PART IV.3. FORMAL GROUPS AND LIE ALGEBRAS

CONTENTS

Introduction 1
1. Formal moduli problems and co-algebras 2
1.1. Co-algebras associated to formal moduli problems 2
1.2. The monoidal structure 3
1.3. The functor of inf-spectrum 5
1.4. An example: vector groups 6
2. Inf-affineness 9
2.1. The notion of inf-affineness 9
2.2. Inf-affineness and co-spectrum 10
2.3. A criterion for being inf-affine 10
3. From formal groups to Lie algebras 12
3.1. The exponential construction 12
3.2. Corollaries of Theorem 3.1.4 12
3.3. Lie algebras and formul moduli problems 13
3.4. The ind-nilpotent version 15
3.5. Base change 16
3.6. An example: split square-zero extensions 17
4. Proof of Theorem 3.1.4 19
4.1. Step 1 19
4.2. Step 2 20
4.3. Step 3 21
5. Modules over formal groups and Lie algebras 23
5.1. Modules over formal groups 23
5.2. An inverse construction 24
5.3. Relation to nil-isomorphisms 25
6. Actions of groups on formal moduli problems 26
6.1. Action of groups vs. Lie algebras 26
6.2. Proof of Theorem 6.1.5 27
6.3. Localization of Lie algebra modules 28
References 31

INTRODUCTION

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1. Formal moduli problems and co-algebras

Our goal in this chapter is to address the following old question: what is exactly the relationship between formal groups and Lie algebras. By a Lie group we understand an object of Grp(FormMod/X), where X is a prestack, and by a Lie algebra an object of LieAlg(IndCoh(X)).

The first step in doing this is the passage between objects of Ptd(FormMod/X) and co-commutative co-algebras in IndCoh(X), and this is what we are doing in this and the next section.

1.1. Co-algebras associated to formal moduli problems. To any scheme (affine or not) we can attach the commutative algebra of global sections of its structure sheaf. This functor is, obviously, contravariant.

It turns out that formal moduli problems are adapted for a dual operation: we send a moduli problem to the co-algebra of sections of its dualizing sheaf. In this subsection we describe this construction.

1.1.1. Let X be an object of ∞Sch_{aff}. We regard the category IndCoh(X) as endowed with the symmetric monoidal structure, given by ⊗.

Recall the category FormMod/X. This this a symmetric monoidal category under the operation of Cartesian product.

The operation of direct image on IndCoh (see [Book-III.3, Proposition 3.1.2(a)]) defines a functor

\[ \text{Distr} : \text{FormMod}_X \rightarrow \text{IndCoh}(X) \]

that sends \( \pi : \mathcal{Y} \rightarrow X \) to

\[ \text{Distr}(\mathcal{Y}) := \pi^\ast_{\text{IndCoh}}(\omega_{\mathcal{Y}}). \]

The functor Distr has a natural symmetric monoidal structure, see [Book-III.3, Corollary 6.1.2].

1.1.2. Note now that the diagonal map upgrades the identity functor on FormMod/X to a functor

\[ \text{FormMod}_X \rightarrow \text{Cocom}(\text{FormMod}_X). \]

Hence, the functor Distr automatically upgrades to a functor

\[ \text{Distr}^\ast_{\text{Cocom}} : \text{FormMod}_X \rightarrow \text{CocomCoalg}(\text{IndCoh}(X)), \]

where CocomCoalg(IndCoh(X)) is the category of co-commutative co-algebras in IndCoh(X).

1.1.3. Similarly, the functor Distr defines a functor

\[ \text{Distr}^\ast_{\text{aug}} : \text{Ptd}(\text{FormMod}_X) \rightarrow \text{IndCoh}(X)_{\omega_X}/, \]

and Distr^\ast_{\text{Cocom}} defines a functor

\[ \text{Distr}^\ast_{\text{Cocom}}^\ast_{\text{aug}} : \text{Ptd}(\text{FormMod}_X) \rightarrow \text{CocomCoalg}^\ast_{\text{aug}}(\text{IndCoh}(X)), \]

where

\[ \text{CocomCoalg}^\ast_{\text{aug}}(\text{IndCoh}(X)) \simeq \text{CocomCoalg}(\text{IndCoh}(X))_{\omega_X}/ \]

is the category of augmented co-commutative co-algebras in IndCoh(X).

We shall denote by Distr^+ the functor Ptd(FormMod/X) \rightarrow IndCoh(X) that sends \( \mathcal{Y} \) to

\[ \ker(\omega_X \rightarrow \text{Distr}(\mathcal{Y})) \simeq \text{Cone}(\text{Distr}(\mathcal{Y}) \rightarrow \omega_X). \]
1.1.4. The following observation will be useful:

**Lemma 1.1.5.**

(a) The functors

\[ \text{Distr} : \text{FormMod}_X \rightarrow \text{IndCoh}(X) \]

and

\[ \text{Distr}^{\text{Cocom}} : \text{FormMod}_X \rightarrow \text{CocomCoalg(IndCoh}(X)) \]

are left Kan extensions of their restrictions to

\[ (\langle <\infty \text{Sch}\text{aff}_{\text{nil-isom to } X} \rangle) \subset \text{FormMod}_X. \]

(b) The functors

\[ \text{Distr}^{\text{aug}} : \text{Ptd(FormMod}_X \rightarrow \text{IndCoh}(X)_{\omega_X/} \]

and

\[ \text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd(IndCoh}_X) \rightarrow \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)), \]

are left Kan extensions of their restrictions to

\[ \text{Ptd}((\langle <\infty \text{Sch}\text{aff}_{\text{nil-isom to } X} \rangle) \subset \text{Ptd(IndCoh}_X). \]

**Proof.** We prove point (b), since point (a) is similar (and simpler).

Since the forgetful functor

\[ \text{oblv}^{\text{Cocom}} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \rightarrow \text{IndCoh}(X)_{\omega_X/} \]

commutes with colimits, it suffices to prove the assertion for the functor

\[ \text{Distr}^{\text{aug}} : \text{Ptd(IndCoh}_X) \rightarrow \text{IndCoh}(X)_{\omega_X/}. \]

The required assertion follows from [Book-IV.2, Corollary 1.5.5], combined with the fact that the category \( \text{Ptd}((\langle <\infty \text{Sch}\text{aff}_{\text{nil-isom to } X} \rangle) \) is contractible (so that colimit in \( \text{IndCoh}(X)_{\omega_X/} \) maps isomorphically to the corresponding colimit in \( \text{IndCoh}(X) \)).

\[ \square \]

**Remark 1.1.6.** Recall (see [Book-IV.1, Sect. D.2]) that for a DG category \( O \), in addition to the category \( \text{Cocom}^{\text{aug}}(O) \), one can consider the category \( \text{Cocom}^{\text{aug}, \text{ind-nilp}} \) of ind-nilpotent commutative coalgebras. This category is endowed with a forgetful functor

\[ \text{res}^{\text{aug}, \text{ind-nilp}} : \text{Cocom}^{\text{aug}, \text{ind-nilp}} \rightarrow \text{Cocom}^{\text{aug}}(O). \]

Using Lemma 1.1.5, one can refine the above functor

\[ \text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd(IndCoh}_X) \rightarrow \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \]

to a functor

\[ \text{Distr}^{\text{Cocom}^{\text{aug}, \text{ind-nilp}}} : \text{Ptd(IndCoh}_X) \rightarrow \text{Cocom}^{\text{aug}, \text{ind-nilp}}(\text{IndCoh}(X)). \]

Namely, for \( Z \in \text{Ptd}((\langle <\infty \text{Sch}\text{aff}_{\text{nil-isom to } X} \rangle) \), the t-structure allows to naturally upgrade the object

\[ \text{Distr}^{\text{Cocom}^{\text{aug}}}(Z) \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \]

to an object

\[ \text{Distr}^{\text{Cocom}^{\text{aug}, \text{ind-nilp}}}(Z) \in \text{Cocom}^{\text{aug}, \text{ind-nilp}}(\text{IndCoh}(X)). \]

1.2. The monoidal structure.
1.2.1. Note that the Cartesian product on $\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X))$ is given by the operation of tensor product. This gives $\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X))$ a symmetric monoidal structure.

Hence, the functor
\[
\text{Distr}^{\text{Cocom}_{\text{aug}}} : \text{Ptd}(\text{FormMod}/X) \to \text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X))
\]
has a natural left-lax symmetric monoidal structure, where $\text{Ptd}(\text{FormMod}/X)$ is regarded as a symmetric monoidal category also with respect to the operation of Cartesian product.

**Lemma 1.2.2.** The left-lax symmetric monoidal structure on the functor $\text{Distr}^{\text{Cocom}_{\text{aug}}}$ is symmetric monoidal.

**Proof.** Follows from the corresponding fact for $\text{Distr}$. □

1.2.3. Recall the notation
\[
\text{CocomBialg}(\text{IndCoh}(X)) := \text{AssocAlg}(\text{CocomCoalg}(\text{IndCoh}(X))) \simeq \text{AssocAlg}(\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X)));
\]
this is the category of associative algebras in $\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X))$.

Recall also that
\[
\text{CocomHopf}(\text{IndCoh}(X)) \subset \text{CocomBialg}(\text{IndCoh}(X))
\]
denotes the full subcategory spanned by group-like objects.

1.2.4. By Lemma 1.2.2, the functor $\text{Distr}^{\text{Cocom}_{\text{aug}}}$ gives rise to a functor
\[
\text{Grp}(\text{FormMod}/X) \simeq \text{Monoid}(\text{Ptd}(\text{FormMod}/X)) \to \text{AssocAlg}(\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X))) = \text{CocomBialg}(\text{IndCoh}(X)),
\]
which in fact factors through
\[
\text{CocomHopf}(\text{IndCoh}(X)) \subset \text{CocomBialg}(\text{IndCoh}(X)).
\]
We denote the resulting functors by $\text{Grp}(\text{Distr}^{\text{Cocom}_{\text{aug}}})$.

1.2.5. The following will be useful in the sequel:

**Lemma 1.2.6.** Let $\mathcal{H}$ be an object of $\text{Grp}(\text{FormMod}/X)$. Then the canonical map
\[
\text{Bar} \circ \text{Grp}(\text{Distr}^{\text{Cocom}_{\text{aug}}})(\mathcal{H}) \to \text{Distr}^{\text{Cocom}_{\text{aug}}} \circ B_X(\mathcal{H})
\]
is an isomorphism.

**Proof.** It is enough to establish the isomorphism in question after applying the forgetful functor $\text{CocomCoalg}_{\text{aug}}(\text{IndCoh}(X)) \to \text{IndCoh}(X)$.

The left-hand side is the geometric realization of the simplicial object of $\text{IndCoh}(X)$ given by
\[
\text{Bar}^\bullet(\text{Distr}(\mathcal{H})) \simeq \text{Distr}(B_X^\bullet(\mathcal{H})).
\]
We can think of $B_X^\bullet(\mathcal{H})$ as the Čech nerve of the map $X \to B_X(\mathcal{H})$. Hence, by [Book-III.3, Proposition 3.3.2(b)], the map
\[
| \text{Distr}(B_X^\bullet(\mathcal{H})) | \to \text{Distr}(B_X(\mathcal{H}))
\]
is an isomorphism, as required. □
1.3. The functor of inf-spectrum. Continuing the parallel with usual algebraic geometry, the functor Spec provides a right adjoint to the functor

\[ \text{Sch} \to (\text{ComAlg(Vect)})^{\text{op}}. \]

In this subsection we will develop its analog for formal moduli problems. This will be a functor, denoted \( \text{Spec}^{\text{inf}} \), right adjoint to \( \text{Distr}^{\text{Cocom}^{\text{aug}}}: \text{Ptd}((\text{FormMod}/X) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)).\]

1.3.1. Starting from \( A \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \), we first define a presheaf on the category \( \text{Ptd}((\text{Sch})_{\text{aff}})_{\text{nil-isom to } X} \), denoted \( \text{Spec}^{\text{inf}}(A)_{\text{nil-isom}} \), by

\[ \text{Maps}(Z, \text{Spec}^{\text{inf}}(A)_{\text{nil-isom}}) := \text{Maps}^{\text{CocomCoalg}^{\text{aug}}}_{\text{IndCoh}(X)}(\text{Distr}^{\text{Cocom}^{\text{aug}}}(Z), A). \]

Let \( \text{Spec}^{\text{inf}}(A) \in (\text{PreStk}_{\text{aff}})/X \) be the left Kan extension of \( \text{Spec}^{\text{inf}}(A)_{\text{nil-isom}} \) under the forgetful functor

\[ \text{Ptd}((\text{Sch})_{\text{aff}})_{\text{nil-isom to } X}^{\text{op}} \to (\text{PreStk}_{\text{aff}})/X^{\text{op}}. \]

We claim that \( \text{Spec}^{\text{inf}}(A) \) is an object of \( \text{Ptd}((\text{FormMod}/X) \), whose restriction under (1.1)

identifies with \( \text{Spec}^{\text{inf}}(A)_{\text{nil-isom}} \).

