda) \to \mathrm{Shv}(\overline{S}^\lambda). \tag{1.5}$$

Since the map $\mathbf{j}_\lambda$ is an open embedding we will also use the notation $(\mathbf{j}_\lambda)^* := (\mathbf{j}_\lambda)^!$. The functors $((\mathbf{j}_\lambda)^*, (\mathbf{j}_\lambda)_*)$ form an adjoint pair with $(\mathbf{j}_\lambda)_*$ being fully faithful.

Since the map $\bar{\mathbf{i}}_\lambda$ is a closed embedding we will also use the notation $(\bar{\mathbf{i}}_\lambda)_! := (\bar{\mathbf{i}}_\lambda)_*$; it is fully faithful. The functors $((\mathbf{i}_\lambda)_!, (\mathbf{i}_\lambda)^!)$ form an adjoint pair.

In particular, the functor $(\mathbf{i}_\lambda)_* \simeq (\bar{\mathbf{i}}_\lambda)_* \circ (\mathbf{j}_\lambda)_*$ is also fully faithful.

1.3.3. The indschemes S^λ and $\overline{S}^\lambda$ are acted on by $\mathfrak{L}(N)$, and the construction in Sect. 1.2 applies to them as well. We denote

$$\mathrm{SI}(\mathrm{Gr}_G)_{\leq\lambda} := \mathrm{Shv}(\overline{S}^\lambda)^{\mathfrak{L}(N)}$$

and

$$\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} := \mathrm{Shv}(S^\lambda)^{\mathfrak{L}(N)}.$$

The functors of (1.4) and (1.5) send the corresponding subcategories

$$\mathrm{SI}(\mathrm{Gr}_G)_{\leq\lambda} \subset \mathrm{Shv}(\overline{S}^\lambda), \quad \mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \subset \mathrm{Shv}(S^\lambda), \quad \mathrm{SI}(\mathrm{Gr}_G) \subset \mathrm{Shv}(\mathrm{Gr}_G)$$

to one another and the same adjunctions hold.

1.3.4. We claim:

Proposition 1.3.5. *The functor of taking the !-fiber at t^λ defines an equivalence*

$$\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \simeq \mathrm{Vect}\,.$$

Proof. Let us write $\mathfrak{L}(N)$ as in (1.2). For each α, set the index category I to be equal to $A_{\alpha/}$, and for $\alpha' \in A_{\alpha/}$ set

$$Y_{\alpha'} = N_{\alpha'} \cdot t^\lambda.$$

Then $\mathrm{Shv}(S^\lambda)^{\mathfrak{L}(N)}$ is the limit

$$\lim_{\alpha\to\alpha'} \mathrm{Shv}(Y_{\alpha'})^{N_\alpha}.$$

But cofinal in the above index category is the subcategory where $\alpha' = \alpha$. Thus, the above limit maps isomorphically to

$$\lim_{\alpha} \mathrm{Shv}(Y_\alpha)^{N_\alpha}.$$

Now, for every α, the functor of taking the fiber at t^λ defines an equivalence

$$\mathrm{Shv}(Y_\alpha)^{N_\alpha} \to \mathrm{Vect};$$

since $\mathrm{Stab}_{N_\alpha}(t^\lambda)$ is unipotent.

□

Corollary 1.3.6. *The inverse equivalence to that of Proposition 1.3.5 sends $\mathsf{e} \in \mathrm{Vect}$ to the dualizing object $\omega_{S^\lambda} \in \mathrm{Shv}(S^\lambda)$.*

Proof. It is clear that ω_{S^λ} belongs to the subcategory $\mathrm{Shv}(S^\lambda)^{\mathfrak{L}(N)}$. Now the assertion follows from the fact that its !-fiber at t^λ identifies with e. □

1.4. **Structure of the semi-infinite category.** In this subsection we will show that the functors introduced in the preceding section admit all the expected adjoints.

1.4.1. We claim:

Proposition 1.4.2. *The partially defined left adjoint* $(\mathbf{j}_\lambda)_!$ *of* $(\mathbf{j}_\lambda)^! : \mathrm{Shv}(\overline{S}^\lambda) \to \mathrm{Shv}(S^\lambda)$ *is defined on the full subcategory* $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \subset S^\lambda$ *and takes values in* $\mathrm{SI}(\mathrm{Gr}_G)_{\leq\lambda} \subset \overline{S}^\lambda$. *Moreover,*

$$(\mathbf{i}_\lambda)_!(\omega_{S^\lambda}) \simeq \mathrm{Av}_!^{\mathfrak{L}(N)}(t^\lambda). \tag{1.6}$$

Proof. By Corollary 1.3.6, to prove the existence of $(\mathbf{j}_\lambda)_!$ it is enough to show that $(\mathbf{j}_\lambda)_!$ is defined on the object ω_{S^λ}, and that the result is N_α-equivariant for any α.

Let $\overset{\circ}{\mathcal{Y}} \overset{\mathbf{j}}{\longrightarrow} \mathcal{Y}$ be an arbitrary open embedding of ind-schemes. Write $\mathcal{Y}$ as in (1.1), and set $\overset{\circ}{Y}_i := \overset{\circ}{\mathcal{Y}} \underset{\mathcal{Y}}{\times} Y_i$

$$\overset{\circ}{Y}_i \overset{\mathbf{j}_i}{\longrightarrow} Y_i.$$

Then

$$\omega_{\overset{\circ}{\mathcal{Y}}} \simeq \underset{i}{\mathrm{colim}}\, \omega_{\overset{\circ}{Y}_i}$$

where we think of $\omega_{\overset{\circ}{Y}_i}$ as an object of $\mathrm{Shv}(\overset{\circ}{\mathcal{Y}})$ via the fully faithful embedding $\mathrm{Shv}(\overset{\circ}{Y}_i) \hookrightarrow \mathrm{Shv}(\overset{\circ}{\mathcal{Y}})$, and

$$(\mathbf{j}_!)(\omega_{\overset{\circ}{\mathcal{Y}}}) \simeq \underset{i}{\mathrm{colim}}\, (\mathbf{j}_i)_!(\omega_{\overset{\circ}{Y}_i}).$$

This implies the existence of $(\mathbf{j}_\lambda)_!(\omega_{S^\lambda})$. The N_α-equivariance follows from the above description: for a given α, take Y_i to be N_α-invariant subschemes of $\overline{S}^\lambda$.

The isomorphism (1.6) follows from the fact that $\mathcal{H}om$ from either side to an object of $\mathrm{SI}(\mathrm{Gr}_G)$ amounts to taking the !-fiber of that object at t^λ.

□

Lemma 1.4.3. *The functor* $(\mathbf{j}_\lambda)_! : \mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \to \mathrm{SI}(\mathrm{Gr}_G)_{\leq\lambda}$ *is fully faithful.*

Proof. Since the right adjoint of $(\mathbf{j}_\lambda)^! \simeq (\mathbf{j}_\lambda)^*$, i.e., $(\mathbf{j}_\lambda)_*$, is fully faithful, it formally follows that so is the left adjoint, i.e., $(\mathbf{j}_\lambda)_!$.

□

Corollary 1.4.4. *The functor* $(\mathbf{i}_\lambda)^! : \mathrm{SI}(\mathrm{Gr}_G) \to \mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$ *admits a* left *adjoint, to be denoted* $(\mathbf{i}_\lambda)_!$. *The functor* $(\mathbf{i}_\lambda)_!$ *is fully faithful.*

Proof. The left adjoint in question is given by $(\mathbf{i}_\lambda)_! := (\bar{\mathbf{i}}_\lambda)_! \circ (\mathbf{j}_\lambda)_!$. □

We also note:

Lemma 1.4.5. *The objects* $(\mathbf{i}_\lambda)_!(\omega_{S^\lambda})$ *are infinitely connective in the t-structure on* $\mathrm{Shv}(\mathrm{Gr}_G)$.

Proof. Write S^λ as the colimit of $N_\alpha \cdot t^\lambda$, as in the proof of Proposition 1.3.5. Then $(\mathbf{i}_\lambda)_!(\omega_{S^\lambda})$ is the colimit of !-extensions of the objects $\omega_{N_\alpha \cdot t^\lambda}$ under the locally closed embeddings $N_\alpha \cdot t^\lambda \hookrightarrow \mathrm{Gr}_G$.

To prove the lemma it suffices to show that for every n, there exists an index α such that for all $\alpha \to \alpha'$, the object $\omega_{N_{\alpha'} \cdot t^\lambda} \in \mathrm{Shv}(N_{\alpha'} \cdot t^\lambda)$ belongs to $(\mathrm{Shv}(N_{\alpha'} \cdot t^\lambda))^{\leq -n}$.

However, this does indeed happen as soon as $\dim(N_\alpha \cdot t^\lambda) \geq n$.

□

1.4.6. For $\lambda \in \Lambda$, let Δ^λ and ∇^λ be the objects of $\mathrm{SI}(\mathrm{Gr}_G)$ equal to

$$(\mathbf{i}_\lambda)_!(\mathsf{e})[-\langle \lambda, 2\check{\rho}\rangle] \text{ and } (\mathbf{i}_\lambda)_*(\mathsf{e})[-\langle \lambda, 2\check{\rho}\rangle],$$

respectively, where we think of e as an object of $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$ via the equivalence of Proposition 1.3.5. The cohomological shift by $[-\langle \lambda, 2\check{\rho}\rangle]$ in our normalization will be explained later.

By construction, the objects Δ^λ are compact in $\mathrm{SI}(\mathrm{Gr}_G)$ (but of course not in $\mathrm{Shv}(\mathrm{Gr}_G)$).

Lemma 1.4.7. *The category* $\mathrm{SI}(\mathrm{Gr}_G)$ *is generated by the objects* Δ^λ.

Proof. The is equivalent to the fact that for $\mathcal{F} \in \mathrm{SI}(\mathrm{Gr}_G)$,

$$\mathcal{H}om(\Delta^\lambda, \mathcal{F}) = 0, \forall \lambda \Rightarrow \mathcal{F} = 0.$$

However,

$$\mathcal{H}om(\Delta^\lambda, \mathcal{F}) = 0 \Leftrightarrow (\mathbf{i}_\lambda)^!(\mathcal{F}) = 0.$$

Since for every closed subscheme $Y \subset \mathrm{Gr}_G$, the intersections $Y \cap S^\lambda$ define a decomposition into locally closed subsets, we obtain that

$$(\mathbf{i}_\lambda)^!(\mathcal{F}) = 0, \forall \lambda \Rightarrow \mathcal{F} = 0.$$

□

By Lemma 1.4.5, the objects Δ^λ are infinitely connective in the t-structure on $\mathrm{Shv}(\mathrm{Gr}_G)$. Combined with Lemma 1.4.7, this implies Proposition 1.2.7.

1.4.8. We claim:

Lemma 1.4.9. *For every* $\lambda \in \Lambda$*, the partially defined left adjoint* $(\mathbf{i}_\lambda)^*$ *of* $(\mathbf{i}_\lambda)_* : \mathrm{Shv}(S^\lambda) \to \mathrm{Shv}(\mathrm{Gr}_G)$ *is defined on the full subcategory* $\mathrm{SI}(\mathrm{Gr}_G) \subset \mathrm{Shv}(\mathrm{Gr}_G)$*, and takes values in* $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \subset \mathrm{Shv}(S^\lambda)$.

Proof. By Lemma 1.4.7, it is enough to show that $(\mathbf{i}_\lambda)^*$ is defined on every $\Delta^{\lambda'}$, and the result belongs to $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$. However, from the explicit description in the proof of Proposition 1.4.2 it follows that $(\mathbf{i}_\lambda)^*(\Delta^{\lambda'})$ equals $\omega_{S^\lambda}[-\langle \lambda, 2\check{\rho}\rangle]$ for $\lambda' = \lambda$, and 0 otherwise.

□

By definition, we obtain:

Lemma 1.4.10. *An object* $\mathcal{F} \in \mathrm{SI}(\mathrm{Gr}_G)$ *is compact if and only if there exist at most finitely many elements* $\lambda \in \Lambda$ *such that* $(\mathbf{i}_\lambda)^*(\mathcal{F}) \neq 0$*, and for every such* λ*, the corresponding object of* $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$ *is compact*[3].

As we shall see in the sequel, the objects ∇^λ are *not* compact.

1.5. **The t-structure on the semi-infinite category.**

1.5.1. We define the t-structure on $\mathrm{SI}(\mathrm{Gr}_G)$ in either of the following (tautologically equivalent) ways:

(i) The subcategory $(\mathrm{SI}(\mathrm{Gr}_G))^{\leq 0}$ is generated under filtered colimits by the objects $\Delta^\lambda[n]$, $n \geq 0$.

(ii) An object $\mathcal{F} \in \mathrm{SI}(\mathrm{Gr}_G)$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G))^{>0}$ if and only if $\mathrm{Maps}(\Delta^\lambda[n], \mathcal{F}) = *$ for $n \geq 0$.

---

[3] I.e., under the equivalence with Vect corresponds to a complex with finitely many cohomologies such that each of these cohomologies is finite-dimensional

1.5.2. Define a t-structure on

$$\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda} \simeq \mathrm{Vect}$$

to be the shift of the standard t-structure on Vect by $[-\langle \lambda, 2\check{\rho}\rangle]$, i.e., the object $\mathsf{e}[-\langle \lambda, 2\check{\rho}\rangle]$ is in the heart.

Then the above t-structure on $\mathrm{SI}(\mathrm{Gr}_G)$ can also be characterized by:

(iii) An object $\mathcal{F} \in \mathrm{SI}(\mathrm{Gr}_G)$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G))^{\leq 0}$ if and only if $(\mathbf{i}_\lambda)^*(\mathcal{F}) \in \mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda})^{\leq 0}$.

(iv) An object $\mathcal{F} \in \mathrm{SI}(\mathrm{Gr}_G)$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G))^{\geq 0}$ if and only if $(\mathbf{i}_\lambda)^!(\mathcal{F}) \in \mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda})^{\geq 0}$.

Remark 1.5.3. The above t-structure on $\mathrm{SI}(\mathrm{Gr}_G)$ (and in particular, the corresponding truncations functors) exists for abstract reasons. Ultimately, all the objects involved are colimits of compact objects of $\mathrm{Shv}(\mathrm{Gr}_G)$, i.e., compact objects in $\mathrm{Shv}(Y_i)$, where Y_i are closed subschemes of Gr_G. However, one should not delude oneself by thinking that the above t-structure can be modeled by a finite-dimensional situation, as is manifested by Theorem 1.5.5 below.

1.5.4. In Sect. 3.3 we will prove:

Theorem 1.5.5. *For $\lambda \neq 0$, the object $(\mathbf{i}_\lambda)^!(\Delta^0)$ belongs to $(\mathrm{SI}(\mathrm{Gr}_G)_\lambda)^{>0}$.*

The meaning of Theorem 1.5.5 is that Δ^0 belongs to $(\mathrm{SI}(\mathrm{Gr}_G))^\heartsuit$ and is *the intermediate extension* of ω_{S^0}, the latter being the unique irreducible object in $\mathrm{SI}(\mathrm{Gr}_G)_{=0}$. In particular, we obtain:

Corollary 1.5.6. *The object $\Delta^0 \in (\mathrm{SI}(\mathrm{Gr}_G))^\heartsuit$ is irreducible.*

In Sect. 5.4 we will prove:

Proposition 1.5.7. *The abelian category $(\mathrm{SI}(\mathrm{Gr}_G))^\heartsuit$ is equivalent to $\mathrm{Rep}(\check{B}^-)$, where $\check{B}^-$ is the (negative) Borel in the Langlands dual $\check{G}$ of G. Under this equivalence, the object Δ^λ correspond to the 1-dimensional representation of $\check{B}^-$ with character $w_0(\lambda)$.*

1.5.8. However, Δ^0 is *not* the object that we are after. In the next section we will construct another object

$$\mathrm{IC}^{\frac{\infty}{2}} \in (\mathrm{SI}(\mathrm{Gr}_G))^\heartsuit$$

that we will call the *semi-infinite IC sheaf*, and study its properties. We will relate $\mathrm{IC}^{\frac{\infty}{2}}$ to two other naturally appearing objects:

(i) In Sect. 3 we will relate $\mathrm{IC}^{\frac{\infty}{2}}$ to the IC sheaf of Drinfeld's compactification.

(ii) In Sect. 5 we will express it as the (dual) baby Verma object.

We will show:

Proposition 1.5.9. *In terms of the equivalence of Proposition 1.5.7, the object $\mathrm{IC}^{\frac{\infty}{2}}$ corresponds to $\mathcal{O}(\check{B}^-/\check{T})$.*

Remark 1.5.10. One may wonder whether the object $\mathrm{IC}^{\frac{\infty}{2}}$ that we will construct can be expressed as the intermediate expression in some t-structure. This is indeed possible via the following procedure:

Instead of the usual affine Grassmannian we consider its version over the Ran space $\mathrm{Ran}(X)$ of a curve X; denote it $\mathrm{Gr}_{G,\mathrm{Ran}}$. It is equipped with an action of the Ran version of the loop group $\mathfrak{L}(G)_{\mathrm{Ran}}$, and in particular $\mathfrak{L}(N)_{\mathrm{Ran}}$. We define the category

$$\mathrm{SI}(\mathrm{Gr}_{G,\mathrm{Ran}}) := (\mathrm{Shv}(\mathrm{Gr}_{G,\mathrm{Ran}}))^{\mathfrak{L}(N)_{\mathrm{Ran}}},$$

and it can be equipped with a t-structure. Let $S^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$ be the locally closed subfunctor equal to the orbit of unit section with respect to $\mathfrak{L}(N)_{\mathrm{Ran}}$. Let $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ be the intermediate extension of the dualizing sheaf on S^0_{Ran}. One can show (and this will be done in [Ga1]) that our $\mathrm{IC}^{\frac{\infty}{2}}$ identifies with the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}$ to

$$\mathrm{Gr}_G = \{x\} \underset{\mathrm{Ran}(X)}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}.$$

2. Construction of the semi-infinite IC sheaf as a colimit

2.1. **The Langlands dual group.** In this subsection we will fix some conventions pertaining to the Lannglands dual of G, to be used in this section and later in the paper.

2.1.1. Let $\check{G}$ be the Langlands dual group of G. We regard as equipped with chosen Borel and Cartan subgroups:

$$\check{T} \subset \check{B}.$$

In particular, we have a well-defined negative Borel $\check{B}^-$, so that $\check{T} = \check{B} \cap \check{B}^-$.

We regard $\check{T}$ as a common torus quotient of $\check{B}$ and $\check{B}^-$ and use characters of $\check{T}$ to produce characters of each. Note, however, that a character that is dominant from the point of view of $\check{B}$ is anti-dominant from the point of view of $\check{B}^-$.

We regard $\mathrm{Rep}(\check{G})$ the *abelian* symmetric monoidal category of $\check{G}$-representations. For $V \in \mathrm{Rep}(\check{G})$ and $\nu \in \Lambda$ (note that Λ is the weight lattice of $\check{T}$) we let $V(\nu)$ denote the corresponding weight space.

2.1.2. We fix representatives of irreducible highest weight representations V^λ, $\lambda \in \Lambda^+$ equipped with *trivializations* of highest weight lines $V^\lambda(\lambda) = (V^\lambda)^{\check{N}}$, i.e., with chosen highest weight vectors

$$v^\lambda \in V^\lambda(\lambda).$$

Let $(v^\lambda)^*$ denote the functional $V^\lambda \to \mathsf{e}$ such that $\langle (v^\lambda)^*, v^\lambda \rangle = 1$ and $(v^\lambda)^*|_{V(\mu)} = 0$ for $\mu \neq \lambda$. We can regard $(v^\lambda)^*$ as a highest weight vector with respect to $\check{B}^-$ in $(V^\lambda)^*$.

2.1.3. In what follows let e^λ denote the 1-dimensional representation of $\check{T}$ on e, given by the character λ.

When we think of V^λ as endowed with the distinguished highest weight vector v^λ, we identify it with

$$\mathrm{Ind}_{\check{B}}^{\check{G}}(\mathbf{e}^\lambda),$$

where $\mathbf{e}^\lambda$ is regarded as a representation of $\check{B}$ via $\check{B} \to \check{T}$, and

$$\mathrm{Ind}_{\check{B}}^{\check{G}} : \mathrm{Rep}(\check{B}) \to \mathrm{Rep}(\check{G})$$

is the functor *left adjoint* to the restriction functor $\mathrm{Res}_{\check{B}}^{\check{G}}$.

When we think of V^λ as endowed with the distinguished covector $(v^\lambda)^*$, we identify it with

$$\mathrm{coInd}_{\check{B}^-}^{\check{G}}(\mathbf{e}^\lambda),$$

where $\mathbf{e}^\lambda$ is regarded as a representation of $\check{B}^-$ via $\check{B}^- \to \check{T}$, and

$$\mathrm{coInd}_{\check{B}^-}^{\check{G}} : \mathrm{Rep}(\check{B}^-) \to \mathrm{Rep}(\check{G})$$

is the functor *right adjoint* to the restriction functor $\mathrm{Res}_{\check{B}^-}^{\check{G}}$.

2.1.4. For each pair $\lambda_1, \lambda_2 \in \Lambda$, we have a *canonical* map

$$V^{\lambda_1} \otimes V^{\lambda_2} \to V^{\lambda_1+\lambda_2}, \tag{2.1}$$

determined by the condition that the diagram

$$\begin{array}{ccc} V^{\lambda_1} \otimes V^{\lambda_2} & \longrightarrow & V^{\lambda_1+\lambda_2} \\ {\scriptstyle (v^{\lambda_1})^* \otimes (v^{\lambda_2})^*}\downarrow & & \downarrow{\scriptstyle (v^{\lambda_1+\lambda_2})^*} \\ \mathsf{e} \otimes \mathsf{e} & \xrightarrow{=} & \mathsf{e} \end{array}$$

commutes.

We also have a canonical map

$$V^{\lambda_1+\lambda_2} \to V^{\lambda_1} \otimes V^{\lambda_2}, \tag{2.2}$$

determined by the condition that the diagram

$$\begin{array}{ccc} V^{\lambda_1+\lambda_2} & \longrightarrow & V^{\lambda_1}\otimes V^{\lambda_2} \\ {\scriptstyle v^{\lambda_1+\lambda_2}}\uparrow & & \uparrow{\scriptstyle v^{\lambda_1}\otimes v^{\lambda_2}} \\ \mathsf{e} & \xrightarrow{=} & \mathsf{e}\otimes\mathsf{e} \end{array}$$

commutes.

