CHAPTER IV.5. INFINITESIMAL DIFFERENTIAL GEOMETRY

Contents

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Introduction

0.1. **What do we mean by 'infinitesimal differential geometry'?** The goal of this chapter is to make sense in the context of derived algebraic geometry of a number of notions of *differential nature* that are standard when working with schemes. These notions include:

- Deformation to the normal cone of a closed embedding;
- The notion of the n-th infinitesimal neighborhood of a scheme embedded into another one;
- The PBW filtration on the universal enveloping algebra of a Lie algebroid (over a smooth scheme);
- The Hodge filtration (a.k.a. de Rham resolution) of the dualizing D-module (again, over a smooth scheme).

A feature of the above objects in the setting of classical schemes is that they are constructed by explicit formulas.

For example, the PBW filtration on the universal enveloping algebra of a Lie algebroid is defined by letting the n-th term of the filtration be generated by n-fold products of sections of the Lie algebroid, a notion that is hard to make sense in the context of higher algebra, and hence derived algebraic geometry.

The de Rham resolution

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \Lambda^n(T(X)) \to ... \to \omega_X \underset{\mathcal{O}_X}{\otimes} T(X) \to \omega_X$$

is also defined by explicitly writing down the differential, something that we cannot do in higher algebra.

But our task is even harder: not only do we want to have the above notions for derived schemes, but we want to have them for objects (and maps) in the category $\mathrm{PreStk}_{\text{laft-def}}$. So, an altogether different method is needed to define these objects.

0.1.1. Continuing with the example of $U(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$, the initial idea of how to produce a filtration is pretty clear: the category of filtered objects in Vect identifies with $\mathrm{QCoh}(\mathbb{A}^1)^{\mathbf{G}_m}$, and similarly, for a DG category $\mathbf{C}$, the category $\mathbf{C}^{\mathrm{Fil}}$ of filtered objects in $\mathbf{C}$ identifies with

$$(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbf{G}_m}.$$

Now, the category $\mathbf{C}^{\mathrm{Fil},\geq 0}$ of *non-negatively* filtered objects identifies with

$$(\mathbf{C}\otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}},$$

where the superscript $\mathbb{A}^1_{\text{left-lax}}$ stands for the structure of left-lax equivariance with respect to the monoid $\mathbb{A}^1$; see Sect. 1.2.3 where this notion is introduced.

For $\mathfrak{L}\in \mathrm{LieAlgbroid}(\mathcal{X})$, we regard $U(\mathfrak{L})$ as an algebra object in the monoidal category $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))$, and we wish to lift it to an object

$$U(\mathfrak{L})^{\mathrm{Fil}}\in \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}\right). \tag{0.1}$$

We shall now explain how to produce such a $U(\mathfrak{L})^{\mathrm{Fil}}$, and this will bring us to the idea of deformation to the normal cone (rather, normal bundle in the present context), central for this chapter.

0.1.2. Our main construction is the following. For $\mathcal{X}\in \mathrm{PreStk}_{\text{laft-def}}$ and $\mathcal{Y}\in \mathrm{FormMod}_{\mathcal{X}/}$ we construct a family

$$\mathcal{Y}_{\mathrm{scaled}}\in \mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1},$$

i.e., a family of objects of $\mathrm{FormMod}_{\mathcal{X}//\mathcal{Y}}$ parameterized by points of $\mathbb{A}^1$.

The fiber $\mathcal{Y}_\lambda$ of this family over $0\neq\lambda\in\mathbb{A}^1$ is be (canonically) isomorphic to the initial $\mathcal{Y}$. Its fiber $\mathcal{Y}_0$ over $0\in\mathbb{A}^1$ identifies canonically with the vector-prestack $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$ (see [Chapter IV.3, Sect. 1.4]), where we can think of $T(\mathcal{X}/\mathcal{Y})[1]$ as the normal to $\mathcal{X}$ in $\mathcal{Y}$.

Crucially, the above $\mathbb{A}^1$-family has the following extra structure: it is *left-lax equivariant* with respect to the monoid $\mathbb{A}^1$ acting on itself by multiplication. Concretely, this means that for $\lambda, a\in\mathbb{A}^1$, we have a system of maps

$$\mathcal{Y}_{a\cdot\lambda}\to\mathcal{Y}_\lambda$$

that satisfy a natural associativity condition.

We will denote the resulting object of $(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$ by $\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$. It is the existence of this object that will allow us to carry out the 'differential' constructions mentioned earlier.

0.1.3. Here is how the deformation $\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ can be used in order to produce the object (0.1).

The datum of $U(\mathfrak{L})$ is encoded by the category $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$, equipped with the forgetful functor $\mathbf{oblv}_{\mathfrak{L}}:\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))\to\mathrm{IndCoh}(\mathcal{X})$.

As will be explained in Sects. 6.2 and 6.3, constructing $U(\mathfrak{L})^{\mathrm{Fil}}$ is equivalent to finding a *right-lax* equivariant extension of the pair $(\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})),\mathbf{oblv}_{\mathfrak{L}})$.

Let $(\mathcal{X}\xrightarrow{f}\mathcal{Y})\in\mathrm{FormMod}_{\mathcal{X}/}$ be the formal moduli problem corresponding to $\mathfrak{L}$. According to [Chapter IV.4, Sect. 4.1.2], we have an identification

$$\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))\simeq\mathrm{IndCoh}(\mathcal{Y})$$

under which the functor $\mathbf{oblv}_{\mathfrak{L}}$ corresponds to $f^!$.

We define the sought-for left-lax equivariant extension for $\mathrm{IndCoh}(\mathcal{Y})$ to be the category $\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}})$, and for the functor $f^!$ to be the pullback along $\mathcal{X}\times\mathbb{A}^1\to\mathcal{Y}_{\mathrm{scaled}}$. The right-lax equivariant structure on $\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}})$ is given by the left-lax equivariant structure on $\mathcal{Y}_{\mathrm{scaled}}$, given by $\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$.

0.2. **The n-th infinitesimal neighborhood and the Hodge filtration.** The deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ is used also for the construction of n-th infinitesimal neighborhoods and of the Hodge filtration on the dualizing D-module (crystal).

0.2.1. The idea of infinitesimal neighborhoods

$$\mathcal{X} = \mathcal{X}^{(0)} \to \mathcal{X}^{(1)} \to ... \to \mathcal{X}^{(n)} \to ... \to \mathcal{Y} \tag{0.2}$$

is that each $\mathcal{X}^{(n)}$ is a square-zero extension of $\mathcal{X}^{(n-1)}$ by means of the object of $\mathrm{IndCoh}(\mathcal{X}^{(n-1)})$ equal to the direct image of $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$ under $\mathcal{X} \to \mathcal{X}^{(n-1)}$. We remind that $T(\mathcal{X}/\mathcal{Y})[1]$ should be thought of as the normal bundle to $\mathcal{X}$ inside $\mathcal{Y}$.

To specify such an extension we need to specify a map

$$\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \to T(\mathcal{X}^{(n-1)})/\mathcal{Y})|_{\mathcal{X}}. \tag{0.3}$$

For example, for $n = 1$, the map (0.3) is the identity. However, for $n \geq 2$ we encounter a problem: which map should it be?

Here the deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ comes to our rescue.

0.2.2. We modify the problem, and instead of the system (0.2), we want to construct its filtered version

$$\mathcal{X} \times \mathbb{A}^1 = \mathcal{X}^{(0)}_{\mathrm{scaled}} \to \mathcal{X}^{(1)}_{\mathrm{scaled}} \to ... \to \mathcal{X}^{(n)}_{\mathrm{scaled}} \to ... \to \mathcal{Y}_{\mathrm{scaled}} \tag{0.4}$$

in $(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$.

In particular, instead of the map (0.3), we now need to construct the map

$$\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}})/\mathcal{Y})|_{\mathcal{X}\times\mathbb{A}^1}, \tag{0.5}$$

in

$$\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0},$$

where $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$ is placed in degree n.

Now, the point is that one can prove that $T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}})/\mathcal{Y})|_{\mathcal{X}\times\mathbb{A}^1}$ belongs to

$$\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0},$$

and its n-th associated graded is isomorphic precisely to $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$, and this gives rise to the desired map (0.5).

0.2.3. When $\mathcal{X} \to \mathcal{Y}$ is a closed embedding, it is intuitively clear what the n-th infinitesmal neighborhood $\mathcal{X}^{(n)}$ of $\mathcal{X}$ in $\mathcal{Y}$ is doing.

But we can apply our construction to any map between objects of $\mathrm{PreStk}_{\text{laft-def}}$. In particular, we can take the map

$$p_{\mathcal{X},\mathrm{dR}} : \mathcal{X} \to \mathcal{X}_{\mathrm{dR}}.$$

What is the n-th infinitesmal neighborhood of $\mathcal{X}$ in $\mathcal{X}_{\mathrm{dR}}$?

A concrete version of this question is the following: consider the filtration on $\omega_{\mathcal{X}_{\mathrm{dR}}}$, whose n-th term is the direct image of $\omega_{\mathcal{X}^{(n)}}$ under $\mathcal{X}^{(n)} \to \mathcal{X}_{\mathrm{dR}}$.

This filtration is the Hodge filtration on $\omega_{\mathcal{X}_{\mathrm{dR}}}$. Its n-th associated graded is

$$\mathbf{ind}_{\mathrm{dR},\mathcal{X}}(\mathrm{Sym}^n(T(\mathcal{X})[1])) \in \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}}) \simeq \mathrm{Crys}(\mathcal{X}).$$

If $\mathcal{X} = X$ is a smooth scheme, this filtration incarnates the de Rham resolution of the dualizing D-module.

0.3. **Constructing the deformation.** We now address the question of how the object

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in (\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$$

is constructed.

0.3.1. To construct $\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ we will use the equivalence of [Chapter IV.1, Theorem 2.3.2], and will instead construct the corresponding ($\mathbb{A}^1$ left-lax equivariant) $\mathbb{A}^1$-family of formal groupoids over $\mathcal{X}$.

This $\mathbb{A}^1$-family of groupoids, denoted $\mathcal{R}^\bullet_{\mathrm{scaled}}$, is constructed by a certain universal procedure, explained to us by J. Lurie.