Indeed, this follows from [Book-IV.2, Corollary 1.5.2(b,c)] and the following assertion:

**Lemma 1.3.2.** Let \( Z'_2 := Z'_1 \sqcup_{Z_1} Z_2 \) be a push-out diagram in \( \text{Ptd}((\text{Sch})_{\text{aff}})_{\text{nil-isom to } X} \), where the map \( Z_1 \to Z'_1 \) is a closed embedding. Then the canonical map

\[ \text{Distr}^{\text{Cocom}^{\text{aug}}}(Z'_1) \sqcup_{\text{Distr}^{\text{Cocom}^{\text{aug}}}(Z'_1)} \text{Distr}^{\text{Cocom}^{\text{aug}}}(Z_2) \to \text{Distr}^{\text{Cocom}^{\text{aug}}}(Z'_2) \]

is an isomorphism.

**Proof.** Since the forgetful functor

\[ \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \to \text{IndCoh}(X)_{\omega X}/ \]

commutes with colimits, it is sufficient to show that the map

\[ \text{Distr}^{\text{aug}}(Z'_1) \sqcup_{\text{Distr}^{\text{aug}}(Z'_1)} \text{Distr}^{\text{aug}}(Z_2) \to \text{Distr}^{\text{aug}}(Z'_2) \]

is an isomorphism in \( \text{IndCoh}(X)_{\omega X}/ \). By Serre duality, this is equivalent to showing that

\[ (\pi'_2)_*(\mathcal{O}_{Z'_2}) \to (\pi'_1)_*(\mathcal{O}_{Z'_1}) \times_{(\pi_1)_*(\mathcal{O}_{Z_1})} (\pi_2)_*(\mathcal{O}_{Z_2}) \]

is an isomorphism in \( \text{QCoh}(X) \), and the latter follows from the assumptions.

1.3.3. We now claim that the assignment

\[ A \mapsto \text{Spec}^{\text{inf}}(A) \]

provides a right adjoint to the functor \( \text{Distr}^{\text{Cocom}^{\text{aug}}} \). Indeed, this follows from [Book-IV.2, Corollaries 1.5.2(a) and 1.5.5].

1.3.4. The following assertion is tautological:

**Lemma 1.3.5.** For \( A \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \), there is a canonical isomorphism

\[ T(\text{Spec}^{\text{inf}}(A)/X)|_X \simeq \text{Prim}(A). \]
1.3.6. Being a right adjoint, the functor Spec$^\text{inf}$ is automatically right-lax symmetric monoidal. Hence, it gives rise to a functor

$$\text{CocomBialg}(\text{IndCoh}(X)) := \text{AssocAlg} \left( \text{CocomCoalg}(\text{IndCoh}(X)) \right) \simeq$$

$$\simeq \text{AssocAlg} \left( \text{CocomCoalg}^\text{aug}(\text{IndCoh}(X)) \right) \to \text{Monoid}(\text{Ptd}(\text{FormMod}_X))$$

and in particular,

$$\text{CocomHopf}(\text{IndCoh}(X)) \to \text{Grp}(\text{FormMod}_X).$$

We shall denote the above functors by Monoid(Spec$^\text{inf}$).

Remark 1.3.7. If instead of the category CocomCoalg$^\text{aug}(\text{IndCoh}(X))$ one works with the category Cocom$^\text{aug,ind-nilp}(\text{IndCoh}(X))$, one obtains a functor

$$\text{Spec}^\text{inf,ind-nilp} : \text{Cocom}^\text{aug,ind-nilp}(\text{IndCoh}(X)) \to \text{Ptd}(\text{FormMod}_X).$$

However, the functors Spec$^\text{inf,ind-nilp}$ and Spec$^\text{inf}$ carry the same information: it follows formally that the functor Spec$^\text{inf}$ factors as the composition

$$\text{CocomCoalg}^\text{aug}(\text{IndCoh}(X)) \to \text{Cocom}^\text{aug,ind-nilp}(\text{IndCoh}(X)) \xrightarrow{\text{Spec}^\text{inf,ind-nilp}} \text{Ptd}(\text{FormMod}_X),$$

where the first arrow is the right adjoint to the forgetful functor

$$\text{res}^* : \text{Cocom}^\text{aug,ind-nilp}(\text{IndCoh}(X)) \to \text{CocomCoalg}^\text{aug}(\text{IndCoh}(X)).$$

Furthermore, it follows from [Book-IV.1, Corollary E.2.2(b)], applied in the case of the cooperad Cocom, and Lemma 1.3.5 that the natural map from Spec$^\text{inf,ind-nilp}$ to the composition

$$\text{Cocom}^\text{aug,ind-nilp}(\text{IndCoh}(X)) \xrightarrow{\text{res}^*} \text{CocomCoalg}^\text{aug}(\text{IndCoh}(X)) \xrightarrow{\text{Spec}^\text{inf}} \text{Ptd}(\text{FormMod}_X)$$

is an isomorphism.

1.4. An example: vector groups. A basic example of a scheme is the scheme attached to a finite-dimensional vector space:

$$\text{Maps}(X, V) = \Gamma(X, \mathcal{O}_X) \otimes V.$$

In this subsection we describe the counterpart of this construction for formal moduli problems.

1.4.1. Let $\mathcal{F}$ be an object of IndCoh($X$) and consider the object

$$\text{Sym}(\mathcal{F}) \in \text{CocomCoalg}^\text{aug}(\text{IndCoh}(X)),$$

where we remind that the monoidal structure on IndCoh($X$) is given by the !-tensor product. See [Book-IV.1, Sect. 1.7.2] for the notation Sym.

Consider the corresponding object

$$\text{Vect}_X(\mathcal{F}) := \text{Spec}^\text{inf}(\text{Sym}(\mathcal{F})) \in \text{Ptd}(\text{FormMod}_X).$$
1.4.2. We claim:

**Proposition 1.4.3.** For $Z \in \text{Ptd}(\langle \leq \infty \rangle_{\text{Sch}_{\text{aff}}})_{\text{nil-isom to } X}$ the natural map

$$\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, \text{Vect}_X(\mathcal{F})) \to \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(Z), \mathcal{F}),$$

given by the projection $\text{Sym}(\mathcal{F}) \to \mathcal{F}$, is an isomorphism.

We remind that the notation $\text{Distr}^+$ was introduced in Sect. 1.1.2.

**Proof.** First, we note that the presheaf on $\text{Ptd}(\langle \leq \infty \rangle_{\text{Sch}_{\text{aff}}})_{\text{nil-isom to } X}$, given by

$$Z \mapsto \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(Z), \mathcal{F}),$$

gives rise to an object of $\text{Ptd}(\text{FormMod}/X)$ for the same reason as $\text{Spec}^{\text{inf}}$ does. Denote this object by $\text{Vect}'_X(\mathcal{F}).$

Hence, in order to prove that the map in question is an isomorphism, by [Book-III.1, Proposition 8.3.2], its suffices to show that the map

$$(1.2) \quad T(\text{Vect}_X(\mathcal{F})/X)|_X \to T(\text{Vect}'_X(\mathcal{F})/X)|_X.$$

is an isomorphism.

By definition, $T(\text{Vect}'_X(\mathcal{F})/X)|_X$ identifies with $\mathcal{F}$. By Lemma 1.3.5,

$$T(\text{Vect}_X(\mathcal{F})/X)|_X \simeq \text{Prim}(\text{Sym}(\mathcal{F})), $$

and the map (1.2) identifies with the canonical map

$$\text{Prim}(\text{Sym}(\mathcal{F})) \to \mathcal{F}. $$

Now, the latter map is an isomorphism by [Book-IV.1, Theorem 1.7.4].

**Remark 1.4.4.** The proof of Proposition 1.4.3 used the somewhat non-trivial isomorphism of [Book-IV.1, Theorem 1.7.4]. However, if instead of the functor $\text{Spec}^{\text{inf}}$, one uses the functor $\text{Spec}^{\text{inf,ind-nilp}}$ (see Remark 1.3.7), then the statement that

$$\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, \text{Spec}^{\text{inf,ind-nilp}}(\text{Sym}(\mathcal{F}))) \to \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(Z), \mathcal{F})$$

is tautological, as $\text{Sym}(\mathcal{F})$ is the co-free object in $\text{Cocom}^{\text{aug,nilp}}(\text{IndCoh}(X))$.

Thus, we can interpret the assertion of Proposition 1.4.3 as saying that the natural map

$$\text{Spec}^{\text{inf,ind-nilp}}(\text{Sym}(\mathcal{F})) \to \text{Spec}^{\text{inf}}(\text{Sym}(\mathcal{F})) = \text{Vect}_X(\mathcal{F})$$

is an isomorphism.

Note that the latter is a particular case of the isomorphism of functors of Remark 1.3.7.

1.4.5. We now claim:

**Proposition 1.4.6.** The co-unit of the adjunction

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Vect}_X(\mathcal{F})) \to \text{Sym}(\mathcal{F})$$

is an isomorphism.

The rest of this subsection is devoted to the proof of the proposition.
1.4.7. Step 1. Suppose for a moment that $\mathcal{F}$ is such that $\mathbb{D}^{\text{Serre}}_X(\mathcal{F}) \in \text{Coh}(X)^{<0}$. In this case, by Proposition 1.4.3,

$$\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, \text{Vect}_X(\mathcal{F})) \simeq \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(Z), \mathcal{F}) \simeq \text{Maps}_{\text{Coh}(X)}(\mathbb{P}^{\text{Serre}}_X(\mathcal{F}), \ker(\pi_* (\mathcal{O}_Z) \to \mathcal{O}_X)) \simeq \text{Maps}_{/X}(Z, \text{Spec} \text{free}_{\text{Com}}(\mathbb{D}^{\text{Serre}}_X(\mathcal{F})))$$

(where $\text{free}_{\text{Com}}$ is taken in the symmetric monoidal category $\text{Qcoh}(X)$), so $\text{Vect}_X(\mathcal{F})$ is a scheme isomorphic to $\text{Spec} \text{free}_{\text{Com}}(\mathbb{D}^{\text{Serre}}_X(\mathcal{F}))$, and the assertion is manifest.

1.4.8. Step 2. Now, we claim that both sides in (1.3), viewed as functors $\text{IndCoh}(X) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$, commute with filtered colimits in $\mathcal{F}$.

The commutation is obvious for the functor $\mathcal{F} \mapsto \text{Sym}(\mathcal{F})$.

Since the functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$ is a left adjoint, it suffices to show that the functor $\mathcal{F} \mapsto \text{Spec}^{\text{inf}}(\text{Sym}(\mathcal{F}))$ commutes with filtered colimits.

By the construction of the functor $\text{Spec}^{\text{inf}}$, it suffices to show that the functor $\mathcal{F} \mapsto \text{Spec}^{\text{inf}}(\text{Sym}(\mathcal{F}))^{\text{nil-isom}}$:

$$\text{IndCoh}(X) \to \text{Funct}\left(\left(\text{Ptd}(\langle \text{Sch}^{\text{aff}}\rangle^{\text{nil-isom}} \text{ to } X\right))^{\text{op}}, \text{Spec}\right)$$

commutes with filtered colimits.

