CHAPTER IV.1. FORMAL MODULI

Contents

Introduction	1
0.1. Groupoids and quotients	1
0.2. Formal groupoids	2
0.3. What else is done in this chapter?	3
1. Formal moduli problems	4
1.1. Formal moduli problems *over* a prestack	4
1.2. Situation over an affine scheme	5
1.3. Formal moduli problems *under* a prestack	6
1.4. Formal moduli problems *under* an affine scheme	7
1.5. The pointed case	8
1.6. Formal groups	9
2. Groupoids	10
2.1. Digression: groupoids and groups over spaces	10
2.2. Groupoids in formal geometry	12
2.3. Taking the quotient by a formal groupoid	13
2.4. Constructing the classifying space of a groupoid	15
2.5. Verification of deformation theory	15

Introduction

In this Chapter we prove one of the main results of this book: the existence of a well-defined procedure of taking a quotient with respect to a formal groupoid.

0.1. **Groupoids and quotients.**

0.1.1. First off, a groupoid in Spc is an object $R^\bullet \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ that is a Segal space such that all of its 1-morphisms are invertible. We shall say that $R^\bullet$ acts on the space $X = R^0$. Sometimes we abuse the notation and instead of $R^\bullet$ write just the space $R := R^1$.

In other words, a groupoid acting on X is a space R, equipped with a pair of projections

$$X \xleftarrow{p_s} R \xrightarrow{p_t} X,$$

and a multiplication map

$$R \underset{p_t, X, p_s}{\times} R \overset{m}{\to} R$$

Date: February 27, 2016.

over $X \times X$, satisfying a homotopy-coherent system of associativity conditions, and such that the map

$$R \underset{p_t,X,p_s}{\times} R \overset{m,\mathrm{id}}{\longrightarrow} R \underset{p_t,X,p_t}{\times} R$$

is an isomorphism.

0.1.2. Given a map $X \to Y$ in $\mathrm{PreStk}_{\mathrm{laft}}$, the Čech nerve construction gives rise to a canonically defined groupoid $R^\bullet$ acting on X with

$$R = X \underset{Y}{\times} X.$$

The above construction is a functor from the category of spaces under X to that of groupoids acting on X.

This functor admits a fully faithful left adjoint that sends $R^\bullet$ to its geometric realization $Y = |R^\bullet|$. The image of this left adjoint is the full subcategory consisting of those $X \to Y$ that induce a surjection on π_0.

0.1.3. The notion of groupoid makes sense in arbitrary ∞-category $\mathcal{C}$ with finite limits, see [Lu1, Sect. 6.1.2].

Namely, given an object $X \in \mathcal{C}$, a groupoid acting on X is a simplicial object $R^\bullet$ of $\mathcal{C}$ with $R^0 = X$ such that for any $X' \in \mathcal{C}$, the object

$$\mathrm{Maps}_{\mathcal{C}}(X', R^\bullet) \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is a groupoid in spaces.

As in the case of $\mathcal{C} = \mathrm{Spc}$, given a map $X \to Y$, we canonically attach to it its Čech nerve, which is a groupoid acting on X.

However, the existence of the left adjoint can only be guaranteed if $\mathcal{C}$ has colimits. This left adjoint will be fully faithful if geometric realizations in $\mathcal{C}$ commute with fiber products.

0.1.4. Thus, we obtain a well-defined notion of groupoid object $\mathcal{R}^\bullet$ in $\mathrm{PreStk}_{\mathrm{laft}}$ acting on a given $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$. Taking its Čech nerve, we obtain a groupoid $\mathcal{R}^\bullet$. The assignment

$$\mathcal{R}^\bullet \to |\mathcal{R}^\bullet|$$

provides a fully faithful left adjoint.

0.2. **Formal groupoids.**

0.2.1. We now modify our problem: instead of the category $\mathrm{PreStk}_{\mathrm{laft}}$, we now consider the category $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. I.e., we impose the condition that our prestacks *admit deformation theory.* In addition, we will restrict to maps between prestacks that are nil-isomorphisms.

Groupoid objects in this context will be called *formal groupoids*; for a given $\mathcal{X}$ we denote the category of formal groupoids over $\mathcal{X}$ by $\mathrm{FormGrpoid}(\mathcal{X})$.

Starting from $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ and an object $\mathcal{R}^\bullet \in \mathrm{FormGrpoid}(\mathcal{X})$, it is *not true* that the prestack $|\mathcal{R}^\bullet|$ admits deformation theory. So, the existence of a fully faithful left adjoint to the Čech nerve construction is not so obvious in this case.

However, the main result of this chapter, Theorem 2.3.2 says:

Theorem 0.2.2. *For $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$, the Čech nerve construction is an equivalence between the category of $\mathcal{Y} \in \text{PreStk}_{\text{laft-def}}$ equipped with a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$ and the category* $\text{FormGrpoid}(\mathcal{X})$.

In other words, this theorem says that, given a formal groupoid $\mathcal{R}$ acting on $\mathcal{X}$, there is a well-defined quotient

$$B_{\mathcal{X}}(\mathcal{R}) \in (\text{PreStk}_{\text{laft}})_{\mathcal{X}/},$$

such that $B_{\mathcal{X}}(\mathcal{R})$ *admits deformation theory* and the map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ is a nil-isomorphism (it is then automatically inf-schematic).

0.2.3. As a particular case of Theorem 0.2.2 we obtain that the loop functor

$$\mathcal{Y} \mapsto \mathcal{G} := \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$$

defines an equivalence between the category of $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$, equipped with a pair of inf-schematic nil-isomorphisms

$$\mathcal{X} \overset{i}{\to} \mathcal{Y} \overset{s}{\to} \mathcal{X}, \quad s \circ i = \text{id}$$

and that of group-objects in the category prestacks $\mathcal{G}$ equipped with an inf-schematic nil-isomorphism $\mathcal{G} \to \mathcal{X}$. We denote the latter category by $\text{Grp}(\text{Form}_{/\mathcal{X}})$, and refer to its objects as *formal groups over* $\mathcal{X}$.