0.3.2. Namely, the prestacks $\mathcal{R}^n_{\mathrm{scaled}}$ are obtained as mapping spaces from a certain universal family of affine schemes $\mathrm{Bifurc}^n_{\mathrm{scaled}}$ over $\mathbb{A}^1$, i.e., for a point $\lambda \in \mathbb{A}^1$, we have

$$\mathcal{R}^n_\lambda = \mathbf{Maps}((\mathrm{Bifurc}^n_{\mathrm{scaled}})_\lambda, \mathcal{X}) \underset{\mathbf{Maps}((\mathrm{Bifurc}^n_{\mathrm{scaled}})_\lambda, \mathcal{Y})}{\times} \mathcal{Y}.$$

The simplicial structure on the assignment $n \mapsto \mathcal{R}^n_{\mathrm{scaled}}$ comes from the structure on the assignment

$$n \mapsto \mathrm{Bifurc}^n_{\mathrm{scaled}}$$

of simplicial object $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ in the category $((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1})^{\mathrm{op}}$.

0.3.3. Once said in the above way, it is clear what $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ must be. For $n = 0$ we have $\mathrm{Bifurc}^0_{\mathrm{scaled}} = \mathbb{A}^1$, because we want $\mathcal{X}^0_\lambda$ to be just $\mathcal{X}$ for any $\lambda \in \mathbb{A}^1$.

For $0 \neq \lambda \in \mathbb{A}^1$ we want $\mathcal{X}^\bullet_\lambda$ to be the Čech nerve of the map $\mathcal{X} \to \mathcal{Y}$. So, $(\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda$ is the groupoid in $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ given by

$$(\mathrm{Bifurc}^n_{\mathrm{scaled}})_\lambda = \underbrace{\mathrm{pt} \sqcup ... \sqcup \mathrm{pt}}_{n+1}.$$

I.e., this is the Čech nerve of the map $\emptyset \to \mathrm{pt}$ in $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$.

Now, since we want $\mathcal{X}^1_0$ to be $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$, we want $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_0$ to be the scheme of dual numbers. From here, it is easy to to guess that all of $\mathrm{Bifurc}^1_{\mathrm{scaled}}$ should be

$$\mathrm{Spec}(k[u,\lambda]/(u-\lambda)\cdot(u+\lambda)),$$

where the variable u corresponds to the projection $\mathrm{Bifurc}^1_{\mathrm{scaled}} \to \mathbb{A}^1$.

The structure on $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_\lambda$ of groupoid in $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ is completely determined by what it is when localized away from $0 = \lambda \in \mathbb{A}^1$.

0.4. **What else is done in this chapter?**

0.4.1. In Sect. 2 we perform the main construction of this chapter–that of the deformation

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in (\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}.$$

0.4.2. In Sect. 3 we translate the construction $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ to the language of Lie algebroids.

We obtain that any Lie algebroid $\mathfrak{L}$ canonically gives rise to a *non-negatively filtered Lie algebroid*, denoted $\mathfrak{L}^{\mathrm{Fil}}$, which technically means an object of

$$\mathrm{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1),$$

equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

The associated graded of $\mathfrak{L}^{\mathrm{Fil}}$, i.e., the fiber of the above family over $0 \in \mathbb{A}^1$ is the *trivial* Lie algebroid corresponding to the object $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(\mathcal{X})$.

We show that the construction $\mathfrak{L} \rightsquigarrow \mathfrak{L}^{\mathrm{Fil}}$ is compatible with the forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

Namely, the object

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}(\mathfrak{L}^{\mathrm{Fil}}) \in (\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})} \simeq (\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}})^{\mathbb{A}^1_{\text{left-lax}}}$$

is the $\mathbb{A}^1$-family, whose value at $\lambda \in \mathbb{A}^1$ is obtained by scaling the original anchor map

$$\mathbf{oblv}_{\mathrm{LieAlgbroids}}(\mathfrak{L}) \to T(\mathcal{X})$$

by λ.

0.4.3. In Sect. 4 we prove the following result: let $\mathcal{H}$ be a formal group over $\mathcal{X}$, and consider the corresponding pointed formal moduli problem $B_{\mathcal{X}}(\mathcal{H})$.

On the one hand, the procedure of deformation to the normal bundle yields the family $(B_{\mathcal{X}}(\mathcal{H}))_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$, which by functoriality is an object of

$$\mathrm{Ptd}\left((\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}\right).$$

On the other hand, using the equivalence

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

and using the canonical deformation of any Lie algebra $\mathfrak{h} \rightsquigarrow \mathfrak{h}^{\mathrm{Fil}}$ (see [Chapter IV.2, Sect. 1.5]), we obtain an object

$$\mathcal{H}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \mathrm{Grp}\left((\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}\right).$$

We prove that there is a canonical isomorphism:

$$B_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}) \simeq (B_{\mathcal{X}}(\mathcal{H}))_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}.$$

I.e., the procedure of deforming a moduli problem, which was defined geometrically via the schemes $\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$, reproduces the procedure of scaling the Lie algebra.

0.4.4. In Sect. 5 we carry out the constriction of the n-th infinitesimal neighborhood of a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$.

We show that the natural map

$$\underset{n}{\operatorname{colim}}\, \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism.

We use the above isomorphism to construct a filtration on $\omega_{\mathcal{Y}}$ whose n-th term is

$$(f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}),$$

where f_n denotes the map $\mathcal{X}^{(n)} \to \mathcal{Y}$. When we interpret $\mathrm{IndCoh}(\mathcal{Y})$ as $\mathfrak{L}$-mod$(\mathrm{IndCoh}(\mathcal{X}))$ for the corresponding Lie algebroid $\mathfrak{L}$, the n-th associated graded of the above filtration is

$$\mathbf{ind}_{\mathfrak{L}}(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])).$$

When $\mathcal{Y} = \mathcal{X}_{\mathrm{dR}}$ we recover the Hodge filtration.

0.4.5. In Sect. 6 we construct the filtration on the universal enveloping algebra of a Lie algebroid.

We also show that the n-th term of the filtration is given by pull-push along

$$\mathcal{X} \xleftarrow{p_s} {}'\mathcal{X}^{(n)} \xrightarrow{p_t} \mathcal{X},$$

where ${}'\mathcal{X}^{(n)}$ denotes the n-th infinitesimal neighborhood of $\mathcal{X}$ under the unit map $\mathcal{X} \to \mathcal{R}$, where $\mathcal{R}$ is the total space of the groupoid corresponding to $\mathfrak{L}$.

0.4.6. Finally, in Sect. 7.1 we apply some elements of the theory developed above to the study of regular embeddings.

We say that a map $f : \mathcal{X} \to \mathcal{Y}$ between objects of $\mathrm{PreStk}_{\mathrm{def}}$ is a regular embedding of relative dimension n if

$$T^*(\mathcal{X}/\mathcal{Y})[-1] \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)$$

belongs to $\mathrm{QCoh}(\mathcal{X})^-$ and is a vector bundle of rank n.

We show that for a regular embedding of relative dimension n, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$

admits a *left* adjoint, to be denoted $f^{\mathrm{IndCoh},*}$, and we prove Grothendiek's formula

$$f^{\mathrm{IndCoh},*} \simeq \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y})) \otimes f^!,$$

where we note that $\mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{X})$ is a line bundle placed in cohomological degree n.

As a corollary, we deduce that for a *schematic smooth map* $g : \mathcal{X} \to \mathcal{Z}$ of relative dimension n, the functor

$$g^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to g_*^{IndCoh}, is defined and we have

$$g^! \simeq \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Z})[1]) \otimes g^{\mathrm{IndCoh},*}.$$

1. Filtrations and the monoid $\mathbb{A}^1$

Let $\mathcal{C}$ be a functor

$$(\mathrm{PreStk})^{\mathrm{op}} \to \infty\text{-Cat}\,.$$

For example, $\mathcal{C}(\mathcal{X}) = \mathrm{QCoh}(\mathcal{X})$ or $\mathcal{C}(\mathcal{X}) = \mathrm{LieAlg}(\mathrm{QCoh}(\mathcal{X}))$.

Suppose now that a prestack $\mathcal{X}$ is acted on by a monoid $\mathcal{G}$. In this section we introduce the notion of what it means for an object $c \in \mathcal{C}(\mathcal{X})$ to be left-lax (resp., right-lax) equivariant with respect to $\mathcal{G}$. This notion generalizes the much more well-known one when $\mathcal{G}$ is a group (and when instead of lax equivariance we have the usual equivariance).

Taking $\mathcal{X} = \mathbb{A}^1$ and $\mathcal{G} = \mathbb{A}^1$, acting on itself by multiplication, we will see that the category

$$(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}$$

(here $\mathbf{C}$ is an arbitrary DG category) is equivalent to that of non-negatively filtered objects in $\mathbf{C}$.

This observation produces a mechanism of creating non-negatively filtered objects from algebraic geometry, as long as we can replace the initial geometric object by an $\mathbb{A}^1$-family, which is left-lax equivariant with respect to the action of $\mathbb{A}^1$ on itself.

1.1. **Equivariance with respect to a monoid.** The notion of equivariance with respect to a group-action is completely standard. The situation with monoids may be less familiar: in fact, there are three different notions of equivariance: right-lax equivariance, left-lax equivariance and just (or strict) equivariance.

1.1.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two ∞-categories, each equipped with an action of a monoid-object of Spc, denoted G. Let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor.

Informally, a structure of *right-lax equivariance* (resp., *left-lax equivariance*) on Φ with respect to G is a homotopy-coherent system of assignments for every point $g \in G$ of a natural transformation $g \circ \Phi \to \Phi \circ g$ (resp., $\Phi \circ g \to g \circ \Phi$), in a way compatible with the monoid structure.

A structure of (strict) *equivariance* is when the above maps are isomorphisms.

If G is a group, then the above maps are automatically isomorphisms.

1.1.2. Formally, the notion of *right-lax equivariance* falls into the paradigm of right-lax module functors between two module categories over a given monoidal category: we can view G as a monoidal ∞-category.

Equivalently, this definition can be formalized as follows. Consider the corresponding simplicial object $G^\bullet$ in Spc, and consider the ∞-category

$$BG := \mathfrak{L}(G^\bullet),$$

see [Chapter A.1, Sect. 1.3.2] for the notation.

I.e., BG is a category with one object, whose monoid of endomorphisms is identified with G.

The datum of action of G on an ∞-category $\mathcal{C}$ is equivalent to that of a co-Cartesian fibration

$$\mathcal{C}_{BG} \to BG,$$

that fits into a pullback diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_{BG} \arrow[d] \\ * \arrow[r] & BG. \end{tikzcd}$$

A structure right-lax equivariance on Φ with respect to G is by definition a datum of extension of Φ to a functor

$$\Phi_{BG} : (\mathcal{C}_1)_{BG} \to (\mathcal{C}_2)_{BG}$$

over BG.

A structure of right-lax equivariance on Φ is a structure of (strict) equivariance if Φ_{BG} sends co-Cartesian arrows to co-Cartesian arrows. Equivalently, this is a natural transformation between the functors

$$BG \to \infty\text{-Cat},$$

classifying the co-Cartesian fibrations $(\mathcal{C}_1)_{BG}$ and $(\mathcal{C}_2)_{BG}$, respectively.