By Proposition 1.4.3, it suffices to show that for $Z \in \text{Ptd}(\langle \text{Sch}^{\text{aff}}\rangle^{\text{nil-isom}} \text{ to } X)$, the functor $\mathcal{F} \mapsto \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(Z), \mathcal{F})$ commutes with filtered colimits. The latter follows from the fact that $\text{Distr}^+(Z) \in \text{Coh}(X) = \text{IndCoh}(X)^{\mathcal{O}}$.

1.4.9. Step 3. According to Step 2, we can assume that $\mathcal{F} \in \text{Coh}(Z)$. Combining with Step 1, it remains to show that if the assertion of the proposition holds for $\mathcal{F}[-1]$, then it also holds for $\mathcal{F}$.

The description of $\text{Vect}_X(\mathcal{F})$, given by Proposition 1.4.3 implies that there is a canonical isomorphism

$$\text{Vect}_X(\mathcal{F}[-1]) \simeq \Omega_X(\text{Vect}_X(\mathcal{F})) \in \text{Grp}(\text{FormMod}/X),$$

and hence

$$B_X(\text{Vect}_X(\mathcal{F}[-1])) \simeq \text{Vect}_X(\mathcal{F}).$$

Note also that we have a canonical isomorphism in $\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$:

$$\text{Bar}(\text{Sym}(\mathcal{F}[-1])) \simeq \text{Sym}(\mathcal{F}),$$

where we regard $\text{Sym}(\mathcal{F}[-1])$ as an object of

$$\text{CocomBialg}(\text{IndCoh}(X)) = \text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)))$$

by viewing $\mathcal{F}[-1] = \Omega(\mathcal{F})$ as an object of $\text{Grp}(\text{IndCoh}(X)) \subset \text{Monoid}(\text{IndCoh}(X))$. 
The following diagram commutes by adjunction

\[
\begin{array}{ccc}
\text{Bar} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}} (\text{Vect}_X (\mathcal{F}[-1])) & \xrightarrow{\sim} & \text{Distr}^{\text{Cocom}^{\text{aug}}} \circ B_X (\text{Vect}_X (\mathcal{F}[-1])) \\
\downarrow & & \downarrow \sim \\
\text{Bar}(\text{Sym}(\mathcal{F}[-1])) & \xrightarrow{\sim} & \text{Distr}^{\text{Cocom}^{\text{aug}}} (\text{Vect}_X (\mathcal{F})) \\
\downarrow & & \downarrow \\
\text{Sym}(\mathcal{F}) & \xrightarrow{\text{id}} & \text{Sym}(\mathcal{F}).
\end{array}
\]

By assumption, the upper left vertical arrow in this diagram is an isomorphism. Hence, so is the lower right vertical arrow.

2. Inf-affineness

In this section we study the notion of inf-affineness, which is a counterpart of the usual notion of affineness in algebraic geometry.

However, this analogy is not perfect: it is not true that the functor $\text{Spec}^{\text{inf}}$ identifies the category $\text{CocomCoalg}(\text{IndCoh}(X))$ with that of inf-affine objects in $\text{Ptd}((\text{Sch}^{\text{aff}})_{\text{nil-isom} to X})$.

2.1. The notion of inf-affineness. Let as before $X \in <\infty \text{Sch}^{\text{aff}}$. We give the following definition:

**Definition 2.1.1.** An object $\mathcal{Y} \in \text{Ptd}(\text{FormMod}_{/X})$ is inf-affine, if the functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$ induces an isomorphism

\[
\text{Maps}_{\text{Ptd}(\text{FormMod}_{/X})}(Z, \mathcal{Y}) \rightarrow \text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}(\text{Distr}^{\text{Cocom}^{\text{aug}}}(Z), \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y})),
\]

where $Z \in \text{Ptd}((<\infty \text{Sch}^{\text{aff}})_{\text{nil-isom} to X})$.

2.1.2. Here are some basic facts related to this notion:

**Proposition 2.1.3.** Any object $\mathcal{Y} \in \text{Ptd}((\text{Sch}^{\text{aff}})_{\text{nil-isom} to X}) \subset \text{Ptd}(\text{FormMod}_{/X})$ is inf-affine.

**Proof of Proposition 2.1.3.** By definition, we need to show that for

\[
\mathcal{Y}_1, \mathcal{Y}_2 \in \text{Ptd}((\text{Sch}^{\text{aff}})_{\text{nil-isom} to X}),
\]

with $\mathcal{Y}_1$ eventually coconnective, the groupoid $\text{Maps}_{\text{Ptd}(\text{FormMod}_{/X})}(\mathcal{Y}_1, \mathcal{Y}_2)$ maps isomorphically to

\[
\text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}(\text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y}_1), \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y}_2)).
\]

The assertion easily reduces to the case when $\mathcal{Y}_2$ is eventually coconnective. In the latter case, Serre duality identifies the above groupoid with

\[
\text{Maps}_{\text{ComAlg}^{\text{aug}}(\text{Qcoh}(X))}(\text{IndCoh}(\omega_{\mathcal{Y}_1}), (\pi_1)_*(\mathcal{O}_{\mathcal{Y}_1})),
\]

and the desired isomorphism is manifest.

□
2.1.4. We claim:

**Lemma 2.1.5.** Let $Y \in \text{Ptd}(\text{FormMod}/X)$ be inf-affine. Then for any $Z \in \text{Ptd}(\text{FormMod}/X)$, the map 

$$\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, Y) \to \text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}(\text{Distr}^{\text{Cocom}^{\text{aug}}}(Z), \text{Distr}^{\text{Cocom}^{\text{aug}}}(Y))$$

is an isomorphism.

**Proof.** Follows from [Book-IV.2, Corollaries 1.5.2(a) and 1.5.5]. \hfill \Box

**Remark 2.1.6.** It follows from Proposition 2.3.4 below, combined with [Book-IV.1, Corollary E.2.2(b)] for the co-operad Cocom that if instead of CocomCoalg^{aug}(\text{IndCoh}(X)) one uses Cocom^{aug,ind-nilp}(\text{IndCoh}(X)), one obtains the same notion of inf-affineness.

2.2. **Inf-affineness and inf-spectrum.** As was mentioned already, it is not true that the functor Spec_{inf} identifies the category CocomCoalg(\text{IndCoh}(X)) with that of inf-affine objects in Ptd(\text{FormMod}/X).

In this subsection we establish several facts that can be said in this direction. A more complete picture is presented in Sect. 3.3.

2.2.1. We note:

**Lemma 2.2.2.** Let $A \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$ be such that the co-unit of the adjunction 

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Spec}_{\text{inf}}(A)) \to A$$

is an isomorphism. Then the object Spec_{inf}(A) is inf-affine.

In particular, combining with Proposition 1.4.6, we obtain:

**Corollary 2.2.3.** For $\mathcal{F} \in \text{IndCoh}(X)$, the object Vect_{X}(\mathcal{F}) \in \text{Ptd}(\text{FormMod}/X)$ is inf-affine.

**Remark 2.2.4.** As we shall see in Sect. 3.3.8, it is not true that the functor Spec_{inf} is fully faithful. I.e., the co-unit of the adjunction 

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Spec}_{\text{inf}}(A)) \to A$$

is not an isomorphism for all $A \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$.

However, we will see that [Book-IV.1, Conjecture 1.6.5(b)] for the symmetric monoidal DG category IndCoh(X) implies that the above map is an isomorphism for $A$ lying in the essential image of the functor Dist^{Cocom^{aug}}.

We shall also see that [Book-IV.1, Conjecture 1.6.5(a)] for IndCoh(X) implies that the essential image of the functor Spec_{inf} lands in the subcategory of Ptd(\text{FormMod}/X) spanned by inf-affine objects.

**Remark 2.2.5.** The same logic shows that [Book-IV.1, Conjecture B.3.4] for the symmetric monoidal DG category IndCoh(X) and the Lie operad, implies that the functor Spec_{inf,ind-nilp} is an equivalence onto the full subcategory of Ptd(\text{FormMod}/X) spanned by objects that are inf-affine.

2.3. **A criterion for being inf-affine.**
2.3.1. Note that we have a commutative diagram
\[
\begin{array}{ccc}
\text{Coh}(X)^{\leq 0} & \xrightarrow{\text{D} \text{Serre}} & \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)), \\
\text{Ptd}((\leq \infty \text{Sch}_{\text{aff}})_{\text{nil-isom to } X}) & \xrightarrow{\text{triv}} & \text{Distr}^{\text{Cocom}^{\text{aug}}(\text{IndCoh}(X))}
\end{array}
\]
where \text{triv}^{\text{Cocom}^{\text{aug}}(\text{IndCoh}(X))} is as in [Book-IV.1, Sect. 1.5.1], and the top horizontal arrow is the functor of split square-zero extension.

Hence, for \( F \in \text{Coh}(X)^{\leq 0} \) and \( Y \in \text{Ptd}(\text{FormMod}/X) \), the functor \( \text{Distr}^{\text{Cocom}^{\text{aug}}(\text{IndCoh}(X))} \) gives rise to a canonically defined map
\[
\text{Maps}_{\text{IndCoh}(X)}(\text{D} \text{Serre}(F), T(Y/X)|_X) \rightarrow \text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}(\text{triv}^{\text{Cocom}^{\text{aug}}(\text{D} \text{Serre}(F))}, \text{Distr}^{\text{Cocom}^{\text{aug}}(Y)}).
\]

2.3.2. Recall (see [Book-IV.1, Sect. 1.5.1]) now that for \( O \in \text{DGCat}^{\text{SymMon}} \), we have a functor
\[
\text{Prim} : \text{CocomCoalg}^{\text{aug}}(O) \rightarrow O,
\]
right adjoint to \( \text{triv}^{\text{Cocom}^{\text{aug}}(\text{IndCoh}(X))} \).

Consider the corresponding functor
\[
\text{Prim} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \rightarrow \text{IndCoh}(X).
\]

We can rewrite (2.1) as a map
\[
\text{Maps}_{\text{IndCoh}(X)}(\text{D} \text{Serre}(F), T(Y/X)|_X) \rightarrow \text{Maps}_{\text{IndCoh}(X)}(\text{D} \text{Serre}(F), \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}(Y)}).
\]

The map (2.2) gives rise to a well-defined map in \( \text{IndCoh}(X) \):
\[
T(Y/X)|_X \rightarrow \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}(Y)}.
\]

2.3.3. We claim:

**Proposition 2.3.4.** For an object \( Y \in \text{Ptd}(\text{FormMod}/X) \) the following conditions are equivalent:

(i) \( Y \) is inf-affine;

(ii) The unit of the adjunction \( Y \rightarrow \text{Spec}^{\text{inf}}(\text{Distr}^{\text{Cocom}^{\text{aug}}(Y)}) \) is an isomorphism;

(iii) The map (2.3) is an isomorphism.

**Proof.** The implication (i) \( \Rightarrow \) (iii) is tautological from the definition of inf-affineness.

Suppose that \( Y \) satisfies (ii). Then for \( Z \in \text{Ptd}((\leq \infty \text{Sch}_{\text{aff}})_{\text{nil-isom to } X}) \) the map
\[
\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, Y) \rightarrow \text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, \text{Spec}^{\text{inf}}(\text{Distr}^{\text{Cocom}^{\text{aug}}(Y)}))
\]
is an isomorphism, while its composition with the adjunction isomorphism
\[
\text{Maps}_{\text{Ptd}(\text{FormMod}/X)}(Z, \text{Spec}^{\text{inf}}(\text{Distr}^{\text{Cocom}^{\text{aug}}(Y)})) \simeq \text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}(\text{Distr}^{\text{Cocom}^{\text{aug}}(Z)}, \text{Distr}^{\text{Cocom}^{\text{aug}}(Y)})
\]
equals the map induced by the functor \( \text{Distr}^{\text{Cocom}^{\text{aug}}(Y)} \). Hence, \( Y \) is inf-affine.
Finally, assume that $\mathcal{Y}$ satisfies (iii). By [Book-III.1, Proposition 8.3.2], in order to show that $\mathcal{Y} \to \text{Spec}^\inf(\text{Distr}^\text{Cocom}^\text{aug}(\mathcal{Y}))/\mathcal{X}$ is an isomorphism in $\text{IndCoh}(\mathcal{X})$, it suffices to show that the map

$$T(\mathcal{Y}/\mathcal{X}|_{\mathcal{X}} \to T(\text{Spec}^\inf(\text{Distr}^\text{Cocom}^\text{aug}(\mathcal{Y}))/\mathcal{X}|_{\mathcal{X}}$$

is an isomorphism in $\text{IndCoh}(\mathcal{X})$.