Thus, to any $\mathcal{G}$ as above, we can attach its classifying prestack $B_{\mathcal{X}}(\mathcal{G})$

$$\mathcal{X} \overset{i}{\to} B_{\mathcal{X}}(\mathcal{G}) \overset{s}{\to} \mathcal{X}$$

where i and s are inf-schematic nil-isomorphisms.

0.3. **What else is done in this chapter?** Let X be an object of $\text{Sch}^{\text{aff}}_{\text{aft}}$. We introduce several notions of formal moduli problems associated with X, and we relate them to notions developed in [Lu6].

0.3.1. A formal moduli problem *over* X is an inf-scheme $\mathcal{Y}$ equipped with a nil-isomorphism $\mathcal{Y} \to X$.

Recall that, by definition, an inf-scheme is a prestack locally almost of finite type. I.e., $\mathcal{Y}$ is encoded by a functor

$$(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/X})^{\text{op}} \to \text{Spc}\,.$$

In Proposition 1.2.2, we show that the data of $\mathcal{Y}$ is completely determined by its values on a much smaller category: namely,

$$({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X} \subset ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/X}$$

that consists of those $S \to X$ that are nil-isomorphisms.

Moreover, the condition that $\mathcal{Y}$ admit deformation theory can also be expressed via the resulting functor

$$(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X})^{\text{op}} \to \text{Spc}\,.$$

0.3.2. Note that when $X = \text{pt} = \text{Spec}(k)$, the category $({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X}$ is the one opposite to that of connective commutative DG algebras A over k that have finitely many cohomologies, and all of whose cohomologies are finite dimensional, with $H^0(A)$ local.

So, a formal moduli problem over pt is the same as a functor on the category of such algebras, subject to some conditions that guarantee deformation theory.

0.3.3. Suppose now that $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. In this case, we study the notion of formal moduli problem *under* X. By definition, this is an inf-scheme $\mathcal{Y}$, equipped with a nil-isomorphism

$$X \to \mathcal{Y}.$$

We show that the data of such $\mathcal{Y}$, viewed as a functor $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc}$, is recovered from its values on a smaller category, namely the category

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X} \subset ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{X/},$$

consisting of those $X \to S$ that are nil-isomorphisms.

0.3.4. Note that when $\mathcal{X} = \mathrm{pt}$, for an object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ to be a formal moduli problem under $\mathcal{X}$ simply means that $\mathcal{Y}$ is an inf-scheme with ${}^{\mathrm{red}}\mathcal{Y} = \mathrm{pt}$.

I.e., formal moduli problems under pt are the same as formal moduli problems over pt.

1. Formal moduli problems

In this section we introduce the notions of formal moduli problem *over* and *under* a given prestack $\mathcal{X}$.

Let us note the following substantial difference between our set-up and that of [Lu6] (in which the case $\mathcal{X} = \mathrm{pt}$):

In the context of *loc.cit.* a formal moduli problem is a functor on the category of connective finite-dimensional commutative DG algebras over k, whose 0-th cohomology is local.

By contrast, our formal moduli problems (for $\mathcal{X} = \mathrm{pt}$) are objects $\mathcal{Y} \in \mathrm{PreStk}_{\text{laft-def}}$ with ${}^{\mathrm{red}}\mathcal{Y} = \mathrm{pt}$, so they can be evaluated on connective commutative finite-dimensional DG algebras over k. The two notions are related by the procedures of restriction and left Kan extension; the fact that these two procedures are inverses of one another is the consequence of [Chapter III.2, Corollary 4.4.6].

1.1. Formal moduli problems *over* a prestack.

Unlike [Lu6], we define the category of formal moduli problems over a given $\mathcal{X}$ to be a full subcategory in $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$. The equivalence of this definition and the one in *loc. cit.* will be established in Sect. 1.2.

1.1.1. Let us fix $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. We define

$$\mathrm{FormMod}_{/\mathcal{X}} \subset (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$$

to be the full subcategory of spanned by those $\mathcal{Y} \to \mathcal{X}$, for which the above map is:

- inf-schematic (see [Chapter III.2, Definition 3.1.5] for what this means);
- a nil-isomorphism (i.e., ${}^{\mathrm{red}}\mathcal{Y} \to {}^{\mathrm{red}}\mathcal{X}$ is an isomorphism).

We shall refer to $\mathrm{FormMod}_{/\mathcal{X}}$ as the category of *formal moduli problems over* $\mathcal{X}$.

1.1.2. Let $\mathcal{X}'$ be an object of $\mathrm{FormMod}_{/\mathcal{X}}$. We will use the notation

$$\mathrm{FormMod}_{\mathcal{X}'/\,/\mathcal{X}}$$

to denote the category

$$(\mathrm{FormMod}_{/\mathcal{X}})_{\mathcal{X}'/}.$$

1.1.3. The following results easily from the definitions:

Lemma 1.1.4. *Let $\mathcal{Y} \to \mathcal{X}$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$. Then $\mathcal{Y} \in \mathrm{FormMod}_{/\mathcal{X}}$ if and only if for every $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, the prestack $S \underset{\mathcal{X}}{\times} \mathcal{Y}$ is an infscheme and ${}^{\mathrm{red}}(S \underset{\mathcal{X}}{\times} \mathcal{Y}) \to {}^{\mathrm{red}}S$ is an isomorphism.*

1.1.5. The following will be useful:

Lemma 1.1.6. *The functor*

$$\mathrm{FormMod}_{/\mathcal{X}} \to \lim_{(Z,x)\in(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}})^{\mathrm{op}}} \mathrm{FormMod}_{/Z}$$

is an equivalence.

1.2. **Situation over an affine scheme.** In this subsection we will assume that that $\mathcal{X} = X \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. We will show that a formal moduli problem over X, viewed as a functor

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$

is completely determined by its value on those $Z \in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X}$ for which ${}^{\mathrm{red}}Z \to {}^{\mathrm{red}}X$ is an isomorphism.

Note that when $X = \mathrm{pt}$, the category $(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}}$ is the same as that of connective finite-dimensional commutative DG algebras over k, whose 0-th cohomology is local. This brings us in contact with the defintion of formal moduli problems in [Lu6].