A structure of left-lax equivariance on Φ is a structure of right-lax equivariance on the functor

$$\Phi^{\mathrm{op}} : \mathcal{C}_1^{\mathrm{op}} \to \mathcal{C}_2^{\mathrm{op}}.$$

1.1.3. It is clear that the composition of functors, each endowed with a structure of right-lax (resp., left-lax) equivariance, has a structure of right-lax (resp., left-lax) equivariance.

It is also clear that if a functor Φ has a structure of right-lax (resp., left-lax) equivariance, then its left (resp., right) adjoint, if it exists, has a natural structure of left-lax (resp., right-lax) equivariance.

1.2. **Equivariance in algebraic geometry.** In this subsection we will adapt the notion of equivariant functor, where instead of just ∞-categories we consider contravariant functors on $\mathrm{Sch}^{\mathrm{aff}}$ with values in ∞-categories.

As a particular case, we will obtain the notion of left-lax or right-lax equivariant quasi-coherent sheaf on a prestack, equipped with an action of a monoid.

1.2.1. Let $\mathcal{C}$ be a presheaf of categories, i.e., a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Cat}\,.$$

Let $\mathcal{G}$ be a monoidal prestack, i.e., a monoid-object in PreStk, equivalently, a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Monoid}(\mathrm{Spc}).$$

Informally, an action of $\mathcal{G}$ on $\mathcal{C}$ is by definition a system of actions of $\mathcal{G}(S)$ on $\mathcal{C}(S)$ for $S \in \mathrm{Sch}^{\mathrm{aff}}$, compatible with pullbacks.

Formally, an action of $\mathcal{G}$ on $\mathcal{C}$ is a datum of a functor

$$\mathcal{C}_{B\mathcal{G}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Cat},$$

equipped with a natural transformation $\mathcal{C}_{B\mathcal{G}} \to B\mathcal{G}$ and a pullback diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_{B\mathcal{G}} \arrow[d] \\ \mathrm{pt} \arrow[r] & B\mathcal{G}, \end{tikzcd}$$

such that for every $S \in \mathrm{Sch}^{\mathrm{aff}}$, the corresponding functor

$$\mathcal{C}_{B\mathcal{G}}(S) \to B\mathcal{G}(S)$$

is a co-Cartesian fibration.

1.2.2. Let $\mathcal{C}_1, \mathcal{C}_2$ be two presheaves of categories, each equipped with an action of $\mathcal{G}$.

For a natural transformation $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$, a datum of *right-lax equivariance* (resp., *left-lax equivariance*) with respect to $\mathcal{G}$ is a compatible system of structures of right-lax equivariance (resp., left-lax equivariance) on the functors

$$\Phi_S : \mathcal{C}_1(S) \to \mathcal{C}_2(S).$$

1.2.3. A particular case of this situation is when $\mathcal{C}_1 = \mathcal{X} \in \mathrm{PreStk}$, i.e., is a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

In this case we can think of a functor Φ from $\mathcal{X}$ to $\mathcal{C} = \mathcal{C}_2$ as a section $p \in \mathcal{C}(\mathcal{X})$, where we regard $\mathcal{C}$ as a functor

$$\mathrm{PreStk}^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}$$

by right Kan-extending the original $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \overset{\mathcal{C}}{\to} \infty\text{-}\mathrm{Cat}$ along $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}$.

Thus, we obtain the notion of a section $p \in \mathcal{C}(\mathcal{X})$ to be *right-lax equivariant* (resp., *left-lax equivariant*, *(strictly) equivariant*) with respect to $\mathcal{G}$.

We will denote the resulting categories by

$$\mathcal{C}(\mathcal{X})^{\mathcal{G}_{\text{right-lax}}}, \quad \mathcal{C}(\mathcal{X})^{\mathcal{G}_{\text{left-lax}}} \text{ and } \mathcal{C}(\mathcal{X})^{\mathcal{G}},$$

respectively. We have the fully faithful embeddings

$$\mathcal{C}(\mathcal{X})^{\mathcal{G}_{\text{right-lax}}} \hookleftarrow \mathcal{C}(\mathcal{X})^{\mathcal{G}} \hookrightarrow \mathcal{C}(\mathcal{X})^{\mathcal{G}_{\text{left-lax}}}.$$

1.2.4. An example of the situation in Sect. 1.2.3 is when

$$\mathcal{C} = \mathrm{QCoh}^*,$$

where the action of $\mathcal{G}$ on QCoh^* is trivial.

Thus, for $\mathcal{X} \in \mathrm{PreStk}$ and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$, a datum of right-lax equivariance (resp., left-lax equivariance) with respect to $\mathcal{G}$ assigns to every $x \in \mathrm{Maps}(S, \mathcal{X})$ and $g \in \mathrm{Maps}(S, \mathcal{G})$ a map

$$x^*(\mathcal{F}) \to (g \cdot x)^*(\mathcal{F}) \quad \text{(resp., } (g \cdot x)^*(\mathcal{F}) \to x^*(\mathcal{F})\text{)},$$

in a way compatible with products of g's and pullbacks $S_1 \to S_2$.

We let $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}_{\text{right-lax}}}$ (resp., $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}_{\text{left-lax}}}$) denote the category of objects in $\mathrm{QCoh}(\mathcal{X})$, equipped with a structure of right-lax (resp., left-lax) equivariance with respect to $\mathcal{G}$. We let $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}}$ be the category of objects equipped with a structure of (strict) $\mathcal{G}$-equivariance.

1.2.5. Here are several other examples of presheaves $\mathcal{C}$ that we will use (for now, all of them have the trivial action of $\mathcal{G}$):

(i) Take $\mathcal{C}(S) = \mathbf{C} \otimes \mathrm{QCoh}(S)$ for a fixed DG category $\mathbf{C}$.

(ii) Take $\mathcal{C}(S) := \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(S))$ for a fixed symmetric monoidal DG category $\mathbf{O}$ and an operad $\mathcal{P}$.

(iii) Take $\mathcal{C}(S) := \mathrm{QCoh}(S)\text{-}\mathbf{mod}$, the category of module categories over $\mathrm{QCoh}(S)$.

(iv) Take $\mathcal{C}(S) := \mathrm{PreStk}_{/S}$. In this case, for a prestack $\mathcal{X}$, a map $p : \mathcal{X} \to \mathcal{C}$ amounts to a prestack $\mathcal{Y}$ over $\mathcal{X}$. Given a $\mathcal{G}$-action on $\mathcal{X}$, a datum of right-lax equivariance on p is equivalent to that of a lift of the given $\mathcal{G}$-action on $\mathcal{X}$ to a $\mathcal{G}$-action on $\mathcal{Y}$. This structure is a (strict) equivariance if and only if the square

$$\begin{CD} \mathcal{G} \times \mathcal{Y} @>{\text{action}}>> \mathcal{Y} \\ @VVV @VVV \\ \mathcal{G} \times \mathcal{X} @>{\text{action}}>> \mathcal{X} \end{CD}$$

is Cartesian.

1.3. **The category of filtered objects.** It is well-known that the formalism of equivariance with respect to the multiplicative group allows to give an algebro-geometric interpretation to the notion of filtered object in a given DG category.

We will review this construction in the present subsection.

1.3.1. Let $\mathbf{C}$ be a DG category. Recall the notation

$$\mathbf{C}^{\mathrm{Fil}} := \mathrm{Funct}(\mathbb{Z}, \mathbf{C}),$$

see [Chapter IV.2, Sect. 1.3].

1.3.2. Consider the presheaf of categories $\mathbf{C} \otimes \mathrm{QCoh}(-)$ from Example (i) in Sect. 1.2.5.

We will take our group-prestack $\mathcal{G}$ to be $\mathbb{G}_m$. We take $\mathcal{X} = \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication.

Proposition-Construction 1.3.3. *There is a canonical equivalence*

$$\mathbf{C}^{\mathrm{Fil}} \simeq \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}. \tag{1.1}$$

Proof. By [Ga3, Theorem 2.2.2], the natural functor

$$\mathbf{C} \otimes \left(\mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$$

is an equivalence. It is equally easy to see that the functor

$$\mathbf{C} \otimes \mathrm{Vect}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{Fil}}$$

is an equivalence. Hence, it is sufficient to treat the case $\mathbf{C} = \mathrm{Vect}$.

We construct the functor

$$\left(\mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m} \to \mathrm{Vect}^{\mathrm{Fil}}$$

as follows. Given $\mathcal{F} \in \left(\mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$ we define the corresponding functor $\mathbb{Z} \to \mathrm{Vect}$ by

$$n \mapsto \Gamma(\mathbb{A}^1, \mathcal{F}(n \cdot \{0\}))^{\mathbb{G}_m},$$

where the superscript $\mathbb{G}_m$ stands for taking $\mathbb{G}_m$-invariants, and $\mathcal{F}(n \cdot \{0\})$ means twisting $\mathcal{F}$ by the corresponding Cartier divisor.

The fact that this functor is an equivalence is a straightforward check.

□

1.3.4. Consider the functor

$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m} \to \mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1) \to \mathbf{C},$$

given by restriction to $\{1\} \in \mathbb{A}^1$. Under the identification (1.1), this functor corresponds to the functor of 'forgetting the filtration'

$$\mathbf{oblv}_{\mathrm{Fil}} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}.$$

1.4. **The category of graded objects.** In this subsection we will consider a variant of the material in Sect. 1.3, where instead of filtered objects we consider graded ones.

1.4.1. Consider the category
$$\mathbf{C}^{\mathrm{gr}} := \mathbf{C}^{\mathbb{Z}}.$$

As in Proposition 1.3.3, we have:
$$\mathbf{C}^{\mathbb{Z}} \simeq \mathbf{C}^{\mathbb{G}_m},$$
i.e., this is the $\mathbb{G}_m$-equivariant category for the presheaf of categories $\mathbf{C}\otimes\mathrm{QCoh}(-)$ over $\mathcal{X} = \mathrm{pt}$.

1.4.2. The forgetful functor
$$\mathbf{C}^{\mathbb{G}_m} \to \mathbf{C}$$
corresponds to the functor of 'forgetting the grading'
$$\mathbf{oblv}_{\mathrm{gr}} : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}.$$

1.4.3. The adjoint functors
$$(\mathrm{gr} \to \mathrm{Fil}) : \mathbf{C}^{\mathrm{gr}} \rightleftarrows \mathbf{C}^{\mathrm{Fil}} : \mathrm{Rees}$$
([Chapter IV.2, Sect. 1.3.3]) correspond to the functors of pullback and push-forward along the projection $\mathbb{A}^1 \to \mathrm{pt}$.