Recall the isomorphism $T(\text{Spec}^\inf(\mathcal{A})/\mathcal{X}|_{\mathcal{X}} \simeq \text{Prim}(\mathcal{A})$ of Lemma 1.3.5.

Now, it is easy to see that the composed map

$$T(\mathcal{Y}/\mathcal{X}|_{\mathcal{X}} \to T(\text{Spec}^\inf(\text{Distr}^\text{Cocom}^\text{aug}(\mathcal{Y}))/\mathcal{X}|_{\mathcal{X}} \simeq \text{Prim}(\text{Distr}^\text{Cocom}^\text{aug}(\mathcal{Y}))$$

equals the map (2.3), implying our assertion. □

3. From formal groups to Lie algebras

In this section we finally spell out the relationship between the categories $\text{Grp}(\text{FormMod}/\mathcal{X})$ and $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$, i.e., formal groups and Lie algebras:

To go from an object of $\text{Grp}(\text{FormMod}/\mathcal{X})$ to $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$, we first attach to it an object $\text{Grp}(\text{LieAlg}(\text{IndCoh}(\mathcal{X})))$ via the functor

$$(\quad [-1] \circ \text{Prim})^{\text{enh}} \circ \text{Distr}^\text{Cocom}^\text{aug}$$

(i.e., we attach to an object of $\text{Grp}(\text{FormMod}/\mathcal{X})$ the corresponding augmented co-commutative co-algebra and use Quillen’s functor $([-1] \circ \text{Prim})^{\text{enh}}$ that maps $\text{CocomCoalg}^\text{aug}$ to $\text{LieAlg}$, and then deloop.

To go from $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$ we use the “exponential map”, incarnated by the functor

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}} : \text{LieAlg} \to \text{CocomHopf}$$

while the latter is canonically isomorphic to the more usual construction given by the functor $U^{\text{Hopf}}$, the universal enveloping algebra, viewed as a co-commutative Hopf algebra.

3.1. The exponential construction. Let as before $\mathcal{X} \in \text{Sch}^{\text{aff}}_{\text{ntr}}$.

3.1.1. We define the functor

$$\exp : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{Grp}(\text{FormMod}/\mathcal{X})$$

to be

$$\mathfrak{h} \mapsto \text{Monoid}(\text{Spec}^\inf) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}).$$

Remark 3.1.2. To bring the above construction closer to the classical idea of the exponential map, let us recall that, according to [Book-IV.1, Theorem 4.1.2], we have a canonical isomorphism in $\text{CocomHopf}(\text{IndCoh}(\mathcal{X}))$

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}) \simeq U^{\text{Hopf}}(\mathfrak{h}).$$

3.1.3. In the next section we will prove:

Theorem 3.1.4. The functor

$$\exp : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{Grp}(\text{FormMod}/\mathcal{X})$$

is an equivalence.

3.2. Corollaries of Theorem 3.1.4.
3.2.1. Recall (see [Book-IV.1, Theorem 2.4.5]) that when $\text{Grp}({\text{Chev}}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h})$ is viewed as an object of $\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$, i.e., if we forget the algebra structure, it is (canonically) isomorphic to $\text{Sym}(\text{obl}v_{\text{Lie}}(\mathfrak{h}))$.

Hence, by Corollary 2.2.3, when we view $\exp(\mathfrak{h})$ as an object of $\text{Ptd}(\text{FormMod}/X)$, it is isomorphic to $\text{Vect}_X(\text{obl}v_{\text{Lie}}(\mathfrak{h}))$, and hence is inf-affine.

Therefore, as a formal consequence of Theorem 3.1.4, we obtain:

**Corollary 3.2.2.** Every object of $\text{Grp}({\text{FormMod}}/X)$, when viewed by means of the forgetful functor as an object of $\text{Ptd}(\text{FormMod}/X)$, is inf-affine.

3.2.3. Furthermore, from Proposition 1.4.6, we obtain that the natural map

$$\text{Grp}({\text{Distr}}^{\text{Cocom}}^{\text{aug}})(\exp(\mathfrak{h})) \rightarrow \text{Grp}({\text{Chev}}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h})$$

is an isomorphism.

Combining with [Book-IV.1, Corollary 3.3.8], we obtain a canonical isomorphism

$$B_{\text{Lie}} \circ \text{Monoid}([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Grp}({\text{Distr}}^{\text{Cocom}}^{\text{aug}})(\exp(\mathfrak{h})) \simeq \mathfrak{h}.$$ 

Let us denote by

(3.1) $$\text{Lie} : \text{Grp}({\text{FormMod}}/X) \rightarrow \text{LieAlg}(\text{IndCoh}(X))$$

the functor

$$\mathcal{H} \mapsto B_{\text{Lie}} \circ \text{Monoid}([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Grp}({\text{Distr}}^{\text{Cocom}}^{\text{aug}})(\mathcal{H}).$$

Hence:

**Corollary 3.2.4.** The functor

$$\text{Lie} : \text{Grp}({\text{FormMod}}/X) \rightarrow \text{LieAlg}(\text{IndCoh}(X))$$

is the inverse of

$$\exp : \text{LieAlg}(\text{IndCoh}(X)) \rightarrow \text{Grp}({\text{FormMod}}/X).$$

3.2.5. By combining Corollary 3.2.2 and Proposition 2.3.4, we obtain:

**Corollary 3.2.6.** There exists a canonical isomorphism of functors

$$\text{Grp}({\text{FormMod}}/X) \rightarrow \text{IndCoh}(X), \quad \text{obl}v_{\text{Lie}} \circ \text{Lie}(\mathcal{H}) \simeq T(\mathcal{H}/X)|_X.$$

In other words, this corollary says that the object of $\text{IndCoh}$ underlying the Lie algebra corresponding to a formal group indeed identifies with the tangent space at the origin.

3.2.7. The upshot of this subsection is that in derived algebraic geometry the passage from the a formal group to its Lie algebra is given by the functor

$$\text{Lie} := B_{\text{Lie}} \circ \text{Monoid}([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Grp}({\text{Distr}}^{\text{Cocom}}^{\text{aug}}).$$

3.3. **Lie algebras and formal moduli problems.** In this subsection we will assume Theorem 3.1.4 and deduce some further corollaries.
3.3.1. First, we claim that there is the following commutative diagram of functors

\[
\begin{array}{ccc}
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) & \xleftarrow{\text{Distr}^{\text{Cocom}^{\text{aug}}}} & \text{Ptd(FormMod}_{/X}) \\
\text{Chev}^{\text{enh}} & & \text{Grp(IndCoh}(X)) \\
\text{LieAlg}(\text{IndCoh}(X)) & \xleftarrow{\text{exp}} & \text{Grp(FormMod}_{/X}).
\end{array}
\]

Indeed, by Theorem 3.1.4, it suffices to construct a functorial isomorphism

\[
\text{Distr}^{\text{Cocom}^{\text{aug}}} \circ \text{B}_{X} \circ \text{exp}(\mathfrak{h}) \simeq \text{Chev}^{\text{enh}}(\mathfrak{h}).
\]

However, by Lemma 1.2.6, the left-hand side identifies with

\[
\text{Bar} \circ \text{Grp}(\text{Distr}^{\text{Cocom}^{\text{aug}}}) \circ \text{exp}(\mathfrak{h}) \simeq \text{Bar} \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \text{Ω}_{\text{Lie}}(\mathfrak{h}) \simeq \text{Chev}^{\text{enh}} \circ \text{B}_{\text{Lie}} \circ \text{Ω}_{\text{Lie}}(\mathfrak{h}) \simeq \text{Chev}^{\text{enh}}(\mathfrak{h}).
\]

3.3.2. By passing to right adjoints in diagram (3.2) we obtain another commutative diagram

\[
\begin{array}{ccc}
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) & \xrightarrow{\text{Spec}^{\text{inf}}} & \text{Ptd(FormMod}_{/X}) \\
\text{([−1]◦Prim)}^{\text{enh}} & & \text{Grp(IndCoh}(X)) \\
\text{LieAlg}(\text{IndCoh}(X)) & \xleftarrow{\text{exp}} & \text{Grp(FormMod}_{/X}).
\end{array}
\]

Remark 3.3.3. The commutative diagrams (3.2) and (3.3) can be summarized as follows: the functor

\[
\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd(FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))
\]

remembers/loses as much information as does the functor

\[
\text{Chev}^{\text{enh}} : \text{LieAlg}(\text{IndCoh}(X)) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)).
\]

However, the functor

\[
\text{Grp}(\text{Distr}^{\text{Cocom}^{\text{aug}}}) : \text{Grp(FormMod}_{/X}) \to \text{CocomBialg}^{\text{aug}}(\text{IndCoh}(X))
\]

is fully faithful, as is the functor

\[
\text{Grp}(\text{Chev}^{\text{enh}}) \circ \text{Ω}_{\text{Lie}} : \text{LieAlg}(\text{IndCoh}(X)) \to \text{CocomBialg}^{\text{aug}}(\text{IndCoh}(X)).
\]

3.3.4. As another corollary of the commutative diagram (3.2), we obtain:

**Corollary 3.3.5.** For \( y \in \text{Ptd(FormMod}_{/X}) \) there is a canonical isomorphism

\[
\text{Distr}^{\text{Cocom}^{\text{aug}}}(y) \simeq \text{Chev}^{\text{enh}}(\text{Lie}(\Omega_{X}(y))).
\]

3.3.6. Let

\[
\mathfrak{h} \to \text{B}_{X}(\mathfrak{h}) : \text{LieAlg}(\text{IndCoh}(X)) \to \text{Ptd(FormMod}_{/X})
\]

denote the functor \( \text{B}_{X} \circ \text{exp} \).

This is the functor that associates to a Lie algebra in \( \text{IndCoh}(X) \) the corresponding moduli problem. By Theorem 3.1.4, this functor is an equivalence, with the inverse being

\[
y \mapsto \text{Lie}(\Omega_{X}(y)).
\]

From Proposition 2.3.4 we obtain:
Corollary 3.3.7. Let \( Y \) be an object of \( \text{Ptd}(\text{FormMod}_X) \), and let \( h \) be the corresponding object of \( \text{LieAlg}(\text{IndCoh}(X)) \), i.e.,
\[
h = \text{Lie}(\Omega_X(\langle Y \rangle)) \text{ and/or } Y := B_X(h).
\]
Then \( Y \) is inf-affine if and only if unit of the adjunction
\[
h \rightarrow ([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Chev}^{\text{enh}}(h)
\]
is an isomorphism.