1.2.1. We have:

Proposition 1.2.2.

(a) *Every* $\mathcal{Y} \in \mathrm{FormMod}_{/X}$*, viewed as a functor*

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$

is the left Kan extension of its restriction to the full subcategory

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})^{\mathrm{op}} \subset (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}}. \tag{1.1}$$

(b) *Let* $\mathcal{Y}_{\text{nil-isom}}$ *be a presheaf on the category* $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X}$*, satisfying:*

- $\mathcal{Y}_{\text{nil-isom}}({}^{\mathrm{red}}X) = *$*;*
- *For a push-out diagram* $S_1 \underset{S}{\sqcup} S'$ *in* $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X}$*, where* $S \to S'$ *has a structure of square-zero extension, the resulting map*

$$\mathcal{Y}_{\text{nil-isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\text{nil-isom}}(S_1) \underset{\mathcal{Y}_{\text{nil-isom}}(S)}{\times} \mathcal{Y}_{\text{nil-isom}}(S')$$

  is an isomorphism.

Then if

$$\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$$

denotes the left Kan extension of $\mathcal{Y}_{\text{nil-isom}}$ *under* (1.1)*, then* $\mathcal{Y} \in \mathrm{FormMod}_{/X}$*.*

(c) *The assignments*

$$\mathcal{Y} \mapsto \mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X}} \text{ and } \mathcal{Y}_{\text{nil-isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

Proof. Point (a) follows from [Chapter III.2, Corollary 4.3.4].

Point (b) follows from [Chapter III.2, Proposition 4.4.5].

Point (c) follows from [Chapter III.2, Corollary 4.4.6]. □

1.2.3. The following assertion will be used extensively in [Chapter IV.3]:

Corollary 1.2.4. *For* $\mathcal{Y} \in \text{FormMod}_{/X}$*, the map*

$$\underset{(Z,f)}{\text{colim}}\, f_*^{\text{IndCoh}}(\omega_Z) \to \omega_{\mathcal{Y}}$$

is an isomorphism, where the colimit is taken over the category

$$(({}^{<\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})_{/\mathcal{Y}}.$$

Proof. By Proposition 1.2.2, the functor

$$(({}^{<\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})_{/\mathcal{Y}} \to ({}^{<\infty}\text{Sch}^{\text{aff}})_{/\mathcal{Y}}$$

is cofinal. Hence, the restriction functor

$$\text{IndCoh}(\mathcal{Y}) \to \underset{(Z,f)}{\lim}\, \text{IndCoh}(Z)$$

is an isomorphism, where the limit is taken over the category $(({}^{<\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})_{/\mathcal{Y}}$.

By [Chapter III.3, Corollary 4.3.4] and [Chapter I.1, Proposition 2.5.7], we obtain that the functors $f_*^{\text{IndCoh}} : \text{IndCoh}(Z) \to \text{IndCoh}(\mathcal{Y})$ define an equivalence

$$\underset{(Z,f)}{\text{colim}}\, \text{IndCoh}(Z) \to \text{IndCoh}(\mathcal{Y}),$$

where the colimit is taken with respect to the $(\text{IndCoh}, *)$-direct image functors.

In particular, we obtain that

$$\omega_{\mathcal{Y}} \simeq \underset{(Z,f)}{\text{colim}}\, f_*^{\text{IndCoh}} \circ f^!(\omega_{\mathcal{Y}}) \simeq \underset{(Z,f)}{\text{colim}}\, f_*^{\text{IndCoh}}(\omega_Z),$$

as required.

□

1.3. **Formal moduli problems *under* a prestack.** In this subsection we consider a prestack

$$\mathcal{X} \in \text{PreStk}_{\text{laft-def}}\,.$$

We will consider another paradigm for formal moduli problems, by looking at prestacks *under* $\mathcal{X}$.

1.3.1. We define the category $\text{FormMod}_{\mathcal{X}/}$ to be the full subcategory of $(\text{PreStk}_{\text{laft}})_{\mathcal{X}/}$ spanned by those $\mathcal{Y} \to \mathcal{X}$, for which:

- $\mathcal{Y} \in \text{PreStk}_{\text{laft-def}}$;
- The map $\mathcal{X} \to \mathcal{Y}$ is a nil-isomorphism.

Note that since in the above definition, the map $\mathcal{X} \to \mathcal{Y}$ is automatically inf-schematic, and so realizes $\mathcal{X}$ as an object of $\text{FormMod}_{/\mathcal{Y}}$.

1.3.2. Let $\mathcal{X}'$ be an object of $\text{FormMod}_{\mathcal{X}/}$. Note that the category

$$(\text{FormMod}_{\mathcal{X}/})_{/\mathcal{X}'}$$

identifies with

$$\text{FormMod}_{\mathcal{X}/\,/\mathcal{X}'}$$

from Sect. 1.1.2.

1.3.3. Note that when $\mathcal{X} = \text{pt}$, there is no difference between $\text{FormMod}_{\mathcal{X}/}$ and $\text{FormMod}_{/\mathcal{X}}$.

1.4. **Formal moduli problems *under* an affine scheme.** In this subsection we specialize to the case when

$$\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}.$$

We will show that a formal moduli problem $\mathcal{Y}$ under X, viewed as a functor

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc},$$

is completely determined by its value on the category of affine schemes Z, equipped with a nil-isomorphism $X \to Z$.

1.4.1. Let

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X} \subset ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{X/}$$

be the full subcategory formed by those $f : X \to Z$, for which f is a nil-isomorphism.

For a prestack $\mathcal{Y}$ under X, consider the functor

$$\mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} * : ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X} \to \mathrm{Spc}$$

that sends $X \to S$ to

$$\mathrm{Maps}(S,\mathcal{Y}) \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *.$$

We claim:

Proposition 1.4.2.
(a) *For $\mathcal{Y} \in \mathrm{FormMod}_{X/}$, viewed as a functor*

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc},$$

the map

$$\mathrm{LKE}_{(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X})^{\mathrm{op}} \to ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}}(\mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *) \to \mathcal{Y}$$

is an isomorphism.