2.1.5. For future reference we note that for a fixed finite-dimensional $V \in \operatorname{Rep}(\check{G})$ and λ deep enough in the dominant cone (i.e., $\lambda \in \lambda_0 + \Lambda^+$ for some λ_0) there exists a *canonical* isomorphism in $\operatorname{Rep}(\check{G})$

$$V^\lambda \otimes V \simeq \underset{\mu\in\Lambda}{\oplus} V^{\lambda+\mu}\otimes V(\mu). \tag{2.3}$$

It is uniquely characterized by the property that for a given $v \in V(\mu)$, the image of

$$v^\lambda \otimes v \in V^\lambda(\lambda)\otimes V(\mu) \subset V^\lambda \otimes V$$

lies in

$$\underset{\mu'\geq\mu}{\oplus} V^{\lambda+\mu}\otimes V(\mu) \subset \underset{\mu\in\Lambda}{\oplus} V^{\lambda+\mu}\otimes V(\mu),$$

and its projection onto the $V^{\lambda+\mu}\otimes V(\mu)$ factor equals $v^{\lambda+\mu}\otimes v$.

2.2. **Recollections on geometric Satake.**

2.2.1. We consider $\operatorname{Sph}(G) := \operatorname{Shv}(\operatorname{Gr}_G)^{\mathfrak{L}^+(G)}$ as a monoidal category with respect to convolution. As such it acts on $\operatorname{Shv}(\operatorname{Gr}_G)$ by right convolutions; denote this action by

$$\mathcal{F} \in \operatorname{Shv}(\operatorname{Gr}_G),\ \mathcal{F}' \in \operatorname{Sph}(G) \mapsto \mathcal{F}\star\mathcal{F}'.$$

2.2.2. We will regard Geometric Satake as a monoidal functor

$$\operatorname{Sat} : \operatorname{Rep}(\check{G}) \to \operatorname{Sph}(G).$$

We will use the following of its properties:

(i) For $\lambda \in \Lambda^+$, we have $\operatorname{Sat}(V^\lambda) = \operatorname{IC}_{\overline{\operatorname{Gr}}^\lambda_G}$, where $\operatorname{Gr}^\lambda_G = \mathfrak{L}^+(G)\cdot t^\lambda$, and $\overline{\operatorname{Gr}}^\lambda_G$ is its closure.

(ii) For $V \in \operatorname{Rep}(\check{G})$ and $\mu \in \Lambda$, the weight space $V(\mu)$ identifies canonically with

$$H_c(S^\mu, (\mathbf{i}_\mu)^*(\operatorname{Sat}(V)))[\langle \mu, 2\check{\rho}\rangle];$$

in particular, the above cohomology sits in a single degree equal to $\langle \mu, 2\check{\rho}\rangle$.

(iii) For $\lambda \in \Lambda^+$ the trivialization of the line $V^\lambda(\lambda)$ corresponds to the identification

$$H_c(S^\lambda, (\mathbf{i}_\lambda)^*(\operatorname{Sat}(V^\lambda)))[\langle \lambda, 2\check{\rho}\rangle] \simeq H_c(\mathfrak{L}^+(N)\cdot t^\lambda, \mathsf{e})[2\langle \lambda, 2\check{\rho}\rangle] \simeq \mathsf{e},$$

where we are using the fact that

$$S^\lambda \cap \overline{\operatorname{Gr}}^\lambda_G = \mathfrak{L}^+(N)\cdot t^\lambda \simeq \mathfrak{L}^+(N)/\operatorname{Ad}_{t^\lambda}(\mathfrak{L}^+(G)),$$

and $\dim(\mathfrak{L}^+(N)/\operatorname{Ad}_{t^\lambda}(\mathfrak{L}^+(G))) = \langle \lambda, 2\check{\rho}\rangle$.

2.2.3. Denote

$$S^{-,\mu} := \mathfrak{L}(N^-) \cdot t^\mu \overset{\mathbf{i}_{-,\mu}}{\hookrightarrow} \mathrm{Gr}_G.$$

Let k_μ (resp., $\mathsf{k}_{-,\mu}$) denote the embeddings of the point

$$S^\mu \overset{\mathsf{k}_\mu}{\hookleftarrow} \mathrm{pt} \overset{\mathsf{k}_{-,\mu}}{\hookrightarrow} S^{-,\mu}.$$

We have the following assertion, which is a hallmark application on Braden's theorem from [Bra]:

Lemma 2.2.4.

(a) *For $\mathcal{F} \in \mathrm{Shv}(S^\mu)$ equivariant with respect to the action of T we have canonical isomorphisms*

$$H_c(S^\mu, \mathcal{F}) \simeq (\mathsf{k}_\mu)^!(\mathcal{F}), \quad H(S^\mu, \mathcal{F}) \simeq (\mathsf{k}_\mu)^*(\mathcal{F}),$$

and similarly for $(S^{-,\mu}, \mathsf{k}_{-,\mu})$.

(b) *For $\mathcal{F} \in \mathrm{Shv}(\mathrm{Gr}_G)$ equivariant with respect to the action of T, we have a canonical isomorphism*

$$(\mathsf{k}_\mu)^! \circ (\mathbf{i}_\mu)^*(\mathcal{F}) \simeq (\mathsf{k}_{-,\mu})^* \circ (\mathbf{i}_{-,\mu})^!(\mathcal{F})$$

We will apply the isomorphism of Lemma 2.2.4 to objects of the form $\mathrm{Sat}(V)$ and thus obtain several alternative expressions for $V(\mu)$.

2.3. **Definition of the semi-infinite IC sheaf.**

2.3.1. Consider Λ^+ with respect to the following (non-standard!) order relation

$$\lambda_1 \leq \lambda_2 \Leftrightarrow \lambda_2 - \lambda_1 \in \Lambda^+.$$

Note that this poset if filtered.

2.3.2. We are going to construct a functor

$$(\Lambda^+, \leq) \to \mathrm{Shv}(\mathrm{Gr}_G)$$

that on the level of objects sends

$$\lambda \in \Lambda^+ \mapsto t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle].$$

2.3.3. For an individual pair $\lambda_1 \leq \Lambda_2$, the corresponding map

$$t^{-\lambda_1} \cdot \mathrm{Sat}(V^{\lambda_1})[\langle \lambda_1, 2\check{\rho}\rangle] \to t^{-\lambda_2} \cdot \mathrm{Sat}(V^{\lambda_2})[\langle \lambda_2, 2\check{\rho}\rangle] \tag{2.4}$$

is defined as follows.

Write $\lambda_2 = \lambda_1 + \lambda$ with $\lambda \in \Lambda^+$. We claim that we have a canonically defined map

$$\delta_{1,\mathrm{Gr}_G} \to t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]. \tag{2.5}$$

Indeed, the datum of a map (2.5) is equivalent to that of a map

$$\delta_{t^\lambda,\mathrm{Gr}_G} \to \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle], \tag{2.6}$$

and the latter amounts to a vector in the !-fiber of $\mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]$ at t^λ.

However, this !-fiber of $\mathrm{Sat}(V^\lambda)$ identifies canonically with e, as t^λ belongs to the smooth locus $\mathrm{Gr}^\lambda_G \subset \overline{\mathrm{Gr}}^\lambda_G$ and $\langle \lambda, 2\check{\rho}\rangle = \dim(\mathrm{Gr}^\lambda_G)$.

We now let (2.4) be the map

$$t^{-\lambda_1} \cdot \mathrm{Sat}(V^{\lambda_1})[\langle \lambda_1, 2\check{\rho}\rangle] \simeq t^{-\lambda_1} \cdot \delta_{1,\mathrm{Gr}_G} \star \mathrm{Sat}(V^{\lambda_1})[\langle \lambda_1, 2\check{\rho}\rangle] \overset{(2.6)}{\longrightarrow}$$
$$\to t^{-\lambda_1} \cdot t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle] \star \mathrm{Sat}(V^{\lambda_1})[\langle \lambda_1, 2\check{\rho}\rangle] \simeq t^{-\lambda_2} \cdot \mathrm{Sat}(V^\lambda \otimes V^{\lambda_1})[\langle \lambda_2, 2\check{\rho}\rangle] \overset{(2.1)}{\longrightarrow}$$
$$\to t^{-\lambda_2} \cdot \mathrm{Sat}(V^{\lambda_2})[\langle \lambda_2, 2\check{\rho}\rangle]$$

2.3.4. It is easy to check that the above assignment defines a functor from $(\Lambda^+, \leq)$ to the *homotopy category* of $\mathrm{Shv}(\mathrm{Gr}_G)$, i.e., that for $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the corresponding two maps

$$t^{-\lambda_1} \cdot \mathrm{Sat}(V^{\lambda_1})[\langle \lambda_1, 2\check{\rho}\rangle] \rightrightarrows t^{-\lambda_3} \cdot \mathrm{Sat}(V^{\lambda_3})[\langle \lambda_3, 2\check{\rho}\rangle]$$

are equal in the homotopy category.

In Sect. 2.7 below we will show that the above assignment uniquely extends to a functor of ∞-categories

$$(\Lambda^+, \leq) \to \mathrm{Shv}(\mathrm{Gr}_G), \quad \lambda \mapsto t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]. \tag{2.7}$$

Assuming the existence of the functor (2.7), we define an object $\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathrm{Gr}_G)$ by

$$\mathrm{IC}^{\frac{\infty}{2}} := \underset{\lambda \in (\Lambda^+, \leq)}{\mathrm{colim}}\, t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle] \tag{2.8}$$

Remark 2.3.5. A conceptual framework for the formation of the colimit (2.8), suggested to us by S. Raskin, will be explained in Sect. 6.3.

2.3.6. Here are the first properties of $\mathrm{IC}^{\frac{\infty}{2}}$:

Proposition 2.3.7.

(a) $\mathrm{IC}^{\frac{\infty}{2}}$ *belongs to* $\mathrm{SI}(\mathrm{Gr}_G) := \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$*;*

(b) $\mathrm{IC}^{\frac{\infty}{2}}$ *is supported on* $\overline{S}^0$*;*

(c) $(\mathbf{i}_0)^!(\mathrm{IC}^{\frac{\infty}{2}}) \simeq \omega_{S^0} \simeq (\mathbf{i}_0)^*(\mathrm{IC}^{\frac{\infty}{2}})$*;*

(d) $\mathrm{IC}^{\frac{\infty}{2}}$ *belongs to* $\mathrm{SI}(\mathrm{Gr}_G)^\heartsuit$.

2.4. **Proof of Proposition 2.3.7.**

2.4.1. For point (a), fix a subgroup N_α as in Sect. 1.2.1. We need to show that $\mathrm{IC}^{\frac{\infty}{2}}$ is N_α-equivariant.

Consider the (cofinal) subset of Λ^+ consisting of those elements λ such that $\mathrm{Ad}_{t^\lambda}(N_\alpha) \subset \mathfrak{L}^+(N)$. Clearly, this subset is cofinal.

It is clear that for every such λ, the object

$$t^{-\lambda} \cdot \mathcal{F}$$

is N_α-equivariant for any $\mathcal{F}$ that is $\mathfrak{L}^+(N)$-equivariant.

This implies that all the terms in the colimit (2.8) corresponding to λ's from this subset are N_α-equivariant.

2.4.2. For point (b) we need to show that if S^μ is in the support of $\mathrm{IC}^{\frac{\infty}{2}}$, then $\mu \in -\Lambda^{\mathrm{pos}}$. However, this follows from the fact that

$$S^\mu \cap \overline{\mathrm{Gr}}^\lambda_G \neq \emptyset \Rightarrow \lambda - \mu \in \Lambda^{\mathrm{pos}}.$$

2.4.3. In view of (b), we have $(\mathbf{i}_0)^!(\mathrm{IC}^{\frac{\infty}{2}}) \simeq (\mathbf{i}_0)^*(\mathrm{IC}^{\frac{\infty}{2}})$. Therefore, using Proposition 1.3.5 and point (a), in order to prove (c), we need to show that

$$(\mathsf{k}_0)^! \circ (\mathbf{i}_0)^!(\mathrm{IC}^{\frac{\infty}{2}}) \simeq \mathsf{e}.$$

The left-hand side is a colimit over $\lambda \in \Lambda^+$ of the !-fibers of

$$t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]$$

at the point $1 \in \mathrm{Gr}_G$.

Each term in this colimit is canonically isomorphic to e. And it is easy to see from the definition of the maps (2.4), that the maps in this system are the identity maps.

2.4.4. For point (d), let us first prove that $\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{SI}(\mathrm{Gr}_G)^{\geq 0}$. We will show that for $0 \neq \mu \in -\Lambda^{\mathrm{pos}}$, we have

$$(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}}) \in (\mathrm{SI}_{=\mu})^{>0}.$$

By Proposition 1.3.5 and point (a), we need to show that the colimit over $\lambda \in \Lambda^+$ of the !-fibers of

$$t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]$$

at t^μ lives in degrees $> \langle \mu, 2\check{\rho}\rangle$.

For every fixed μ, this is the same as the !-fiber of $\mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]$ at $t^{\lambda+\mu}$. Now the desired estimate follows from the fact that the !-restriction of $\mathrm{Sat}(V^\lambda)$ to $\mathrm{Gr}_G^{\lambda+\mu}$ is concentrated in strictly positive perverse degrees, while

$$\dim(\mathrm{Gr}_G^{\lambda+\mu}) = \langle \lambda+\mu, 2\check{\rho}\rangle.$$

2.4.5. Let us now show that $\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{SI}(\mathrm{Gr}_G)^{\leq 0}$. I.e., we need to show that for $\mu \in -\Lambda^{\mathrm{pos}}$, we have

$$(\mathbf{i}_\mu)^*(\mathrm{IC}^{\frac{\infty}{2}}) \in (\mathrm{SI}_{=\mu})^{\leq 0}.$$

In fact, we will see that $(\mathbf{i}_\mu)^*(\mathrm{IC}^{\frac{\infty}{2}})$ belongs to $(\mathrm{SI}_{=\mu})^{\heartsuit}$ for all $\mu \in -\Lambda^{\mathrm{pos}}$. (We will also see in Proposition 2.5.4 that these objects are non-zero, which implies that $\mathrm{IC}^{\frac{\infty}{2}}$ is *not* the minimal extension from S^0.)

By Proposition 1.3.5 and point (a), we need to show that the colimit over $\lambda \in \Lambda^+$ of

$$(\mathsf{k}_\mu)^! \circ (\mathbf{i}_\mu)^* \left(t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]\right)$$

lives degree $\langle \mu, 2\check{\rho}\rangle$.

We claim this happens for each term in the colimit. Indeed, using Lemma 2.2.4(a), we rewrite

$$(\mathsf{k}_\mu)^! \circ (\mathbf{i}_\mu)^*(t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)) \simeq H_c(S^{\mu+\lambda}, (\mathbf{i}_{\mu+\lambda})^*(\mathrm{Sat}(V^\lambda))).$$

Now, by Sect. 2.2.2(ii), the latter cohomology indeed lives in single degree equal to $\langle \mu+\lambda, 2\check{\rho}\rangle$.
□[Proposition 2.3.7]

2.5. Fibers of the semi-infinite IC sheaf.

2.5.1. We will now derive some information on the shape of $(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}})$ and $(\mathbf{i}_\mu)^*(\mathrm{IC}^{\frac{\infty}{2}})$.

We claim:

Proposition 2.5.2. *For $\mu \in -\Lambda^{\mathrm{pos}}$, we have*

$$(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}}) \simeq \omega_{S^\mu}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathrm{Sym}(\check{\mathfrak{g}}/\check{\mathfrak{b}}^-[-2])(-\mu).$$

Proof. We need to show that the colimit over $\lambda \in \Lambda^+$ of !-fibers at $t^{\lambda+\mu}$ of $\mathrm{Sat}(V^\lambda)[\langle \lambda+\mu, 2\check{\rho}\rangle]$, under the maps (2.4) identifies with $\mathrm{Sym}((\check{\mathfrak{g}}/\check{\mathfrak{b}}^-)[-2])(-\mu)$. This calculation will be performed in Sect. 4.2.4. □

2.5.3. We now claim:

Proposition 2.5.4. *For $\mu \in -\Lambda^{\mathrm{pos}}$ there is a canonical isomorphism*

$$(\mathbf{i}_\mu)^*(\mathrm{IC}^{\frac{\infty}{2}}) \simeq \omega_{S^\mu}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathcal{O}(\check{N})(\mu),$$

where $\mathcal{O}_{\check{N}}$ denotes the algebra of regular functions on $\check{N}$, viewed as a T-representation via the adjoint action.

Proof. We need to show that the colimit over $\lambda \in \Lambda^+$ of

$$(\mathsf{k}_\mu)^! \circ (\mathbf{i}_\mu)^* \left(t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda+\mu, 2\check{\rho}\rangle]\right)$$

identifies canonically with $\mathcal{O}(\check{N})(\mu)$.

By Sect. 2.2.2(ii), each term in the colimit identifies canonically with $V^\lambda(\lambda+\mu)$. Furthermore, by Sect. 2.6 below, the transition maps are given as follows:

For $\lambda_2 = \lambda_1 + \lambda$, the corresponding map $V^{\lambda_1}(\lambda_1+\mu) \to V^{\lambda_2}(\lambda_2+\mu)$ is given by

$$V^{\lambda_1}(\lambda_1+\mu) \simeq V^\lambda(\lambda) \otimes V^{\lambda_1}(\lambda_1+\mu) \to (V^\lambda \otimes V^{\lambda_1})(\lambda_2+\mu) \overset{(2.1)}{\longrightarrow} V^{\lambda_2}(\lambda_2+\mu).$$

However, we claim that this colimit indeed identifies with $\mathcal{O}(\check{N})(\mu)$. Namely, we map each $V^\lambda(\lambda+\mu)$ to $\mathcal{O}(\check{N})(\mu)$ by the matrix coefficient map, which sends $v \in V^\lambda(\lambda+\mu)$ to the function

$$n \mapsto \langle (v^\lambda)^*, n \cdot v\rangle.$$

□

2.6. **Tensor product maps in terms of geometric Satake.** In this subsection we will explain the geometry that stands behind the claim made in the course of the proof of Proposition 2.5.4 above.

2.6.1. Let V be a representation of $\check{G}$, and $\lambda \in \Lambda^+$ and $\mu \in \Lambda$. Consider the map

$$V(\mu) \simeq V^\lambda(\lambda) \otimes V(\mu) \to (V^\lambda \otimes V)(\lambda+\mu). \tag{2.9}$$

In this subsection we will describe it in terms of geometric Satake.

2.6.2. Let Conv_G denote the convolution diagram, i.e.,

$$\mathrm{Conv}_G = \mathfrak{L}(G) \overset{\mathfrak{L}^+(G)}{\times} \mathrm{Gr}_G,$$

where $\overset{\mathfrak{L}^+G}{\times}$ means taking the quotient with respect to the diagonal action of $\mathfrak{L}^+(G)$. We will think of Conv_G as fibered over Gr_G via

$$\mathrm{pr} : \mathfrak{L}(G) \overset{\mathfrak{L}^+(G)}{\times} \mathrm{Gr}_G \to \mathfrak{L}(G)/\mathfrak{L}^+(G) = \mathrm{Gr}_G$$

with typical fiber Gr_G. We will write Conv_G as a "twisted product"

$$\mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G,$$

where the first factor is the "base copy" Gr_G and the second factor is the "fiber copy" of Gr_G.

For any $\mathcal{F} \in \mathrm{Shv}(\mathrm{Gr}_G)$ and $\mathcal{F}' \in \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)} = \mathrm{Sph}(G)$, we can form their twisted external product

$$\mathcal{F} \widetilde{\boxtimes} \mathcal{F}' \in \mathrm{Shv}(\mathrm{Conv}_G).$$

The action of $\mathfrak{L}(G)$ on Gr_G defines yet another map

$$\mathrm{act} : \mathrm{Conv}_G \to \mathrm{Gr}_G.$$

This map is (ind)-proper. For $\mathcal{F}, \mathcal{F}'$ as above, their convolution product $\mathcal{F} \star \mathcal{F}'$ is by definition

$$\mathrm{act}_!(\mathcal{F} \widetilde{\boxtimes} \mathcal{F}') \simeq \mathrm{act}_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{F}').$$

2.6.3. Using Sect. 2.2.2 and Lemma 2.2.4(b), for V, λ and μ as above, we need to describe the map

$$H(S^{-,\mu},(\mathbf{i}_{-,\mu})^!(\mathrm{Sat}(V)))[\langle\mu,2\check\rho\rangle]\to H(S^{-,\mu+\lambda},(\mathbf{i}_{-,\mu+\lambda})^!(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}\star\mathrm{Sat}(V)))[\langle\mu+\lambda,2\check\rho\rangle]\simeq$$
$$\simeq H(\mathrm{act}^{-1}(S^{-,\mu+\lambda}),(\widetilde{\mathbf{i}}_{-,\mu+\lambda})^*(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}\widetilde{\boxtimes}\mathrm{Sat}(V)))[\langle\mu+\lambda,2\check\rho\rangle], \tag{2.10}$$

where $\widetilde{\mathbf{i}}_{-,\mu+\lambda}$ denotes the locally closed embedding

$$\mathrm{act}^{-1}(S^{-,\mu+\lambda})\hookrightarrow \mathrm{Gr}_G\widetilde{\times}\mathrm{Gr}_G=\mathrm{Conv}_G\,.$$

We now notice that

$$\mathrm{act}^{-1}(S^{\mu+\lambda})\cap(\overline{\mathrm{Gr}}^\lambda_G\widetilde{\times}\mathrm{Gr}_G)$$

contains as a closed subscheme the preimage with respect to pr of

$$S^{-,\lambda}\cap\overline{\mathrm{Gr}}^\lambda_G=\{t^\lambda\}.$$

This subscheme is isomorphic to a copy of $S^{-,\mu}$, and the !-pullback of

$$(\widetilde{\mathbf{i}}_{-,\mu+\lambda})^*(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}\widetilde{\boxtimes}\mathrm{Sat}(V))[\langle\mu+\lambda,2\check\rho\rangle]$$

to it identifies with $(\mathbf{i}_{-,\mu})^!(\mathrm{Sat}(V))[\langle\mu,2\check\rho\rangle]$.

We claim that the resulting map

$$H(S^{-,\mu},(\mathbf{i}_{-,\mu})^!(\mathrm{Sat}(V)))[\langle\mu,2\check\rho\rangle]\to H(\mathrm{act}^{-1}(S^{-,\mu+\lambda}),(\widetilde{\mathbf{i}}_{-,\mu+\lambda})^*(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}\widetilde{\boxtimes}\mathrm{Sat}(V)))[\langle\mu+\lambda,2\check\rho\rangle]$$

is the desired map (2.10).