3.3.8. Let \( A \) be an object of \( \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \). From the diagrams (3.2) and (3.3) we obtain that the co-unit of the adjunction
\[
\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Spec}^{\text{inf}}(A) \rightarrow A
\]
identifies with the map
\[
\text{Chev}^{\text{enh}} \circ ([-1] \circ \text{Prim})^{\text{enh}}(A) \rightarrow A.
\]

In particular, we obtain that if [Book-IV.1, Conjecture 1.6.5(b)] holds for the symmetric monoidal DG category \( \text{IndCoh}(X) \), i.e., if the map (3.5) is an isomorphism for \( A \) lying in the essential image of the functor
\[
\text{Chev}^{\text{enh}} : \text{LieAlg}(\text{IndCoh}(X)) \rightarrow \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)),
\]
then the map (3.4) is an isomorphism for \( A \) lying in the essential image of the functor
\[
\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_X) \rightarrow \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)).
\]

Similarly, suppose that [Book-IV.1, Conjecture 1.6.5(a)] holds for the symmetric monoidal DG category \( \text{IndCoh}(X) \), i.e., if the map
\[
h \rightarrow ([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Chev}^{\text{enh}}(h)
\]
is an isomorphism for \( h \) lying in the essential image of the functor
\[
([-1] \circ \text{Prim})^{\text{enh}} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \rightarrow \text{LieAlg}(\text{IndCoh}(X)).
\]
Then, by Corollary 3.3.7, any \( Y \in \text{Ptd}(\text{FormMod}_X) \) lying in the essential image of the functor
\[
\text{Spec}^{\text{inf}} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \rightarrow \text{Ptd}(\text{FormMod}_X),
\]
is inf-affine.

3.4. The ind-nilpotent version. For completeness, let us explain what happens to the picture in Sect. 3.3 if instead of the adjoint functors
\[
\text{Distr}^{\text{Cocom}^{\text{aug}}, \text{ind-nilp}} : \text{Ptd}(\text{FormMod}_X) \rightleftharpoons \text{CocomCoalg}^{\text{aug}, \text{ind-nilp}}(\text{IndCoh}(X)) : \text{Spec}^{\text{inf}, \text{ind-nilp}}.
\]
3.4.1. First, we have the commutative diagrams

\[
\begin{array}{c}
\text{CocomCoalg}^{\text{aug, ind-nilp}}(\text{IndCoh}(X)) \xleftarrow{\text{Chev}^{\text{enh, ind-nilp}}} \text{LieAlg}(\text{IndCoh}(X)) \\
\downarrow \sim \\
\text{Grp(FormMod}/X) \xrightarrow{\sim} \Omega_X
\end{array}
\]

\[
\begin{array}{c}
\text{Ptd(FormMod}/X) \xrightarrow{\text{Distr}} \text{CocomCoalg}^{\text{aug, ind-nilp}}(\text{IndCoh}(X))
\end{array}
\]

and

\[
\begin{array}{c}
\text{Spec}^{\text{inf, ind-nilp}}(\text{IndCoh}(X)) \xrightarrow{\text{exp}} \text{Grp(FormMod}/X)
\end{array}
\]

3.4.2. Let us now assume the validity of [Book-IV.1, Conjecture D.3.4] for the co-operad Cocom and the symmetric monoidal DG category IndCoh(X).

From it we obtain:

**Conjecture 3.4.3.** The functor

\[
\text{Spec}^{\text{inf, ind-nilp}} : \text{CocomCoalg}^{\text{aug, ind-nilp}}(\text{IndCoh}(X)) \to \text{Ptd(FormMod}/X)
\]

is fully faithful.

3.5. Base change.

3.5.1. Let \( f : X' \to X \) be a map in \((<\infty_{\text{Sch}_{\text{aff}}}^{\text{nil-isom}} \xrightarrow{\sim} X, \text{and consider the corresponding functor}

\[
\begin{array}{c}
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \xrightarrow{f^!} \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X')).
\end{array}
\]

The following diagram commutes by construction

\[
\begin{array}{c}
Ptd(\text{FormMod}/X) \xrightarrow{\text{Dist}_{\text{CocomCoalg}^{\text{aug}}}} Ptd(\text{FormMod}/X')
\end{array}
\]

Hence, by adjunction, for \( A \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \) we have a canonically defined map

\[
\begin{array}{c}
X' \times_X \text{Spec}^{\text{inf}}(A) \to \text{Spec}^{\text{inf}}(f^!(A)).
\end{array}
\]

**Remark 3.5.2.** It follows from Lemma 4.2 below and diagram (3.3), that the natural transformation (3.8) is an isomorphism if \( f \) is proper.
3.5.3. The following is immediate from Lemma 2.2.2

**Lemma 3.5.4.** Assume that $A$ is such that both maps

$$\text{Distr}^{\text{Cocom}^\text{aug}} \circ \text{Spec}^{\text{inf}}(A) \rightarrow A \text{ and } \text{Distr}^{\text{Cocom}^\text{aug}} \circ \text{Spec}^{\text{inf}}(f^!(A)) \rightarrow f^!(A)$$

are isomorphisms. Then the map (3.8) is an isomorphism for $A$.

**Corollary 3.5.5.** For $\mathcal{F} \in \text{IndCoh}(X)$, the canonical map

$$X' \times \text{Vect}_{X}(\mathcal{F}) \rightarrow \text{Vect}_{X}(f^!(\mathcal{F}))$$

is an isomorphism.

3.5.6. By combining Corollary 3.5.5 and [Book-IV.1, Theorem 2.4.5], we obtain:

**Corollary 3.5.7.** For $\mathfrak{h} \in \text{LieAlg}(\text{IndCoh}(X))$, the canonical map

$$X' \times \exp_{X}(\mathfrak{h}) \rightarrow \exp_{X}(f^!(\mathfrak{h}))$$

is an isomorphism.

3.5.8. Observe now that Corollary 3.5.7 allows to define the functor

$$\exp : \text{LieAlg}(\text{IndCoh}(X)) \rightarrow \text{Grp}(\text{FormMod}/X)$$

for an arbitrary $X \in \text{PreStk}_{\text{left}}$.

Indeed, this follows from the fact that the functors

$$\text{LieAlg}(\text{IndCoh}(X)) \rightarrow \lim_{X \in ((\leq \text{aff}^{\text{aff}})/X)^{\text{op}}} \text{LieAlg}(\text{IndCoh}(X))$$

and

$$\text{Grp}(\text{FormMod}/X) \rightarrow \lim_{X \in ((\leq \text{aff}^{\text{aff}})/X)^{\text{op}}} \text{Grp}(\text{FormMod}/X)$$

are both equivalences, see [Book-IV.2, Lemma 1.1.5] for the latter statement.

Furthermore, from Theorem 3.1.4, we obtain:

**Corollary 3.5.9.** For any $X \in \text{PreStk}_{\text{left}}$, the above functor

$$\exp : \text{LieAlg}(\text{IndCoh}(X)) \rightarrow \text{Grp}(\text{FormMod}/X)$$

is an equivalence.

3.6. **An example: split square-zero extensions.** In [Book-III.1, Sect. 2.1] we discussed the functor of **split square-zero extension**

$$\text{RealSplitSqZ} : (\text{QCoh}(X)^{\leq 0})^{\text{op}} \rightarrow \text{Ptd}((\text{Sch}^{\text{aff}})/X), \quad X \in \text{Sch}^{\text{aff}}.$$

In this subsection we will extend this construction to the case of arbitrary objects $X \in \text{PreStk}_{\text{left-def}}$, where instead of $\text{QCoh}(-)^{\leq 0}$ we use all of $\text{IndCoh}(X)$. 
3.6.1. For $X \in \text{PreStk}_{\text{laft-def}}$ consider the functor

$$\text{RealSplitSqZ} : \text{IndCoh}(X) \to \text{Ptd}((\text{PreStk}_{\text{laft-def}})/X),$$

defined as follows:

We send $\mathcal{F} \in \text{IndCoh}(X)$ to

$$B_X \circ \exp \circ \text{free}_{\text{Lie}}(\mathcal{F}[-1]) \in \text{Ptd}((\text{Sch}^{\text{aff}})/X) \in \text{Ptd}((\text{FormMod})/X) \subset \text{Ptd}((\text{PreStk}_{\text{laft-def}})/X).$$

Let us say the same in words: we create the free Lie algebra on $\mathcal{F}[-1]$, then we consider the corresponding object of $\text{Grp}((\text{Sch}^{\text{aff}})/X)$, and then take the formal classifying space of the latter.

By construction, we have a commutative diagram:

$$\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{\text{RealSplitSqZ}} & \text{Ptd}((\text{FormMod})/X) \\
\text{LieAlg}(\text{IndCoh}(X)) & \xrightarrow{\exp} & \text{Grp}((\text{FormMod})/X) \\
\end{array}$$

(3.9)

3.6.2. We claim that the functor $\text{RealSplitSqZ}$ can also be described as a left adjoint:

**Proposition 3.6.3.** The functor $\text{RealSplitSqZ}$ is the left adjoint of the functor

$$\text{Ptd}((\text{PreStk}_{\text{laft-def}})/X) \to \text{IndCoh}(X), \quad \mathcal{Y} \mapsto T(\mathcal{Y}/X)|_X.$$

**Proof.** Given $\mathcal{Y} \in \text{Ptd}((\text{PreStk}_{\text{laft-def}})/X)$ and $\mathcal{F} \in \text{IndCoh}(X)$ we need to establish a canonical isomorphism

$$\text{Maps}_{\text{Ptd}((\text{PreStk}_{\text{laft-def}})/X)}(\text{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}) \simeq \text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}[-1], T(\mathcal{Y}/X)|_X).$$

(3.10)

Note that the left-hand side receives an isomorphism from $\text{Maps}(\text{RealSplitSqZ}(\mathcal{F}[-1]), \mathcal{Y}|_X)$, where $\mathcal{Y}|_X$ is the formal completion of $\mathcal{Y}$ along the map $X \to \mathcal{Y}$. So, with no restriction of generality, we can assume that $\mathcal{Y} \in \text{Ptd}((\text{FormMod})/X)$.

In this case, by [Book-IV.2, Theorem 1.6.4], we can further rewrite the left-hand side in (3.10) as

$$\text{Maps}_{\text{Grp}((\text{FormMod})/X)}(\exp \circ \text{free}_{\text{Lie}}(\mathcal{F}[-1]), \Omega_X(\mathcal{Y})),$$

and then as

$$\text{Maps}_{\text{LieAlg}(\text{IndCoh}(X))}(\text{free}_{\text{Lie}}(\mathcal{F}[-1]), \text{Lie}(\Omega_X(\mathcal{Y}))) \simeq \text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}[-1], \text{oblv}_{\text{Lie}} \circ \text{Lie}(\Omega_X(\mathcal{Y}))).$$

However, by Corollary 3.2.6, we have

$$\text{oblv}_{\text{Lie}} \circ \text{Lie}(\Omega_X(\mathcal{Y})) \simeq T(\Omega_X(\mathcal{Y})/X)|_X \simeq T(\mathcal{Y}/X)|_X[-1].$$

Thus, the left-hand side in (3.10) identifies with

$$\text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}[-1], T(\mathcal{Y}/X)|_X[-1]),$$

as required. \qed
Remark 3.6.4. The above verification of the adjunction can be summarized by the commutative diagram

\[
\begin{array}{ccc}
\text{IndCoh}(X) & \overset{T(-/X)|_{X}}{\leftarrow} & \text{Ptd}(\text{FormMod}_{/X}) \\
\cong & & \\
\text{LieAlg}(\text{IndCoh}(X)) & \overset{\text{Lie}}{\leftarrow} & \text{Grp}(\text{FormMod}_{/X}).
\end{array}
\]

3.6.5. As a corollary of Proposition 3.6.3, we obtain:

**Corollary 3.6.6.** The monad on $\text{IndCoh}(X)$, given by the composition

\[(\cdot - 1) \circ T(-/X)|_{X} \circ (\text{RealSplitSqZ} \circ \cdot 1)\]

is canonically isomorphic to $\text{oblv}_{\text{Lie}} \circ \text{free}_{\text{Lie}}$.

4. Proof of Theorem 3.1.4

4.1. **Step 1.** In this subsection we will prove that the functor $\text{exp}$ defines an equivalence from $\text{LieAlg}(\text{IndCoh}(X))$ to the full subcategory of $\text{Grp}(\text{FormMod}_{/X})$, spanned by objects that are inf-affine when viewed as objects of $\text{Ptd}(\text{FormMod}_{/X})$ (i.e., after forgetting the group structure). We denote this category by $\text{Grp}(\text{FormMod}_{/X})'$.