(b) *Let $\mathcal{Y}_{\text{nil-isom}}$ be a presheaf on $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X}$, satisfying:*

- *$\mathcal{Y}_{\text{nil-isom}}(X) = *$;*
- *For a push-out diagram $S_1 \underset{S}{\sqcup} S'$ in $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X}$, where $S \to S'$ has a structure of square-zero extension, the resulting map*

$$\mathcal{Y}_{\text{nil-isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\text{nil-isom}}(S_1) \underset{\mathcal{Y}_{\text{nil-isom}}(S)}{\times} \mathcal{Y}_{\text{nil-isom}}(S')$$

is an isomorphism.

Then if $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ denotes the left Kan extension of $\mathcal{Y}_{\text{nil-isom}}$ under

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X})^{\mathrm{op}} \to ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}},$$

then the canonical map $X \to \mathcal{Y}$ makes $\mathcal{Y}$ into an object of $\mathrm{FormMod}_{X/}$.

(c) *The assignments*

$$\mathcal{Y} \mapsto \mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} * \text{ and } \mathcal{Y}_{\text{nil-isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

Proof. Applying [Chapter III.2, Corollary 4.3.4, Proposition 4.4.5 and Corollary 4.4.6], it is enough to evaluate the functor $\mathcal{Y}$ as in the proposition on the subcategory

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}) \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} \{{}^{\mathrm{red}}X\}.$$

The assertion of the proposition follows now from the fact that the forgetful functor

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X} \to ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}) \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} \{{}^{\mathrm{red}}X\}$$

admits a left adjoint, given by

$$S \mapsto S \underset{{}^{\mathrm{red}}X}{\sqcup} X.$$

□

1.4.3. As a corollary, we obtain:

Corollary 1.4.4. *For $\mathcal{Y} \in \mathrm{FormMod}_{X/}$, the map*

$$\underset{(Z,f)}{\mathrm{colim}}\, f^{\mathrm{IndCoh}}_*(\omega_Z) \to \omega_{\mathcal{Y}}$$

is an isomorphism, where the colimit is taken over the category

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom from } X})_{/\mathcal{Y}}.$$

Proof. Same as that of Corollary 1.2.4. □

1.5. The pointed case.

For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ we consider the category

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) = \mathrm{FormMod}_{\mathcal{X}/\,/\mathcal{X}}$$

of pointed objects in $\mathrm{FormMod}_{/\mathcal{X}}$.

1.5.1. By definition, $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ is the category of diagrams

$$(\pi : \mathcal{Y} \rightleftarrows \mathcal{X} : s), \quad \pi \circ s = \mathrm{id}$$

with the map π being an inf-schematic nil-isomorphism.

Note also that if $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$, then

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq (\mathrm{Ptd}(\mathrm{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}.$$

Combining Propositions 1.2.2 and 1.4.2, we obtain:

Corollary 1.5.2. *For $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ we have:*

(a) *Any $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, viewed as a functor*

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$

receives an isomorphism from the left Kan extension along

$$(\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X}))^{\mathrm{op}} \to (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}}$$

of

$$\mathcal{Y}_{\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X})} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *.$$

(b) *Let $\mathcal{Y}_{\text{nil-isom}}$ be a presheaf on $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X})$, satisfying:*

- $\mathcal{Y}_{\text{nil-isom}}(X) = *$;

- *For a push-out diagram* $S_1 \underset{S}{\sqcup} S'$ *in* $\operatorname{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X})$*, where* $S \to S'$ *has a structure of square-zero extension, the resulting map*
$$\mathcal{Y}_{\text{nil-isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\text{nil-isom}}(S_1) \underset{\mathcal{Y}_{\text{nil-isom}}(S)}{\times} \mathcal{Y}_{\text{nil-isom}}(S')$$
is an isomorphism.

Then if $\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$ *denotes the left Kan extension of* $\mathcal{Y}_{\text{nil-isom}}$ *along*
$$(\operatorname{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X}))^{\mathrm{op}} \to (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}},$$
then the canonical map $X \to \mathcal{Y}$ *makes* $\mathcal{Y}$ *into an object of* $\operatorname{Ptd}(\mathrm{FormMod}_{/X})$*.*

(c) *The assignments*
$$\mathcal{Y} \mapsto \mathcal{Y}|_{\operatorname{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X})} \underset{\operatorname{Maps}(X,\mathcal{Y})}{\times} * \text{ and } \mathcal{Y}_{\text{nil-isom}} \mapsto \mathcal{Y}$$
are mutually inverse equivalences of categories.

Remark 1.5.3. The informal meaning of this corollary is that in order to 'know' an object $\mathcal{Y} \in \operatorname{Ptd}(\mathrm{FormMod}_{/X})$ as a prestack, it is is enough to test it on affine schemes S, equipped with a nil-isomorphism to X *and* a section of this nil-isomorphisms.

1.5.4. As in the case of Corollary 1.2.4, from Corollary 1.5.2, we obtain:

Corollary 1.5.5. *For* $\mathcal{Y} \in \operatorname{Ptd}(\mathrm{FormMod}_{/X})$*, the map*
$$\underset{(Z,f)}{\operatorname{colim}} f_*^{\mathrm{IndCoh}}(\omega_Z) \to \omega_{\mathcal{Y}}$$
is an isomorphism, where the colimit is taken over the category
$$(\operatorname{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to } X}))_{/\mathcal{Y}}.$$

1.6. **Formal groups.** In this subsection we let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$, and we introduce formal groups over $\mathcal{X}$ as group-objects in the category of formal moduli problems.

In [Chapter IV.3] we will show that the category of formal groups identifes with that of Lie algebras in $\operatorname{IndCoh}(\mathcal{X})$, thereby generalizing the main result in [Lu6].

1.6.1. Let $\operatorname{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ be the category of monoid-objects in $\operatorname{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, and let
$$\operatorname{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \operatorname{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$$
be the full subcategory spanned by group-like objects.