Indeed, this follows from the construction of the monoidal structure on the functor Sat.

2.7. **Construction at the level of ∞-categories.** In this subsection we carry out the construction of $\mathrm{IC}^{\frac{\infty}{2}}$ at the level of ∞-categories.

2.7.1. Consider the following paradigm. Let $\mathcal{A}$ be a monoidal ∞-category, and let $\mathcal{C}$ be a right-lax bi-module ∞-category, which means that we have the maps

$$a_1\star(a_2\star c)\to(a_1\star a_2)\star c \text{ and } (c\star a_2)\star a_1\to c\star(a_2\star a_1),\quad a_1,a_2\in\mathcal{A},\ c\in\mathcal{C}$$

(satisfying a coherent system of higher compatibilities) but these maps are not necessarily isomorphisms.

Let $c\in\mathcal{C}$ be a lax central object. By this we mean that we are given a family of maps

$$a\star c\overset{\phi(a,c)}{\longrightarrow}c\star a,\quad a\in A$$

that make the diagrams

$$\begin{array}{ccc} a_1\star(a_2\star c) & \xrightarrow{\phi(a_2,c)} & a_1\star(c\star a_2)\\ \downarrow & & \downarrow\sim\\ (a_1\star a_2)\star c & & (a_1\star c)\star a_2\\ {\scriptstyle\phi(a_1\star a_2,c)}\downarrow & & \downarrow{\scriptstyle\phi(a_1,c)}\\ c\star(a_1\star a_2) & \longleftarrow & (c\star a_1)\star a_2 \end{array} \tag{2.11}$$

commute, along with a coherent system of higher compatibilities.

2.7.2. Assume for a moment that the monoidal category $\mathcal{A}$ is ordinary (i.e., the mapping spaces are discrete). Assume also that for any $a_1,...,a_n\in\mathcal{A}$, the space

$$\mathrm{Maps}\,(a_1\star(a_2\star...(a_n\star c)...)),c\star(a_1\star....a_n))$$

is discrete. In this case, the datum of the maps $\phi(a,c)$ and the commutativity of the diagrams (2.11) at the level of homotopy categories uniquely extends to the ∞-level.

2.7.3. Suppose now that in the situation of Sect. 2.7.1, the left action of $\mathcal{A}$ on $\mathcal{C}$ is genuine (i.e., not lax), and that every object $a \in \mathcal{A}$ is left dualizable. I.e., there exists an object $a^\vee \in \mathcal{A}$ equipped with a unit map

$$\mathbf{1}_\mathcal{A} \to a^\vee \star a$$

with the corresponding universal property.

Let $\mathcal{A}^\sim$ be the ∞-category that has the same objects as $\mathcal{A}$, but where we set $\mathrm{Maps}_{\mathcal{A}^\sim}(a_1, a_2)$ to be the ∞-groupoid underlying the category

$$(a \in \mathcal{A}, a_2 \simeq a \star a_1),$$

and where the composition is given by

$$(a'' \in \mathcal{A}, a_3 \simeq a'' \star a_2) \circ (a' \in \mathcal{A}, a_2 \simeq a' \star a_1) = (a'' \star a', a_3 \simeq (a'' \star a') \star a_1).$$

In this case, we can consider the functor

$$\mathcal{A}^\sim \to \mathcal{C}$$

that at the level of objects sends $a \in \mathcal{A}$ to

$$a^\vee \star c \star a$$

and at the level of morphisms sends $(a \in \mathcal{A}, a_2 \simeq a \star a_1)$ to

$$a_1^\vee \star c \star a_1 \simeq a_1^\vee \star (\mathbf{1}_\mathcal{A} \star c) \star a_1 \to a_1^\vee \star ((a^\vee \star a) \star c) \star a_1 \simeq (a_1^\vee \star a^\vee) \star (a \star c) \star a_1 \overset{\phi(a,c)}{\longrightarrow}$$
$$\to (a_1^\vee \star a^\vee) \star (c \star a) \star a_1 \to (a \star a_1)^\vee \star c \star (a \star a_1) \simeq a_2^\vee \star c \star a_2.$$

2.7.4. We apply the above paradigm to $\mathcal{A} = \Lambda^+$, viewed as a monoidal structure given by the structure of semi-group, and $\mathcal{C} = \mathrm{Shv}(\mathrm{Gr}_G)$.

The left action of Λ on $\mathrm{Shv}(\mathrm{Gr}_G)$ is given by

$$\lambda \star \mathcal{F} = t^\lambda \cdot \mathcal{F}.$$

The right action is obtained by composing the right-lax functor

$$\Lambda^+ \to \mathrm{Rep}(\check{G}), \quad \lambda \to V^\lambda[\langle \lambda, 2\check{\rho} \rangle]$$

(with the right-lax monoidal structure given by the maps (2.1)) with the right convolution action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}(\mathrm{Gr}_G)$ obtained from Geometric Satake.

We take $c = \delta_{1,\mathrm{Gr}_G}$. The map

$$t^\lambda \cdot \delta_{1,\mathrm{Gr}_G} = \delta_{t^\lambda,\mathrm{Gr}_G} \to \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}[\langle \lambda, 2\check{\rho} \rangle] = \delta_{1,\mathrm{Gr}_G} \star \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho} \rangle] \tag{2.12}$$

is given by the identification

$$H_{\{t^\lambda\}}(\mathrm{Gr}_G, \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})[\langle \lambda, 2\check{\rho} \rangle] \simeq H_{\{t^\lambda\}}(\mathrm{Gr}^\lambda_G, \mathrm{IC}_{\mathrm{Gr}^\lambda_G})[\langle \lambda, 2\check{\rho} \rangle] \simeq H_{\{t^\lambda\}}(\mathrm{Gr}^\lambda_G, \omega_{\mathrm{Gr}^\lambda_G}) \simeq \mathsf{e}.$$

We claim that c indeed has a unique weak central structure which at the level of objects is (2.12). To prove this, we will show that we are in the situation of Sect. 2.7.2.

Indeed, we need to check that

$$\mathrm{Maps}(\delta_{t^\lambda,\mathrm{Gr}_G}, \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}[\langle \lambda, 2\check{\rho} \rangle])$$

is discrete. However, as we have just seen the above mapping space identifies with e.

2.8. **Another way to view** $\mathrm{IC}^{\frac{\infty}{2}}$. In this subsection we will write $\mathrm{IC}^{\frac{\infty}{2}}$ as a colimit in the abelian category $(\mathrm{SI}(\mathrm{Gr}_G))^\heartsuit$.

2.8.1. First, we make the following observation:

Proposition 2.8.2. *The action of* $\mathrm{Sph}(G)^{\heartsuit}$ *on* $\mathrm{SI}(\mathrm{Gr}_G)$ *is t-exact.*

Proof. Let $\mathcal{F}$ be an object of $(\mathrm{SI}(\mathrm{Gr}_G))^{\heartsuit}$. To prove that the functor $-\star\mathcal{F}$ is right t-exact on $\mathrm{SI}(\mathrm{Gr}_G)$, it is enough to show that the objects $\Delta^{\lambda}\star\mathcal{F}$ are connective. I.e., we need to check that $(\mathbf{i}_\mu)^*(\Delta^{\lambda}\star\mathcal{F})$ is a connective object of $(\mathrm{SI}(\mathrm{Gr}_G))_{=\mu}$ for every μ.

In fact, we claim that

$$(\mathbf{i}_\mu)^*(\Delta^{\lambda}\star\mathcal{F})\simeq H_c(S^{\lambda-\mu},(\mathbf{i}_{\lambda-\mu})^*(\mathcal{F}))[\langle\mu-\lambda,2\check{\rho}\rangle]\otimes\omega_{S^\mu}[\langle-\mu,2\check{\rho}\rangle],$$

or equivalently

$$(\mathbf{i}_\mu)^*((\mathbf{i}_\lambda)_!(\omega_{S^\lambda})\star\mathcal{F})\simeq H_c(S^{\mu-\lambda},(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F}))H_c(S^{\lambda-\mu},(\mathbf{i}_{\lambda-\mu})^*(\mathcal{F}))\otimes\omega_{S^\mu}.$$

Indeed, $(\mathbf{i}_\mu)^*((\mathbf{i}_\lambda)_!(\omega_{S^\lambda})\star\mathcal{F})$ ia given as a !-direct image under the map

$$\mathrm{act}: S^\lambda\widetilde{\boxtimes}S^{\mu-\lambda}\to S^\mu$$

of $\omega_{S^\lambda}\widetilde{\boxtimes}(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F})$ and is isomorphic to $\omega_{S^\mu}\otimes V$ for some $V\in\mathrm{Vect}$. In order to calculate V we compute

$$V\simeq H_c(S^\mu,\omega_{S^\mu}\otimes V)\simeq\; H_c\left(S^\mu,\mathrm{act}_!(\omega_{S^\lambda}\widetilde{\boxtimes}(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F}))\right)\simeq$$

$$\simeq H_c(S^\lambda\widetilde{\boxtimes}S^{\mu-\lambda},\omega_{S^\lambda}\widetilde{\boxtimes}(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F}))\simeq H_c(S^\lambda,\omega_{S^\lambda})\otimes H_c(S^{\mu-\lambda},(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F}))\simeq H_c(S^{\mu-\lambda},(\mathbf{i}_{\mu-\lambda})^*(\mathcal{F})),$$

as required.

In order to show that $-\star\mathcal{F}$ is left t-exact we either swap ! and * in the above argument, or argue as follows:

We note that for $\mathcal{F}\in\mathrm{Sph}(G)$, both the left (and also right) adjoint of $-\star\mathcal{F}$ is given by $-\star\mathbb{D}(\mathcal{F}^\tau)$, where $\mathbb{D}$ denotes Verdier duality, and τ is the anti-involution on $\mathrm{Sph}(G)$, induced by the inversion on $\mathfrak{L}(G)$. Now, since the left adjoint is right t-exact (by the above), the right adjoint is left t-exact. □

2.8.3. Consider the objects

$$\Delta^{-\lambda}\star\mathrm{Sat}(V^\lambda)\in\mathrm{SI}(\mathrm{Gr}_G).$$

By Proposition 2.8.2 and assuming Theorem 1.5.5, we obtain that they all lie in $(\mathrm{SI}(\mathrm{Gr}_G))^{\heartsuit}$.

Note that by (1.6), we have

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda}\star\mathrm{Sat}(V^\lambda))[-\langle\lambda,2\check{\rho}\rangle]\simeq\mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{t^{-\lambda}})\star\mathrm{Sat}(V^\lambda)\simeq\Delta^{-\lambda}\star\mathrm{Sat}(V^\lambda)\in\mathrm{SI}(\mathrm{Gr}_G),$$

where the first isomorphism is due to the fact that the action of $\mathrm{Sph}(G)$ by right convolutions is given by proper maps.

In particular, applying $\mathrm{Av}_!^{\mathfrak{L}(N)}$ to the maps (2.4) give rise to a functor

$$(\Lambda^+,\le)\to(\mathrm{SI}(\mathrm{Gr}_G))^{\heartsuit},\quad\lambda\mapsto\Delta^{-\lambda}\star\mathrm{Sat}(V^\lambda).$$

We claim:

Proposition 2.8.4. *There exists a canonical isomorphism*

$$\mathrm{IC}^{\frac{\infty}{2}}\simeq\underset{\lambda\in(\Lambda^+,\le)}{\mathrm{colim}}\,\Delta^{-\lambda}\star\mathrm{Sat}(V^\lambda)\in\mathrm{SI}(\mathrm{Gr}_G).$$

Proof. Since $\mathrm{IC}^{\frac{\infty}{2}}$ is already $\mathfrak{L}(N)$-equivariant, we have

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(\mathrm{IC}^{\frac{\infty}{2}})\simeq\mathrm{IC}^{\frac{\infty}{2}}.$$

Now, the functor $\mathrm{Av}_!^{\mathfrak{L}(N)}$, being a left adjoint, commutes with colimits, and hence

$$\mathrm{Av}_!^{\mathfrak{L}(N)}\left(\underset{\lambda\in(\Lambda^+,\le)}{\mathrm{colim}}\,t^{-\lambda}\cdot\mathrm{Sat}(V^\lambda)\right)\simeq\underset{\lambda\in(\Lambda^+,\le)}{\mathrm{colim}}\,\Delta^{-\lambda}\star\mathrm{Sat}(V^\lambda)\in\mathrm{SI}(\mathrm{Gr}_G).$$

□

3. Relation to the IC sheaf of Drinfeld's compactification

In this section we let X be a smooth and complete curve with a marked point $x \in X$.

3.1. **Recollections on Drinfeld's compactification.**

3.1.1. We let $\overline{\mathrm{Bun}}_N$ Drinfeld's relative compactification of the stack Bun_N along the fibers of the map $\mathrm{Bun}_N \to \mathrm{Bun}_G \times \mathrm{pt}/T$.

I.e., $\overline{\mathrm{Bun}}_N$ is the algebraic stack that classifies triples $(\mathcal{P}_G, \kappa)$, where:

(i) $\mathcal{P}_G$ is a G-bundle on X;

(ii) κ is a *Plücker* data, i.e., a system of non-zero maps

$$\kappa^{\check{\lambda}} : \mathcal{O}_X \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G},$$

(here $\mathcal{V}^{\check{\lambda}}$ denotes the Weyl module with highest weight $\check{\lambda} \in \check{\Lambda}^+$) that satisfy Plücker relations, i.e., for $\check{\lambda}_1$ and $\check{\lambda}_2$ the diagram

$$\begin{array}{ccc} \mathcal{O}_X \otimes \mathcal{O}_X & \xrightarrow{\kappa^{\check{\lambda}_1} \otimes \kappa^{\check{\lambda}_2}} & \mathcal{V}^{\check{\lambda}_1}_{\mathcal{P}_G} \otimes \mathcal{V}^{\check{\lambda}_2}_{\mathcal{P}_G} \\ \sim \big\uparrow & & \big\uparrow \\ \mathcal{O}_X & \xrightarrow{\kappa^{\check{\lambda}_1+\check{\lambda}_2}} & \mathcal{V}^{\check{\lambda}_1\check{\lambda}_2}_{\mathcal{P}_G} \end{array}$$

must commute.

3.1.2. We let $(\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$ an ind-algebraic stack, which is a version of $\overline{\mathrm{Bun}}_N$, where we allow the maps $\kappa^{\check{\lambda}}$ to have poles at x.

For each $\lambda \in \Lambda$, we let

$$(\overline{\mathrm{Bun}}_N)_{\leq \lambda\cdot x} \overset{\bar{\imath}_\lambda}{\hookrightarrow} (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}$$

be the closed substack, where we bound the order of pole of $\kappa^{\check{\lambda}}$ by the integer $\langle \lambda, \check{\lambda}\rangle$. In particular,

$$\overline{\mathrm{Bun}}_N = (\overline{\mathrm{Bun}}_N)_{\leq 0\cdot x}.$$

3.1.3. We let

$$(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x} \overset{\jmath_\lambda}{\hookrightarrow} (\overline{\mathrm{Bun}}_N)_{\leq \lambda\cdot x}$$

be the open substack, where we require that $\kappa^{\check{\lambda}}$ have a pole of order exactly $\langle \lambda, \check{\lambda}\rangle$ and where we forbid it to have zeroes on $X - x$. For $\lambda = 0$ we have

$$(\overline{\mathrm{Bun}}_N)_{=0\cdot x} = \mathrm{Bun}_N;$$

denote the corresponding map $\mathrm{Bun}_N \hookrightarrow \overline{\mathrm{Bun}}_N$ simply by $\jmath$.

The stack $(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}$ is smooth for any λ; in fact

$$(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x} \simeq \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{\mathcal{P}^0_T(-\lambda\cdot x)\},$$

where $\mathcal{P}^0_T$ denotes the trivial T-bundle on X.

Denote

$$\imath_\lambda = \bar{\imath}_\lambda \circ \jmath_\lambda.$$

3.1.4. Set
$$\Delta^\lambda_{\mathrm{glob}} := (\imath_\lambda)_!(\omega_{(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}})[-\dim((\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x})]$$
and
$$\nabla^\lambda_{\mathrm{glob}} := (\imath_\lambda)_*(\omega_{(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}})[-\dim((\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x})].$$

In the above formula,
$$\dim((\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}) = (g-1)\cdot\dim(N) + \langle\lambda, 2\check{\rho}\rangle.$$

Thus, $\omega_{(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}}[-\dim((\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x})]$ is the IC sheaf on $(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}$.

We let $\mathrm{IC}_{\mathrm{glob}}$ denote the intersection cohomology sheaf on $\overline{\mathrm{Bun}}_N$, viewed as an object of $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x})$.

3.1.5. Hecke modifications of the underlying G-bundle define a (right) action of the monoidal category $\mathrm{Sph}(G)$ on $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x})$. We denote this action by
$$\mathcal{F} \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x}),\ \mathcal{F}' \in \mathrm{Sph}(G) \ \mapsto\ \mathcal{F}\star\mathcal{F}'.$$

3.1.6. We have a tautological map
$$\pi : \mathrm{Gr}_G \to (\overline{\mathrm{Bun}}_N)_{\infty\cdot x}.$$

Indeed, if $\mathcal{P}_G$ is a G-bundle on X, equipped with a trivialization on $X - x$, the tautological reduction to N of the trivial bundle defines a Plücker data on $\mathcal{P}_G$.

Note that the preimage of $(\overline{\mathrm{Bun}}_N)_{\leq\lambda\cdot x}$ (resp., $(\overline{\mathrm{Bun}}_N)_{=\lambda\cdot x}$) under π equals $\overline{S}^\lambda$ (resp., S^λ).

3.1.7. Thus, we have the pullback functor
$$\pi^! : \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x}) \to \mathrm{Shv}(\mathrm{Gr}_G)$$
and its partially defined left adjoint $\pi_!$.

Both these functors intertwine the actions of Sph_G on $\mathrm{Shv}(\mathrm{Gr}_G)$ and $\mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{\infty\cdot x})$. This is because the convolution action is given by proper maps.

3.2. **Statement of the result.**

3.2.1. The main result of this paper is the following:

Theorem 3.2.2. *There exists a canonical isomorphism*
$$\mathrm{IC}^{\frac{\infty}{2}} \simeq \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1)\cdot\dim(N)].$$

3.2.3. It is clear from the definitions that for $\lambda \in \Lambda$, we have a canonical isomorphism
$$\nabla^\lambda \simeq \pi^!(\nabla^\lambda_{\mathrm{glob}})[(g-1)\cdot\dim(N)].$$

In addition, by adjunction, we obtain a map
$$\Delta^\lambda \to \pi^!(\Delta^\lambda_{\mathrm{glob}})[(g-1)\cdot\dim(N)]. \tag{3.1}$$

Along with Theorem 3.2.2, we will prove:

Theorem 3.2.4. *The map* (3.1) *is an isomorphism.*

3.3. **Description of fibers.**

3.3.1. Note also that Theorem 3.2.2 gives an answer to the long-standing question of how to see geometrically an isomorphism between
$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, H_{\{t^{\lambda+\mu}\}}(\mathrm{Gr}_G, \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})$$
and !-fibers of $(\imath_\mu)^!(\mathrm{IC}_{\mathrm{glob}})$.

Indeed, the two sides identify with the !-fiber at t^μ of the two sides of the isomorphism of Theorem 3.2.2.

3.3.2. From Theorems 3.2.2 and 3.2.4 one can obtain explicit descriptions of the objects

$$(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}}) \text{ and } (\mathbf{i}_\mu)^!(\Delta^0).$$

3.3.3. The following was established in [BFGM, Theorem 1.12]:

Proposition 3.3.4. *For $\mu \in -\Lambda^{\mathrm{pos}}$, there exists a canonical isomorphism*

$$(\imath_\mu)^!(\mathrm{IC}_{\mathrm{glob}}) \simeq \omega_{(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}}[-\dim((\overline{\mathrm{Bun}}_N)_{=\mu\cdot x})] \otimes \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu).$$

Combining this with Theorem 3.2.2, we obtain:

Corollary 3.3.5. *There exists a canonical isomorphism*

$$(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}}) \simeq \omega_{S^\mu}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu).$$

Remark 3.3.6. Note that Proposition 2.5.2 gives another description of $(\mathbf{i}_\mu)^!(\mathrm{IC}^{\frac{\infty}{2}})$, namely as

$$\omega_{S^\mu}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathrm{Sym}(\check{\mathfrak{g}}/\check{\mathfrak{b}}^-[-2])(-\mu).$$

Presumably, the two identifications are related by isomorphism

$$\check{\mathfrak{n}}^- \simeq \check{\mathfrak{n}} \simeq \check{\mathfrak{g}}/\check{\mathfrak{b}}^-,$$

where the first arrow is the Cartan involution. However, the author was not able to prove it.

3.3.7. The next assertion follows as a combination of Proposition 3.3.4 and [BG, Theorem 6.6]:

Proposition 3.3.8. *For $\mu \in -\Lambda^{\mathrm{pos}}$, the object*

$$(\imath_\mu)^!(\Delta^0_{\mathrm{glob}}) \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)_{=\mu\cdot x})$$

has a canonical filtration with subquotients of the form

$$\omega_{(\overline{\mathrm{Bun}}_N)_{=\mu\cdot x}}[-\dim((\overline{\mathrm{Bun}}_N)_{=\mu\cdot x})] \otimes \mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu_1) \otimes \mathrm{C}^\cdot(\check{\mathfrak{n}})(\mu_2), \quad \mu_1+\mu_2=\mu.$$

In the above proposition, $\mathrm{C}^\cdot(\check{\mathfrak{n}})$ denotes the cohomological Chevalley complex of the Lie algebra $\check{\mathfrak{n}}$.

Note that Proposition 3.3.8 immediately implies Theorem 1.5.5. Indeed, for $\mu \neq 0$, the complexes

$$\mathrm{Sym}(\check{\mathfrak{n}}^-[-2])(\mu_1) \otimes \mathrm{C}^\cdot(\check{\mathfrak{n}})(\mu_2), \quad \mu_1+\mu_2=\mu$$

are concentrated is strictly positive degrees.

3.4. **Construction of the map.** In this subsection we begin the proof of Theorem 3.2.2 by constructing the map $\mathrm{IC}^{\frac{\infty}{2}} \to \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1)\cdot\dim(N)]$.