1.4.4. Consider the functor of the 'associated graded'
$$\text{ass-gr} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{gr}},$$
see [Chapter IV.2, Sect. 1.3.4].

In terms of the identification
$$\mathbf{C}^{\mathrm{Fil}} \simeq \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}, \tag{1.2}$$
the functor ass-gr corresponds to
$$\mathcal{F} \mapsto (\mathrm{Id}_{\mathbf{C}} \otimes i_0)^*(\mathcal{F}),$$
where $i_0 : \{0\} \to \mathbb{A}^1$.

1.5. **Positive and negative filtrations.** It turns out that if in the discussion in Sect. 1.3, we replace the group $\mathbb{G}_m$ by the monoid $\mathbb{A}^1$, the corresponding extra structure will single out non-negative filtrations (in the case of left-lax equivariance) or non-positive filtrations (in the case of right-lax equivariance).

1.5.1. Consider again the presheaf of categories from Example (i) in Sect. 1.2.5.

Let us now take the monoid-prestack $\mathcal{G} = \mathbb{A}^1$, where $\mathbb{A}^1$ is a monoid with respect to the operation of *multiplication*. We take $\mathcal{X} = \mathbb{A}^1$, where $\mathbb{A}^1$ acts on itself by multiplication. We consider $\mathbb{G}_m$ as a sub-monoid of $\mathbb{A}_1$.

We have:

Lemma 1.5.2.
(a) *The forgetful functor*
$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$$
is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{Fil},\geq 0} \subset \mathbf{C}^{\mathrm{Fil}}$.
(b) *The forgetful functor*
$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{right-lax}}} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{Fil},\leq 0} \subset \mathbf{C}^{\mathrm{Fil}}$.

1.5.3. We also have the following graded analog of Lemma 1.5.2:

Lemma 1.5.4.

(a) *The forgetful functor*

$$\mathbf{C}^{\mathbb{A}^1_{\text{left-lax}}} \to \mathbf{C}^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{gr},\geq 0} \subset \mathbf{C}^{\mathrm{gr}}$.

(b) *The forgetful functor*

$$\mathbf{C}^{\mathbb{A}^1_{\text{right-lax}}} \to \mathbf{C}^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{gr},\leq 0} \subset \mathbf{C}^{\mathrm{gr}}$.

1.6. **Scaling the structure of a $\mathcal{P}$-algebra.** In this subsection we make a digression and explain that the construction in [Chapter IV.2, Sect. 1.4] of endowing an algebra B over an operad $\mathcal{P}$ with a filtration can be viewed as the operation of 'scaling' the structure maps $\mathcal{P}(n) \otimes B^{\otimes n} \to B$.

1.6.1. Let $\mathbf{O}$ be a symmetric monoidal category, and let $\mathcal{P}$ be an operad. Recall the presheaf of categories

$$\mathcal{C}(S) := \mathbf{O} \otimes \mathrm{QCoh}(S),$$

endowed with the trivial action of a monoid $\mathcal{G}$.

The operad $\mathcal{P}$ defines an algebra object in the monoidal category of $\mathcal{G}$-equivariant endomorphisms of $\mathcal{C}$. It follows that for a prestack $\mathcal{X}$, equipped with an action of $\mathcal{G}$, the forgetful functor

$$(\mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})))^{\mathbb{A}^1_{\text{left-lax}}} \to (\mathcal{C}(\mathcal{X}))^{\mathbb{A}^1_{\text{left-lax}}}$$

is monadic, with the corresponding monad being given by the action of $\mathcal{P}$ on $(\mathcal{C}(\mathcal{X}))^{\mathbb{A}^1_{\text{left-lax}}}$ as a symmetric monoidal DG category. Hence, we obtain an identification

$$(\mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})))^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}).$$

1.6.2. We apply this to $\mathcal{G} = \mathcal{X} = \mathbb{A}^1$, acting on itself by multiplication. Thus, we obtain an identification

$$(\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathcal{P}\text{-Alg}((\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}).$$

Using Lemma 1.5.2(a), we identify

$$\mathcal{P}\text{-Alg}((\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}) \simeq \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

Thus, we obtain a canonical equivalence:

$$(\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}). \tag{1.3}$$

1.6.3. Recall the functor

$$\mathrm{AddFil} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

see [Chapter IV.2, Sect. 1.4.2].

We obtain that it gives rise to a functor

$$\mathrm{Scale}^{\mathbb{A}^1_{\text{left-lax}}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to (\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\text{left-lax}}}.$$

Composing with the forgetful functor

$$(\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\text{left-lax}}} \to \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)),$$

we obtain a functor

$$\mathrm{Scale} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)).$$

Sometimes we will use the short-hand notation

$$B_{\mathrm{scaled}} := \mathrm{Scale}(B) \text{ and } B_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} := \mathrm{Scale}^{\mathbb{A}^1_{\text{left-lax}}}(B).$$

1.6.4. The functor Scale has the following properties:

- $\mathbf{oblv}_{\mathcal{P}}(B_{\mathrm{scaled}}) \simeq \mathbf{oblv}_{\mathcal{P}}(B) \otimes \mathcal{O}_{\mathbb{A}^1}$;
- $i_\lambda^*(B_{\mathrm{scaled}}) \simeq B$ for any $0 \neq \lambda \in \mathbb{A}^1$;
- $i_0^*(B_{\mathrm{scaled}}) \simeq \mathbf{triv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathcal{P}}(B)$.

Remark 1.6.5. One can endow the functor Scale with a structure of *associativity* with respect to the monoid structure on $\mathbb{A}^1$. This gives rise to a *non-trivial* action of the monoid $\mathbb{A}^1$ on presheaf of categories $\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(-))$.

2. Deformation to the normal bundle

In this section we introduce a key construction that deforms a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$ to its normal bundle. It is a derived analog of the deformation of a closed embedding to its normal cone.

In subsequent sections, this procedure will give rise to naturally defined filtrations on various objects constructed out of Lie algebroids (e.g., the universal enveloping algebra of a Lie algebroid).

The geometric input into the main construction in this section, explained in Sect. 2.2, was suggested to us by J. Lurie.

2.1. **The idea.** Before we give the formal construction, let us explain its idea. It is the following: given a nil-closed embedding $\mathcal{X} \to \mathcal{Y}$ we will construct the deformation of $\mathcal{Y}$ to the normal bundle by deforming the corresponding groupoid $\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$.

The sought-for $\mathbb{A}^1$-family of groupoids of $\mathcal{X}$ will obtained by mapping into the original $\mathcal{X}$ a particular $\mathbb{A}^1$-family of groupoids in the category $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, denoted

$$(\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda, \quad \lambda \in \mathbb{A}^1.$$

In this subsection we will informally describe what this family looks like.

We draw the reader's attention to the fact that groupoid objects in the category $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ are not very familiar gadgets.

2.1.1. For any $\lambda \in \mathbb{A}^1$, the scheme $(\mathrm{Bifurc}^0_{\mathrm{scaled}})_\lambda$ of objects in $(\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda$ is just pt. The scheme $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_\lambda$ of 1-morphisms is described as follows: if $\lambda \neq 0$, then

$$(\mathrm{Bifurc}^1_{\mathrm{scaled}})_\lambda = \{\lambda\} \sqcup \{-\lambda\} \subset \mathbb{A}^1.$$

I.e., $(\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda$ is the free groupoid in $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ with the scheme of objects being pt.

When $\lambda = 0$, then $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_0$ is the scheme of dual numbers $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$. I.e., we should think of $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ the limit of $\{\lambda\} \cup \{-\lambda\}$ as $\lambda \to 0$.

2.1.2. For any $\lambda \in \mathbb{A}^1$ and $\mathcal{X} \in \mathrm{PreStk}$, we obtain a groupoid object in PreStk by considering the prestack of maps from $(\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda$ into it:

$$\mathbf{Maps}((\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda, \mathcal{X}).$$

Note that for $\lambda \neq 0$, the groupoid $\mathbf{Maps}((\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda, \mathcal{X})$ is just $\mathcal{X} \times \mathcal{X} \rightrightarrows \mathcal{X}$.

However, for $\lambda = 0$, the groupoid $\mathbf{Maps}((\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})_\lambda, \mathcal{X})$ is the total space of the tangent complex on $\mathcal{X}$.

2.2. **A family of co-groupoids.** We will now spell out the construction described above in a more formal way.

2.2.1. In the category of connective DG algebras, consider the Čech nerve, denoted $A^\bullet$, of the map

$$k \to 0.$$

Explicitly, $A^0 = k$, $A^1 = k \oplus k$ and, in general, $A^i = \underbrace{k \oplus ... \oplus k}_{i+1}$. In particular, all A^i are classical.

2.2.2. We now claim that the groupoid $A^\bullet$ in classical commutative algebras can be naturally lifted to one in the category of *non-negatively filtered* classical commutative algebras, denoted $(A^{\mathrm{Fil}})^\bullet$.

Indeed, we define the filtration on A^i to be

$$(A^i)_n = \begin{cases} A \text{ for } n \geq 1 \\ k \text{ for } n = 0. \end{cases}$$

In particular, $(A^0)_n = A^0$ for all n.

Remark 2.2.3. Note that $(A^{\mathrm{Fil}})^1$ is the the filtered algebra that we used in [Chapter IV.2, Sect. 1.4.1] in order to construct a canonical filtration on algebras over operads.

2.2.4. Note that

$$\text{ass-gr}((A^{\mathrm{Fil}})^\bullet) \simeq k \oplus \epsilon(B^\bullet(k)), \quad \epsilon^2 = 0, \deg(\epsilon) = 1. \tag{2.1}$$

In other words, $\text{ass-gr}((A^{\mathrm{Fil}})^\bullet)$ is the classifying space simplicial object in the category of classical augmented commutative algebra, corresponding to the commutative group object

$$k[\epsilon]/\epsilon^2, \quad \deg(\epsilon) = 1.$$

2.2.5. Applying the equivalence of (1.3), we turn the groupoid $(A^{\mathrm{Fil}})^\bullet$ in the category of non-negatively filtered commutative connective DG algebras into a groupoid, denoted

$$A^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$$

in the category of commutative connective algebras in $\mathrm{QCoh}(\mathbb{A}^1)$, equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

Denote by $A^\bullet_{\mathrm{scaled}}$ the groupoid in the category of commutative connective algebras in $\mathrm{QCoh}(\mathbb{A}^1)$, obtained from $A^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ by forgetting the left-lax equivariance with respect to $\mathbb{A}^1$.