Lemma 1.6.2. *The inclusion* $\operatorname{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \operatorname{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ *is an equivalence.*

Proof. We need to show that for $\mathcal{H} \in \operatorname{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ the map
$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H} \to \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H}, \quad (h_1, h_2) \mapsto (h_1, h_1 \cdot h_2)$$
is an isomorphism.

This follows from [Chapter III.1, Corollary 8.3.6], applied to the Cartesian square
$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h \mapsto (1,h)} & \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle (h_1,h_2)\mapsto(h_1,h_1\cdot h_2)} \\ \mathcal{H} & \xrightarrow{h \mapsto (1,h)} & \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H}. \end{array}$$
□

1.6.3. We have a naturally defined functor

$$\Omega_{\mathcal{X}} : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}), \quad \mathcal{Y} \mapsto \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}. \tag{1.2}$$

In Sect. 2.3.4 we will prove:

Theorem 1.6.4. *The functor* $\Omega_{\mathcal{X}}$ *of* (1.2) *is an equivalence.*

In what follows we shall denote by $B_{\mathcal{X}}$ the functor

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}),$$

inverse to $\Omega_{\mathcal{X}}$.

1.6.5. Combining Theorem 1.6.4 with [Chapter IV.3, Corollary 3.6.3] (which will be proved independently), we obtain:

Corollary 1.6.6. *The category* $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ *contains sifted colimits, and the functor*

$$(\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto T(\mathcal{X}/\mathcal{Y}) : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

preserves sifted colimits.

Remark 1.6.7. Note, however, that the forgetful functor

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}})$$

does *not* preserve sifted colimits.

1.6.8. Assume for a moment that $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$. Consider the category $\mathrm{FormMod}_{\mathcal{X}/}$. We note that even before we knew that the category $\mathrm{FormMod}_{\mathcal{X}/}$ contains sifted colimits, we could conclude that forgetful functor

$$(\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto (\mathcal{X} \to \mathcal{Y}) : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{FormMod}_{\mathcal{X}/}$$

preserves colimits. This follows from the fact that the above functor admits a right adjoint, given by

$$(\mathcal{X} \to \mathcal{Y}') \mapsto (\mathcal{X} \to \mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}').$$

2. Groupoids

In this section we introduce the notion of formal groupoid over a given object $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. We show that if $\mathcal{X}$ admits deformation theory, then there is a well-defined procedure of taking a quotient by a formal groupoid, that produces another object of $\mathrm{PreStk}_{\text{laft-def}}$.

2.1. **Digression: groupoids and groups over spaces.** In this subsection we review the definition of the notion of *groupoid* acting on a space in the category Spc.

2.1.1. Given a space X, recall the category $\mathrm{Grpoid}(X)$ of groupoids acting on X (see [Lu1], Sect. 6.1.2). By definition, $\mathrm{Grpoid}(X)$ is a full subcategory in the category of simplicial spaces, equipped with an identification of the space of 0-simplices with X. A simplicial object $R^\bullet$ belongs to $\mathrm{Grpoid}(X)$ if the following two conditions are satisfied:

- For every $n \geq 2$, the map

  $$R^n \to R^1 \underset{X}{\times} ... \underset{X}{\times} R^1,$$

  given by the product of the maps

  $$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

is an isomorphism. [1]

- The map $R^2 \to R^1 \underset{X}{\times} R^1$, given by the product of the maps $[1] \to [2]$

$$0 \mapsto 0, 1 \mapsto 1 \text{ and } 0 \mapsto 0, 1 \mapsto 2,$$

is an isomorphism.

We shall symbolically depict a groupoid via a diagram

$$\begin{array}{ccc} & R & \\ {}^{p_s}\swarrow & & \searrow^{p_t} \\ X & & X, \end{array} \tag{2.1}$$

while properly we should be thinking about the entire simplicial object $R^\bullet$ of Spc with $R^0 = X$ and $R^1 = R$.

The category $\text{Grpoid}(X)$ contains an initial object, the identity groupoid, where all $R^i = X$. We shall denote it by diag_X.

The category $\text{Grpoid}(X)$ also contains a final object, namely $X \times X$.

2.1.2. The following assertion will be used repeatedly:

Lemma 2.1.3. *The forgetful functor* $\text{Grpoid}(X) \to \text{Spc}$ *that sends a groupoid to* R^n *for any* n *preserves sifted colimits.*

Proof. Follows from the fact that if I is a sifted index category and

$$i \mapsto R_i^\bullet$$

is an I family of objects of $\text{Grpoid}(X)$, the map

$$\underset{i}{\text{colim}}\, (R_i^1 \underset{X}{\times} ... \underset{X}{\times} R_i^1) \to (\underset{i}{\text{colim}}\, R_i^1) \underset{X}{\times} ... \underset{X}{\times} (\underset{i}{\text{colim}}\, R_i^1)$$

is an isomorphism.

□

2.1.4. Given a groupoid R acting on X we can consider the quotient space

$$X/R := |R^\bullet|,$$

which receives a natural map from X:

$$X = R^0 \to |R^\bullet| \to X/R.$$

Vice versa, given a space Y under X, we construct the groupoid over X by $R := X \underset{Y}{\times} X$, i.e., $R^\bullet$ is the Čech nerve of the above map $X \to Y$.

It is clear that the two functors

$$\text{Grpoid}(X) \rightleftarrows \text{Spc}_{X/}$$

are adjoint to one another.

We have:

Lemma 2.1.5. *The above two functors define equivalences between the category* $\text{Grpoid}(X)$ *and the full subcategory* $\text{Spc}_{X/,\text{surj}}$ *of* $\text{Spc}_{X/}$ *spanned by objects* $i : X \to Y$*, for which the map* i *is surjective on* π_0.

[1] If we impose just this condition, the corresponding category is that of *Segal objects* (a.k.a. *category-objects*) acting on X, denoted $\text{Seg}(X)$.