4.1.1. Recall that by Proposition 1.4.6, for $\mathfrak{h} \in \text{LieAlg}(\text{IndCoh}(X))$ the canonical map

\[(4.1) \quad \text{Grp}(\text{Distr}^{\text{Cocom}}}^{\text{aug}})(\text{exp}(\mathfrak{h})) \rightarrow \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h})
\]

is an isomorphism.

In particular, $\text{exp}(\mathfrak{h})$ is inf-affine.

4.1.2. We claim that the functor $\text{Lie}$ of Sect. 3.1, restricted to $\text{Grp}(\text{FormMod}_{/X})'$, provides a right adjoint to $\text{exp}$. In other words, we claim that for $\mathfrak{h} \in \text{LieAlg}(\text{IndCoh}(X))$ and $\mathfrak{H}' \in \text{Grp}(\text{FormMod}_{/X})'$, there is a canonical isomorphism:

\[
\text{Maps}_{\text{Grp}(\text{FormMod}_{/X})'}(\text{exp}(\mathfrak{h}), \mathfrak{H}') \simeq \text{Maps}_{\text{LieAlg}(\text{IndCoh}(X))}(\mathfrak{h}, \text{Lie}(\mathfrak{H}')).
\]

By Lemma 2.1.5 and (4.1), we rewrite the left-hand side as

\[
\text{Maps}_{\text{CocomHopf}(\text{IndCoh}(X))}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}), \text{Grp}(\text{Distr}^{\text{Cocom}}}^{\text{aug}})(\mathfrak{H}'))
\]

and, further, using [Book-IV.1, Sect. 2.4.3] as

\[
\text{Maps}_{\text{LieAlg}(O)}(\mathfrak{h}, B_{\text{Lie}} \circ \text{Monoid}([-1] \circ \text{Prim})^{\text{enh}} \circ \text{Grp}(\text{Distr}^{\text{Cocom}}}^{\text{aug}})(\mathfrak{H}'))
\]

as required.

4.1.3. We claim that the unit of the adjunction

\[
\text{Id} \rightarrow \text{Lie} \circ \text{exp}
\]

is an isomorphism.

Indeed, this follows from (4.1) and [Book-IV.1, Corollary 2.4.8].
4.1.4. Hence, it remains to show that the functor Lie, restricted to $\text{Grp}(\text{FormMod}/_X)'$, is conservative. I.e., we need to show that if $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a map in $\text{Grp}(\text{FormMod}/_X)'$, such that $\text{Lie}(\mathcal{H}_1) \rightarrow \text{Lie}(\mathcal{H}_2)$ is an isomorphism, then the original map is also an isomorphism.

More generally, we claim that if $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ is a map between two inf-affine objects of $\text{Ptd}(\text{FormMod}/_X)$, such that

$$\text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y}_1) \rightarrow \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y}_2)$$

is an isomorphism in $\text{IndCoh}(X)$, then the original map is also an isomorphism.

Indeed, this follows from Proposition 2.3.4 and [Book-III.1, Proposition 8.3.2].

4.2. Step 2. In this subsection we will reduce the assertion of Theorem 3.1.4 to the case when $X$ is reduced.

4.2.1. Taking into account Step 1, the assertion of Theorem 3.1.4 is equivalent to the fact that every object $\mathcal{H} \in \text{Grp}(\text{FormMod}/_X)$ is inf-affine, when we consider it as an object of $\text{Ptd}(\text{FormMod}/_X)$.

Thus, by Proposition 2.1.3, we need to show that for any $\mathcal{H} \in \text{Grp}(\text{FormMod}/_X)$, the canonical map

$$T(\mathcal{H}/_X) \rightarrow \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{H})$$

is an isomorphism.

4.2.2. Let $f : X' \rightarrow X$ be a map in $(^{<\infty}\text{Sch}_{\text{aff}}')/_X$. We have a symmetric monoidal functor

$$f^! : \text{IndCoh}(X) \rightarrow \text{IndCoh}(X'),$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\text{IndCoh}(X) & \overset{f^!}{\longrightarrow} & \text{IndCoh}(X') \\
\text{triv}_{\text{Cocom}} & & \text{triv}_{\text{Cocom}} \\
\downarrow & & \downarrow \\
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) & \overset{f^!}{\longrightarrow} & \text{Cocom}^{\text{aug}}(\text{IndCoh}(X'))
\end{array}$$

Hence, by adjunction, we obtain a natural transformation:

$$(4.2) \quad f^! \circ \text{Prim}_X \rightarrow \text{Prim}_{X'} \circ f^!.$$  

We claim:

**Lemma 4.2.3.** Assume that $f$ is proper. Then the natural transformation $(4.2)$ is an isomorphism

*Proof.* Follows by the $(f^! \circ \text{IndCoh}, f^!)$-adjunction from the commutative diagram

$$\begin{array}{ccc}
\text{IndCoh}(X) & \overset{f^\text{IndCoh}}{\longrightarrow} & \text{IndCoh}(X') \\
\text{triv}_{\text{Cocom}} & & \text{triv}_{\text{Cocom}} \\
\downarrow & & \downarrow \\
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) & \overset{f^\text{IndCoh}}{\longrightarrow} & \text{Cocom}^{\text{aug}}(\text{IndCoh}(X')).
\end{array}$$  

□
4.2.4. Let \( i \) denote the canonical map \( \text{red} X \to X \). From Lemma 4.2.3 we obtain that for \( Y \in \text{Ptd}(\text{FormMod}/X) \), we have a commutative diagram with vertical arrows being isomorphisms

\[
\begin{array}{ccc}
T(Y/X) & \xrightarrow{i^!} & i^!(\text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(y)) \\
\downarrow & & \downarrow \\
T(Y'/X') & \xrightarrow{i^!} & \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(y'),
\end{array}
\]

where \( Y' := X' \times_X Y \).

Since the functor \( i^! \) is conservative (see [Book-II.1, Corollary 6.1.5]), we obtain that if

\[
T(Y'/X') \mid_{X'} \to \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(y')
\]
is an isomorphism, then so is

\[
T(Y/X) \mid_{X} \to \text{Prim} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(y).
\]

Hence, the assertion of Theorem 3.1.4 for \( \text{red} X \) implies that for \( X \).

4.3. **Step 3.**

4.3.1. We will now show that the functor \( \text{exp} \) is essentially surjective onto the entire category \( \text{Grp}(\text{FormMod}/X) \). By Step 2, we can assume that \( X \) is reduced.

4.3.2. For \( \mathcal{H} \in \text{Grp}(\text{FormMod}/X) \) set

\[
Y := B_X(\mathcal{H}) \in \text{Ptd}(\text{FormMod}/X).
\]

Using [Book-IV.2, Corollary 1.5.2(a)], we can write

\[
Y \simeq \text{colim}_{\alpha \in A} Z_\alpha,
\]

where the index category \( A \) is

\[
(\text{Ptd}(\langle \text{Sch}^{\text{aff}}_{\text{nil-isom to } X} \rangle) / Y',
\]

and where the colimit is taken in the category \( \text{PreStk}_{\text{aff}} \).

We have the following crucial observation:

**Lemma 4.3.3.** If the scheme \( X \) is reduced, then the category \( A \) is sifted.

**Proof.** We claim that the diagonal functor \( A \to A \times A \) admits a left adjoint. Namely, it is given by sending

\[
Z_1, Z_2 \to Z_1 \uplus_X Z_2,
\]

see [Book-III.1, Proposition 7.2.2].

NB: the fact that \( X \) is reduced is used to ensure that the maps \( X \to Z_i \) are closed (and hence, nilpotent embeddings).
4.3.4. Set
\[ \mathcal{H}_\alpha := \Omega_X(Z_\alpha). \]

Since \( \mathcal{H}_\alpha \) is a scheme, it is inf-affine, by Proposition 2.1.3. Hence, there exists a canonically defined functor
\[ A \to \text{LieAlg}(\text{IndCoh}(X)), \quad \alpha \mapsto h_\alpha, \]
so that \( \mathcal{H}_\alpha = \exp(h_\alpha) \).

Set
\[ h := \colim_{\alpha \in A} h_\alpha \in \text{LieAlg}(\text{IndCoh}(X)). \]

We are going to construct an isomorphism \( \mathcal{H} \simeq \exp(h) \).

4.3.5. By [Book-IV.2, Theorem 1.6.4], it suffices to construct an isomorphism
\[ \mathcal{Y} \simeq B_X(\exp(h)) \]
in \( \text{Ptd}(\text{FormMod}/X) \).

We let \( \mathcal{Y} \to B_X(\exp(h)) \) be the map, given by the compatible system of maps
\[ Z_\alpha \to B_X(\exp(h)) \]
that correspond under the equivalence \( \Omega_X \) to the maps
\[ \mathcal{H}_\alpha \to \exp(h). \]

To prove that the resulting map \( \mathcal{Y} \to B_X(\exp(h)) \) is an isomorphism, by [Book-III.1, Proposition 8.3.2], it suffices to show that the induced map
\[ T(\mathcal{Y}/X)|_X \to T(B_X(\exp(h))/X)|_X \]
is an isomorphism in \( \text{IndCoh}(X) \).

4.3.6. We have a commutative diagram
\[
\begin{array}{ccc}
\colim_{\alpha \in A} T(Z_\alpha/X)|_X & \xrightarrow{\text{id}} & \colim_{\alpha \in A} T(Z_\alpha/X)|_X \\
\downarrow & & \downarrow \\
T(\mathcal{Y}/X)|_X & \longrightarrow & T(B_X(\exp(h))/X)|_X.
\end{array}
\]

We note that the left vertical arrow is an isomorphism by (the “laft” version of) [Book-III.1, Proposition 2.5.3].

Hence, it remains to show that the right vertical arrow is an isomorphism.

4.3.7. The corresponding map
\[ \colim_{\alpha \in A} T(Z_\alpha/X)|_X[-1] \to T(B_X(\exp(h))/X)|_X[-1] \]
identifies with
\[ \colim_{\alpha \in A} T(\mathcal{H}_\alpha/X)|_X \to T(\exp(h)/X)|_X, \]
and, further, by Proposition 2.3.4, with
\[ \text{(4.3)} \quad \colim_{\alpha \in A} \text{obl}_\text{Lie}(h_\alpha) \to \text{obl}_\text{Lie}(h). \]
Recall that by the assumption on $X$, the category $A$ is sifted (see Lemma 4.3.3). Hence, in the commutative diagram

\[
\begin{array}{ccc}
\text{colim}_{\alpha \in A} \text{oblv}_{\text{Lie}}(h_{\alpha}) & \longrightarrow & \text{oblv}_{\text{Lie}}(h) \\
\downarrow & & \downarrow \text{id} \\
\text{oblv}_{\text{Lie}}(\text{colim}_{\alpha \in A} h_{\alpha}) & \sim & \text{oblv}_{\text{Lie}}(h)
\end{array}
\]

the vertical arrows are isomorphisms.

Hence, the map (4.3) is an isomorphism, as required.

5. Modules over formal groups and Lie algebras

In the previous sections we have constructed an equivalence between formal groups and Lie algebras. In this section we will show that under this equivalence, the datum of action of a formal group on a given object of IndCoh is equivalent to that of action of the corresponding Lie algebra.

5.1. Modules over formal groups.

5.1.1. Let $\mathcal{H}$ be an object of $\text{Grp}((\text{FormMod}_{\text{laft}})/X)$. We define the category $\mathcal{H}\text{-mod}(\text{IndCoh}(X))$ as

\[
\text{Tot}\left(\text{IndCoh}^!(B^\bullet(\mathcal{H}))\right).
\]

Denote $\mathfrak{h} := \text{Lie}(\mathcal{H})$. Recall our definition of modules over a Lie algebra, see [Book-IV.1, Sect. 3.5].