Explicitly,

$$A^0_{\mathrm{scaled}} = k[u],$$

and

$$A^1_{\mathrm{scaled}} = \mathrm{Spec}(k[u,\epsilon]/(u-\epsilon)\cdot(u+\epsilon)).$$

The two maps

$$A^1_{\mathrm{scaled}} \rightrightarrows A^0_{\mathrm{scaled}}$$

are given by

$$\epsilon \mapsto u \text{ and } \epsilon \mapsto -u,$$

respectively.

The degeneracy map $A^0_{\mathrm{scaled}} \to A^1_{\mathrm{scaled}}$ is $u \mapsto u$. The inverse for the groupoid is the map

$$A^1_{\mathrm{scaled}} \to A^1_{\mathrm{scaled}}, \quad u \mapsto u, \epsilon \mapsto -\epsilon.$$

2.2.6. Passing to spectra (in the sense of algebraic geometry)), we obtain a groupoid object, denoted

$$\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{right-lax}}}$$

in the category

$$\left(((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\text{right-lax}}}\right)^{\mathrm{op}}.$$

Let

$$s,t : \mathrm{Bifurc}^0_{\mathrm{scaled},\mathbb{A}^1_{\text{right-lax}}} \rightrightarrows \mathrm{Bifurc}^1_{\mathrm{scaled},\mathbb{A}^1_{\text{right-lax}}}$$

denote the two face maps (source and target, respectively).

Let $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ denote the groupoid object in the category

$$\left((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1}\right)^{\mathrm{op}},$$

obtained from $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{right-lax}}}$ by forgetting the the left-lax equivariance with respect to $\mathbb{A}^1$.

Note that by construction, for any $0 \neq \lambda \in \mathbb{A}^1$, we have

$$(\mathrm{Bifurc}^i_{\mathrm{scaled}})_\lambda = \underbrace{\mathrm{pt} \sqcup ... \sqcup \mathrm{pt}}_{i+1},$$

and the groupoid structure is that of the Čech nerve of the map

$$\emptyset \to \mathrm{pt},$$

viewed as a morphism in $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$.

Remark 2.2.7. Note that according to Sect. 1.2.5, Example (iv), the datum of upgrading of $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ to $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{right-lax}}}$ amounts simply to the action of the monoid $\mathbb{A}^1$ on $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$, compatible with the projection to $\mathbb{A}^1$.

2.3. **The canonical deformation of a groupoid.** We will now use $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ to deform the Čech nerve of a nil-isomorphism in $\mathrm{PreStk}_{\text{laft-def}}$ to the total space of its relative tangent complex.

2.3.1. Let $\mathcal{X} \to \mathcal{Y}$ be a map in PreStk, and let $\mathcal{R}^\bullet$ be its Čech nerve. Consider the $\mathbb{A}^1$-family of simplicial objects of PreStk equal to

$$\mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}(\mathcal{X} \times \mathrm{Bifurc}^\bullet_{\mathrm{scaled}}) \underset{\mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}(\mathcal{Y}\times \mathrm{Bifurc}^\bullet_{\mathrm{scaled}})}{\times} (\mathcal{Y} \times \mathbb{A}^1), \tag{2.2}$$

where the notation Weil is as in Sect. A.1.

I.e., for an affine scheme S, a point of the space of maps from S to i-simplices of (2.2) consists of the data of:

- a map $S \to \mathbb{A}^1$;
- a map $y : S \to \mathcal{Y}$;
- a map $S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to \mathcal{X}$;
- an identification of the composition $S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to \mathcal{X} \to \mathcal{Y}$ with the map

$$S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to S \xrightarrow{y} \mathcal{Y}.$$

Note that, by construction, the simplicial object (2.2) in $(\mathrm{PreStk})_{/\mathbb{A}^1}$ is augmented by $\mathcal{Y}$.

By Sects. A.2.3 and A.2.1, we have:

Lemma 2.3.2. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ belong to* $\mathrm{PreStk}_{\mathrm{laft}}$ *(resp.,* $\mathrm{PreStk}_{\text{laft-def}}$*). Then the same will be true for the terms of the simplicial prestack* (2.2).

2.3.3. Assume now that $\mathcal{X}$ and $\mathcal{Y}$ belong to $\mathrm{PreStk}_{\text{laft-def}}$ and that $\mathcal{X} \to \mathcal{Y}$ is a nil-isomorphism. Denote the simplicial prestack (2.2) by $\mathcal{R}^\bullet_{\mathrm{scaled}}$.

Denote

$$\mathcal{R}_{\mathrm{scaled}} := \mathcal{R}^1_{\mathrm{scaled}}.$$

We have a canonical (unit) map $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{R}^\bullet_{\mathrm{scaled}}$, and it is clear that this map is a nil-isomorphism. We claim:

Lemma 2.3.4. *The simplicial object $\mathcal{R}^\bullet_{\mathrm{scaled}}$ is a groupoid object of* $(\mathrm{PreStk}_{\text{laft-def}})_{/\mathbb{A}^1}$.

Proof. We need to show that for any $n \geq 2$, the canonical map

$$\mathcal{R}^n_{\mathrm{scaled}} \to \underbrace{\mathcal{R}_{\mathrm{scaled}} \underset{\mathcal{X}\times\mathbb{A}^1}{\times} ... \underset{\mathcal{X}\times\mathbb{A}^1}{\times} \mathcal{R}_{\mathrm{scaled}}}_{n \text{ times}} \tag{2.3}$$

is an isomorphism.

By [Chapter III.1, Proposition 8.3.2], it is enough to show that the map in question induces an isomorphism of the tangent spaces along the unit section.

By (A.4), for an affine scheme S and a point

$$S \xrightarrow{x,\lambda} \mathcal{X} \times \mathbb{A}^1,$$

the pullback of the tangent space of the left-hand side of (2.3) relative to $\mathcal{Y}$ identifies with

$$T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^n_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^n_{\mathrm{scaled}}}),$$

while that of the right-hand side with

$$T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \underset{T_x(\mathcal{X}/\mathcal{Y})\otimes k[u]}{\times} \cdots$$
$$\cdots \underset{T_x(\mathcal{X}/\mathcal{Y})\otimes k[u]}{\times} T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}).$$

Now, the required assertion follows from the fact that

$$\Gamma(\mathrm{Bifurc}^n_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^n_{\mathrm{scaled}}}) \simeq \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \underset{k[u]}{\times} \cdots \underset{k[u]}{\times} \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}).$$

□

2.3.5. Let us calculate the fiber $\mathcal{R}^\bullet_0$ of $\mathcal{R}^\bullet_{\mathrm{scaled}}$ over $0 \in \mathbb{A}^1$. First, by Sect. 2.2.4, the groupoid $\mathcal{R}^\bullet_0$ is actually a group, and furthermore a commutative group-object in $\mathrm{FormMod}_{/\mathcal{X}}$.

We now claim:

Proposition 2.3.6. *The commutative group-object* $\mathcal{R}^\bullet_0 \in \mathrm{FormMod}_{/\mathcal{X}}$ *identifies canonically with* $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y}))$.

Proof. We will consider both sides as functors on the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

The commutative group-object $\mathcal{R}^\bullet_0 \in \mathrm{FormMod}_{/\mathcal{X}}$ assigns to $\mathcal{Z} \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ the space equal to the fiber of the restriction map

$$* \underset{\mathrm{Maps}(\mathcal{Z},\mathcal{X})}{\times} \mathrm{Maps}(\mathcal{Z}\times\mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{X}) \underset{\mathrm{Maps}(\mathcal{Z}\times\mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{Y})}{\times} \mathrm{Maps}(\mathcal{Z},\mathcal{Y}) \to$$
$$\to * \underset{\mathrm{Maps}(\mathcal{X},\mathcal{X})}{\times} \mathrm{Maps}(\mathcal{X}\times\mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{X}) \underset{\mathrm{Maps}(\mathcal{X}\times\mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{Y})}{\times} \mathrm{Maps}(\mathcal{X},\mathcal{Y}), \tag{2.4}$$

where the map in the above formula is given by restriction along $\mathcal{X} \to \mathcal{Z}$, and where the commutative group structure coming from the structure of commutative group on

$$\mathrm{Spec}(k[\lambda]/\lambda^2) \in ((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{pt}\,/})^{\mathrm{op}}.$$

By [Chapter IV.3, Corollary 3.6.7], the commutative group-object $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y}))$ in the category $\mathrm{FormMod}_{/\mathcal{X}}$ assigns to

$$(\mathcal{Z} \overset{\pi}{\to} \mathcal{X}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

the space

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathrm{coFib}(\omega_{\mathcal{X}} \to \pi^{\mathrm{IndCoh}}_*(\omega_{\mathcal{Z}})), T(\mathcal{X}/\mathcal{Y})). \tag{2.5}$$

Note that

$$\mathcal{Z}\times\mathrm{Spec}(k[\lambda]/\lambda^2) \simeq \mathrm{RealSplitSqZ}(\omega_{\mathcal{Z}}),$$

see [Chapter IV.3, Sect. 3.7] for the notation.

Hence, we can rewrite the fiber of the map (2.4) as

$$\mathrm{Fib}\left(\mathrm{Maps}_{\mathcal{Z}/\,/\mathcal{Y}}(\mathrm{RealSplitSqZ}(\omega_{\mathcal{Z}}),\mathcal{X}) \to \mathrm{Maps}_{\mathcal{X}/\,/\mathcal{Y}}(\mathrm{RealSplitSqZ}(\omega_{\mathcal{X}}),\mathcal{X})\right),$$

and further as

$$\mathrm{Fib}\left(\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Z})}(\omega_{\mathcal{Z}}, T(\mathcal{X}/\mathcal{Y})|_{\mathcal{Z}}) \to \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\omega_{\mathcal{X}}, T(\mathcal{X}/\mathcal{Y}))\right),$$

identifies with (2.5), as required.

□

2.4. **Deformation of a formal moduli problem to the normal bundle.** We will now use the deformation

$$\mathcal{R}^\bullet \rightsquigarrow \mathcal{R}^\bullet_{\text{scaled}}$$

to construct the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}^\bullet_{\text{scaled}}.$$

2.4.1. Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft-def}}$ and let $\mathcal{Y}$ be an object of $\text{FormMod}_{\mathcal{X}/}$.

Consider the formal groupoid $\mathcal{R}^\bullet_{\text{scaled}}$ over $\mathcal{X}$ (and relative to $\mathcal{Y}\times\mathbb{A}^1$). Applying [Chapter IV.1, Theorem 2.3.2], we obtain an object

$$\mathcal{Y}_{\text{scaled}} \in \text{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1},$$

i.e., an $\mathbb{A}^1$-family of objects $\mathcal{Y}_{\text{scaled}} \in \text{FormMod}_{\mathcal{X}/}$.