2.1.6. Given a space X, consider the category

$$\mathrm{Grp}(\mathrm{Spc}_{/X}),$$

of group-objects in the category of spaces over X. We have:

$$\mathrm{Grp}(\mathrm{Spc}_{/X}) \simeq \mathrm{Grpoid}(X)_{/\,\mathrm{diag}_X},$$

where $\mathrm{diag}_X \in \mathrm{Grpoid}(X)$ is the identity groupoid.

Consider also the category $\mathrm{Ptd}(\mathrm{Spc}_{/X})$, which is the same as the category of of retraction diagrams

$$i : X \rightleftarrows Y : s,\ \ s \circ i \simeq \mathrm{id}_X\,. \tag{2.2}$$

We will also use the notation

$$\mathrm{Spc}_{X//X}$$

for the above category.

We have a natural functor

$$\mathrm{Grp}(\mathrm{Spc}_{/X}) \to \mathrm{Ptd}(\mathrm{Spc}_{/X}), \tag{2.3}$$

given by

$$G \mapsto B_X(G),$$

where $B_X(G)$ is the relative classifying space over X.

We also have the adjoint loop functor

$$\Omega_X : \mathrm{Ptd}(\mathrm{Spc}_{/X}) \to \mathrm{Grp}(\mathrm{Spc}_{/X}).$$

Lemma 2.1.7. *The functors B_X and Ω_X define an equivalence between $\mathrm{Grp}(\mathrm{Spc}_{/X})$ and the full subcategory $\mathrm{Ptd}(\mathrm{Spc}_{/X})_{\mathrm{isom}}$ of $\mathrm{Ptd}(\mathrm{Spc}_{/X})$, spanned by those objects*

$$i : X \rightleftarrows Y : s$$

for which the map i is an isomorphism on π_0.

Note that the condition on an object $(i : X \rightleftarrows Y : s) \in \mathrm{Ptd}(\mathrm{Spc}_{/X})$ to belong to

$$\mathrm{Ptd}(\mathrm{Spc}_{/X})_{\mathrm{isom}} \subset \mathrm{Ptd}(\mathrm{Spc}_{/X})$$

is equivalent to the map i being a surjection on π_0.

2.2. **Groupoids in formal geometry.** In this subsection we render the set-up of Sect. 2.1 into the context of algebraic geometry.

2.2.1. The definitions of Sect. 2.1 carry over automatically to the algebro-geometric setting. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, we consider the categories

$$(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}},\ \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}),\ \mathrm{Grp}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}),\ \mathrm{Grpoid}_{\mathrm{laft}}(\mathcal{X}),\ \text{ and } \mathrm{Seg}_{\mathrm{laft}}(\mathcal{X})$$

and their full subcategories

$$\mathrm{FormMod}_{/\mathcal{X}},\ \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}),\ \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}),\ \mathrm{FormGrpoid}(\mathcal{X}),\ \text{ and } \mathrm{FormSeg}(\mathcal{X})$$

formed by objects that are formal as prestacks over $\mathcal{X}$.

Note, however, that as in Lemma 1.6.2, one shows that the inclusion

$$\mathrm{FormGrpoid}(\mathcal{X}) \hookrightarrow \mathrm{FormSeg}(\mathcal{X})$$

is an equivalence.

2.2.2. *Examples.* Let $\mathcal{X} \to \mathcal{Y}$ be a map between objects of $\mathrm{PreStk}_{\mathrm{laft}}$. To it we attach the object of $\mathrm{Grpoid}_{\mathrm{laft}}(\mathcal{X})$, namely, the Čech nerve of this map. Thus, the corresponding

$$\mathcal{R} \rightrightarrows \mathcal{X}$$

is given by

$$\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}.$$

If the above map $\mathcal{X} \to \mathcal{Y}$ is an inf-schematic nil-isomorphism, then $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$.

2.2.3. As in Lemma 2.1.3, we obtain:

Corollary 2.2.4. *The category* $\mathrm{FormGrpoid}(\mathcal{X})$ *contains sifted colimits, and the functors*

$$\mathrm{FormGrpoid}(\mathcal{X}) \to \mathrm{FormMod}_{\mathcal{X}/}, \quad \mathrm{FormGrpoid}(\mathcal{X}) \rightrightarrows \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

and

$$\mathrm{FormGrpoid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$

that send a groupoid to

$$(\mathcal{X} \overset{\mathrm{unit}}{\to} \mathcal{R}), \quad (\mathcal{X} \underset{p_s}{\overset{\mathrm{unit}}{\rightleftarrows}} \mathcal{R}), \quad (\mathcal{X} \underset{p_t}{\overset{\mathrm{unit}}{\rightleftarrows}} \mathcal{R}),$$

and $T(\mathcal{X}/\mathcal{R})$*, respectively, commute with sifted colimits.*

Proof. Let I be a filtered index category and

$$i \mapsto \mathcal{R}_i^\bullet$$

be an I-family of objects of $\mathrm{FormGrpoid}(\mathcal{X})$. It follows from Corollary 1.6.6, Sect. 1.6.8 and [Chapter III.1, Proposition 8.3.2] that for any n, the colimit

$$\underset{i}{\mathrm{colim}}\, (\mathcal{R}_i^1 \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} \mathcal{R}_i^1)$$

exists and the map

$$\underset{i}{\mathrm{colim}}\, (\mathcal{R}_i^1 \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} \mathcal{R}_i^1) \to (\underset{i}{\mathrm{colim}}\, \mathcal{R}_i^1) \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} (\underset{i}{\mathrm{colim}}\, \mathcal{R}_i^1)$$

is an isomorphism.