3.4.1. By adjunction, the datum of a map

$$\mathrm{IC}^{\frac{\infty}{2}} \to \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1)\cdot\dim(N)]$$

is equivalent to that of a map

$$\pi_!(\mathrm{IC}^{\frac{\infty}{2}}) \to \mathrm{IC}_{\mathrm{glob}}[(g-1)\cdot\dim(N)]. \tag{3.2}$$

We will first construct a map

$$\pi_!(t^{-\lambda}\cdot \mathrm{Sat}(V^\lambda))[\langle\lambda, 2\check{\rho}\rangle] \to \mathrm{IC}_{\mathrm{glob}}[(g-1)\cdot\dim(N)] \tag{3.3}$$

for an individual λ.

3.4.2. Note that for $\mathcal{F}'\in \operatorname{Sph}(G)$, the functor adjoint to $-\star\mathcal{F}'$ is given by

$$-\star\mathbb{D}((\mathcal{F}')^{\tau}),$$

where $\mathbb{D}$ denotes Verdier duality on $\operatorname{Sph}(G)=\operatorname{Shv}(\operatorname{Gr}_G)^{\mathfrak{L}^+(G)}$ and τ is the anti-involution on $\operatorname{Sph}(G)$ induced by the inversion on $\mathfrak{L}(G)$.

Thus, the functor adjoint to convolution with $\operatorname{Sat}(V^\lambda)=\operatorname{IC}_{\overline{\operatorname{Gr}}^\lambda_G}$ is given by convolution with $\operatorname{IC}_{\overline{\operatorname{Gr}}^{-\lambda}_G}$. Therefore, by adjunction, the datum of a map (3.3) is equivalent to that of a map

$$\delta_{\pi(t^{-\lambda})}[\langle\lambda,2\check\rho\rangle]\to \operatorname{IC}_{\operatorname{glob}}\star\operatorname{IC}_{\overline{\operatorname{Gr}}^{-\lambda}_G}[(g-1)\cdot\dim(N)],$$

or which is the same, of a vector in the !-fiber of

$$\operatorname{IC}_{\operatorname{glob}}\star\operatorname{IC}_{\overline{\operatorname{Gr}}^{-\lambda}_G}[(g-1)\cdot\dim(N)-\langle\lambda,2\check\rho\rangle]$$

at the point $\pi(t^{-\lambda})\in(\overline{\operatorname{Bun}}_N)_{\infty\cdot x}$.

We claim that the fiber in question identifies canonically with e.

3.4.3. Indeed, consider the convolution morphism

$$\begin{array}{ccc}(\overline{\operatorname{Bun}}_N)_{\infty\cdot x}\widetilde{\times}\operatorname{Gr}_G & \xrightarrow{\operatorname{pr}} & (\overline{\operatorname{Bun}}_N)_{\infty\cdot x}\\ \downarrow{\scriptstyle\operatorname{act}} & & \\ (\overline{\operatorname{Bun}}_N)_{\infty\cdot x}. & & \end{array}$$

It is easy to see that the intersection

$$\operatorname{act}^{-1}(\pi(t^{-\lambda}))\cap(\overline{\operatorname{Bun}}_N\widetilde{\times}\overline{\operatorname{Gr}}^{-\lambda}_G)$$

coincides with its open subset

$$\operatorname{act}^{-1}(\pi(t^{-\lambda}))\cap(\operatorname{Bun}_N\widetilde{\times}\operatorname{Gr}^{-\lambda}_G) \tag{3.4}$$

and identifies canonically with $S^\lambda\cap\operatorname{Gr}^\lambda_G$.

The restriction of

$$\operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)]\widetilde{\boxtimes}\operatorname{IC}_{\overline{\operatorname{Gr}}^{-\lambda}_G}[\langle\lambda,2\check\rho\rangle]$$

to $\operatorname{Bun}_N\widetilde{\times}\operatorname{Gr}^{-\lambda}_G$ identifies with $\omega_{\operatorname{Bun}_N\widetilde{\times}\operatorname{Gr}^{-\lambda}_G}$, and hence its further !-restriction to (3.4) identifies with

$$\omega_{S^\lambda\cap\operatorname{Gr}^\lambda_G}.$$

Thus, the !-fiber of

$$\operatorname{IC}_{\operatorname{glob}}\star\operatorname{IC}_{\overline{\operatorname{Gr}}^{-\lambda}_G}[(g-1)\cdot\dim(N)-\langle\lambda,2\check\rho\rangle]$$

at $\pi(t^{-\lambda})\in(\overline{\operatorname{Bun}}_N)_{\infty\cdot x}$ identifies with

$$H(S^\lambda\cap\operatorname{Gr}^\lambda_G,\omega_{S^\lambda\cap\operatorname{Gr}^\lambda_G})[-2\langle\lambda,2\check\rho\rangle],$$

and the latter is indeed canonically e, since

$$S^\lambda\cap\operatorname{Gr}^\lambda_G=\mathfrak{L}^+(N)/\operatorname{Ad}_{t^\lambda}(\mathfrak{L}^+(N)),$$

and is isomorphic to an affine space of dimension $\langle\lambda,2\check\rho\rangle$.

3.4.4. It is easy to see from the construction that for $\lambda_2=\lambda_1+\lambda$, the corresponding maps
$$\pi_!(t^{-\lambda_1}\cdot \operatorname{Sat}(V^{\lambda_1}))[\langle\lambda_1,2\check\rho\rangle]\to \operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)]$$
and
$$\pi_!(t^{-\lambda_2}\cdot \operatorname{Sat}(V^{\lambda_2}))[\langle\lambda_2,2\check\rho\rangle]\to \operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)]$$
are compatible with the morphism (2.4).

We claim that these maps combine to give rise to a uniquely defined map
$$\pi_!(\operatorname{IC}^{\frac{\infty}{2}})=\underset{\lambda\in\Lambda^+}{\operatorname{colim}}\,\pi_!(t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda}))[\langle\lambda,2\check\rho\rangle]\to \operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)].$$

This follows from the next general observation.

3.4.5. Let $\mathcal{C}$ be an ∞-category, and
$$I\to\mathcal{C},\quad i\mapsto c_i$$
a diagram of objects, where I is some index ∞-category.

Let c' be another object of $\mathcal{C}$. Let us be given a system of maps $\phi_i:c_i\to c$, so that for every arrow $i_1\to i_2$ in I, the two maps
$$c_{i_1}\to c_{i_2}\overset{\phi_{i_2}}{\to} c' \text{ and } c_{i_1}\overset{\phi_{i_1}}{\to} c'$$
agree on the homotopy category of $\mathcal{C}$.

We have:

Lemma 3.4.6. *Suppose that for every i, the space* $\operatorname{Maps}(c_i,c')$ *is discrete. Then the maps ϕ_i uniquely combine to a map*
$$\underset{i\in I}{\operatorname{colim}}\,c_i\to c'.$$

3.4.7. The conditions of the above lemma are applicable in our situation since we have identified
$$\mathcal{M}aps_{\operatorname{Shv}((\overline{\operatorname{Bun}}_N)_{\infty\cdot x})}(\pi_!(t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda}))[\langle\lambda,2\check\rho\rangle],\operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)])$$
with e.

3.4.8. For future use, we will need to following description of the composite map
$$t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda})[\langle\lambda,2\check\rho\rangle]\to \pi^!(\operatorname{IC}_{\operatorname{glob}})[(g-1)\cdot\dim(N)]\to \pi^!(\nabla^0_{\operatorname{glob}})[(g-1)\cdot\dim(N)]\simeq(\mathbf{j}_0)_*(\omega_{S^0}).$$

Namely, unwinding the definitions, we obtain that this map equals
$$t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda})[\langle\lambda,2\check\rho\rangle]\to(\mathbf{j}_0)_*\circ(\mathbf{j}_0)^*(t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda})[\langle\lambda,2\check\rho\rangle])\to(\mathbf{j}_0)_*(\omega_{S^0}),$$
where the first arrow is the unit of the $((\mathbf{j}_0)^*,(\mathbf{j}_0)_*)$-adjunction, and the second comes from the identification
$$(\mathbf{j}_0)^*(t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda})[\langle\lambda,2\check\rho\rangle])\simeq(\mathbf{j}_0)^!(t^{-\lambda}\cdot \operatorname{Sat}(V^{\lambda})[\langle\lambda,2\check\rho\rangle])\simeq\omega_{S^0\cap t^{-\lambda}\cdot\operatorname{Gr}^\lambda_G},$$
where $S^0\cap t^{-\lambda}\cdot\operatorname{Gr}^\lambda_G$ is a closed subscheme of S^0 and we regard $\omega_{S^0\cap t^{-\lambda}\cdot\operatorname{Gr}^\lambda_G}$ as an object of $\operatorname{Shv}(S^0)$.

3.5. **Strategy of proofs of Theorems 3.2.2 and 3.2.4.**

3.5.1. In Sect. 3.8 we will prove:

Proposition 3.5.2. *The objects* $\pi^!(\operatorname{IC}_{\operatorname{glob}})$ *and* $\pi^!(\Delta^0_{\operatorname{glob}})$ *belong to* $\operatorname{Shv}(\operatorname{Gr}_G)^{\mathfrak{L}(N)}=:\operatorname{SI}(\operatorname{Gr}_G)$.

Let us assume this proposition and proceed with the proofs of Theorems 3.2.2 and 3.2.4.

3.5.3. In order to prove Theorem 3.2.2 it is sufficient to show that for any $\mu\in-\Lambda^{\operatorname{pos}}$, the map
$$(\mathbf{i}_\mu)^*(\operatorname{IC}^{\frac{\infty}{2}})\to(\mathbf{i}_\mu)^*\circ\pi^!(\operatorname{IC}_{\operatorname{glob}}[(g-1)\cdot\dim(N)]),$$
induced by (3.2) is an isomorphism.

In order to prove Theorem 3.2.4, it suffices to show that
$$(\mathbf{i}_\mu)^*\circ\pi^!(\Delta^0_{\operatorname{glob}})=0$$
for $\mu\neq0$.

3.5.4. Taking into account Proposition 1.3.5 and using Lemma 2.2.4, we obtain that Theorems 3.2.2 and 3.2.4 would follow once we establish the next assertion:

Proposition 3.5.5. *For any* $\mu \in -\Lambda^{\mathrm{pos}}$ *we have:*

(a) *The map*

$$H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^!(\mathrm{IC}^{\frac{\infty}{2}})) \to H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^! \circ \pi^!(\mathrm{IC}_{\mathrm{glob}}[(g-1)\cdot \dim(N)])),$$

induced by (3.2)*, is an isomorphism.*

(b) $H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^! \circ \pi^!(\Delta^0_{\mathrm{glob}})) = 0$ *for* $\mu \neq 0$*.*

3.6. **Recollections about the Zastava space.**

3.6.1. Consider the fiber product

$$\overline{\mathrm{Bun}}_N \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}^{-\mu}. \tag{3.5}$$

Here $\mathrm{Bun}_{B^-}^{-\mu}$ denotes the connected component of Bun_{B^-} equal to

$$\mathrm{Bun}_{B^-} \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_T^{-\mu},$$

where $\mathrm{Bun}_T^{-\mu}$ corresponds to T-bundles of degree $-\mu$.

The Zastava space $\mathcal{Z}^\mu$ is by definition the open subset in (3.5), corresponding to the condition that (generic) reduction of $\mathcal{P}_G$ to N and the (genuine) reduction of $\mathcal{P}_G$ to B^- are transversal at the generic point of X. By construction, $\mathcal{Z}^\mu$ is an algebraic stack, but it is in fact a quasi-projective scheme.

Let $\mathfrak{q}$ denote the forgetful map

$$\mathcal{Z}^\mu \to \overline{\mathrm{Bun}}_N.$$

We denote by $\overset{\circ}{\mathcal{Z}}{}^\mu \subset \mathcal{Z}^\mu$ the open subset equal to the preimage of $\mathrm{Bun}_N \subset \overline{\mathrm{Bun}}_N$ under $\mathfrak{q}$. By a slight abuse of notation we denote by $\jmath$ the corresponding open embedding. The scheme $\overset{\circ}{\mathcal{Z}}{}^\mu$ is smooth.

3.6.2. Let us think of a point of Bun_{B^-} as a triple $(\mathcal{P}_G, \mathcal{P}_T, \kappa^-)$, where $\mathcal{P}_G$ is a G-bundle on X, $\mathcal{P}_T$ is a T-bundle on X of degree μ, and κ^- is a non-degenerate Plücker data, i.e., surjective maps

$$\kappa^{-,\check{\lambda}} : \widetilde{\mathcal{V}}^{\check{\lambda}}_{\mathcal{P}_G} \to \check{\lambda}(\mathcal{P}_T),$$

satisfying Plücker relations. In the above formula $\widetilde{\mathcal{V}}^{\check{\lambda}}$ is the dual Weyl module with highest weight $\check{\lambda}$.

A point of (3.5) belongs to $\mathcal{Z}^\mu$ if and only if for every $\check{\lambda}$, the composite map

$$\mathcal{O}_X \overset{\kappa^{\check{\lambda}}}{\to} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to \widetilde{\mathcal{V}}^{\check{\lambda}}_{\mathcal{P}_G} \overset{\kappa^{-,\check{\lambda}}}{\longrightarrow} \check{\lambda}(\mathcal{P}_T^f)$$

is non-zero.

In this case, the datum of zeroes of the above composite maps is encoded by a point of X^μ. Here for an element $\mu \in -\Lambda^{\mathrm{pos}}$ equal to $\underset{i}{\Sigma} - n_i \cdot \alpha_i$ we let

$$X^\mu = \prod_i X^{(n_i)}.$$

We denote by $\mathfrak{s}$ the resulting map

$$\mathcal{Z}^\mu \to X^\mu.$$

3.6.3. Denote by $\mathfrak{F}^\mu$ the "central fiber" of $\mathcal{Z}^\mu$ over X^μ, i.e., the preimage of the point $\mu \cdot x \in X^\mu$. Set

$$\overset{\circ}{\mathfrak{F}}{}^\mu := \mathfrak{F}^\mu \cap \overset{\circ}{\mathcal{Z}}{}^\mu.$$

According to [BFGM, Proposition 2.6], we have a canonical isomorphism

$$\mathfrak{F}^\mu \simeq \overline{S}^0 \cap S^{-,\mu}$$

so that $\overset{\circ}{\mathfrak{F}}{}^\mu$ corresponds to $S^0 \cap S^{-,\mu}$.

It follows from the construction that under this identification the composite map

$$\mathfrak{F}^\mu \hookrightarrow \mathcal{Z}^\mu \xrightarrow{\mathfrak{q}} \overline{\mathrm{Bun}}_N$$

equals the map

$$\overline{S}^0 \cap S^{-,\mu} \hookrightarrow \overline{S}^0 \xrightarrow{\pi} \overline{\mathrm{Bun}}_N.$$

3.6.4. The following assertion was implicit in [BFGM, Sect. 3.4]. For completeness, we will supply a proof in Sect. 3.9:

Proposition 3.6.5.

(a) *We have a (canonical) isomorphism*

$$\mathfrak{q}^!(\mathrm{IC}_{\mathrm{glob}})[(g-1)\cdot \dim(N)] \simeq \mathrm{IC}_{\mathcal{Z}^\mu}[-\langle \mu, 2\check{\rho}\rangle],$$

extending the tautological isomorphism over $\overset{\circ}{\mathcal{Z}}{}^\mu$.

(b) *The map* $\jmath_!(\omega_{\overset{\circ}{\mathcal{Z}}{}^\mu}) \to \mathfrak{q}^! \circ \jmath_!(\omega_{\mathrm{Bun}_N})$ *is an isomorphism.*

In addition, the following was established in [BFGM, Sect. 5]:

Proposition 3.6.6.

(a) *The cohomology* $H_{\{\mathfrak{F}^\mu\}}(\mathcal{Z}^\mu, \mathrm{IC}_{\mathcal{Z}^\mu})$ *is concentrated in cohomological degree zero.*

(b) *The map*

$$H_{\{\mathfrak{F}^\mu\}}(\mathcal{Z}^\mu, \mathrm{IC}_{\mathcal{Z}^\mu}) \to H_{\{\overset{\circ}{\mathfrak{F}}{}^\mu\}}(\overset{\circ}{\mathcal{Z}}{}^\mu, \mathrm{IC}_{\overset{\circ}{\mathcal{Z}}{}^\mu})) \simeq H_{\{\overset{\circ}{\mathfrak{F}}{}^\mu\}}(\overset{\circ}{\mathcal{Z}}{}^\mu, \omega_{\overset{\circ}{\mathcal{Z}}{}^\mu}))[\langle \mu, 2\check{\rho}\rangle] \simeq \\ \simeq H(\overline{S}^0 \cap S^{-,\mu}, \omega_{\overline{S}^0 \cap S^{-,\mu}})[\langle \mu, 2\check{\rho}\rangle] \tag{3.6}$$

induces an isomorphism in (the lowest) cohomological degree 0.

3.6.7. *Proof of Proposition 3.5.5(b).* We will now deduce Proposition 3.5.5(b) from Proposition 3.6.5(b).

Indeed, from Sect. 3.6.3, we obtain that the expression

$$H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^! \circ \pi^!(\Delta^0_{\mathrm{glob}}))$$

identifies with the !-fiber at $\mu \cdot x \in X^\mu$ of $\mathfrak{s}_* \circ \mathfrak{q}^!(\Delta^0_{\mathrm{glob}})$. Now, using Proposition 3.6.5 (b) and base change along $\{\mu \cdot x\} \to X^\mu$, we obtain that it suffices to show that

$$\mathfrak{s}_* \circ \jmath_!(\omega_{\overset{\circ}{\mathcal{Z}}{}^\mu}) = 0.$$

However, this follows from the fact that there a $\mathbb{G}_m$-action on $\mathcal{Z}^\mu$ that preserves the projection $\mathfrak{s}$ that contracts $\mathcal{Z}^\mu$ to the canonical section

$$X^\mu \simeq\to \mathcal{Z}^\mu$$

(see [BFGM, Sect. 5.1]) whose image lies outside of $\overset{\circ}{\mathcal{Z}}{}^\mu$, unless $\mu = 0$.

3.7. **Proof of Proposition 3.5.5(a).**

3.7.1. For $\lambda \in \Lambda$ consider the map

$$t^{-\lambda} \cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}[\langle \lambda, 2\check{\rho} \rangle] \to \mathrm{IC}^{\frac{\infty}{2}} \to \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1) \cdot \dim(N)],$$

and the corresponding map

$$H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^!(t^{-\lambda} \cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})[\langle \lambda, 2\check{\rho} \rangle]) \to H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^! \circ \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1) \cdot \dim(N)]). \tag{3.7}$$

By Sect. 2.2.2(ii) and Lemma 2.2.4, the left-hand side is concentrated in single cohomology degree $\langle \mu, 2\check{\rho} \rangle$.

The right-hand side is also is concentrated in single cohomology degree $\langle \mu, 2\check{\rho} \rangle$, by Propositions 3.6.5(a) and 3.6.6(a). Thus, it suffices to show that the map (3.7) induces an isomorphism on the $\langle \mu, 2\check{\rho} \rangle$ cohomology.

3.7.2. First, we note that the intersection

$$S^{-,\mu} \cap (t^{-\lambda} \cdot \overline{\mathrm{Gr}}^\lambda_G)$$

contains as an open subset

$$S^0 \cap S^{-,\mu} \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G),$$

which is dense in every irreducible component of the (top) dimension $-\langle \mu, 2\check{\rho} \rangle$. Moreover, for a fixed μ and λ large, the inclusion

$$S^0 \cap S^{-,\mu} \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G) \subset S^0 \cap S^{-,\mu}$$

is an equality.

Hence, we obtain a map

$$H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^!(t^{-\lambda} \cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})[\langle \lambda, 2\check{\rho} \rangle]) \to H(S^{-,\mu} \cap S^0 \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G), \omega_{S^{-,\mu} \cap S^0 \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G)}),$$

which is injective at the level of the $\langle \mu, 2\check{\rho} \rangle$ cohomology, and this injection is an isomorphism for λ large.

3.7.3. Taking into account Proposition 3.6.6(b), it suffices to show that the following diagram commutes

$$\begin{array}{ccc} H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^!(t^{-\lambda} \cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})[\langle \lambda, 2\check{\rho} \rangle]) & \longrightarrow & H(S^0 \cap S^{-,\mu} \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G), \omega_{S^{-,\mu} \cap S^0 \cap (t^{-\lambda} \cdot \mathrm{Gr}^\lambda_G)}) \\ {\scriptstyle (3.7)}\downarrow & & \downarrow \\ H(S^{-,\mu}, (\mathbf{i}_{-,\mu})^! \circ \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1) \cdot \dim(N)]) & \underset{(3.6)}{\longrightarrow} & H(\overline{S}^0 \cap S^{-,\mu}, \omega_{\overline{S}^0 \cap S^{-,\mu}}). \end{array}$$

However, this follows from the description of the map

$$t^{-\lambda} \cdot \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}[\langle \lambda, 2\check{\rho} \rangle] \to \pi^!(\mathrm{IC}_{\mathrm{glob}})[(g-1) \cdot \dim(N)] \to \pi^!(\nabla^0_{\mathrm{glob}}) \simeq (\mathbf{j}_0)_*(\omega_{S^0})$$

in Sect. 3.4.8.

3.8. **Proof of equivariance.** In this subsection we will prove Proposition 3.5.2.

3.8.1. Let y be a point on X different from x. In what follows we will use the subscript x in $\mathrm{Gr}_{G,x}$, S^λ_x, π_x, $\mathfrak{L}(N)_x$, $\mathfrak{L}^+(N)_x$ to emphasize the dependence on x. We will also consider the corresponding objects at y.

3.8.2. Let $(\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y} \subset \overline{\mathrm{Bun}}_N$ be the open substack, where we forbid the maps $\kappa^{\check{\lambda}}$ to have a zero at y. Clearly, the map

$$\pi_x : \overline{S}^0_x \to \overline{\mathrm{Bun}}_B$$

has its image in $(\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y}$.

In addition, we can consider the map

$$\pi_{x,y} : \overline{S}^0_x \times S^0_y \to (\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y}.$$

Its restriction to $\overline{S}^0_x \times 1 \subset \overline{S}^0_x \times S^0_y$ is the original map π_x.

We will prove:

Proposition 3.8.3. *For $\mathcal{F} = \mathrm{IC}_{\mathrm{glob}}$ or $\mathcal{F} = \Delta^0_{\mathrm{glob}}$, the object $\pi^!_{x,y}(\mathcal{F})$ is equivariant with respect to $\mathfrak{L}(N)_y$ acting on the second factor in $\overline{S}^0_x \times S^0_y$.*

Let us deduce Proposition 3.5.2 from Proposition 3.8.3:

Proof. Let $\mathcal{F} \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y})$ be such that $\pi^!_{x,y}(\mathcal{F})$ is $\mathfrak{L}(N)_y$-equivariant. We claim that in this case $\pi^!_x(\mathcal{F})$ is $\mathfrak{L}(N)_x$-equivariant.