2.4.2. By construction, the fiber $\mathcal{Y}_{\{\lambda\}}$ at $0 \neq \lambda \in \mathbb{A}^1$ identifies with the original $\mathcal{Y}$.

On the other hand, by Proposition 2.3.6 the fiber $\mathcal{Y}_0$ at $0 \in \mathbb{A}^1$ identifies canonically with

$$\text{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]),$$

where $T(\mathcal{X}/\mathcal{Y})[1] =: N(\mathcal{X}/\mathcal{Y})$ can be thought of as the normal bundle to $\mathcal{X}$ in $\mathcal{Y}$.

2.4.3. *Example.* Let us take $\mathcal{Y} = \mathcal{X}_{\text{dR}}$. Then the object

$$(\mathcal{X}_{\text{dR}})_{\text{scaled}} \in \text{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}$$

is the Dolbeault degeneration of $\mathcal{X}_{\text{dR}}$ to $\text{Vect}_{\mathcal{X}}(T(\mathcal{X})[1])$.

Remark 2.4.4. One can show that when $\mathcal{X} = X$ is a classical scheme, and $\mathcal{Y}$ is the obtained as the formal completion of a classical scheme Y along a *regular* closed embedding $X \to Y$, then $\mathcal{Y}_{\text{scaled}}$ is a nil-schematic ind-scheme (i.e., formal scheme) equal to the formal completion along $X \times \mathbb{A}^1$ of the scheme given by the usual deformation of Y to the normal cone.

2.5. **The action of the monoid $\mathbb{A}^1$.** Above to any $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$ we have assigned an $\mathbb{A}^1$-family $\mathcal{Y}_{\text{scaled}}$ of objects of $\text{FormMod}_{\mathcal{X}/}$. However, this family possesses an extra structure: that of left-lax equivariance with respect to the monoid $\mathbb{A}^1$.

According to Sect. 1.5, this is exactly the kind of structure that allows to endow linear objects attached to $\mathcal{Y}$ with a non-negative filtration. The latter observation will be extensively used in the sequel.

2.5.1. Recall that by construction, the groupoid

$$\text{Bifurc}^\bullet_{\text{scaled}} \in \left((\text{Sch}^{\text{aff}})_{/\mathbb{A}^1}\right)^{\text{op}}$$

could be naturally upgraded to

$$\text{Bifurc}^\bullet_{\text{scaled},\mathbb{A}^1_{\text{right-lax}}} \in \left(((\text{Sch}^{\text{aff}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\text{right-lax}}}\right)^{\text{op}}.$$

Consider now the functor

$$\text{FormMod}_{\mathcal{X}\times-//\mathcal{Y}\times-} : (\text{Sch}^{\text{aff}}_{\text{aft}})^{\text{op}} \to \infty\text{-Cat}, \quad S \mapsto \text{FormMod}_{\mathcal{X}\times S//\mathcal{Y}\times S}.$$

By transport of structure, we obtain that for $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$, the object

$$\mathcal{Y}_{\text{scaled}} \in \text{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1},$$

viewed as a natural transformation

$$\mathbb{A}^1 \to \mathrm{FormMod}_{\mathcal{X}\times -//\mathcal{Y}\times -}$$

has a natural structure of *left-lax equivariance* with respect to $\mathbb{A}^1$, where the target presheaf of categories $\mathrm{FormMod}_{\mathcal{X}\times -//\mathcal{Y}\times -}$ is endowed with the trivial action of $\mathbb{A}^1$.

Thus, we obtain a well-defined object

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

2.5.2. Restricting along $\{0\} \to \mathbb{A}^1$, we obtain the object

$$\mathcal{Y}_{0,\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}/}\right)^{\mathbb{A}^1_{\text{left-lax}}},$$

that according to Proposition 2.3.6 identifies with

$$\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]),$$

where $T(\mathcal{X}/\mathcal{Y})$ is regarded as an object of

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},=1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

3. The canonical filtration on a Lie algebroid

In this section we will show that any Lie algebroid on $\mathcal{X}$ gives rise, in a canonical way, to a *filtered Lie algebroid*. This construction is a generalization of the construction in Sect. 1.6 that assigns to a Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$ (or any symmetric monoidal DG category) a filtered Lie algebra.

The associated graded of this filtration will yield the trivial Lie algebroid, and this fact will be subsequently used to establish various properties of formal moduli problems.

When working in the setting of classical algebraic geometry, the above filtered structure can be constructed 'by hand'. However, in the context of derived algebraic geometry we will use the deformation to the normal bundle to produce it.

3.1. **Deformation to the normal bundle and Lie algebroids.** In this subsection we will adapt the material of Sect. 2.5 to the language of Lie algebroids.

3.1.1. Consider the presheaf of categories

$$\mathrm{LieAlgbroid}(\mathcal{X}\times -/-) : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \infty\text{-Cat}, \quad S \mapsto \mathrm{LieAlgbroid}(\mathcal{X}\times S/S).$$

We obtain that for any $\mathfrak{L} \in \mathrm{LieAlgbroid}(\mathcal{X})$ there is a canonically defined object

$$\mathfrak{L}^{\mathrm{Fil}} \in \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

Moreover, this assignment is functorial in $\mathfrak{L}$. We denote the resulting functor

$$\mathrm{LieAlgbroid}(\mathcal{X}) \to \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}$$

by AddFil.

3.1.2. Let us denote by ass-gr the functor

$$\left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}\to(\mathrm{LieAlgbroid}(\mathcal{X}))^{\mathbb{A}^1_{\text{left-lax}}},$$

given by taking the fiber at $0\in\mathbb{A}^1$.

By Sect. 2.4.2, the composite functor

$$\text{ass-gr}\circ\mathrm{AddFil}:\mathrm{LieAlgbroid}(\mathcal{X})\to\mathrm{LieAlgbroid}(\mathcal{X})$$

equals the composition

$$\mathrm{LieAlgbroid}(\mathcal{X})\overset{\mathbf{oblv}_{\mathrm{LieAlgbroid}}}{\longrightarrow}\mathrm{IndCoh}(\mathcal{X})\overset{\deg=1}{\longrightarrow}\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}}\simeq\mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}\to$$
$$\overset{\mathbf{triv}_{\mathrm{Lie}}}{\longrightarrow}\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}})\simeq\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))^{\mathbb{A}^1_{\text{left-lax}}}\overset{\mathrm{diag}}{\longrightarrow}\mathrm{LieAlgbroid}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

Remark 3.1.3. As in Remark 1.6.5, one can show that the above functor

$$\mathrm{AddFil}:\mathrm{LieAlgbroid}(\mathcal{X})\to\left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}$$

is part of a richer structure. Namely, the functor

$$\mathrm{LieAlgbroid}(\mathcal{X}\times -/-):(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}\to\infty\text{-Cat}$$

carries a canonical action of the monoid $\mathbb{A}^1$.

3.2. **Compatibility with the forgetful functor.** We shall now study how the above canonical filtration on a Lie algebroid is compatible with the forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}:\mathrm{LieAlgbroid}(\mathcal{X})\to\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

3.2.1. Consider presheaf of categories

$$\mathrm{IndCoh}(X\times -)_{/T(\mathcal{X})|_{\mathcal{X}\times -}},\quad(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}\to\infty\text{-Cat},\quad S\mapsto\mathrm{IndCoh}(X\times S)_{/T(\mathcal{X})|_{\mathcal{X}\times S}}.$$

The functor $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ defines a natural transformation

$$\mathrm{LieAlgbroid}(\mathcal{X}\times -/-)\to\mathrm{IndCoh}(X\times -)_{/T(\mathcal{X})|_{\mathcal{X}\times -}}.$$

We endow $\mathrm{IndCoh}(X\times -)_{/T(\mathcal{X})|_{\mathcal{X}\times -}}$ with the trivial action of the monoid $\mathbb{A}^1$, and the above natural transformation is (obviously) $\mathbb{A}^1$-equivariant.

In particular, we obtain a functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}:\left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}\to\left(\mathrm{IndCoh}(X\times\mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

3.2.2. Note that by Lemma 1.5.2(a), the category $\left(\mathrm{IndCoh}(X\times\mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}$ identifies with

$$(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}. \tag{3.1}$$

Above we view $T(\mathcal{X})$ as an object of $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$ via the functor $(\mathrm{gr}\to\mathrm{Fil})\circ(\deg=0)$, i.e.,

$$\mathrm{IndCoh}(\mathcal{X})\simeq\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0,\leq 0}\subset\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}.$$

3.2.3. Recall now that in [Chapter IV.4, Sect. 5.3.5], we defined a functor

$$\mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

Namely, for $\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X}) \in \mathrm{IndCoh}(X)_{/T(\mathcal{X})}$, the underlying object $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X}))_{\text{scaled}}$ of $\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}$ is given by

$$\mathcal{F}|_{\mathcal{X}\times\mathbb{A}^1} \xrightarrow{\gamma_{\text{scaled}}} T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1},$$

where the value of γ_{scaled} over $\lambda \in \mathbb{A}^1$ equals $\lambda \cdot \gamma$.

The structure of left-lax $\mathbb{A}^1$-equivariance on $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X}))_{\text{scaled}}$ is defined naturally. Denote this functor by AddFil.

3.2.4. In terms of the identification (3.1), the functor AddFil sends $\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X})$ to

$$(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 1)(\mathcal{F}) \xrightarrow{\gamma} (\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 0)(T(\mathcal{X})).$$

Here for $i \geq 0$, we recall that $(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = i)$ denotes the functor

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq i,\leq i} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq i} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$$

(i.e., we take an object of $\mathrm{IndCoh}(\mathcal{F})$ and place it in degree i).

3.2.5. The goal of this section is to establish the following:

Proposition 3.2.6. *The following diagram of functors commutes:*

$$\begin{tikzcd}
\mathrm{LieAlgbroid}(\mathcal{X}) \arrow[r, "\mathrm{AddFil}"] \arrow[d, "\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}"'] & \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \arrow[d, "\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}"] \\
\mathrm{IndCoh}(X)_{/T(\mathcal{X})} \arrow[r, "\mathrm{AddFil}"] & \left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}.
\end{tikzcd} \tag{3.2}$$

Remark 3.2.7. Note that by adjunction from the commutative diagram (3.2), we obtain a diagram that commutes *up to a natural transformation*:

$$\begin{tikzcd}
\mathrm{LieAlgbroid}(\mathcal{X}) \arrow[r, "\mathrm{AddFil}"] & \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \\
\mathrm{IndCoh}(X)_{/T(\mathcal{X})} \arrow[r, "\mathrm{AddFil}"] \arrow[u, "\mathbf{free}_{\mathrm{LieAlgbroid}}"] & \left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}. \arrow[u, "\mathbf{free}_{\mathrm{LieAlgbroid}}"']
\end{tikzcd} \tag{3.3}$$

We note, however, that the above natural transformation is *not* an isomorphism. Indeed, the two circuits give a different result even after applying the functor ass-gr.