□

2.2.5. *Ind-coherent sheaves equivariant with respect to a groupoid.* Let $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$; let $\mathcal{R}$ be an object of $\mathrm{FormGrpoid}(\mathcal{X})$, and let $\mathcal{R}^\bullet$ be the corresponding simplicial object of $\mathrm{PreStk}_{\mathrm{laft}}$. We define the category

$$\mathrm{IndCoh}(\mathcal{X})^{\mathcal{R}}$$

of *ind-coherent sheaves equivariant with respect to a* $\mathcal{R}$ to be

$$\mathrm{Tot}(\mathrm{IndCoh}(\mathcal{R}^\bullet)).$$

By [Chapter III.3, Proposition 3.3.3(b)], we have:

Proposition 2.2.6. *Let* $\mathcal{R}$ *be the formal groupoid corresponding to a map* $\mathcal{X} \to \mathcal{Y}$ *in* $\mathrm{PreStk}_{\mathrm{laft}}$*, which is an inf-schematic nil-isomorphism* $\mathcal{X} \to \mathcal{Y}$*. Then the pullback functor defines an equivalence*

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathcal{R}}.$$

2.3. **Taking the quotient by a formal groupoid.** In this subsection we state one of the main results of this book: namely that in the world of prestacks admitting deformation theory there is a well-defined procedure of taking the quotient by a formal groupoid.

2.3.1. Assume that $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$. Recall the category $\mathrm{FormMod}_{\mathcal{X}/}$, see Sect. 1.3.1. We have a naturally defined functor

$$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{FormGrpoid}(\mathcal{X}), \tag{2.4}$$

namely, $\mathcal{Y} \mapsto \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$, see Sect. 2.2.2.

The main result of this section is the following:

Theorem 2.3.2. *The functor* (2.4) *is an equivalence.*

2.3.3. *An example.* Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\text{laft-def}}$. Consider the formal completion $\mathcal{Y}^\wedge_{\mathcal{X}}$ of $\mathcal{Y}$ along $\mathcal{X}$, i.e.,

$$\mathcal{Y}^\wedge_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}.$$

Then the map $\mathcal{X} \to \mathcal{Y}^\wedge_{\mathcal{X}}$ defines an object of $\mathrm{FormMod}_{\mathcal{X}/}$.

Consider the groupoid

$$\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X},$$

(see Sect. 2.2.2) and its formal completion along the diagonal map,

$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^\wedge.$$

It is easy to see that

$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^\wedge \simeq \mathcal{X} \underset{\mathcal{Y}^\wedge_{\mathcal{X}}}{\times} \mathcal{X},$$

where the latter is an object of $\mathrm{FormGrpoid}(\mathcal{X})$ by Sect. 2.2.2.

Thus, the formal completion $\mathcal{Y}^\wedge_{\mathcal{X}}$ can be recovered from $(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^\wedge$ by taking the functor inverse to that in Theorem 2.3.2.

2.3.4. Note that Theorem 2.3.2 implies Theorem 1.6.4:

Proof. To prove Theorem 1.6.4 we can assume that $\mathcal{X} \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. In particular, we can assume that $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$. Then the required assertion follows from Theorem 2.3.2 by noting that

$$\mathrm{Grp}(\mathrm{FormMod}_{\mathcal{X}}) = \mathrm{FormGrpoid}(\mathcal{X})_{/\,\mathrm{diag}_{\mathcal{X}}},$$

and

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) = (\mathrm{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}.$$

□

2.3.5. As another formal consequence of Theorem 2.3.2, combined with Corollary 2.2.4, we obtain:

Corollary 2.3.6. *The category* $\mathrm{FormMod}_{\mathcal{X}/}$ *contains sifted colimits, and the functor*

$$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{IndCoh}(\mathcal{X}), \quad (\mathcal{X} \to \mathcal{Y}) \mapsto T(\mathcal{X}/\mathcal{Y})$$

commutes with sifted colimits.

We emphasize again that the forgetful functor

$$\mathrm{FormMod}_{\mathcal{X}/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$

does *not* commute with sifted colimits.

However, from [Chapter III.1, Corollary 7.2.8], we obtain:

Corollary 2.3.7. *The forgetful functor*

$$\mathrm{FormMod}_{\mathcal{X}/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$

commutes with filtered *colimits.*

2.4. **Constructing the classifying space of a groupoid.** In this subsection we will begin the proof of Theorem 2.3.2. In fact, we will explicitly construct the inverse functor.

2.4.1. For $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$ we define an object $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\mathrm{laft}}$ as follows:

For $Z \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we let $\mathrm{Maps}(Z, B_{\mathcal{X}}(\mathcal{R}))$ be the groupoid consisting of the following data:

$$\{(\widetilde{\mathcal{Z}} \to Z) \in \mathrm{FormMod}_{/Z},\ \widetilde{\mathcal{Z}} \to \mathcal{X},\ \text{a map of groupoids } \widetilde{\mathcal{Z}} \underset{Z}{\times} \widetilde{\mathcal{Z}} \to \mathcal{R}\},$$

where we require that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Z}} \underset{Z}{\times} \widetilde{\mathcal{Z}} & \longrightarrow & \mathcal{R} \\ \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{Z}} & \longrightarrow & \mathcal{X} \end{array}$$

be Cartesian, where the vertical arrows are either of the projections.

2.4.2. We have a tautological map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ that sends $Z \to \mathcal{X}$ to

$$\widetilde{\mathcal{Z}} := Z \underset{\mathcal{X}}{\times} \mathcal{R},$$

where the fiber product $Z \underset{\mathcal{X}}{\times} \mathcal{R}$ is formed using the map $p_1 : \mathcal{R} \to \mathcal{X}$, and the map $Z \underset{\mathcal{X}}{\times} \mathcal{R} \to \mathcal{X}$ corresponds to $p_2 : \mathcal{R} \to \mathcal{X}$.

2.4.3. Let us show that the map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ makes $\mathcal{X}$ into an object of $\mathrm{FormMod}_{/B_{\mathcal{X}}(\mathcal{R})}$. Indeed, for a given map $Z \to B_{\mathcal{X}}(\mathcal{R})$, the fiber product

$$Z \underset{B_{\mathcal{X}}(\mathcal{R})}{\times} \mathcal{X}$$

identifies with $\widetilde{\mathcal{Z}}$.

The latter observation also implies that

$$\mathcal{X} \underset{B_{\mathcal{X}}(\mathcal{R})}{\times} \mathcal{X} \simeq \mathcal{R}. \tag{2.5}$$

2.4.4. We claim that it suffices to show that the functor $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\text{laft-def}}$. Indeed, let us assume this for a moment and conclude the proof of the theorem.