Let $N_{X-(x,y)}$ be the group ind-scheme of maps $(X - \{x, y\}) \to N$. Laurent expansion defines a closed embedding

$$N_{X-(x,y)} \to \mathfrak{L}(N)_x \times \mathfrak{L}(N)_y.$$

We have three actions of $N_{X-(x,y)}$ on $\overline{S}^0_x \times S^0_y$. One is via the map $N_{X-(x,y)} \to \mathfrak{L}(N)_x$ and the first factor; another is via the map $N_{X-(x,y)} \to \mathfrak{L}(N)_y$ and the second factor; and the third is diagonal.

It is clear that the map $\pi_{x,y}$ is $N_{X-(x,y)}$-invariant with respect to the *diagonal* action. In particular, for any $\mathcal{F} \in \mathrm{Shv}((\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y})$, we have

$$\pi^!_{x,y}(\mathcal{F}) \in \mathrm{Shv}(\overline{S}^0_x \times S^0_y)^{N_{X-(x,y)},\mathrm{diag}}.$$

Note, however, that the condition on $\mathcal{F}$ implies that $\pi^!_{x,y}(\mathcal{F})$ is equivariant with respect to the $N_{X-(x,y)}$-action via the second factor. Combined with diagonal equivariance, we obtain that $\pi^!_{x,y}(\mathcal{F})$ is $N_{X-(x,y)}$-equivariant with respect to the action on the first factor.

In particular, we obtain that $\pi^!_x(\mathcal{F})$ is $N_{X-(x,y)}$-equivariant with respect to the action on the first factor.

We claim that any object $\mathcal{F}' \in \mathrm{Shv}(\overline{S}^0_x)$ with this property is $\mathfrak{L}(N)_x$-equivariant. Indeed, it is sufficient to show that the !-restriction of $\mathcal{F}'$ to any S^λ_x is $\mathfrak{L}(N)_x$-equivariant. However, this follows from the fact that the action of $N_{X-(x,y)}$ on S^λ_x is transitive. □

3.8.4. We now prove Proposition 3.8.3[4]:

Proof. The data of $\{\kappa^{\check{\lambda}}\}$ that does not have zeroes at y defines a reduction of the G-bundle $\mathcal{P}_G$ to N on the formal neighborhood of y. Thus, we can consider the $\mathfrak{L}^+(N)_y$-torsor over $(\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y}$, denoted

$$(\overline{\mathrm{Bun}}_N)^{\mathrm{level}_y}$$

that classifies the data $(\mathcal{P}_G, \kappa, \epsilon)$, where ϵ is the datum of trivialization of the resulting N bundle on the formal neighborhood of y.

The usual regluing procedure defines an action on $(\overline{\mathrm{Bun}}_N)^{\mathrm{level}_y}$ of the group ind-scheme $\mathfrak{L}(N)_y$. By functoriality, for $\mathcal{F} = \mathrm{IC}_{\mathrm{glob}}$ or $\mathcal{F} = \Delta^0_{\mathrm{glob}}$, the pullback of $\mathcal{F}$ to $(\overline{\mathrm{Bun}}_N)^{\mathrm{level}_y}$ is $\mathfrak{L}(N)_y$-equivariant.

[4] In the proof below we will use sheaves on stacks and schemes of infinite-type. As this may cause a feeling of discomfort, we note that everything can be rephrased by choosing finite level structures and thus dealing only with stacks locally of finite type.

We have a commutative diagram

$$\begin{array}{ccc} \overline{S}^0_x \times \mathfrak{L}(N)_y & \xrightarrow{\widetilde{\pi}_{x,y}} & (\overline{\mathrm{Bun}}_N)^{\mathrm{level}_y} \\ \downarrow & & \downarrow \\ \overline{S}^0_x \times S^0_y & \xrightarrow{\pi_{x,y}} & (\overline{\mathrm{Bun}}_N)^{\mathrm{good}_y}, \end{array}$$

where the map $\widetilde{\pi}_{x,y}$ is $\mathfrak{L}(N)_y$-equivariant. Hence, the pullback of $\pi^!_{x,y}(\mathcal{F})$ along

$$\overline{S}^0_x \times \mathfrak{L}(N)_y \to \overline{S}^0_x \times S^0_y$$

is $\mathfrak{L}(N)_y$-equivariant. This implies that $\pi^!_{x,y}(\mathcal{F})$ itself was $\mathfrak{L}(N)_y$-equivariant.

□

3.9. **Proof of Proposition 3.6.5.**

3.9.1. Let us first assume that $-\mu$ is *sufficiently dominant*, by which we mean that $\check{\alpha}(-\mu) > 2(g-1)$ for every root $\check{\alpha}$ of G. In this case, the projection

$$\mathrm{Bun}^{-\mu}_{B^-} \to \mathrm{Bun}_G$$

is smooth.

Hence, in this case, the map

$$\mathsf{q} : \mathcal{Z}^\mu \to \overline{\mathrm{Bun}}_N$$

is also smooth, and the assertion is evident.

We will reduce the general case to the one above, using the factorization property of the Zastava spaces over the configuration spaces.

3.9.2. For a pair of elements $\mu, \lambda \in -\Lambda^{\mathrm{pos}}$, let

$$(X^\mu \times X^\lambda)_{\mathrm{disj}} \subset X^\mu \times X^\lambda$$

be the open subset corresponding to the locus when the two $-\Lambda^{\mathrm{pos}}$-valued divisors have disjoint support.

The *factorization property* (see [BFGM, Proposition 2.4]) says that there is a canonical isomorphism

$$\mathcal{Z}^{\mu+\lambda} \underset{X^{\mu+\lambda}}{\times} (X^\mu \times X^\lambda)_{\mathrm{disj}} \simeq (\mathcal{Z}^\mu \times \mathcal{Z}^\lambda) \underset{X^\mu \times X^\lambda}{\times} (X^\mu \times X^\lambda)_{\mathrm{disj}}. \tag{3.8}$$

3.9.3. In what follows we will need a particular property of the isomorphism (3.8) that follows from its definition.

Let $(\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{good}}$ be the open subset of the product

$$\overline{\mathrm{Bun}}_N \times X^\lambda,$$

where we forbid the maps $\kappa^{\check{\lambda}}$ to have a zero at the support of the point of X^λ.

As in Sect. 3.8.4, we consider the group-scheme $\mathfrak{L}^+(N)_{X^\lambda}$ (over X^λ), the group ind-scheme $\mathfrak{L}(N)_{X^\lambda}$, and the $\mathfrak{L}^+(N)_{X^\lambda}$-torsor

$$(\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{level}} \to (\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{good}}.$$

The action of $\mathfrak{L}^+(N)_{X^\lambda}$ on $(\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{level}}$ extends to that of $\mathfrak{L}(N)_{X^\lambda}$. We fix a group subscheme N'_{X^λ}

$$\mathfrak{L}^+(N)_{X^\lambda} \subset N'_{X^\lambda} \subset \mathfrak{L}(N)_{X^\lambda},$$

pro-smooth over X^λ, and consider the quotient stack $(\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{level}}/N'_{X^\lambda}$. It comes equipped with a smooth projection

$$(\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{good}} \to (\overline{\mathrm{Bun}}_N \times X^\lambda)^{\mathrm{level}}/N'_{X^\lambda}.$$

Then for N'_{X^λ} large enough the following diagram is commutative:

$$
\begin{array}{ccc}
(\mathcal{Z}^\mu \times \overset{\circ}{\mathcal{Z}}{}^\lambda) \underset{X^\mu\times X^\lambda}{\times} (X^\mu \times X^\lambda)_{\operatorname{disj}} & \xrightarrow{(3.8)} & \mathcal{Z}^{\mu+\lambda} \underset{X^{\mu+\lambda}}{\times} (X^\mu \times X^\lambda)_{\operatorname{disj}} \\
\Big\downarrow{\scriptstyle \operatorname{id}\times\mathfrak{s}} & & \Big\downarrow{\scriptstyle \mathfrak{q}} \\
(\mathcal{Z}^\mu \times X^\lambda) \underset{X^\mu\times X^\lambda}{\times} (X^\mu \times X^\lambda)_{\operatorname{disj}} & & (\overline{\operatorname{Bun}}_N \times X^\lambda)^{\operatorname{good}} \\
\Big\downarrow{\scriptstyle \mathfrak{q}} & & \Big\downarrow \\
(\overline{\operatorname{Bun}}_N \times X^\lambda)^{\operatorname{good}} & \longrightarrow & (\overline{\operatorname{Bun}}_N \times X^\lambda)^{\operatorname{level}}/N'_{X^\lambda}.
\end{array}
\tag{3.9}
$$

The idea is that a point $z \in \overset{\circ}{\mathcal{Z}}{}^\lambda$ modifies the (generalized) N-bundle at the points of the support of the divisor $\mathfrak{s}(z)$.

3.9.4. Now, given any $\mu \in -\Lambda^{\operatorname{pos}}$, we can find $\lambda \in -\Lambda^{\operatorname{pos}}$ so that $-(\mu+\lambda)$ is sufficiently dominant (as in Sect. 3.9.1), so that

$$\mathcal{Z}^{\mu+\lambda} \to \overline{\operatorname{Bun}}_N$$

is smooth.

The assertion of Proposition 3.6.5 follows by chasing over the diagram (3.9). For example, point (a) is obtained as follows:

It suffices to show that the pullback of the IC sheaf along the composite left vertical map is isomorphic to the IC sheaf (up to a cohomological shift).

Since the bottom horizontal arrow in (3.9) is smooth, it suffices to show that the pullback of $\operatorname{IC}_{(\overline{\operatorname{Bun}}_N\times X^\lambda)^{\operatorname{level}}/N'_{X^\lambda}}$ along the counter-clockwise circuit in (3.9) is isomorphic to the IC sheaf (up to a cohomological shift).

Since the diagram (3.9) is commutative, this is equivalent to showing that the pullback of $\operatorname{IC}_{(\overline{\operatorname{Bun}}_N\times X^\lambda)^{\operatorname{level}}/N'_{X^\lambda}}$ along the clockwise circuit in (3.9) is isomorphic to the IC sheaf (up to a cohomological shift).

Since the top horizontal arrow and the lower right vertical arrows in (3.9) are smooth, it suffices to show that the pullback of the IC sheaf along

$$\mathcal{Z}^{\mu+\lambda} \underset{X^{\mu+\lambda}}{\times} (X^\mu \times X^\lambda)_{\operatorname{disj}} \to (\overline{\operatorname{Bun}}_N \times X^\lambda)^{\operatorname{good}}$$

is isomorphic to the IC sheaf (up to a cohomological shift).

Since X^λ is smooth, it suffices to show that the pullback of the IC sheaf along the composite map

$$\mathcal{Z}^{\mu+\lambda} \underset{X^{\mu+\lambda}}{\times} (X^\mu \times X^\lambda)_{\operatorname{disj}} \to (\overline{\operatorname{Bun}}_N \times X^\lambda)^{\operatorname{good}} \to \overline{\operatorname{Bun}}_N$$

is isomorphic to the IC sheaf (up to a cohomological shift). But this follows from the fact that the map

$$\mathcal{Z}^{\mu+\lambda} \xrightarrow{\mathfrak{q}} \overline{\operatorname{Bun}}_N$$

is smooth.

4. Digression: (dual) baby Verma objects

In this section we summarize the construction of (dual) baby Verma objects, following [ABBGM] and [FG]. These are objects of the category $\operatorname{Shv}(\operatorname{Gr}_G)^I$ that have a particular property (the Hecke property) with respect to convolutions with objects of the form $\operatorname{Sat}(V)$, $V \in \operatorname{Rep}(\check{G})$.

4.1. **The Iwahori category on the affine Grassmannian.**

4.1.1. Consider the category

$$\mathrm{Shv}(\mathrm{Gr}_G)^I,$$

where $I \subset \mathfrak{L}^+(G)$ is the Iwahori subgroup[5].

This category carries an action by right convolutions by $\mathrm{Sph}(G) = \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}$ and a commuting action of

$$\mathcal{H}(G) := \mathrm{Shv}(\mathrm{Fl}_G)^I$$

by left convolutions.

4.1.2. For an element $\widetilde{w}$ of the extended affine Weyl group W^{aff}, we let

$$j_{\widetilde{w},!} \text{ and } j_{\widetilde{w},*}$$

denote the corresponding standard and costandard objects in $\mathcal{H}(G)^{\heartsuit}$.

The key fact is that there are canonical isomorphisms

$$j_{\widetilde{w},!} \star j_{\widetilde{w}^{-1},*} \simeq \delta_{1,\mathrm{Fl}_G} \simeq j_{\widetilde{w}^{-1},*} \star j_{\widetilde{w},!}. \tag{4.1}$$

4.1.3. Another crucial observation that there are symmetric monoidal functors

$$\Lambda \rightrightarrows \mathcal{H}(G)^{\heartsuit},$$

uniquely characterized by the property that they send

$$(\lambda \in \Lambda^+) \mapsto j_{\lambda,*} \text{ and } (\lambda \in \Lambda^+) \mapsto j_{\lambda,!},$$

respectively.

Using (4.1), we obtain that the first functor sends $\lambda \in -\Lambda^+$ to $j_{\lambda,!}$ and the second functor sends $\lambda \in -\Lambda^+$ to $j_{\lambda,*}$.

The above two symmetric monoidal functors are intertwined by the automorphism induced by the action of $w_0 \in W$ and the automorphism of $\mathcal{H}(G)$ given by

$$\mathcal{F} \mapsto j_{w_0,!} \star \mathcal{F} \star j_{w_0,*}.$$

Indeed, for $\lambda \in \Lambda^+$ we have:

$$j_{w_0,!} \star j_{\lambda,*} \star j_{w_0,*} \simeq j_{w_0(\lambda),*}.$$

In particular, the auto-equivalence of $\mathrm{Shv}(\mathrm{Gr}_G)^I$, given by

$$\mathcal{F} \mapsto j_{w_0,!} \star \mathcal{F}$$

intertwines the $\mathrm{Rep}(\check{T})$-action on $\mathrm{Shv}(\mathrm{Gr}_G)^I$ given

$$\mathsf{e}^{\lambda} \star \mathcal{F} := j_{\lambda,*} \star \mathcal{F}, \quad \lambda \in \Lambda^+ \tag{4.2}$$

and the action given by

$$\mathsf{e}^{\lambda} \star \mathcal{F} := j_{w_0(\lambda),*} \star \mathcal{F}, \quad \lambda \in \Lambda^+. \tag{4.3}$$

4.2. **Recollections on the [ABG] theory.** For the remainder of this paper, we will change our conventions, and for a group H, we let $\mathrm{Rep}(H)$ denote the symmetric monoidal DG category of its representation (rather than the corresponding abelian category).

4.2.1. Consider the derived stack

$$\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-.$$

Note that its underlying classical stack is $\mathrm{pt}/\check{B}^-$.

[5]We will use a slightly renormalized version of $\mathrm{Shv}(\mathrm{Gr}_G)^I$, where we declare compact objects to be the ones that map to compact objects under the forgetful functor $\mathrm{Shv}(\mathrm{Gr}_G)^I \to \mathrm{Shv}(\mathrm{Gr}_G)$. This is done in order to avoid the singular support condition on coherent sheaves on the spectral side. We are grateful to J. Campbell for catching this imprecision.

4.2.2. The following theorem is established in [ABG]:

Theorem 4.2.3. *There exists a canonically defined equivalence of categories*

$$\mathrm{Sat}^I : \mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-) \simeq \mathrm{Shv}(\mathrm{Gr}_G)^I$$

with the following properties:

(i) *The action of* $\mathrm{Rep}(\check{G})$ *on* $\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$ *arising from the projection*

$$\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^- \to \mathrm{pt}/\check{B}^- \to \mathrm{pt}/\check{G}$$

corresponds to the action of $\mathrm{Rep}(\check{G})$ *on* $\mathrm{Shv}(\mathrm{Gr}_G)^I$ *via* $\mathrm{Sat} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}(G)$ *and right convolutions.*

(ii) *The action of* $\mathrm{Rep}(\check{T})$ *on* $\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$ *arising from the projection*

$$\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^- \to \mathrm{pt}/\check{B}^- \to \mathrm{pt}/\check{T}$$

corresponds to the action on $\mathrm{Rep}(\check{G})$ *on* $\mathrm{Shv}(\mathrm{Gr}_G)^I$ *given by* (4.2).

(iii) *The object*

$$\mathcal{O}_{\mathrm{pt}/\check{B}^-} \in \mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$$

corresponds under Sat^I *to* $\delta_{1,\mathrm{Gr}_G} \in \mathrm{Shv}(\mathrm{Gr}_G)^I$.

(iv) *For* $\lambda \in \Lambda^+$ *the morphism*

$$(v^\lambda)^* : V^\lambda \to \mathsf{e}^\lambda$$

in $\mathrm{Rep}(\check{B}^-)$ *corresponds under* Sat^I *to the natural map of perverse sheaves*

$$\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G} \to j_{\lambda,*} \star \delta_{1,\mathrm{Gr}_G}.$$

4.2.4. Let us use Theorem 4.2.3 to supply calculation from the proof of Proposition 2.5.2 that

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, H_{\{t^{\lambda+\mu}\}}(\mathrm{Gr}_G, \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G})$$

identifies canonically with $\mathrm{Sym}^n(\mathfrak{g}/\mathfrak{b}^-[-2])(-\mu)$.

Proof. By Verdier duality, the !-fiber of $\mathrm{Sat}(V^\lambda)[\langle\lambda+\mu, 2\check{\rho}\rangle]$ at $t^{\lambda+\mu}$ identifies with

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Gr}_G)^I}(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}, j_{\lambda+\mu,*} \star \delta_{1,\mathrm{Gr}_G}), \tag{4.4}$$

tensored with e over $\mathrm{Sym}(\mathfrak{t}[-2])$, the latter being the I-equivariant cohomology of the point.

Using Theorem 4.2.3, we rewrite the expression in (4.4) as

$$\mathcal{H}om_{\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-)}(\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^\lambda) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}, \mathrm{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^{\lambda+\mu}) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}). \tag{4.5}$$

Moreover, by unwinding the definitions, we obtain that the transition map for $\lambda_2 = \lambda_1 + \lambda$

$$H_{\{t^{\lambda_1+\mu}\}}(\mathrm{Gr}_G, \mathrm{Sat}(V^{\lambda_1})[\langle\lambda_1+\mu, 2\check{\rho}\rangle] \to H_{\{t^{\lambda_2+\mu}\}}(\mathrm{Gr}_G, \mathrm{Sat}(V^{\lambda_2})[\langle\lambda_2+\mu, 2\check{\rho}\rangle]$$

is induced by the map

$$\mathcal{H}om_{\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-)}(\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^{\lambda_1}) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}, \mathrm{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^{\lambda_1+\mu}) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}) \to$$

$$\to \mathcal{H}om_{\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-)}(\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^{\lambda_1} \otimes V^\lambda) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}, \mathrm{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^{\lambda_1+\mu}) \otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}(V^\lambda) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}) \to$$

$$\to \mathcal{H}om_{\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-)}(\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^{\lambda_2}) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}, \mathrm{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^{\lambda_2}) \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}),$$

where second arrow is induced by the maps

$$V^{\lambda_2} \to V^{\lambda_1} \otimes V^\lambda$$

and

$$\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^\lambda) \to \mathrm{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^\lambda).$$

We rewrite (4.5) as

$$\mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda+\mu}\otimes \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2]))), \tag{4.6}$$

and the transition maps are given by

$$\begin{aligned}\mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda_1}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda_1+\mu}\otimes \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2]))) \to \\ \to \mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda_1}\otimes V^{\lambda}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda_1+\mu}\otimes \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2]))\otimes V^{\lambda}) \simeq \\ \simeq \mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda_1}\otimes V^{\lambda}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda_1+\mu}\otimes \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2])\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}(V)^{\lambda})) \to \\ \to \mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda_2}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda_2+\mu}\otimes \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2]))).\end{aligned}$$

Finally, we claim that the colimit of the expressions (4.6) over $\lambda\in\Lambda^+$ identifies with

$$\mathrm{Sym}(\mathfrak{g}/\mathfrak{n}^-[-2])(-\mu).$$

Indeed, for a given integer $m\geq 0$, let λ_0 be such that for every weight ν that appears in $\mathrm{Sym}^n(\mathfrak{g}/\mathfrak{n}^-)$, we have $\lambda_0+\nu\in\Lambda^+$. Then for all $\lambda\in\lambda_0+\Lambda^+$, we have

$$\mathcal{H}om_{\mathrm{Rep}(\check{G})}(V^{\lambda}, \mathrm{coInd}^{\check{G}}_{\check{B}^-}(\mathsf{e}^{\lambda+\mu}\otimes \mathrm{Sym}^n(\mathfrak{g}/\mathfrak{n}^-[-2])))\simeq \mathrm{Sym}^n(\mathfrak{g}/\mathfrak{n}^-[-2])(-\mu).$$

□

4.3. Hecke patterns.

4.3.1. Let $\mathcal{C}$ be a DG category, acted on by $\mathrm{Rep}(M)$, where M as an algebraic group. We denote by $\mathrm{Hecke}_M(\mathcal{C})$ the corresponding Hecke category

$$\mathrm{Hecke}_M(\mathcal{C}) := \mathcal{C}\underset{\mathrm{Rep}(M)}{\otimes}\mathrm{Vect}.$$

We have a tautological functor

$$\mathbf{ind}_{\mathrm{Hecke}_M}:\mathcal{C}\to \mathrm{Hecke}_M(\mathcal{C}),$$

which admits a continuous right adjoint, denoted $\mathbf{oblv}_{\mathrm{Hecke}_M}$. The comonad $\mathbf{oblv}_{\mathrm{Hecke}_M}\circ \mathbf{ind}_{\mathrm{Hecke}_M}$ on $\mathcal{C}$ is given by the action of the left regular representation object $\mathcal{O}(M)\in\mathrm{Rep}(M)$.