I.e., the structure of filtered Lie algebroid on filtration on $\mathbf{free}_{\mathrm{LieAlgbroid}}(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X}))$, given by the construction in [Chapter IV.4, Sect. 5.3] is *different* from the canonical filtration that exists on an arbitrary Lie algebroid, given by the construction in Sect. 3.1.

3.3. **Proof of Proposition 3.2.6.**

3.3.1. For $\mathfrak{L} \in \mathrm{LieAlgbroid}(\mathcal{X})$ the object

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathrm{AddFil}(\mathfrak{L}) \in \Big(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\Big)^{\mathbb{A}^1_{\text{left-lax}}}$$

can be described as follows.

Let $\mathcal{Y}$ be the object of $\mathrm{FormMod}_{\mathcal{X}/}$, corresponding to $\mathfrak{L}$. Consider the corresponding prestack

$$\mathcal{R}_{\mathrm{scaled}} := \mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{X} \times \mathrm{Bifurc}^1_{\mathrm{scaled}}) \underset{\mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{Y}\times\mathrm{Bifurc}^1_{\mathrm{scaled}})}{\times} (\mathcal{Y} \times \mathbb{A}^1),$$

equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

Consider the object

$$T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X}\times\mathbb{A}^1} \in \mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1),$$

where $\mathcal{R}_{\mathrm{scaled}} \to \mathcal{X} \times \mathbb{A}^1$ is induced by the map $t : \mathbb{A}^1 \to \mathrm{Bifurc}^1_{\mathrm{scaled}}$. The map $s : \mathbb{A}^1 \to \mathrm{Bifurc}^1_{\mathrm{scaled}}$ induces a map

$$T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X}\times\mathbb{A}^1} \to T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1},$$

and the resulting object of $\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)_{/\mathrm{IndCoh}(X\times\mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}}$ naturally lifts to

$$\Big(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\Big)^{\mathbb{A}^1_{\text{left-lax}}},$$

which is our $\mathbf{oblv}_{\mathrm{LieAlgbroid}} \circ \mathrm{AddFil}(\mathfrak{L})$.

We need to show that the above object is obtained from the tautological map

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})$$

by the scaling procedure of Sect. 3.2.3.

3.3.2. We identify

$$T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X}\times\mathbb{A}^1} \simeq \mathrm{Fib}\left(T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{Y} \times \mathbb{A}^1)|_{\mathcal{X}\times\mathbb{A}^1} \to T(\mathcal{X} \times \mathbb{A}^1/\mathcal{Y} \times \mathbb{A}^1)\right),$$

which, by (A.4), identifies with

$$T(\mathcal{X}/\mathcal{Y}) \otimes \mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{t^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right),$$

and its map to

$$\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}} \simeq T(\mathcal{X}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$$

identifies with

$$\begin{aligned} T(\mathcal{X}/\mathcal{Y}) \otimes \mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{t^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right) \to \\ \to T(\mathcal{X}/\mathcal{Y}) \otimes \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{s^*} \\ \to T(\mathcal{X}/\mathcal{Y}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \to T(\mathcal{X}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}). \end{aligned}$$

3.3.3. Thus, we need to show that the composite arrow

$$\mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \overset{t^*}{\to} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right) \to$$
$$\to \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \overset{s^*}{\to} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}),$$

viewed as an object of

$$(\mathrm{QCoh}(\mathbb{A}^1)_{/\mathcal{O}_{\mathbb{A}^1}})^{\mathbb{A}^1_{\text{left-lax}}},$$

is obtained by the scaling procedure of Sect. 3.2.3 from the identity map $\mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$.

3.3.4. We identify the above map with

$$\mathrm{Fib}((A^{\mathrm{Fil}})^1 \to k) \to (A^{\mathrm{Fil}})^1 \to k,$$

where $(A^{\mathrm{Fil}})^1$ is the filtered algebra from Sect. 2.2.2. Here the two maps $k \oplus k \simeq A^1 \rightrightarrows k \oplus k$ are the projection on the first and the second copy of k

The resulting map in $\mathrm{Vect}^{\mathrm{Fil},\geq 0}$ is

$$(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 1)(k) \to (\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 0)(k),$$

as required.

□

4. The case of groups

In this section we will show that assignment $\mathfrak{L} \rightsquigarrow \mathfrak{L}^{\mathrm{Fil}}$ of Sect. 3.1 reproduces in the case of Lie algebras the construction of scaling the Lie algebra structure

$$\mathfrak{h} \rightsquigarrow \mathfrak{h}^{\mathrm{Fil}},$$

see [Chapter IV.2, Sect. 1.4.3].

This is not altogether obvious, since the filtration in the case of Lie algebras was produced purely algebraically, and in the case of Lie algebroids we used geometry (specifically, the co-simplicial scheme $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$).

4.1. **Deformation to the normal cone in the *pointed* case.** In thus subsection we will consider the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$$

when $\mathcal{Y}$ is an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. Denote

$$\mathcal{H} := \Omega_{\mathcal{X}}(\mathcal{Y}).$$

In this case, by functoriality, $\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ is an object of

$$\left(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

4.1.1. Consider the corresponding object

$$\mathcal{H}_{\mathrm{scaled}} := \Omega_{\mathcal{X}}(\mathcal{Y}_{\mathrm{scaled}})$$

in $\left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\right)$.

By functoriality, it lifts to an object

$$\mathcal{H}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

4.1.2. Applying the functor

$$\mathrm{Lie}_{\mathcal{Z}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}))$$

(see [Chapter IV.3, Sect. 3.6]), we obtain an object

$$\mathrm{Lie}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}) \in \left(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1))\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

Using the equivalence of (1.3), we regard it as an object, denoted

$$\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}}) \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}\geq 0}).$$

We claim:

Theorem 4.1.3. *The above object* $\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}}) \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}\geq 0})$ *identifies canonically with the object* $(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}$ *of* [Chapter IV.2, Sect. 1.4.3].

4.1.4. It will follow from the proof that the isomorphism in Theorem 4.1.3 is compatible with the corresponding forgetful functors.

Namely, it follows from Proposition 3.2.6 that there is a canonical isomorphism

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}})) \simeq (\mathrm{gr}\to\mathrm{Fil})\circ(\deg=1)\circ\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})). \tag{4.1}$$

In addition, by construction,

$$\mathbf{oblv}_{\mathrm{Lie}}((\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}) \simeq (\mathrm{gr}\to\mathrm{Fil})\circ(\deg=1)\circ\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})). \tag{4.2}$$

Now, the isomorphisms (4.1) and (4.2) are compatible via the isomorphism of Theorem 4.1.3.

4.1.5. Translating to the language of Lie algebroids we obtain:

Corollary 4.1.6. *The following diagram canonically commutes*

$$\begin{array}{ccc}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\mathrm{AddFil}} & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}} \\
{\scriptstyle\mathrm{diag}}\downarrow & & \downarrow{\scriptstyle\mathrm{diag}} \\
\mathrm{LieAlgebroids}(\mathcal{X}) & \xrightarrow{\mathrm{AddFil}} & \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}.
\end{array}$$

4.1.7. The compatibility in Sect. 4.1.4 amounts to the fact that the data of commutativity of the outer square in

$$\begin{array}{ccc}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\mathrm{AddFil}} & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}} \\
{\scriptstyle\mathrm{diag}}\downarrow & & \downarrow{\scriptstyle\mathrm{diag}} \\
\mathrm{LieAlgebroids}(\mathcal{X}) & \xrightarrow{\mathrm{AddFil}} & \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \\
{\scriptstyle\mathbf{oblv}_{\mathrm{LieAlgrboid}\,/T(\mathcal{X})}}\downarrow & & \downarrow{\scriptstyle\mathbf{oblv}_{\mathrm{LieAlgrboid}\,/T(\mathcal{X})}} \\
\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xrightarrow{\mathrm{AddFil}} & \left(\mathrm{IndCoh}(X\times\mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}},
\end{array}$$

equals one in the outer square of

$$\begin{tikzcd}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \arrow[r, "\mathrm{AddFil}"] \arrow[d] & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}} \arrow[d] \\
\mathrm{IndCoh}(\mathcal{X}) \arrow[r, "(\mathrm{gr}\to\mathrm{Fil})\circ(\deg=1)"] \arrow[d] & \mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1)^{\mathbb{A}^1_{\text{left-lax}}} \arrow[d] \\
\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \arrow[r, "\mathrm{AddFil}"] & \left(\mathrm{IndCoh}(X\times\mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}},
\end{tikzcd}$$

where the lower vertical arrows are given by

$$\mathcal{F}\mapsto(\mathcal{F}\overset{0}{\to}T(\mathcal{X})).$$

Remark 4.1.8. By adjunction, the next diagram commutes *up to a natural transformation*:

$$\begin{tikzcd}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \arrow[r, "\mathrm{AddFil}"] & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}} \\
\mathrm{LieAlgebroids}(\mathcal{X}) \arrow[r, "\mathrm{AddFil}"] \arrow[u, "\text{ker-anch}"] & \left(\mathrm{LieAlgbroid}(\mathcal{X}\times\mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}. \arrow[u, "\text{ker-anch}"']
\end{tikzcd}$$

We note, however, that this natural transformation is *not* an isomorphism.

4.1.9. The rest of this section is devoted to the proof of Theorem 4.1.3.

4.2. **A digression: category objects and group-objects.** We will now explain a general categorical paradigm that will be used in the proof of Theorem 4.1.3.

4.2.1. Let $\mathcal{C}$ be a pointed category with finite limits. Let $c^\bullet$ be a Segal-object (a.k.a., category-object) of $\mathcal{C}$; see [Chapter II.2, Sect. 5.1.1] for what this means.

On the one hand, we consider the simplicial object of $\mathcal{C}$ equal to

$$'c^\bullet := * \underset{c^\bullet}{\times} c^0, \tag{4.3}$$

where $c^0\to c^n$ is given by the degeneracy map.

It is easy to see that $'c^\bullet$ is a groupoid-object in $\mathcal{C}$ with $'c^0 = *$, i.e., it defines a structure of group-object on $'d := {}'c^1$.