The goal of this subsection is to prove the following:

**Proposition-Construction 5.1.2.** There exists a canonical equivalence of categories

\[
(5.1) \quad \mathcal{H}\text{-mod}(\text{IndCoh}(X)) \simeq \mathfrak{h}\text{-mod}(\text{IndCoh}(X))
\]

that commutes with the forgetful functor to $\text{IndCoh}(X)$.

The rest of this subsection is devoted to the proof of Proposition 5.1.2.

With no restriction of generality, we can assume that $X = X \in <\infty \text{Sch}_{\text{aff}}$.

**Remark 5.1.3.** For the proof of this proposition, the identification

\[
\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}) \simeq U(\mathfrak{h})
\]

of [Book-IV.1, Theorem 4.1.2] now becomes crucial.

5.1.4. Let $\pi^\bullet$ denote the augmentation $B^\bullet(\mathcal{H}) \rightarrow X$. Note that by definition, the simplicial object of CocomCoalg(IndCoh($X$)), given by

\[
(\pi^\bullet)^{\text{IndCoh}}(\omega_{B^\bullet(\mathcal{H})}),
\]

identifies with $\text{Bar}^\bullet(\text{Grp}(\text{Distr}_{\text{Cocom}^{\text{avg}}}(\mathcal{H})))$.

The functor $\pi^\bullet_{\text{IndCoh}}$ defines a map of simplicial categories

\[
\text{IndCoh}_{\text{laft}}(B^\bullet(\mathcal{H})) \rightarrow (\pi^\bullet)^{\text{IndCoh}}(\omega_{B^\bullet(\mathcal{H})})\text{-comod}(\text{IndCoh}(X)).
\]

By passing to right adjoints, we obtain a functor between the corresponding co-simplicial categories

\[
\text{IndCoh}^!(B^\bullet(\mathcal{H})) \rightarrow (\pi^\bullet)^{\text{IndCoh}}(\omega_{B^\bullet(\mathcal{H})})\text{-comod}(\text{IndCoh}(X))^R
\]
see [Book-IV.1, Sect. A.2.2] for the notation.

Applying totalization, we obtain a functor

\[ \text{Tot} \left( \text{IndCoh}(B^\bullet(\mathcal{H})) \right) \to \text{Tot} \left( (\pi^\bullet)^\text{IndCoh}_*(\omega_{B^\bullet(\mathcal{H})}) \right) \]

Finally, taking into account:

- [Book-IV.1, Proposition-Construction A.2.3],
- the isomorphism \( \text{Grp}(\text{Distr}_{\text{Cocom}}^{\text{aug}})(\mathcal{H}) \simeq \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}) \),
- the isomorphism \( \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}) \simeq U^{\text{Hopf}}(\mathfrak{h}) \) of [Book-IV.1, Theorem 5.1.2],

we obtain a functor

\[ (5.2) \quad \mathcal{H}\text{-mod}(\text{IndCoh}(X)) \simeq \text{Tot} \left( \text{IndCoh}(B^\bullet(\mathcal{H})) \right) \simeq \text{Tot} \left( (\pi^\bullet)^\text{IndCoh}_*(\omega_{B^\bullet(\mathcal{H})}) \right) \]

5.1.5. We will now show that the functor (5.2) is an equivalence. In other words, we need to show that the functor

\[ \text{Tot}(\text{IndCoh}(B^\bullet(\mathcal{H}))) \rightarrow \text{Tot} \left( (\pi^\bullet)^\text{IndCoh}_*(\omega_{B^\bullet(\mathcal{H})}) \right) \]

is an equivalence.

This follows from the next general assertion:

**Lemma 5.1.6.** Let \( C_i^\bullet \rightarrow C_2^\bullet \) be a functor between co-simplicial categories. Assume that:

- The arrows in the semi co-simplicial categories underlying \( C_i^\bullet \), \( i = 1, 2 \) admit left adjoints (let \( C_i^{\bullet, L} \) denote the resulting semi-simplicial categories obtained by passing to the left adjoints);
- The semi co-simplicial categories underlying satisfy the monadic Beck-Chevalley condition;
- The lax semi-simplicial map \( C_1^{\bullet, L} \rightarrow C_2^{\bullet, L} \) is semi-simplicial;
- The functor \( C_0^0 \rightarrow C_2^0 \) is an equivalence.

Then the resulting functor

\[ \text{Tot}(C_1^\bullet) \rightarrow \text{Tot}(C_2^\bullet) \]

is an equivalence.

5.2. **An inverse construction.** We shall now give an explicit description of the functor

\[ \mathfrak{h}\text{-mod}(\text{IndCoh}(X)) \rightarrow \mathcal{H}\text{-mod}(\text{IndCoh}(X)) \]

inverse to (5.2).

5.2.1. Note that the datum of lifting an object \( M \in \text{IndCoh}(X) \) to that of \( \mathcal{H}\text{-mod}(\text{IndCoh}(X)) \) is equivalent to a functorial assignment for every \((Z,f) \in (\infty\text{Sch}^{\text{aff}}_{\text{aff}})/X \) of action of the group \( \text{Maps}_{/X}(Z,\mathcal{H}) \) on the object \( f^!(M) \in \text{IndCoh}(Z) \).
5.2.2. Set $h_Z := f^!(\mathfrak{h}) \in \text{LieAlg}(\text{IndCoh}(Z))$. We have

$$f^!(\text{Grp}((\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}(\mathfrak{h}))) \simeq U^{\text{Hopf}}(h_Z) \in \text{CocomHopf}(\text{IndCoh}(Z)).$$

By [Book-IV.1, Proposition-Construction A.2.3 and Remark A.2.4], the datum of action of $h$ on $M$ gives rise to that of action of the group

$$\text{Maps}_{\text{Cocom}(\text{IndCoh}(Z))} (A', U(h_Z))$$
on the $A'$-comodule $A' \otimes f^!(M)$ for any $A' \in \text{Cocom}(\text{IndCoh}(Z))$. Here the group structure on the above space of maps comes from the structure on $U(h_Z)$ of group-object in $\text{Cocom}(\text{IndCoh}(Z))$, given by the object $U^{\text{Hopf}}(h_Z)$.

We take $A' := \omega(Z)$, i.e., the unit object in $\text{IndCoh}(Z)$. Hence, starting from $M \in h$-mod, we obtain an action of the group $\text{Maps}_{\text{Cocom}(\text{IndCoh}(Z))} (\omega_Z, U(h_Z))$ on $f^!(M)$. Here the group structure on the above space of maps comes from the structure on $U(h_Z)$ of group-object in $\text{Cocom}(\text{IndCoh}(Z))$, given by the object $U^{\text{Hopf}}(h_Z)$.

NB: we emphasize that $\text{Maps}_{\text{Cocom}(\text{IndCoh}(Z))} (\omega_Z, U(h_Z))$ is taken in the non-augmented category!

5.2.3. To complete the construction, it suffices to construct a homomorphism of groups

$$\text{(5.3)} \quad \text{Maps}_{/X}(Z, \mathcal{H}) \to \text{Maps}_{\text{Cocom}(\text{IndCoh}(Z))} (\omega_Z, U(h_Z)).$$

Set

$$\mathcal{H}_Z := Z \times \mathcal{H} \in \text{Grp}((\text{FormMod}_{/X})_{/Z}).$$

We have

$$\text{Maps}_{/X}(Z, \mathcal{H}) \simeq \text{Maps}_{/Z}(Z, \mathcal{H}_Z).$$

Applying the functor

$$\text{Distr}^{\text{Cocom}} : \text{FormMod}_{/X}_{/Z} \to \text{Cocom}(\text{IndCoh}(Z))$$
(note that we are now working in the non-augmented category), we obtain a group homomorphism

$$\text{Maps}_{/Z}(Z, \mathcal{H}_Z) \to \text{Maps}_{\text{Cocom}(\text{IndCoh}(Z))} (\omega_Z, \text{Distr}^{\text{Cocom}}(\mathcal{H}_Z)).$$

Taking into account the isomorphism of objects of $\text{CocomCoalg}(\text{IndCoh}(Z))$

$$\text{Distr}^{\text{Cocom}}(\mathcal{H}_Z) \simeq \text{oblv}_{\text{Grp}} \circ (\text{Grp}((\text{Chev}^{\text{enh}}) \circ \Omega))(h_Z) \simeq \text{oblv}_{\text{Grp}} \circ U^{\text{Hopf}}(h_Z) \simeq U(h_Z),$$

which is compatible with the group structures, we obtain the desired map (5.3).

5.3. Relation to nil-isomorphisms.

5.3.1. Let

$$\pi : Y \overset{s}{\Rightarrow} X : \text{s}$$
be an object of $\text{Ptd}((\text{FormMod}_{/X})_{/X})$, and set $\mathcal{H} = \Omega_X(Y)$.

Set $\mathcal{K} := \Omega_X(Y)$. By [Book-III.3, Proposition 3.3.2(b)], there is a canonical equivalence

$$\text{IndCoh}(Y) \simeq \text{Tot}(\text{IndCoh}(B^+(\mathcal{K}))) = \mathcal{K}\text{-mod}(\text{IndCoh}(X)),$$

and thus

$$\text{IndCoh}(Y) \simeq h\text{-mod}(\text{IndCoh}(X)).$$

Under this identification, the forgetful functor $\text{oblv}_{\text{h}} : h\text{-mod}(\text{IndCoh}(X)) \to \text{IndCoh}(X)$ corresponds to $s^!$, and the functor

$$\text{triv}_{\text{h}} : \text{IndCoh}(X) \to h\text{-mod}(\text{IndCoh}(X))$$
corresponds to $\pi^!$. 
In particular, we obtain:

**Corollary 5.3.2.** The monad $s^1 \circ s^\text{IndCoh}$ on $\text{IndCoh}(\mathcal{X})$ is canonically isomorphic to the monad $U(\mathfrak{h}) \otimes (-)$, where $\mathfrak{h} := \text{Lie}(\mathcal{H})$.

5.3.3. For $\mathcal{Y}$ as above, the functor $\pi^! \text{IndCoh} : \text{IndCoh}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{X})$, being the left adjoint of $\pi^!$, identifies with $\text{coinv}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})) \to \text{IndCoh}(\mathcal{X})$.

Furthermore, the functor $\pi^\text{IndCoh}$ naturally lifts to a functor $\text{IndCoh}(\mathcal{Y}) \to \text{Distr}^\text{Cocom}(\mathcal{Y})\text{-comod}(\text{IndCoh}(\mathcal{X}))$, and the latter can be identified with $\text{coinv}^{\text{enh}}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})) \to \text{Chev}^{\text{enh}}(\mathfrak{h})\text{-comod}(\text{IndCoh}(\mathcal{X}))$, see [Book-IV.1, Equation (3.7)] for the notation.

### 6. Actions of groups on formal moduli problems

The goal of this section is to make precise the following idea: an action of a Lie algebra on a prestack is equivalent to that of action of the corresponding formal group. The first difficulty that we have to grapple with is what we mean by an action of a Lie algebra on a prestack? For now we will skirt this question by considering free Lie algebras; we will return to it in [Book-IV.4, Sect. 7].

#### 6.1. Action of groups vs. Lie algebras.

6.1.1. Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft}}$. Let $\mathcal{H} \in \text{Grp}((\text{FormMod}_{\text{laft}})/\mathcal{X})$; denote $\mathfrak{h} := \text{Lie}(\mathcal{H})$.

Let $\pi : \mathcal{Y} \to \mathcal{X}$ be an object of $(\text{PreStk}_{\text{laft}})/\mathcal{X}$, equipped with an action of $\mathcal{H}$. Let us assume that $\mathcal{Y}$ admits deformation theory relative to $\mathcal{X}$ (see [Book-III.1, Sect. 7.1.6] for what this means).