4.3.2. Since $\mathrm{Rep}(M)$ is *rigid* (see [GR, Chapter 1]), we an canonically identify

$$\mathrm{Hecke}_M(\mathcal{C}) := \mathcal{C}\underset{\mathrm{Rep}(M)}{\otimes}\mathrm{Vect}\simeq \mathrm{Funct}_{\mathrm{Rep}(M)}(\mathrm{Vect},\mathcal{C}),$$

see Proposition 9.4.8 in *loc.cit.*

Under this identification, the functor $\mathbf{oblv}_{\mathrm{Hecke}_M}$ corresponds to the forgetful functor

$$\mathrm{Funct}_{\mathrm{Rep}(M)}(\mathrm{Vect},\mathcal{C})\to \mathrm{Funct}_{\mathrm{Rep}(M)}(\mathrm{Rep}(M),\mathcal{C})\simeq\mathcal{C}.$$

This point of view allows to think of objects of $\mathrm{Hecke}_M(\mathcal{C})$ as "Hecke eigen-objects": these are objects $c\in\mathcal{C}$ equipped with a system of isomorphisms

$$c\star V\simeq \underline{V}\otimes c,\quad V\in\mathrm{Rep}(M)$$

(here $\underline{V}$ denotes the vector space underlying V) that are associative in the same sense as in Sect. 2.7.1.

The functor $\mathbf{ind}_{\mathrm{Hecke}_M}$ sends $c\in\mathcal{C}$ to $c\star\mathcal{O}(M)$, with the Hecke structure induced by that on $\mathcal{O}(M)$:

$$\mathcal{O}(M)\otimes V\simeq \underline{V}\otimes\mathcal{O}(M).$$

4.3.3. We apply this for M being $\check{G}$ or $\check{T}$. In addition, we will the following two variants.

For $\mathcal{C}$ equipped with an action of $\operatorname{Rep}(\check{G})$ we will denote by $\overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ the category

$$\mathcal{C} \underset{\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}(\check{T}).$$

The category $\overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ is acted on by $\operatorname{Rep}(\check{T})$ and is related to $\operatorname{Hecke}_{\check{G}}(\mathcal{C})$ by the formula

$$\operatorname{Hecke}_{\check{G}}(\mathcal{C}) = \operatorname{Hecke}_{\check{T}}(\mathcal{C})(\overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})).$$

We have the corresponding pair of adjoint functors

$$\mathbf{ind}_{\operatorname{Hecke}_T} : \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C}) \rightleftarrows \operatorname{Hecke}_{\check{G}}(\mathcal{C}) : \mathbf{oblv}_{\operatorname{Hecke}_T}.$$

4.3.4. We can think of $\overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ as the category of Λ-graded Hecke eigen-objects. I.e., an object of $\overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ is a collection of objects $\{c_\lambda \in \mathcal{C}, \lambda \in \Lambda\}$, equipped with a system of isomorphisms

$$c_\lambda \star V \simeq \bigoplus_\mu V(\mu) \otimes c_{\lambda+\mu}, \quad V \in \operatorname{Rep}(\check{G}) \tag{4.7}$$

that are associative in the same sense as in Sect. 2.7.1.

In terms of this description, the functor $\mathbf{ind}_{\operatorname{Hecke}_T} : \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C}) \to \operatorname{Hecke}_{\check{G}}(\mathcal{C})$ sends

$$\{c_\lambda\} \mapsto \underset{\lambda}{\oplus}\, c_\lambda,$$

and the functor

$$\mathbf{oblv}_{\operatorname{Hecke}_T} : \operatorname{Hecke}_{\check{G}}(\mathcal{C}) \to \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$$

sends c to

$$\{c_\lambda\},\ c_\lambda = c \text{ for all } \lambda.$$

4.3.5. Suppose now that $\mathcal{C}$ carries an action of $\operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G})$. We will consider the category

$$\operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}) := \mathcal{C} \underset{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}(\check{T}).$$

We can think of its objects as $c \in \mathcal{C}$, equipped with a collection of isomorphisms

$$c \star V \simeq \operatorname{Res}^{\check{G}}_{\check{T}}(V) \star c, \quad V \in \operatorname{Rep}(\check{G}) \tag{4.8}$$

(here convolution on the right denotes the action of $\operatorname{Rep}(\check{G})$ and convolution on the right denotes the action of $\operatorname{Rep}(\check{T})$), that are associative in the same sense as in Sect. 2.7.1.

4.3.6. We have the tautological functor

$$\Phi : \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C}) = \mathcal{C} \underset{\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}(\check{T}) \to \mathcal{C} \underset{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}(\check{T}) = \operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}),$$

which admits a continuous right adjoint

$$\Psi : \operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C}). \tag{4.9}$$

Explicitly, for an object c of $\operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C})$ as in Sect. 4.3.5, the object $\Psi(c) \in \overset{\bullet}{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ consists of

$$c_\lambda = \mathsf{e}^\lambda \star c,$$

with the Hecke structure supplied by (4.8).

4.3.7. In addition, the functor

$$\mathbf{ind}_{\mathrm{Hecke}_{\check{T}}} : \mathcal{C} \to \mathrm{Hecke}_{\check{T}}(\mathcal{C})$$

induces a functor

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathbf{ind}_{\mathrm{Hecke}_{\check{T}}}) : \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Hecke}_{\check{T}}(\mathcal{C})) \simeq \mathrm{Hecke}_{\check{G}}(\mathrm{Hecke}_{\check{T}}(\mathcal{C})).$$

Explicitly, for an object c of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C})$, the corresponding object of $\mathrm{Hecke}_{\check{T}}(\mathcal{C})$ is

$$\mathbf{ind}_{\mathrm{Hecke}_{\check{T}}}(c) := \underset{\lambda \in \Lambda}{\oplus} \mathsf{e}^{\lambda} \star c,$$

with the Hecke structure with respect to $\check{G}$ given by (4.8).

4.3.8. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) & \xrightarrow{\Psi} & \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C}) \\ {\scriptstyle \mathrm{Hecke}_{\check{G},\check{T}}(\mathbf{ind}_{\mathrm{Hecke}_{\check{T}}})} \big\downarrow & & \big\downarrow {\scriptstyle \mathbf{ind}_{\mathrm{Hecke}_{\check{T}}}} \\ \mathrm{Hecke}_{\check{G}}(\mathrm{Hecke}_{\check{T}}(\mathcal{C})) & \xrightarrow{\mathrm{Hecke}_{\check{G}}(\mathbf{oblv}_{\mathrm{Hecke}_{\check{T}}})} & \mathrm{Hecke}_{\check{G}}(\mathcal{C}). \end{array} \tag{4.10}$$

4.4. **The (dual) baby Verma object: coherent side.**

4.4.1. Consider $\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$ as a category endowed with a pair of commuting actions of $\mathrm{Rep}(\check{G})$ and $\mathrm{Rep}(\check{T})$, obtained from the maps

$$\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^- \to \mathrm{pt}/\check{B}^-$$

and

$$\mathrm{pt}/\check{B}^- \to \mathrm{pt}/\check{G} \text{ and } \mathrm{pt}/\check{B}^- \to \mathrm{pt}/\check{T},$$

respectively.

Consider the corresponding categories

$$\mathrm{Hecke}_{\check{G}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)),\ \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)),$$

$$\mathrm{Hecke}_{\check{T}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)) \text{ and } \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)).$$

Direct image along the closed embedding $\mathrm{pt}/\check{B}^- \to \check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-$ defines functors

$$\mathrm{QCoh}(\check{G}/\check{B}^-) \simeq \mathrm{Hecke}_{\check{G}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-)) \to \mathrm{Hecke}_{\check{G}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)); \tag{4.11}$$

$$\mathrm{QCoh}(\check{T}\backslash\check{G}/\check{B}^-) \simeq \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-)) \to \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)); \tag{4.12}$$

$$\mathrm{QCoh}(\check{G}/\check{N}^-) \simeq \mathrm{Hecke}_{\check{T}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-)) \to \mathrm{Hecke}_{\check{T}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)) \tag{4.13}$$

and

$$\mathrm{QCoh}((\check{G}/\check{N}^-)/\mathrm{Ad}_{\check{T}}) \simeq \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-)) \to \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)). \tag{4.14}$$

The diagram (4.10) is induced by the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}((\check{G}/\check{N}^-)/\mathrm{Ad}_{\check{T}}) & \longrightarrow & \mathrm{QCoh}(\check{T}\backslash\check{G}/\check{B}^-) \\ \big\downarrow & & \big\downarrow \\ \mathrm{QCoh}(\check{G}/\check{N}^-) & \longrightarrow & \mathrm{QCoh}(\check{G}/\check{B}^-), \end{array}$$

where we take inverse images along the vertical arrows and direct images along the horizontal arrows.

4.4.2. The inclusion $\check{N}^- \hookrightarrow \check{G}$ induces a map

$$\text{pt}/\check{T} \to (\check{G}/\check{N}^-)/\operatorname{Ad}_{\check{T}}. \tag{4.15}$$

Taking the direct image of $\mathcal{O}_{\text{pt}/\check{T}}$ along this map and applying the functor (4.14) we obtain an object that we denote

$$\mathcal{M}_{\check{G},\check{T}} \in \text{Hecke}_{\check{G},\check{T}}(\text{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)).$$

Note that its image under the functor Ψ (4.9) is the object of $\overset{\bullet}{\text{Hecke}}_{\check{G}}(\text{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-))$, to be denoted $\overset{\bullet}{\mathcal{M}}_{\check{G}}$, obtained by means of (4.12) from the direct image of $\mathcal{O}_{\text{pt}/\check{T}}$ under the map

$$\text{pt}/\check{T} \to \check{T}\backslash\check{G}/\check{B}^-.$$

Finally, the object

$$\mathcal{M}_{\check{G}} := \mathbf{ind}_{\text{Hecke}_T}(\overset{\bullet}{\mathcal{M}}) \in \text{Hecke}_{\check{G}}(\text{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-))$$

is obtained by means of (4.11) from the sky-scraper

$$\mathsf{e}_{1,\check{G}/\check{B}^-} \in \text{QCoh}(\check{G}/\check{B}^-).$$

The objects $\mathcal{M}_{\check{G},\check{T}}$, $\mathcal{M}_{\check{G}}$ and $\overset{\bullet}{\mathcal{M}}_{\check{G}}$ are the various incarnations of what we call the "(dual) baby Verma object"; the origin of the name will be explained shortly.

4.4.3. For future use, let us describe explicitly the object of $\text{Hecke}_{\check{G},\check{T}}(\text{QCoh}(\text{pt}/\check{B}^-))$ from which we obtained $\mathcal{M}_{\check{G},\check{T}}$ as a direct image in terms of Sect. 4.3.5. Unwinding the definition, we obtain that as an object of $\text{Rep}(\check{B}^-)$ it identifies with $\mathcal{O}(\check{B}^-/\check{T})$. The isomorphisms (4.8) are given as follows:

For any $W \in \text{Rep}(\check{B}^-)$, we have a canonical isomorphism

$$\begin{aligned}
&W \otimes \mathcal{O}(\check{B}^-/\check{T}) = \\
&= (W \otimes \mathcal{O}(\check{B}^-))^{\check{T}_{\text{right}}} \simeq \\
&\simeq (\mathcal{O}(\check{B}^-) \otimes W)^{\check{T}_{\text{right-diag}}} \simeq \\
&\simeq \underset{\mu}{\oplus} \mathcal{O}(\check{B}^-)(-\mu) \otimes W(\mu) \simeq \\
&\simeq \underset{\mu}{\oplus} (\mathcal{O}(\check{B}^-/\check{T}) \otimes \mathsf{e}^\mu) \otimes W(\mu) = \\
&= \mathcal{O}(\check{B}^-/\check{T}) \otimes \text{Res}^{\check{B}^-}_{\check{T}}(W),
\end{aligned}$$

where:

–In the second line we view $W \otimes \mathcal{O}(\check{B}^-)$ as an object of $\text{Rep}(\check{B}^-)$ diagonally via the given action on W and an action on $\mathcal{O}(\check{B}^-)$ by left translations, and as such acted on by $\check{T}$ via right translations on the $\mathcal{O}(\check{B}^-)$-factor;

–In the third line we view $\mathcal{O}(\check{B}^-) \otimes W$ as an object of $\text{Rep}(\check{B}^-)$ via the action of $\check{B}^-$ by left translations on the $\mathcal{O}(\check{B}^-)$-factor, and as such acted on diagonally by $\check{T}$ via $\check{T} \to \check{B}^-$ and right translations on the $\mathcal{O}(\check{B}^-)$-factor and the given action on W;

–In the fourth line $\mathcal{O}(\check{B}^-)(-\mu)$ means the $(-\mu)$ weight space with respect to the action of $\check{T}$ via $\check{T} \to \check{B}^-$ and right translations;

–The isomorphism between the fourth and the fifth lines is induced by the identification

$$\mathcal{O}(\check{B}^-)(-\mu) \simeq \mathcal{O}(\check{B}^-/\check{T}) \otimes \mathsf{e}^\mu,$$

given by multiplication by the character $\check{B}^- \to \check{T} \overset{\mu}{\to} \mathbb{G}_m$.

4.4.4. We note that the above object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-))$ can be written as

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^{\lambda}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda})^*),$$

where the transition maps for $\lambda_2=\lambda_1+\lambda$ are given as follows

$$\mathsf{e}^{\lambda_1}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda_1})^*)\simeq \mathsf{e}^{\lambda_1}\otimes \mathsf{e}^{\lambda}\otimes \mathsf{e}^{-\lambda}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda_1})^*)\to$$
$$\to \mathsf{e}^{\lambda_2}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda})^*)\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda_1})^*)\to \mathsf{e}^{\lambda_2}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda_2})^*),$$

where the second arrow is given by the identification $\mathsf{e}^{-\lambda}\simeq ((V^{\lambda})^*)^{\check{N}^-}$, and the last arrow by (2.2).

The Hecke structure is given as follows. For a given finite-dimensional $V\in \mathrm{Rep}(\check{G})^{\heartsuit}$ and $\lambda\gg 0$, we start with the isomorphism (2.3), from which we produce a canonical isomorphism

$$(V^{\lambda})^*\otimes V\simeq \underset{\mu}{\oplus}\, (V^{\lambda+\mu})^*\otimes V(-\mu). \tag{4.16}$$

Hence, for a fixed V as above and $\lambda\gg 0$, we have

$$\mathsf{e}^{\lambda}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda})^*)\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}(V)\simeq \underset{\mu}{\oplus}\, \left(\mathsf{e}^{-\mu}\otimes V(-\mu)\right)\otimes \left(\mathsf{e}^{\lambda+\mu}\otimes \mathrm{Res}^{\check{G}}_{\check{B}^-}((V^{\lambda+\mu})^*)\right),$$

as required.

Remark 4.4.5. In Sect. 6.3 we will give a more conceptual explanation of the above presentation the above object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{QCoh}(\mathrm{pt}/\check{B}^-))$.

4.5. **The geometric (dual) baby Verma object.**

4.5.1. We now consider the category $\mathrm{Shv}(\mathrm{Gr}_G)^I$, equipped with the action of $\mathrm{Rep}(\check{G})$ via Sat and a commuting action of $\mathrm{Rep}(\check{T})$ given by (4.2).

Applying the equivalence Sat^I of Theorem 4.2.3, from the objects $\mathcal{M}_{\check{G},\check{T}}$, $\mathcal{M}_{\check{G}}$ and $\overset{\bullet}{\mathcal{M}}_{\check{G}}$ we obtain the objects

$$\mathcal{F}_{\check{G},\check{T}}\in \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^I);$$
$$\overset{\bullet}{\mathcal{F}}_{\check{G}}\in \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathrm{Shv}(\mathrm{Gr}_G)^I);$$
$$\mathcal{F}_{\check{G}}\in \mathrm{Hecke}_{\check{G}}(\mathrm{Shv}(\mathrm{Gr}_G)^I).$$

Remark 4.5.2. The object $\overset{\bullet}{\mathcal{F}}_{\check{G}}$ was introduced in [ABBGM] so that under the equivalence between $\overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathrm{Shv}(\mathrm{Gr}_G)^{I^0})$ and the (regular block of) category of modules over the small quantum group, it corresponds to the *dual baby Verma module*.

The object $\mathcal{F}_{\check{G}}$ was introduced in [FG], so that the functor of global sections of critically twisted D-modules sends it to the Verma module over affine algebra at the critical level (reduced modulo the ideal of the center that corresponds to regular opers).

4.5.3. The object $\mathcal{F}_{\check{G},\check{T}}$ and its derivatives (i.e., $\overset{\bullet}{\mathcal{F}}_{\check{G}}$, $\mathcal{F}_{\check{G}}$) can be described explicitly using Sect. 4.4.4. Namely, the underlying object of $\mathrm{Shv}(\mathrm{Gr}_G)^I$ is

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, j_{\lambda,*}\star \mathrm{Sat}((V^{\lambda})^*),$$

where the transition maps for $\lambda_2=\lambda_1+\lambda$ are given by

$$j_{\lambda_1,*}\star \mathrm{Sat}((V^{\lambda_1})^*)\to j_{\lambda_1,*}\star j_{\lambda,*}\star j_{-\lambda,!}\star \mathrm{Sat}((V^{\lambda_1})^*)\to$$
$$\to j_{\lambda_2,*}\star \mathrm{Sat}((V^{\lambda})^*)\star \mathrm{Sat}((V^{\lambda_1})^*)\to j_{\lambda_2,*}\star \mathrm{Sat}((V^{\lambda_2})^*),$$

where the second arrow comes the (monoidal) dual of the canonical map $\mathrm{IC}_{\overline{\mathrm{Gr}}^{\lambda}_G}\to j_{\lambda,*}\star \delta_{1,\mathrm{Gr}_G}$, and the last arrow is given by (2.2).

The Hecke structure is given by identifying for a fixed finite-dimensional $V \in \mathrm{Rep}(G)^\heartsuit$ and $\lambda \gg 0$

$$j_{\lambda,*} \star \mathrm{Sat}((V^\lambda)^*) \star \mathrm{Sat}(V) \simeq j_{\lambda,*} \star \mathrm{Sat}((V^\lambda)^* \otimes V) \overset{(4.16)}{\simeq}$$
$$\simeq \underset{\mu}{\oplus} V(-\mu) \otimes j_{\lambda,*} \star \mathrm{Sat}((V^{\lambda+\mu})^*) \simeq \underset{\mu}{\oplus} (j_{-\mu,!} \otimes V(-\mu)) \star \Big(j_{\lambda+\mu,*} \star \mathrm{Sat}((V^{\lambda+\mu})^*)\Big).$$

Remark 4.5.4. In Sect. 6.3 we will present a point of view from which the presentation of (the object of $\mathrm{Shv}(\mathrm{Gr}_G)^I$ underlying) $\mathcal{F}_{\check{G},\check{T}}$ as a colimit is automatic, along with its Hecke structure.

4.6. **The twist by w_0.**

4.6.1. Note that the Weyl group W acts on $\mathrm{Rep}(T)$, preserving the forgetful functor $\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(T)$. Hence, its also acts on the category $\overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C})$.

Explicitly, given an object $\{c_\lambda\} \in \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C})$ and choosing a representative $g_w \in W$ of a given element $w \in W$, define a new object $\{c^w_\lambda\} \in \overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C})$ as follows:

We set $c^w_\lambda := c_{w(\lambda)}$. The Hecke data for $\{c^w_\lambda\}$ is given by the composing the Hecke data (4.7) for $\{c_\lambda\}$ and the maps

$$g_w : V(\mu) \to V(\mathrm{w}(\mu)), \quad V \in \mathrm{Rep}(\check{G}).$$

If we modify the choice of g_w by an element $t \in \check{T}$, the corresponding two objects $\{c^w_\lambda\}$ are isomorphed by means of acting by $\lambda(t)$ on the λ-component.

4.6.2. Given an element $w \in W$ we can consider the category $\mathrm{Hecke}_{\check{G},\check{T},w}(\mathcal{C})$, which is defined in the same way as $\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C})$, but where we modify the $\mathrm{Rep}(\check{T})$-action by applying the automorphism w (the restriction functor $\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{T})$ stays intact).

Choosing a representative $g_w \in G$ of w, there exists a canonical equivalence

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \to \mathrm{Hecke}_{\check{G},\check{T},w}(\mathcal{C}).$$

Explicitly, given an object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C})$, the corresponding object of $\mathrm{Hecke}_{\check{G},\check{T},w}(\mathcal{C})$ has the same underlying object $c \in \mathcal{C}$, and the new data of (4.8) is obtained by acting on the old data of (4.8) by using g_w to isomorph

$$\mathrm{Res}^{\check{G}}_{\check{T}} \text{ and } \mathrm{Res}^{\check{G}}_{\check{T},w},$$

where $\mathrm{Res}^{\check{G}}_{\check{T},w}$ is the composite

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Res}^{\check{G}}_{\check{T}}}{\longrightarrow} \mathrm{Rep}(\check{T}) \overset{w}{\longrightarrow} \mathrm{Rep}(\check{T}).$$

4.6.3. Note that the geometric Satake isomorphism provides a canonical representative g_{w_0} of the element $w_0 \in W$. Indeed, finding a representative for w_0 is equivalent to trivializing the lower weight line in every V^λ in a way compatible with the maps (2.1) and (2.2).

However, by Sect. 2.2.2, we have

$$V^\lambda(w_0(\lambda)) \simeq H_c(S^{w_0(\lambda)}, \mathbf{i}^*_{w_0(\lambda)}(\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}))[\langle w_0(\lambda), 2\rho\rangle].$$

Now, the intersection $S^{w_0(\lambda)} \cap \overline{\mathrm{Gr}}^\lambda_G$ consists of a single point $t^{w_0(\lambda)}$, hence the above cohomology identifies canonically with e.

4.6.4. Note also that a choice of a representative of w_0 defines a system of identifications

$$V^{-w_0(\lambda)} \simeq (V^\lambda)^*, \tag{4.17}$$

compatible with the maps (2.1) and (2.2). The isomorphism is uniquely fixed by the condition that the trivialization of the highest weight line in $V^{-w_0(\lambda)}$ corresponds to the trivialization of the lowest weight line in V^λ.