First, (2.5) implies that the functor

$$\mathcal{R} \mapsto B_{\mathcal{X}}(\mathcal{R})$$

is the right inverse to the functor (2.4).

For $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$ we have a tautological map

$$\mathcal{Y} \mapsto B_{\mathcal{X}}(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}), \tag{2.6}$$

which becomes an isomorphism after applying the functor (2.4). However, it follows from [Chapter III.1, Proposition 8.3.2] that the functor (2.4) is conservative.

2.5. **Verification of deformation theory.** In this subsection we will prove that the object $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\mathrm{laft}}$ constructed in Sect. 2.4.1, admits deformation theory.

2.5.1. Let Z be an object of $Z \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, equipped with a map to $B_{\mathcal{X}}(\mathcal{R})$. We will now construct a certain object of $\mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}}$, which we will later identify with the pro-cotangent space to $B_{\mathcal{X}}(\mathcal{R})$ at our given point $Z \to B_{\mathcal{X}}(\mathcal{R})$.

Consider the Čech nerve $\widetilde{\mathcal{Z}}^\bullet$ of the corresponding map $\widetilde{\mathcal{Z}} \to Z$, and consider the resulting map of simplicial prestacks

$$\widetilde{\mathcal{Z}}^\bullet \to \mathcal{R}^\bullet.$$

Let

$$T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \in \mathrm{Tot}\left(\mathrm{Pro}(\mathrm{QCoh}(\widetilde{\mathcal{Z}}^\bullet)^-)^{\mathrm{fake}}_{\mathrm{laft}}\right)$$

be the corresponding relative pro-cotangent complex (see [Chapter III.1, Sect. 4.3.1]), which receives a canonically defined map from the pullback of $T^*(Z)$.

By nil-descent for $\mathrm{Pro}(\mathrm{QCoh}(-)^-)^{\mathrm{fake}}_{\mathrm{laft}}$ with respect to $\widetilde{\mathcal{Z}} \to Z$ (see [Chapter III.3, Corollary 3.3.5]), we obtain that $T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet)$ gives rise to a canonically defined object, denoted,

$$'T^*(Z/B_{\mathcal{X}}(\mathcal{R})) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}},$$

which receives a map from $T^*(Z)$. Set

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z := \mathrm{Fib}\left(T^*(Z) \to {}'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))\right).$$

We will show that the above object

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}},$$

identifies with the pro-cotangent space of $B_{\mathcal{X}}(\mathcal{R})$ at the above point $Z \to B_{\mathcal{X}}(\mathcal{R})$.

2.5.2. We need to show that, given a square-zero extension $Z \hookrightarrow Z'$, corresponding to

$$\gamma : T^*(Z) \to \mathcal{I}[1], \quad \mathcal{I} \in \mathrm{Coh}(Z)^{\leq 0},$$

the groupoid of extensions of the initial map $Z \to B_{\mathcal{X}}(\mathcal{R})$ to a map $Z' \to B_{\mathcal{X}}(\mathcal{R})$, identifies canonically with groupoid of factorizations of γ as

$$T^*(Z) \to {}'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \to \mathcal{I}[1].$$

This will show that $B_{\mathcal{X}}(\mathcal{R})$ admits pro-cotangent spaces that are indeed identified with ones constructed in Sect. 2.5.1, and that $B_{\mathcal{X}}(\mathcal{R})$ is infinitesimally cohesive. The fact that $B_{\mathcal{X}}(\mathcal{R})$ admits a pro-cotangent complex (i.e., that the formation of pro-cotangent spaces is compatible with pullback) will follow from the construction in Sect. 2.5.1.

2.5.3. For $Z \hookrightarrow Z'$ as above, by [Chapter III.1, Proposition 10.3.5], the datum of a prestack $\mathcal{Z}' \to Z'$, equipped with a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Z}} & \longrightarrow & \widetilde{\mathcal{Z}}' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \end{array}$$

is equivalent to that of a map $T^*(\widetilde{\mathcal{Z}}) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}}[1]$ (in the category $\mathrm{Pro}(\mathrm{QCoh}(\widetilde{\mathcal{Z}})^-)^{\mathrm{fake}}_{\mathrm{laft}}$), and a homotopy between the composition

$$T^*(Z)|_{\widetilde{\mathcal{Z}}} \to T^*(\widetilde{\mathcal{Z}}) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}}[1]$$

and $\gamma|_{\widetilde{\mathcal{Z}}}$. Moreover, by [Chapter III.1, Proposition 10.4.2], such $\mathcal{Z}'$ is automatically an inf-scheme.

The same discussion applies to each term of the Čech nerve $\widetilde{\mathcal{Z}}^\bullet$.

Furthermore, by [Chapter III.1, Proposition 10.2.6] the datum of a compatible system of maps from the Čech nerve $\widetilde{\mathcal{Z}}'^\bullet$ to $\mathcal{R}^\bullet$, extending the initial system $\widetilde{\mathcal{Z}}^\bullet \to \mathcal{R}^\bullet$, is equivalent to a compatible system of factorizations of the resulting maps

$$T^*(\widetilde{\mathcal{Z}}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1]$$

as

$$T^*(\widetilde{\mathcal{Z}}^\bullet) \to T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1].$$

2.5.4. Hence, we obtain that the datum of extension of the initial map $Z \to B_{\mathcal{X}}(\mathcal{R})$ to a map $Z' \to B_{\mathcal{X}}(\mathcal{R})$ is equivalent to that of a compatible family of maps

$$T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1],$$

and homotopies between

$$T^*(Z)|_{\widetilde{\mathcal{Z}}^\bullet} \to T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1]$$

and $\gamma|_{\widetilde{\mathcal{Z}}^\bullet}$.

By nil-descent for $\mathrm{Pro}(\mathrm{QCoh}(-)^-)^{\mathrm{fake}}_{\mathrm{laft}}$ with respect to $\widetilde{\mathcal{Z}} \to Z$, the latter datum is equivalent to that of factorizations of γ as

$$T^*(Z) \to {}'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \to \mathcal{I}[1],$$

as desired.