6.1.2. We claim that the data of action gives rise to a map in $\text{IndCoh}(\mathcal{Y})$:

$$
\pi^!(\text{oblv}_{\text{Lie}}(\mathfrak{h})) \to T(\mathcal{Y}/\mathcal{X}).
$$

Indeed, if $\text{act}$ denotes the action map $\mathcal{H} \times \mathcal{Y} \to \mathcal{Y}$, then we have a canonically map

$$
\mathcal{H} \times_\mathcal{X} \mathcal{Y} \to \mathcal{Y},
$$

then we have a canonically map

$$
T((\mathcal{H} \times_\mathcal{X} \mathcal{Y})/\mathcal{X}) \to \text{act}^!(T(\mathcal{Y}/\mathcal{X})).
$$

Pulling back along the unit section of $\mathcal{H}$, and composing with the canonical map

$$
\pi^!(T(\mathcal{H}/\mathcal{X})|_{\mathcal{X}}) \to T(\mathcal{H} \times_\mathcal{X} \mathcal{Y})|_\mathcal{Y},
$$

and using the isomorphism $T(\mathcal{H}/\mathcal{X})|_{\mathcal{X}} \simeq \text{oblv}_{\text{Lie}}(\mathfrak{h})$ of Corollary 3.2.6, we obtain the desired map

$$
\pi^!(\text{oblv}_{\text{Lie}}(\mathfrak{h})) \simeq \pi^!(T(\mathcal{H}/\mathcal{X})|_{\mathcal{X}}) \to T(\mathcal{H} \times_\mathcal{X} \mathcal{Y})|_\mathcal{Y} \to \text{act}^!(T(\mathcal{Y}/\mathcal{X}))|_\mathcal{Y} \simeq T(\mathcal{Y}/\mathcal{X}).
$$
6.1.3. Assume now that $h$ is of the form $\text{free}_{\text{Lie}}(\mathcal{F})$ for some $\mathcal{F} \in \text{IndCoh}(X)$. Note that by adjunction we have a canonical map

$$\mathcal{F} \rightarrow \text{oblv}_{\text{Lie}} \circ \text{free}_{\text{Lie}}(\mathcal{F}).$$

Composing with (6.1), we obtain a map

$$\pi^!(\mathcal{F}) \rightarrow T(\mathcal{Y}/X).$$

6.1.4. The above construction defines a map from the groupoid of actions of $H$ on $Y$ to

$$\text{Maps}_{\text{IndCoh}(Y)}(\pi^!(\mathcal{F}), T(\mathcal{Y}/X)).$$

The goal of this section is to prove the following assertion:

**Theorem 6.1.5.** For $Y$ and $\mathcal{F}$ as above the map from groupoid of data of actions of $H$ on $Y$ to

$$\text{Maps}_{\text{IndCoh}(Y)}(\pi^!(\mathcal{F}), T(\mathcal{Y}/X))$$

is an isomorphism.

6.2. **Proof of Theorem 6.1.5.**

6.2.1. **Idea of proof.** The statement of the theorem readily reduces to the case when $X = X \in <\infty\text{Sch}_{\text{aff}}$.

Let $(\pi^!)^R$ denote the (discontinuous) right adjoint of $\pi^! : \text{IndCoh}(X) \rightarrow \text{IndCoh}(Y)$, so that

$$\text{Maps}_{\text{IndCoh}(Y)}(\pi^!(\mathcal{F}), T(\mathcal{Y}/X)) \simeq \text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/X))).$$

Starting from $Y$ as above, we will construct an object

$$\text{Aut}_{\text{inf}}^{{}\mathcal{Y}/X} \in \text{Grp}(\text{FormMod}_{\text{aff}}/X),$$

such that for any $\mathcal{H}' \in \text{Grp}(\text{FormMod}_{\text{aff}}/X)$, the data of action of $\mathcal{H}'$ on $Y$ is equivalent to that of a homomorphism

$$\mathcal{H}' \rightarrow \text{Aut}_{\text{inf}}^{{}\mathcal{Y}/X}.$$ 

Moreover, we will show that the map

$$\text{oblv}_{\text{Lie}} \left(\text{Lie}(\text{Aut}_{\text{inf}}^{{}\mathcal{Y}/X})\right) \rightarrow (\pi^!)^R(T(\mathcal{Y}/X)),$$

arising by adjunction from (6.2), is an isomorphism.

This will prove Theorem 6.1.5, since the functor Lie is an equivalence.

6.2.2. By [Book-IV.2, Proposition 1.2.2], in order to construct $\text{Aut}_{\text{inf}}^{{}\mathcal{Y}/X}$ as an object of

$$\text{Monoid}(\text{FormMod}_{\text{aff}}/X),$$

it suffices to define it as a presheaf with values in $\text{Monoid}(\text{Spc})$ on the category

$$<\infty\text{Sch}_{\text{aff}}\text{-nil-isom to } X,$$

so that it satisfies the deformation theory conditions of [Book-IV.2, Proposition 1.2.2(b)].

For $Z \in <\infty\text{Sch}_{\text{aff}}\text{-nil-isom to } X$, we set

$$\text{Maps}_{\mathcal{X}}(Z, \text{Aut}_{\text{inf}}^{{}\mathcal{Y}/X}) := \text{Maps}_{\mathcal{X}}(Z \times \mathcal{Y}, \mathcal{Y}) \simeq \text{Maps}_{Z/\mathcal{X}}(\mathcal{Y}_Z, \mathcal{Y}_Z),$$

(here $\mathcal{Y}_Z := Z \times \mathcal{Y}$) with the natural structure of monoid.

The deformation theory conditions of [Book-IV.2, Proposition 1.2.2(b)] follow from the fact that $\mathcal{Y}$ admits deformation theory.
Remark 6.2.3. When we evaluate the prestack $\text{Aut}^\inf(Y/X)$, constructed above, on an arbitrary $Z \in (\text{Sch}_{\text{aff}})/X$, it is not necessarily true that its value will map isomorphically to

$$\text{Maps}_{\mathcal{X}}(Z \times Y, Y).$$

So, the functor $\text{Aut}^\inf(Y/X)$ is only sensible when evaluated on $Z \in (\leq \infty \text{Sch}_{\text{aff}})_{\text{nil-isom}}$ to $X$.

6.2.4. Thus, we have constructed $\text{Aut}^\inf(Y/X)$ as an object of $\text{AssocAlg}((\text{FormMod}_{\text{laft}})/X)$. It belongs to $\text{Grp}((\text{FormMod}_{\text{laft}})/X)$ by [Book-IV.2, Lemma 1.6.2].

It remains to show that the map (6.2) is an isomorphism. By construction, for $\mathcal{F} \in \text{Coh}(X)$ such that $D^\text{Serre}_X(\mathcal{F}) \in \text{Coh}(X) \leq 0$, we have

$$\text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}, T(\text{Aut}^\inf(Y/X))|_X) \simeq \text{Maps}_{\mathcal{X}}(\text{RealSplitSqZ}(D^\text{Serre}_X(\mathcal{F})) \times Y, Y).$$

By the deformation theory of $Y$, the latter maps isomorphically to

$$\text{Maps}_{\text{IndCoh}(Y)}(\pi^!(\mathcal{F}), T(Y/X)), $$

and by adjunction, further (still isomorphically) to

$$\text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}, (\pi^!)^R(T(Y/X))), $$

Furthermore, it follows from the construction that the resulting map

$$\text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}, T(\text{Aut}^\inf(Y/X))|_X) \rightarrow \text{Maps}_{\text{IndCoh}(X)}(\mathcal{F}, (\pi^!)^R(T(Y/X)))$$

is the one induced by (6.2).

This implies the required assertion, as $\text{IndCoh}(X)$ is generated by the above objects of $\text{Coh}(X)$ under colimits.

6.3. Localization of Lie algebra modules. In this subsection we show how to construct crystals on a given prestack starting from modules over a Lie algebra that acts on this prestack.

6.3.1. Let $f : Y \rightarrow X$ and $\mathfrak{g}$ be as in Sect. 6.1.1. Consider the prestack

$$Y_{\text{dR}} := Y_{\text{dR}} \times X_{\text{dR}},$$

see [Book-III.4, Sect. 3.3.2] for the notation.

Recall also the notation

$$/X\text{Crys}(Y) := \text{IndCoh}(Y_{\text{dR}} \times X_{\text{dR}}).$$

In this subsection we will construct the localization functor

$$\text{Loc}_{\mathfrak{h}, Y/X} : \mathfrak{h}\text{-mod}(\text{IndCoh}(X)) \rightarrow /X\text{Crys}(Y).$$
6.3.2. The action of $\mathcal{H}$ on $\mathcal{Y}$ defines an object
$$\mathcal{H} \times \mathcal{Y} \in \text{FormGrpoid}(\mathcal{Y}),$$
see [Book-IV.2, Sect. 2.2.1] for the notation.

By [Book-IV.2, Theorem 2.3.2], the corresponding quotient
$$\mathcal{Y}/\mathcal{H} \in \text{FormMod}_{\mathcal{Y}}$$
is well-defined.

We have canonically defined maps of prestacks
$$\begin{array}{ccc}
\mathcal{Y}/\mathcal{H} & \xrightarrow{f/\mathcal{H}} & B_{X}(\mathcal{H}) \\
g & & \\
\mathcal{Y}/\mathcal{dR} & & \\
\end{array}$$

6.3.3. We define the sought-for functor $\text{Loc}_{\mathcal{H}, \mathcal{Y}/X}$ as
$$g^{\text{IndCoh}} \circ (f/\mathcal{H})^{!},$$
where we identify
$$\text{IndCoh}(B_{X}(\mathcal{H})) \simeq \mathfrak{h}\text{-mod}(\text{IndCoh}(X))$$by means of (5.4).

6.3.4. Note that the functor $\text{Loc}_{\mathcal{H}, \mathcal{Y}/X}$ is by construction the left of the (in general, discontinuous) functor
$$((f/\mathcal{H})^{\text{IndCoh}})^{R} \circ g^{t} : \mathcal{X}\text{Crys}(\mathcal{Y}) \rightarrow \mathfrak{h}\text{-mod}(\text{IndCoh}(X)).$$

We claim that the functor $((f/\mathcal{H})^{\text{IndCoh}})^{R} \circ g^{t}$ makes the following diagram commutative:
$$\begin{array}{ccc}
\mathcal{X}\text{Crys}(\mathcal{Y}) & \xrightarrow{\text{oblv}_{/\mathcal{dR}, \mathcal{Y}}} & \text{IndCoh}(\mathcal{Y}) \\
((f/\mathcal{H})^{\text{IndCoh}})^{R} \circ g^{t} & & (f')^{R} \\
\mathfrak{h}\text{-mod}(\text{IndCoh}(X)) & \xrightarrow{\text{oblv}_{\mathcal{H}}} & \text{IndCoh}(X),
\end{array}$$
where $\text{oblv}_{/\mathcal{dR}, \mathcal{Y}}$ is by definition the $!$-pullback functor along
$$p_{/\mathcal{dR}, \mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y}/\mathcal{dR}.$$

6.3.5. Indeed, we need to establish the commutativity of the diagram
$$\begin{array}{ccc}
\text{IndCoh}(\mathcal{Y}) & \xrightarrow{(f/\mathcal{H})^{\text{IndCoh}}^{R}} & \text{IndCoh}(\mathcal{X}) \\
\uparrow & & \uparrow \\
\text{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xrightarrow{((f/\mathcal{H})^{\text{IndCoh}})^{R}} & \text{IndCoh}(B_{X}(\mathcal{H})),
\end{array}$$
where the vertical arrows are given by $!$-pullback.

However, this follows by passing to right adjoints in the commutative diagram,
$$\begin{array}{ccc}
\text{IndCoh}(\mathcal{Y}) & \xleftarrow{f^{\text{IndCoh}}^{R}} & \text{IndCoh}(\mathcal{X}) \\
\downarrow & & \downarrow \\
\text{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xleftarrow{(f/\mathcal{H})^{\text{IndCoh}}^{R}} & \text{IndCoh}(B_{X}(\mathcal{H})),
\end{array}$$
given by base-change.
References

[Book-II.1] [Book-II.2] [Book-III.1] [Book-III.2] [Book-III.3] [Book-III.4] [Book-IV.1] [Book-IV.2] [Book-IV.4]