The following is easy to verify:

Lemma 4.6.5. *For any choice of a representative g_{w_0} of w_0, a fixed finite-dimensional $V \in \operatorname{Rep}(\check{G})$ and $\lambda \gg 0$, the following diagram commutes:*

$$\begin{array}{ccc} (V^\lambda)^* \otimes V & \xrightarrow{(4.16)} & \underset{\mu}{\oplus} (V^{\lambda+\mu})^* \otimes V(-\mu) \\ \downarrow & & \downarrow \\ V^{-w_0(\lambda)} \otimes V & \xrightarrow{\sim} & \underset{\mu}{\oplus} (V^{-w_0(\lambda+\mu)})^* \otimes V(-w_0(\mu)), \end{array}$$

where the right vertical arrow is given by the maps

$$V(-\mu) \xrightarrow{g_{w_0}} V(-w_0(\mu)).$$

4.6.6. Applying the procedure of Sect. 4.6.2 to the object $\mathcal{F}_{\check{G},\check{T}}$ for the above choice of representative of w_0, we obtain an object, denoted

$$\mathcal{F}_{\check{G},\check{T},w_0} \in \operatorname{Hecke}_{\check{G},\check{T},w_0}(\operatorname{Shv}(\operatorname{Gr}_G)^I).$$

4.6.7. From Lemma 4.6.5, we obtain that $\mathcal{F}_{\check{G},\check{T},w_0}$ can be explicitly described as follows. The underlying object of $\operatorname{Shv}(\operatorname{Gr}_G)^I$ is given by

$$\underset{\lambda\in\Lambda^+}{\operatorname{colim}}\, j_{-w_0(\lambda),*} \star \operatorname{Sat}(V^\lambda). \tag{4.18}$$

The transition maps for $\lambda_2 = \lambda_1 + \lambda$ are given by

$$\begin{aligned} j_{-w_0(\lambda_1),*} \star \operatorname{Sat}(V^{\lambda_1}) \simeq j_{-w_0(\lambda_1),*} \star j_{-w_0(\lambda),*} \star j_{w_0(\lambda),!} \star \operatorname{Sat}(V^{\lambda_1}) \to \\ j_{-w_0(\lambda_2),*} \star \operatorname{Sat}(V^\lambda) \star \operatorname{Sat}(V^{\lambda_1}) \to j_{-w_0(\lambda_2),*} \star \operatorname{Sat}(V^{\lambda_w}), \end{aligned} \tag{4.19}$$

where the second arrow comes from the monoidal dual of the map $\operatorname{IC}_{\overline{\operatorname{Gr}}^\lambda_{-w_0(\lambda)}} \to j_{-w_0(\lambda),*}$ via the identification

$$\operatorname{Sat}((V^\lambda)^*) \simeq \operatorname{Sat}(V^{-w_0(\lambda)}) \simeq \operatorname{IC}_{\overline{\operatorname{Gr}}^\lambda_{-w_0(\lambda)}}.$$

The Hecke structure is given by the system isomorphisms

$$\begin{aligned} & j_{-w_0(\lambda),*} \star \operatorname{Sat}(V^\lambda) \star \operatorname{Sat}(V) \simeq j_{-w_0(\lambda),*} \star \operatorname{Sat}(V^\lambda \otimes V) \simeq \\ & \simeq \underset{\mu}{\oplus} V(\mu) \otimes j_{-w_0(\lambda),*} \star \operatorname{Sat}(V^{\lambda+\mu}) \simeq \underset{\mu}{\oplus} \left(j_{w_0(\mu),!} \times V(\mu)\right) \star \left(j_{-w_0(\lambda+\mu),*} \star \operatorname{Sat}(V^{\lambda+\mu})\right). \end{aligned}$$

4.6.8. Finally, we perform one more modification. We replace the action of $\operatorname{Rep}(\check{T})$ given by (4.2) by that given by (4.3). Denote the resulting variant of $\operatorname{Hecke}_{\check{G},\check{T}}(\operatorname{Shv}(\operatorname{Gr}_G)^I)$ by $\operatorname{Hecke}'_{\check{G},\check{T}}(\operatorname{Shv}(\operatorname{Gr}_G)^I)$.

We note that the functor

$$\mathcal{F} \mapsto j_{w_0,!} \star \mathcal{F}$$

defines an equivalence

$$\operatorname{Hecke}_{\check{G},\check{T},w_0}(\operatorname{Shv}(\operatorname{Gr}_G)^I) \to \operatorname{Hecke}'_{\check{G},\check{T}}(\operatorname{Shv}(\operatorname{Gr}_G)^I).$$

We denote by

$$\mathcal{F}'_{\check{G},\check{T}} := j_{w_0,!} \star \mathcal{F}_{\check{G},\check{T},w_0}$$

the resulting object of $\operatorname{Hecke}'_{\check{G},\check{T}}(\operatorname{Shv}(\operatorname{Gr}_G)^I)$.

5. The semi-infinite IC sheaf as a (dual) baby Verma object

In this section we will relate our semi-infinite cohomology sheaf $\mathrm{IC}^{\frac{\infty}{2}}$ with the (dual) baby Verma object from the previous section (in its incarnation as $\mathcal{F}'_{\check{G},\check{T}}$).

5.1. **Hecke structure on the semi-infinite IC sheaf.**

5.1.1. Let us return to our main object of study, namely, the object

$$\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}.$$

Note that it naturally belongs to the category $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$, and we will consider it as such[6].

5.1.2. We note that the category $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$ is acted on by $\mathrm{Rep}(\check{T})$ by translations

$$\mathsf{e}^\lambda \star \mathcal{F} := t^\lambda \cdot \mathcal{F}[-\langle \lambda, 2\rho\rangle]. \tag{5.1}$$

In addition, it is acted on by $\mathrm{Rep}(\check{G})$ via Geometric Satake and convolutions on the right. So we are in the situation of Sect. 4.3.5.

5.1.3. We claim that our object $\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$ naturally upgrades to an object of

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}).$$

5.1.4. In order to construct this structure, we recall that $\mathrm{IC}^{\frac{\infty}{2}}$ lives in the heart of the t-structure, and so do the convolutions

$$t^\lambda \cdot \mathrm{IC}^{\frac{\infty}{2}} \text{ and } \mathrm{IC}^{\frac{\infty}{2}} \star \mathrm{Sat}(V),\ V \in \mathrm{Rep}(\check{G})^\heartsuit.$$

Hence, when constructing the Hecke structure, we are staying within the abelian category, and it is enough to make the construction at the level of the homotopy categories.

Now, the required Hecke structure on

$$\mathrm{IC}^{\frac{\infty}{2}} := \underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]$$

is given by

$$\left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda)[\langle \lambda, 2\check{\rho}\rangle]\right) \star \mathrm{Sat}(V) \simeq \left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, t^{-\lambda} \cdot \mathrm{Sat}(V^\lambda \otimes V)[\langle \lambda, 2\check{\rho}\rangle]\right) \overset{(2.3)}{\simeq} \tag{5.2}$$
$$\simeq \bigoplus_\mu V(\mu) \otimes \left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, t^{-\lambda} \cdot \mathrm{Sat}(V^{\lambda+\mu})[\langle \lambda, 2\check{\rho}\rangle]\right) \simeq$$
$$\simeq \bigoplus_\mu V(\mu) \otimes t^\mu \cdot \left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, t^{-\lambda-\mu} \cdot \mathrm{Sat}(V^{\lambda+\mu})[\langle \lambda+\mu, 2\check{\rho}\rangle]\right)[-\langle \mu, 2\check{\rho}\rangle].$$

The compatibility of these isomorphisms with tensor products in $\mathrm{Rep}(\check{G})$ at the level of the homotopy categories is straightforward.

5.2. **The semi-infinite vs Iwahori equivalence.** The results of this subsection are borrowed from [Ras, Theorem 6.2.1]. They are applicable for $\mathrm{Shv}(\mathrm{Gr}_G)$ replaced by any category acted on by $\mathfrak{L}(G)$.

---

[6] When forming $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$ starting from $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$, we apply the same renormalization procedure as we did for $\mathrm{Shv}(\mathrm{Gr}_G)^I$.

5.2.1. Consider the forgetful functor

$$\mathrm{Shv}(\mathrm{Gr}_G)^I \to \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}. \tag{5.3}$$

It admits a right adjoint, given by *-averaging with respect to I "modulo" $\mathfrak{L}^+(T)$; we denote it by $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}$.

Proposition 5.2.2. *The functor* $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(\mathcal{F})$ *restricted to*

$$(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)} := \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)} \subset \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$$

defines an equivalence

$$(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)} \to \mathrm{Shv}(\mathrm{Gr}_G)^I.$$

This equivalence intertwines the $\mathrm{Rep}(\check{T})$*-action on* $(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)}$*, given by* (5.1) *and the* $\mathrm{Rep}(\check{T})$*-action on* $\mathrm{Shv}(\mathrm{Gr}_G)^I$*, given by* (4.3).

The rest of this subsection is devoted to the proof of this proposition. The argument essentially mimics the proof of the corresponding statement in the theory of $\mathfrak{p}$-adic groups, due to J. Bernstein.

5.2.3. We claim that the functor

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(\mathcal{F}) : \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)} \to \mathrm{Shv}(\mathrm{Gr}_G)^I$$

admits a left adjoint.

More precisely, we claim that the partially defined functor

$$\mathrm{Av}_!^{\mathfrak{L}(N)} : \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)} \to \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$$

(see Sect. 1.2.5) is defined on objects lying in the essential image of the forgetful functor (5.3). Then it automatically provides the required left adjoint.

To show this, we pick a particular sequence of subgroups $N_\alpha \subset \mathfrak{L}(N)$. Namely, we take the indexing set to be Λ^+ (with the order relation from Sect. 2.3.1) and we set

$$N_\lambda := \mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}^+(N)).$$

Then it is easy to see that for $\mathcal{F} \in \mathrm{Shv}(\mathrm{Gr}_G)^I$ we have a canonical identification

$$\mathrm{Av}_!^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}^+(N))}(\mathcal{F}) \simeq t^{-\lambda} \cdot j_{\lambda,!} \star \mathcal{F}[\langle\lambda, 2\check{\rho}\rangle],$$

where the right-hand side is viewed as an object of $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$.

5.2.4. Note that by construction, for $\lambda \in \Lambda^+$ and $\mathcal{F}$ as above, we have a canonical identification

$$t^\lambda \cdot \mathrm{Av}_!^{\mathfrak{L}(N)}(\mathcal{F})[-\langle\lambda, 2\check{\rho}\rangle] \simeq \mathrm{Av}_!^{\mathfrak{L}(N)}(j_{\lambda,!} \star \mathcal{F}).$$

This shows that the functor $\mathrm{Av}_!^{\mathfrak{L}(N)}$ intertwines the $\mathrm{Rep}(\check{T})$-action on $(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)}$, given by (5.1) and the $\mathrm{Rep}(\check{T})$-action on $\mathrm{Shv}(\mathrm{Gr}_G)^I$, given by (4.3).

5.2.5. We now check that the unit of the adjunction, namely, the natural transformation

$$\mathcal{F} \to \mathrm{Av}_*^{I/\mathfrak{L}^+(T)} \circ \mathrm{Av}_!^{\mathfrak{L}(N)}(\mathcal{F}), \quad \mathcal{F} \in \mathrm{Shv}(\mathrm{Gr}_G)^I \tag{5.4}$$

is an isomorphism.

Indeed, for every $\lambda \in \Lambda^+$ and $\mathcal{F}' \in \mathrm{Shv}(\mathrm{Gr}_G)^I$ we have a canonical identification

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(t^{-\lambda} \cdot \mathcal{F}') \simeq j_{-\lambda,*} \star \mathcal{F}'[-\langle\lambda, 2\check{\rho}\rangle].$$

So,

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)} \circ \mathrm{Av}_!^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}^+(N))}(\mathcal{F}) \simeq j_{-\lambda,*} \star j_{\lambda,!} \star \mathcal{F} \simeq \mathcal{F},$$

and hence

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)} \circ \mathrm{Av}_!^{\mathfrak{L}(N)}(\mathcal{F}) \simeq \underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathrm{Av}_*^{I/\mathfrak{L}^+(T)} \circ \mathrm{Av}_!^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}^+(N))}(\mathcal{F}) \simeq \underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathcal{F} \simeq \mathcal{F}.$$

Furthermore, by unwinding the definitions it is easy to see that with respect to the above identification $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)} \circ \mathrm{Av}_!^{\mathfrak{L}(N)}(\mathcal{F}) \simeq \mathcal{F}$, the map (5.4) is the identity map $\mathcal{F} \to \mathcal{F}$.

5.2.6. Finally, let us show that the functor $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}$ is conservative on $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$.

Let $\mathcal{F}$ be a non-zero object of $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$. Since it is non-zero as an object of $\mathrm{Shv}(\mathrm{Gr}_G)$, there exists $\mathcal{F}' \in \mathrm{Shv}(\mathrm{Gr}_G)$ equivariant with respect to some congruence subgroup of $\mathfrak{L}^+(G)$ such that

$$\mathcal{M}aps_{\mathrm{Shv}(\mathrm{Gr}_G)}(\mathcal{F}', \mathcal{F}) \neq 0.$$

By assumption $\mathcal{F}'$ is equivariant with respect to $\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}_1^+(N^-))$ for $\lambda \gg 0$ (here $\mathfrak{L}_1^+(N^-)$ denotes the first congruence subgroup of $\mathfrak{L}^+(N)$). Hence,

$$\mathrm{Av}_*^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}^+(N^-))}(\mathcal{F}) \neq 0.$$

Note, however that since $\mathcal{F}$ belongs to $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$ and

$$I = \mathfrak{L}_1^+(N^-) \cdot \mathfrak{L}^+(T) \cdot \mathfrak{L}^+(N),$$

we have

$$\mathrm{Av}_*^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}_1^+(N^-))}(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Gr}_G)^{\mathrm{Ad}_{t^{-\lambda}}(I)}.$$

Now, for any $\mathcal{F}'' \in \mathrm{Shv}(\mathrm{Gr}_G)^{\mathrm{Ad}_{t^{-\lambda}}(I)}$, we have

$$t^\lambda \cdot \mathcal{F}'' \in \mathrm{Shv}(\mathrm{Gr}_G)^I$$

and

$$\mathrm{Av}_*^{\mathfrak{L}_1^+(N^-)}(\mathcal{F}'') \simeq j_{-\lambda,*} \star (t^\lambda \cdot \mathcal{F}'')[-\langle \lambda, 2\check{\rho}\rangle].$$

Hence, for $\mathcal{F}$ as above,

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(\mathcal{F}) \simeq \mathrm{Av}_*^{\mathfrak{L}_1^+(N^-)}(\mathcal{F}) \simeq \mathrm{Av}_*^{\mathfrak{L}_1^+(N^-)} \circ \mathrm{Av}_*^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}_1^+(N^-))}(\mathcal{F}) \simeq$$
$$\simeq j_{-\lambda,*} \star \left(t^\lambda \cdot \mathrm{Av}_*^{\mathrm{Ad}_{t^{-\lambda}}(\mathfrak{L}_1^+(N^-))}(\mathcal{F})\right)[-\langle \lambda, 2\check{\rho}\rangle],$$

which is non-zero, because the functor $j_{-\lambda,*} \star -$ is conservative (it is in fact a self-equivalence of $\mathrm{Shv}(\mathrm{Gr}_G)^I$).

5.3. **The semi-infinite IC sheaf as the baby Verma object.** We are now ready to state the second main result of this paper:

Theorem 5.3.1. *Under the equivalence of Proposition 5.2.2, the object*

$$\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)})$$

goes over to $\mathcal{F}'_{\check{G},\check{T}}[\dim(G/B)]$.

The rest of this subsection is devoted to the proof of Theorem 5.3.1.

5.3.2. Since the stated isomorphism takes place in the heart of $\mathrm{Shv}(\mathrm{Gr}_G)^I$ (with respect to the perverse t-structure), and since the actions of $\mathrm{Rep}(\check{G})$ and $\mathrm{Rep}(\check{T})$ are given by t-exact functors, it is enough to establish it at the level of the homotopy categories.

5.3.3. According to Sect. 4.6.7, the object $\mathcal{F}'_{\check{G},\check{T}}$ can be described as follows. The underlying object of $\mathrm{Shv}(\mathrm{Gr}_G)^I$ is

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, j_{-\lambda\cdot w_0,*}\star \mathrm{Sat}(V^\lambda). \tag{5.5}$$

Here we are using the fact that for λ regular, we have

$$j_{w_0,!}\star j_{-w_0(\lambda),*}\simeq j_{-\lambda\cdot w_0,*},$$

which follows from the isomorphism

$$j_{-w_0(\lambda),*}\simeq j_{w_0,*}\star j_{-\lambda\cdot w_0,*},$$

which in turn follows from the fact that

$$-w_0(\lambda)=w_0\cdot(-\lambda\cdot w_0) \text{ and } \ell(-w_0(\lambda))=\ell(w_0)+\ell(-\lambda\cdot w_0)$$

for λ dominant and regular.

The transition maps are induced by (4.18), and the Hecke structure is induced by (4.19).

5.3.4. The object of $\mathrm{Shv}(\mathrm{Gr}_G)^I$, underlying $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(\mathrm{IC}^{\frac{\infty}{2}})$ is given by

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathrm{Av}_*^{I/\mathfrak{L}^+(T)}\left(t^{-\lambda}\cdot \mathrm{Sat}(V^\lambda)\right)[\langle\lambda,2\check{\rho}\rangle].$$

We note that for any $\mathcal{F}\in\mathrm{Shv}(\mathrm{Gr}_G)^I$, we have a canonical isomorphism

$$\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(t^{-\lambda}\cdot\mathcal{F})[\langle\lambda,2\check{\rho}\rangle]\simeq j_{-\lambda,*}\star\mathcal{F}.$$

Note also that for λ regular we have

$$j_{-\lambda,*}\simeq j_{-\lambda\cdot w_0,*}\star j_{w_0,*}.$$

Hence, for any $\mathcal{F}'\in\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}$, we have a canonical isomorphism

$$j_{-\lambda,*}\star\mathcal{F}'\simeq j_{-\lambda\cdot w_0,*}\star\mathcal{F}'[\dim(G/B)].$$

Hence, we obtain that $\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}(\mathrm{IC}^{\frac{\infty}{2}})$ is given by

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, j_{-\lambda\cdot w_0,*}\star\mathrm{Sat}(V^\lambda)[\dim(G/B)], \tag{5.6}$$

where the transition maps are induced by (2.4), and the Hecke structure by (5.2).

5.3.5. However, by unwinding the definitions, we see that the transition maps in (5.6) coincide with those in (5.5), and that the Hecke structures match up.

5.4. **Description of the abelian category.** In this subsection we will prove Propositions 1.5.7 and 1.5.9.

5.4.1. Consider the functor

$$(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I/\mathfrak{L}^+(T)}}{\longrightarrow} \mathrm{Shv}(\mathrm{Gr})^I \overset{j_{w_0,*}\star-[-\dim(G/B)]}{\longrightarrow} \mathrm{Shv}(\mathrm{Gr})^I \overset{\mathrm{Sat}^I}{\longrightarrow} \mathrm{IndCoh}(\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-). \tag{5.7}$$

According to Theorem 4.2.3 and Proposition 5.2.2, the above functor is an equivalence. Note that by construction, it sends the object $\Delta^\lambda\in(\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)}$ to the object

$$\mathsf{e}^{w_0(\lambda)}\otimes\mathcal{O}_{\mathrm{pt}/\check{B}^-}\in\mathrm{IndCoh}(\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-).$$

We will prove:

Proposition 5.4.2. *Under the equivalence* (5.7)*, the abelian category* $((\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)})^\heartsuit$ *goes over to*

$$(\mathrm{IndCoh}(\mathrm{pt}/\check{B}^-))^\heartsuit\simeq(\mathrm{IndCoh}(\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-))^\heartsuit.$$

We will also prove:

Proposition 5.4.3. *The inclusion* $((\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)})^{\heartsuit} \hookrightarrow (\mathrm{SI}(\mathrm{Gr}_G))^{\heartsuit}$ *is an equivalence.*

Note that the combination of these two propositions implies (1.5.7). Note also that Proposition 1.5.9 follows this and Theorem 5.3.1.

5.4.4. *Proof of Proposition 5.4.2.* To prove the inclusion $\supset$ it suffices to show that the objects

$$\mathsf{e}^{w_0(\lambda)} \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-} \in \mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$$

go over to objects in $((\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)})^{\heartsuit}$. But, as was noted above, they correspond to objects $\Delta^\lambda \in (\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)}$, which do belong to $((\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)})^{\heartsuit}$ by Corollary 1.5.6.

Vice versa, by Corollary 1.5.6, every object of $((\mathrm{SI}(\mathrm{Gr}_G))^{\mathfrak{L}^+(T)})^{\heartsuit}$ admits a filtration with subquotients of the form Δ^λ. Its image in $\mathcal{O}_{\mathrm{pt}/\check{B}^-} \in \mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-)$ thus admits a filtration with subquotients of the form $\mathsf{e}^{w_0(\lambda)} \otimes \mathcal{O}_{\mathrm{pt}/\check{B}^-}$, and hence belongs to

$$(\mathrm{IndCoh}(\mathrm{pt}/\check{B}^-))^{\heartsuit} \simeq (\mathrm{IndCoh}(\check{\mathfrak{n}}^- \underset{\check{\mathfrak{g}}}{\times} \{0\}/\check{B}^-))^{\heartsuit}.$$

□

5.4.5. *Proof of Proposition 5.4.3.* Since $\mathrm{SI}(\mathrm{Gr}_G)$ is generated by $\mathfrak{L}^+(T)$-equivariant objects, every object $\mathcal{F}$ of $(\mathrm{SI}(\mathrm{Gr}_G))^{\heartsuit}$ is $\mathfrak{L}^+(T)$-*monodromic.* Hence, $\mathcal{F}$ admits an action of the algebra $\mathrm{Sym}(\mathfrak{t})$ by endomorphisms, and $\mathcal{F}$ is $\mathfrak{L}^+(T)$-equivariant if and only if this action is trivial.

The Cousin decomposition of $\mathcal{F}$ with respect to the strata S^λ defines on $\mathcal{F}$ a functorial filtration, indexed by the poset Λ, with the subquotient corresponding to a given $\lambda \in \Lambda$ isomorphic to a direct sum of copies of Δ^λ. The action of $\mathrm{Sym}(\mathfrak{t})$ an any such subquotient is trivial. However, since the Δ^λ's are pairwise non-isomorphic, the action of $\mathrm{Sym}(\mathfrak{t})$ on $\mathcal{F}$ is therefore also trivial.

□

6. The Drinfeld-Plücker formalism

The contents of this section follow a suggestion of S. Raskin. We will present another, in a sense more functorial, point of view that leads to the construction of $\mathrm{IC}^{\frac{\infty}{2}}$ and the baby Verma modules.

6.1. **Drinfeld-Plücker structures.**

6.1.1. Consider the scheme $\check{G}/\check{N}^-$. It is known to be quasi-affine, and let $\overline{\check{G}/\check{N}^-}$ be its affine closure, (i.e., the spectrum of the algebra of global functions), so that we have an open embedding

$$\check{G}/\check{N}^- \hookrightarrow \overline{\check{G}/\check{N}^-}. \tag{6.1}$$

By transport of structure, $\overline{\check{G}/\check{N}^-}$ is acted on by $\check{G} \times \check{T}$. Thus, we can regard $\mathcal{O}(\overline{\check{G}/\check{N}^-})$ as a commutative algebra object in the symmetric monoidal category $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$.

6.1.2. Note that

$$\overline{\check{G}/\check{N}^-} \simeq \underset{\lambda \in \Lambda^+}{\oplus} V^\lambda \otimes \mathsf{e}^{-\lambda},$$

with the product given by

$$\left(V^{\lambda_1} \otimes \mathsf{e}^{-\lambda_1}\right) \otimes \left(V^{\lambda_2} \otimes \mathsf{e}^{-\lambda_2}\right) \to V^{\lambda_1+\lambda_2} \otimes \mathsf{e}^{-\lambda_1-\lambda_2},$$

induced by (2.1).

6.1.3. Let $\mathcal{C}$ be a DG category equipped with an action of $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$. We shall say that an object $c \in \mathcal{C}$ is equipped with a *Drinfeld-Plücker structure* if it is equipped with an action of $\mathcal{O}(\overline{\check{G}/\check{N}^-})$, viewed as an algebra object in $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$.

We denote the category of objects of $\mathcal{C}$ equipped with a Drinfeld-Plücker structure by

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C}).$$

6.1.4. Using the above presentation of $\overline{\check{G}/\check{N}^-}$, we can think of a Drinfeld-Plücker structure on c as a system of maps

$$c \star V^\lambda \to \mathsf{e}^\lambda \star c, \quad \lambda \in \Lambda^+ \tag{6.2}$$

that make the diagrams

$$\begin{array}{ccc}
(c \star V^{\lambda_1}) \star V^{\lambda_2} & \xrightarrow{\sim} & c \star (V^{\lambda_1} \star V^{\lambda_2}) \\
\downarrow & & \downarrow \\
(\mathsf{e}^{\lambda_1} \star c) \star V^{\lambda_2} & & c \star V^{\lambda_1+\lambda_2} \\
\sim\downarrow & & \downarrow(2.2) \\
\mathsf{e}^{\lambda_1} \star (c \star V^{\lambda_2}) & & \mathsf{e}^{\lambda_1+\lambda_2} \star c \\
\downarrow & & \downarrow\sim \\
\mathsf{e}^{\lambda_1} \star (\mathsf{e}^{\lambda_2} \star c) & \xrightarrow{\sim} & \mathsf{e}^{\lambda_1+\lambda_2} \star c
\end{array}$$

satisfying a coherent system of higher compatibilities (note that the latter is automatic if the corresponding mapping spaces are discrete).

Remark 6.1.5. Note that we can rewrite the maps (6.2) as

$$\mathsf{e}^{-\lambda} \star c \to c \star (V^\lambda)^*, \quad \lambda \in \Lambda^+.$$

Then a Drinfeld-Plücker structure on c is the same as the structure of *lax central object*, in the terminology of (2.7). Here our monoidal category is Λ^+, its left action on $\mathcal{C}$ is via the monoidal functor

$$\Lambda^+ \to \mathrm{Rep}(\check{T}), \quad \lambda \mapsto \mathsf{e}^{-\lambda},$$

and the lax right action is via

$$\Lambda^+ \to \mathrm{Rep}(\check{G}), \quad \lambda \mapsto (V^\lambda)^*.$$

6.1.6. By a similar token, instead of $\overline{\check{G}/\check{N}^-}$, we can consider $\overline{\check{G}/\check{N}}$. We denote the corresponding category by $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathcal{C})$.

Let ${}_{w_0}\mathcal{C}$ be obtained from $\mathcal{C}$ by modifying the action of $\mathrm{Rep}(\check{T})$ on $\mathcal{C}$ by applying the automorphism w_0 to $\check{T}$. A choice of a representative $g_{w_0} \in \check{G}$ of w_0 defines an equivalence

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}({}_{w_0}\mathcal{C}) \simeq \mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathcal{C}). \tag{6.3}$$

Remark 6.1.7. As above, objects of $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathcal{C})$ can be thought of as objects $c \in \mathcal{C}$ equipped with a system of maps

$$\mathsf{e}^\lambda \star c \to c \star V^\lambda, \quad \lambda \in \Lambda^+,$$

equipped with a system of compatibilities. I.e., they are the same as lax central objects in the terminology of(2.7), where the left action of Λ^+ is given by

$$\Lambda^+ \to \mathrm{Rep}(\check{T}), \quad \lambda \mapsto \mathsf{e}^\lambda$$

and the lax right action is via

$$\Lambda^+ \to \mathrm{Rep}(\check{G}), \quad \lambda \mapsto V^\lambda.$$

6.2. **From Drinfeld-Plücker structures to Hecke objects.**

6.2.1. The monad associated with the functor

$$\mathcal{C} \simeq \mathcal{C} \underset{\operatorname{Rep}(\check{T})}{\otimes} \operatorname{Rep}(\check{T}) \to \mathcal{C} \underset{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}(\check{T}) =: \operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \tag{6.4}$$

and its right adjoint, is given by the action of $\mathcal{O}(\check{G})$, viewed as an object of $\operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G})$.

Hence, by the Barr-Beck-Lurie theorem, we can identify

$$\operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \simeq \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}),$$

so that the right adjoint to (6.4) corresponds to the forgetful functor

$$\mathbf{oblv}_{\mathcal{O}(\check{G})} : \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}) \to \mathcal{C}.$$

6.2.2. The composite map

$$\check{G} \to \check{G}/\check{N}^- \to \overline{\check{G}/\check{N}^-}$$

defines a pair of adjoint functors

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C}) \rightleftarrows \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}).$$

We will now describe the left adjoint

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C}) \to \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}) = \operatorname{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \tag{6.5}$$

explicitly.

6.2.3. We will first describe the composition

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C}) \to \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}) \overset{\mathbf{oblv}_{\mathcal{O}(\check{G})}}{\longrightarrow} \mathcal{C}. \tag{6.6}$$

By Sect. 2.7, for $c \in \mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C})$, we have a well-defined functor

$$\Lambda^+ \to \mathcal{C}, \quad \lambda \mapsto \mathsf{e}^\lambda \star c \star (V^\lambda)^*,$$

where the transition maps for $\lambda_2 = \lambda_1 + \lambda$ are given by

$$\mathsf{e}^{\lambda_1} \star c \star (V^{\lambda_1})^* \simeq \mathsf{e}^{\lambda_1+\lambda_2} \star \mathsf{e}^{-\lambda_2} \star c \star (V^{\lambda_1})^* \to$$
$$\to \mathsf{e}^{\lambda_1+\lambda_2} \star c \star (V^{\lambda_2})^* \star c \star (V^{\lambda_1})^* \to \mathsf{e}^{\lambda_1+\lambda_2} \star c \star (V^{\lambda_1+\lambda_2})^*.$$

Moreover, this construction is functorial in $c \in \mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C})$. We claim:

Proposition 6.2.4. *The functor* (6.6) *identifies with*

$$c \mapsto \underset{\lambda\in\Lambda^+}{\operatorname{colim}}\, \mathsf{e}^\lambda \star c \star (V^\lambda)^*.$$

Proof. Note that we have:

$$\mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\mathcal{C}) \simeq \mathcal{C} \underset{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}{\otimes} \mathcal{O}(\overline{\check{G}/\check{N}^-})\text{-mod}(\operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G}))$$

and

$$\mathcal{O}(\check{G})\text{-mod}(\mathcal{C}) \simeq \mathcal{C} \underset{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}{\otimes} \mathcal{O}(\check{G})\text{-mod}(\operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G})).$$

Hence, the assertion of the proposition reduces to the universal situation of $\mathcal{C} = \operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G})$ and $c = \mathcal{O}(\overline{\check{G}/\check{N}^-})$. Thus, we need to construct an isomorphism

$$\underset{\lambda\in\Lambda^+}{\operatorname{colim}}\, \mathsf{e}^\lambda \otimes \mathcal{O}(\overline{\check{G}/\check{N}^-}) \otimes (V^\lambda)^* \simeq \mathcal{O}(\check{G})$$

as objects of $\operatorname{Rep}(\check{T}) \otimes \operatorname{Rep}(\check{G})$.

Since the index category Λ^+ is filtered, the left-hand side belongs to the heart of the t-structure. So, the computation we need to perform is inside the abelian category.

The map from the left-hand side to the right-hand side is given by embedding $\mathcal{O}(\overline{\check{G}/\check{N}^-})$ into $\mathcal{O}(G)$, and multiplying by the matrix coefficient function of v^λ against an element of $(V^\lambda)^*$. The fact that the resulting map is an isomorphism can be seen as follows:

Fix $\mu \in \Lambda^+$, and let us calculate $\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, -)$ into both sides. In the right hand side, we have

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, \mathcal{O}(\check{G})) \simeq (V^\mu)^*,$$

viewed as a $\check{T}$-representation. In the left-hand side, we have the colimit over $\lambda \in \Lambda^+$ of

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, \mathsf{e}^\lambda \otimes \mathcal{O}(\overline{\check{G}/\check{N}^-}) \otimes (V^\lambda)^*),$$

also viewed as a $\check{T}$-representation.

Now, for λ large relative to μ, using (2.3), we have:

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, \mathsf{e}^\lambda \otimes \mathcal{O}(\overline{\check{G}/\check{N}^-}) \otimes (V^\lambda)^*) \simeq \mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu \otimes V^\lambda, \mathcal{O}(\overline{\check{G}/\check{N}^-})) \otimes \mathsf{e}^\lambda \overset{(2.3)}{\simeq}$$
$$\simeq \bigoplus_\nu (V^\mu(\nu))^* \otimes \mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^{\lambda+\nu}, \mathcal{O}(\overline{\check{G}/\check{N}^-})) \otimes \mathsf{e}^\lambda \simeq \bigoplus_\nu (V^\mu(\nu))^* \otimes \mathsf{e}^{-\lambda-\nu} \otimes \mathsf{e}^\lambda \simeq$$
$$\simeq \bigoplus_\nu (V^\mu(\nu))^* \otimes \mathsf{e}^{-\nu} \simeq (V^\mu)^*.$$

Moreover, for such λ, the map

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, \mathsf{e}^\lambda \otimes \mathcal{O}(\overline{\check{G}/\check{N}^-}) \otimes (V^\lambda)^*) \to \mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V^\mu, \mathcal{O}(\check{G}))$$

induced by $\mathsf{e}^\lambda \otimes \mathcal{O}(\overline{\check{G}/\check{N}^-}) \otimes (V^\lambda)^* \to \mathcal{O}(\check{G})$, is the identity map $(V^\mu)^* \to (V^\mu)^*$.

This establishes the desired isomorphism.

□

6.2.5. Next, we wish to describe the structure of an object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathcal{C}) \simeq \mathcal{O}(\check{G})\text{-mod}(\mathcal{C})$ on the colimit $\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^\lambda \star c \star (V^\lambda)^*$. We claim that for an individual finite-dimensional $V \in \mathrm{Rep}(\check{G})^\heartsuit$, the corresponding isomorphism

$$\left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^\lambda \star c \star (V^\lambda)^*\right) \star V \simeq \mathrm{Res}^{\check{G}}_{\check{T}}(V) \star \left(\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^\lambda \star c \star (V^\lambda)^*\right)$$

is given as follows:

Let us take λ large enough so that (2.3) applies. Then we have:

$$\left(\mathsf{e}^\lambda \star c \star (V^\lambda)^*\right) \star V \overset{(4.16)}{\simeq}$$
$$\simeq \bigoplus_\mu \mathsf{e}^\lambda \star c \star (V^{\lambda+\mu})^* \otimes V(-\mu) \simeq \bigoplus_\mu (\mathsf{e}^{-\mu} \otimes V(-\mu)) \star \left(\mathsf{e}^{\lambda+\mu} \star c \star (V^{\lambda+\mu})^*\right) \simeq$$
$$\simeq \mathrm{Res}^{\check{G}}_{\check{T}}(V) \star \left(\mathsf{e}^{\lambda+\mu} \star c \star (V^{\lambda+\mu})^*\right),$$

To establish this description, as in the proof of Proposition 6.2.4, it is enough to do in the universal case of $\mathcal{C} = \mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ and $c = \mathcal{O}(\overline{\check{G}/\check{N}^-})$, in which case it becomes an elementary verification.

6.2.6. We have an obvious analog of Proposition 6.2.4 when we replace $\check{G}/\check{N}^-$ by $\check{G}/\check{N}$. Namely, the corresponding functor

$$\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathcal{C}) \to \mathcal{O}(\check{G})\text{-mod}(\mathcal{C}) \overset{\mathbf{oblv}_{\mathcal{O}(\check{G})}}{\longrightarrow} \mathcal{C}$$

is given by

$$c \mapsto \underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, \mathsf{e}^{-\lambda} \star c \star V^\lambda,$$

and the Hecke structure is given by

$$\left(\mathsf{e}^{-\lambda}\star c\star V^{\lambda}\right)\star V \overset{(2.3)}{\simeq}$$

$$\simeq \underset{\mu}{\oplus}\, \mathsf{e}^{-\lambda}\star c\star V^{\lambda+\mu}\otimes V(\mu) \simeq \underset{\mu}{\oplus}\, (\mathsf{e}^{\mu}\otimes V(\mu))\star \left(\mathsf{e}^{-\lambda-\mu}\star c\star V^{\lambda+\mu}\right)\simeq$$

$$\simeq \operatorname{Res}^{\check{G}}_{\check{T}}(V)\star \left(\mathsf{e}^{-\lambda-\mu}\star c\star V^{\lambda+\mu}\right),$$

6.3. **Construction of $\mathrm{IC}^{\frac{\infty}{2}}$ and other objects from the Drinfeld-Plücker picture.** We will now apply the above discussion and give a conceptual explanation of the colimit formations that we encountered earlier in the paper.

6.3.1. Let $\mathcal{C}$ be a category equipped with an action of $\operatorname{Rep}(\check{B}^-)$. We endow it with actions of $\operatorname{Rep}(\check{G})$ and $\operatorname{Rep}(\check{T})$ coming from the restriction functors

$$\operatorname{Rep}(\check{G})\to \operatorname{Rep}(\check{B}^-)\leftarrow \operatorname{Rep}(\check{T}).$$

Then any object of $\mathcal{C}$ is automatically equipped with a Drinfeld-Plucker structure: indeed, this structure comes from the canonical system of maps

$$\operatorname{Res}^{\check{G}}_{\check{B}^-}(V^{\lambda})\to \operatorname{Res}^{\check{T}}_{\check{B}^-}(\mathsf{e}^{\lambda}),$$

given by the covectors $(v^{\lambda})^*$.

6.3.2. First, take $\mathcal{C}=\operatorname{Rep}(\check{B}^-)$. Applying the above construction to $c=\mathsf{e}$, and applying the functor (6.5) we obtain an object of

$$\operatorname{Hecke}_{\check{G},\check{T}}(\operatorname{Rep}(\check{B}^-))\simeq \operatorname{QCoh}((\check{G}/\check{N}^-)/\operatorname{Ad}_{\check{T}})$$

equal to the image of $\mathcal{O}_{\mathrm{pt}/\check{T}}$ along the map

$$\mathrm{pt}/\check{T}\to \operatorname{QCoh}((\check{G}/\check{N}^-)/\operatorname{Ad}_{\check{T}}).$$

This is the object, considered in Sect. 4.4.2. Its description as a colimit in Sect. 4.4.4 coincides with one given by Proposition 6.2.4.

6.3.3. Next we take $\mathcal{C}=\operatorname{IndCoh}(\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-)$, equipped with an action of $\operatorname{Rep}(\check{B}^-)$, coming from the projection

$$\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-\to \mathrm{pt}/\check{B}^-.$$

Take c equal to the direct image of the structure sheaf along the closed embedding

$$\{0\}/\check{B}^-\to \check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-.$$

The corresponding object of

$$\operatorname{Hecke}_{\check{G},\check{T}}(\operatorname{IndCoh}(\check{\mathfrak{n}}^-\underset{\check{\mathfrak{g}}}{\times}\{0\}/\check{B}^-))$$

is the object that we denoted $\mathcal{M}_{\check{G},\check{T}}$.

6.3.4. Next, we take $\mathcal{C}=\operatorname{Shv}(\operatorname{Gr}_G)^I$, equipped with an action of $\operatorname{Rep}(\check{G})$ coming right convolutions by $\operatorname{Sph}(G)$ and $\operatorname{Sat}:\operatorname{Rep}(\check{G})\to\operatorname{Sph}(G)$. We equip it with the action of $\operatorname{Rep}(\check{T})$ given by (4.2). (Note that according to Theorem 4.2.3, this situation is equivalent to the one in Sect. 6.3.3.)

We take $c=\delta_{1,\operatorname{Gr}_G}$. It is equipped with a Drinfeld-Plucker structure by means of

$$\mathrm{IC}_{\overline{\operatorname{Gr}}^{\lambda}_G}\to j_{\lambda,*}\star\delta_{1,\operatorname{Gr}_G}.$$

The corresponding object of $\operatorname{Hecke}_{\check{G},\check{T}}(\operatorname{Shv}(\operatorname{Gr}_G)^I)$ is the object that in Sect. 4.5 we denoted by $\mathcal{F}_{\check{G},\check{T}}$. Its presentation as a colimit in Sect. 4.5.3 is the one given by Proposition 6.2.4.

6.3.5. We again take $\mathcal{C} = \mathrm{Shv}(\mathrm{Gr}_G)^I$, but equip it with the $\mathrm{Rep}(\check{T})$-action, given by (4.3). We claim that the object δ_{1,Gr_G} canonically lifts to an object of

$$\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^I).$$

The corresponding maps are given by

$$j_{\lambda,!} \star \delta_{1,\mathrm{Gr}_G} \to \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}.$$

The resulting object of

$$\mathrm{Hecke}'_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^I)$$

is what we denoted by $\mathcal{F}'_{\check{G},\check{T}}[\dim(G/B)]$. The cohomological shift comes from the identification

$$j_{w_0,!} \star \delta_{1,\mathrm{Gr}_G} \simeq \delta_{1,\mathrm{Gr}_G}[-\dim(G/B)].$$

The description of $\mathcal{F}'_{\check{G},\check{T}}$, given in Sect. 5.3.3, coincides with the one given by Proposition 6.2.4.

6.3.6. Let us now take $\mathcal{C} = \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$. It is equipped with an action of $\mathrm{Rep}(\check{G})$ coming right convolutions by $\mathrm{Sph}(G)$ and $\mathrm{Sat} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}(G)$. It is equipped with an action of $\mathrm{Rep}(\check{T})$ coming from he action of $\Lambda \simeq \mathfrak{L}(T)/\mathfrak{l}^+(T)$ by left translations with a cohomological shift:

$$\mathsf{e}^\lambda \cdot \mathcal{F} := t^\lambda \cdot \mathcal{F}[-\langle \lambda, 2\check{\rho}\rangle].$$

Take $c = \delta_{1,\mathrm{Gr}_G}$; it upgrades to an object of $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}^+(T)})$ via the maps

$$\delta_{t^\lambda,\mathrm{Gr}_G}[-\langle \lambda, 2\check{\rho}\rangle] \to \mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}.$$

The resulting object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}^+(T)})$ is our $\mathrm{IC}^{\frac{\infty}{2}}$, equipped with the Hecke structure as in Sect. 5.1.

6.3.7. Finally, let us take $\mathcal{C} = \mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$. The above $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ action on $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$ induces one on $\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$.

We take $c = \Delta_0 \simeq \mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{1,\mathrm{Gr}_G})$. The structure on δ_{1,Gr_G} of object in the category $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}^+(T)})$ induces a structure on Δ_0 of object in $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}(N)\cdot\mathfrak{l}^+(T)})$.

The resulting object of $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}(N)\cdot\mathfrak{l}^+(T)})$ is again $\mathrm{IC}^{\frac{\infty}{2}}$.

6.3.8. It is clear from the construction that the structure of objects of $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^{\mathfrak{l}(N)\cdot\mathfrak{l}^+(T)})$ and $\mathcal{O}(\overline{\check{G}/\check{N}})\text{-mod}(\mathrm{Shv}(\mathrm{Gr}_G)^I)$ on Δ_0 and δ_{1,Gr_G}, respectively, match up under the equivalence of Proposition 5.2.2.

This gives a "formula-free" proof of Theorem 5.3.1.

References

[ABG] S. Arkhipov, R. Bezrukavnikov and V. Ginzburg, *Quantum groups, the loop Grassmannian and the Springer resolution*, JAMS **17** (2004), 595–678.

[ABBGM] S. Arkhipov, R. Bezrukavnikov, A. Braverman, D. Gaitsgory and I. Mirković, *Modules over the quantum group and semi-infinite flag manifold*, Transformation Groups **10** (2005), 279–362.

[BFGM] A. Braverman, M. Finkelberg, D. Gaitsgory and I. Mirkovic, *Intersection cohomology of Drinfeld compactifications*, Selecta Math. (N.S.) **8** (2002), 381–418.

[BG] A. Braverman, D. Gaitsgory, *Deformations of local systems and Eisenstein series*, GAFA **17** (2008), 1788–1850.

[BK] A. Bouthier and K. Kazhdan, *Faisceaux pervers sur les espaces d'arcs–I: le cas d'égales caractéristiques*, arXiv:1509.02203.

[Bra] T. Braden, *Hyperbolic localization of Intersection Cohomology*, Transformation Groups **8** (2003), no. 3, 209–216.

[FG] E. Frenkel and D. Gaitsgory, *Localization of $\widehat{\mathfrak{g}}$-modules on the affine Grassmannian*, joint with E. Frenkel, Ann. of Math. **170** (2009), 1339–1381.

[FFKM] B. Feigin, M. Finkelberg, A. Kuznetsov, I. Mirković *Semi-infinite flags II*, The AMS Translations **194** (1999), 81–148.

[Ga] D. Gaitsgory, *Eisenstein series and quantum groups*, arXiv: 1505.02329

[Ga1] D. Gaitsgory, *The semi-infinite intersection cohomology sheaf–II*, in preparation.

[GLys] D. Gaitsgory and S. Lysenko, *Twisted Whittaker category and quantum groups*, in preparation.

[GR] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*, available from http://www.math.harvard.edu/∼gaitsgde/GL.

[Lu] J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.
[Lus] G. Lusztig, *Hecke algebras and Jantzen?s generic decomposition patterns*, Adv. in Math. **37** (1980), 121–164.
[Ras] S. Raskin, *Chiral principal series categories II: the factorizable Whittaker category*, available from http://math.mit.edu/∼sraskin/