4.2.2. On the other hand, consider the group-objects $\Omega(c^1)$ and $\Omega(c^0)$. The 'target' map $t: c^1\to c^0$ defines a homomorphism $\Omega(c^1)\to\Omega(c^0)$. Define

$$''d := \mathrm{Fib}(\Omega(c^1)\to\Omega(c^0)).$$

We claim:

Proposition 4.2.3. *Under the above circumstances, there is a canonical isomorphism of group-objects in $\mathcal{C}$*

$$'d \simeq {}''d.$$

Proof. Consider the category-object in $\mathrm{Grp}(\mathcal{C})$ given by $\Omega(c^\bullet)$. We can regard it as a group-object in the category of category-objects in $\mathcal{C}$ and as such it acts on $'c^\bullet$. This action defines an action of the group-object $\Omega(c^1)$ on the object of $\mathcal{C}$ underlying $'c^1$.

The action of the group $\mathrm{Fib}(\Omega(c^1) \to \Omega(c^0))$ on $'c^1$ has an additional structure: it commutes with the action of the group-object $'c^1$ on itself by *right* translations.

This defines a homomorphism $''d \to {}'d$. At the level of the underlying objects of $\mathcal{C}$, this map is the map

$$* \underset{* \underset{c^0}{\times} *}{\times} (* \underset{c^1}{\times} *) \to * \underset{c^1}{\times} * \to * \underset{c^1}{\times} c^0,$$

which is an isomorphism since the degeneracy map $c^0 \to c^1$ is a right inverse to $t : c^1 \to c^0$.
□

4.3. **Proof of Theorem 4.1.3.**

4.3.1. *Step 1.* We claim that the object $\mathcal{H}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\right)^{\mathbb{A}^1_{\text{left-lax}}}$ is given by

$$\mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{H}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}) \underset{\mathcal{H}\times\mathbb{A}^1}{\times} (\mathcal{X}\times\mathbb{A}^1), \tag{4.4}$$

with its natural left-lax equivariant structure with respect to $\mathbb{A}^1$, and the map

$$\mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{H}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}) \to \mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^0_{\mathrm{scaled}}}(\mathcal{H}\times\mathrm{Bifurc}^0_{\mathrm{scaled}}) = \mathcal{H}\times\mathbb{A}^1$$

is induced by $t : \mathrm{Bifurc}^0_{\mathrm{scaled}} \to \mathrm{Bifurc}^1_{\mathrm{scaled}}$.

Indeed, we apply the setting of Sect. 4.2 to the category

$$\mathcal{C} := \left(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1}))\right)^{\mathbb{A}^1_{\text{left-lax}}}$$

and

$$c^\bullet := \mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{Y}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}).$$

Then the object $'d$ of Sect. 4.2.1 identifies with

$$\mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}(\mathcal{X}\times\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}) \underset{\mathrm{Weil}_{\mathbb{A}^1}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}(\mathcal{Y}\times\mathrm{Bifurc}^\bullet_{\mathrm{scaled}})}{\times} (\mathcal{Y}\times\mathbb{A}^1) = \mathcal{H}_{\mathrm{scaled}}.$$

This is while the object $''d$ of Sect. 4.2.1 identifies with (4.4).

Note that the group structure on (4.4) is induced by that on $\mathcal{H}$ (i.e., the groupoid structure on $\mathrm{Bifurc}^1_{\mathrm{scaled}}$ is not involved).

4.3.2. *Step 2.* From the commutativity of the diagram (A.7), we obtain a canonical identification of objects of objects of $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})\otimes\mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}$

$$\mathrm{Lie}_{\mathcal{X}\times\mathbb{A}^1}(\mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathcal{H}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}) \underset{\mathcal{H}\times\mathbb{A}^1}{\times} (\mathcal{X}\times\mathbb{A}^1)) \simeq$$
$$\simeq \mathrm{Fib}\left(\mathrm{Weil}_{\mathcal{X}\times\mathbb{A}^1}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})|_{\mathcal{X}\times\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \to \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})|_{\mathcal{X}\times\mathbb{A}^1}\right).$$

Now, it follows from Remark 2.2.3 that the latter expression is canonically isomorphic to $(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}$.

5. Infinitesimal neighborhoods

Let $X \to Y$ be a closed embedding of classical schemes. In this case we can consider the n-infinitesimal neighborhood of X inside Y (it corresponds to the n-th power of the defining X in Y).

However, the derived version of this construction is not so evident (what do we mean by the n-th power of an ideal?).

In this section we will define the corresponding derived version in the general context of formal moduli problems. The key tool will be deformation to the normal bundle from Sect. 2.

5.1. **The n-th infinitesimal neighborhood.** Let $\mathcal{X} \to \mathcal{Y}$ be a map objects of $\mathrm{PreStk}_{\text{laft-def}}$. In this subsection we will construct a sequence of objects

$$\mathcal{X} = \mathcal{X}^{(0)} \to \mathcal{X}^{(1)} \to ... \to \mathcal{X}^{(n)} \to ... \to \mathcal{Y},$$

with $\mathcal{X}^{(n)} \in \mathrm{FormMod}_{/\mathcal{X}}$.

The prestacks $\mathcal{X}^{(n)}$ will generalize the construction of the 'n-th infinitesimal neighborhood of X in Y' for a closed embedding of classical schemes $X \to Y$. (In the case when the embedding is regular, the derived construction will agree with the classical construction. However, in general, even if both X and Y are classical, the derived n-th infinitesimal neighborhood will have a non-trivial derived structure.)

It will follow from the construction that $\mathcal{X}^{(1)}$ is the square-zero extension corresponding to the map $T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})$, i.e.,

$$\mathrm{RealSqZ}(T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})),$$

see [Chapter IV.4, Sect. 5.1.1].

5.1.1. We will construct the objects $\mathcal{X}^{(n)}$ inductively, starting from $n = 0$. In fact, we will construct their filtered enhancements, denoted

$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

Let

$$\mathcal{X}^{(n)}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}$$

be the object obtained from $\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ by forgetting the structure of left-lax equivariance with respect to $\mathbb{A}^1$, so that $\mathcal{X}^{(n)}$ is the fiber of $\mathcal{X}^{(n)}_{\mathrm{scaled}}$ at $1 \in \mathbb{A}^1$.

Set $\mathcal{X}^{(0)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} := \mathcal{X} \times \mathbb{A}^1$. Assume that $\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$, equipped with a map

$$\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$$

has been constructed.

5.1.2. Consider the object
$$T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}} \in \operatorname{IndCoh}(\mathcal{X}).$$
It canonically lifts to an object in $\operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0} \in \operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil}}$, denoted $(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}$. Namely, we consider
$$T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}})|_{\mathcal{X}\times\mathbb{A}^1} \in \operatorname{IndCoh}(\mathcal{X}\times\mathbb{A}^1),$$
equipped with the natural structure of left-lax equivariance with respect to $\mathbb{A}^1$, and thus giving rise to the sought-for
$$(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}} \in \operatorname{IndCoh}(\mathcal{X}\times\mathbb{A}^1)^{\mathbb{A}^1_{\text{left-lax}}} \simeq \operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}.$$

We will prove:

Theorem 5.1.3. *The object* $(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}$ *belongs to* $\operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n}$*, and the* n*-th term of the filtration identifies canonically with*
$$\operatorname{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1].$$

Here, by a slight abuse of notation, we denote by $\operatorname{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]$ the object of
$$\operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n} \subset \operatorname{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$$
that should properly be denoted
$$(\mathrm{gr}\to\mathrm{Fil})\circ(\deg = n)(\operatorname{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]).$$

5.1.4. Let i_{n-1} denote the map $\mathcal{X}\to\mathcal{X}^{(n-1)}$, and let $i_{n-1,\mathrm{scaled}}$ denote the map
$$\mathcal{X}\times\mathbb{A}^1 \to \mathcal{X}^{(n-1)}_{\mathrm{scaled}}.$$

Assuming Theorem 5.1.3, we obtain a canonically defined map
$$(i_{n-1,\mathrm{scaled}})^{\mathrm{IndCoh}}_*(\operatorname{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}}).$$

5.1.5. We let $\mathcal{X}^{(n)}_{\mathrm{scaled}}$ denote the square-zero extension of $\mathcal{X}^{(n-1)}_{\mathrm{scaled}}$ corresponding to the composite map
$$(i_{n-1,\mathrm{scaled}})^{\mathrm{IndCoh}}_*(\operatorname{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}}) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}).$$

By transport of structure, the object
$$\mathcal{X}^{(n)}_{\mathrm{scaled}} \in \operatorname{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}$$
lifts to an object
$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\operatorname{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}},$$
the map $\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}$ is equipped with an extension to a map
$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}}.$$

5.2. **Computing the colimit.** In this subsection we will show that the colimit of the n-th infinitesimal neighborhoods recovers the ambient prestack.

5.2.1. Recall that according to [Chapter IV.1, Corollary 2.3.6], the category $\mathrm{FormMod}_{\mathcal{X}/}$ admits sifted, and in particular, filtered colimits. Consider the object

$$\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)} \in \mathrm{FormMod}_{\mathcal{X}/},$$

which is equipped with a canonically defined map to $\mathcal{Y}$.

Proposition 5.2.2. *The map*

$$\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism in $\mathrm{FormMod}_{\mathcal{X}/}$.

Proof. By [Chapter III.1, Proposition 8.3.2], it suffices to show that the map in question induces an isomorphism at the level of tangent spaces

$$T(\mathcal{X}/\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)}) \to T(\mathcal{X}/\mathcal{Y}).$$

By [Chapter IV.1, Corollary 2.3.6], the natural map

$$\underset{n}{\mathrm{colim}}\, T(\mathcal{X}/\mathcal{X}^{(n)}) \to T(\mathcal{X}/\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)})$$

is an isomorphism.

Hence, it suffices to show that the colimit

$$\underset{n}{\mathrm{colim}}\, T(\mathcal{X}^{(n)}/\mathcal{Y})|_{\mathcal{X}}$$

vanishes.

However, this follows from Theorem 5.1.3. Indeed, the above colimit lifts to an object of $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$, which belongs to $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n}$ for any n.

□

By combining with [Chapter IV.1, Corollary 2.3.7], we obtain:

Corollary 5.2.3. *The map*

$$\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism in $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$.

5.2.4. By combining Proposition 5.2.2 with [Chapter IV.3, Corollary 5.3.3(b)] (or, alternatively, just using Corollary 5.2.3) above, we obtain:

Corollary 5.2.5. *For* $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$, *there is a canonical isomorphism*

$$\underset{n}{\mathrm{colim}}\, (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}) \to \omega_{\mathcal{Y}},$$

where f_n *denotes the map* $\mathcal{X}^{(n)} \to \mathcal{Y}$.

5.3. **The Hodge filtration (a.k.a., de Rham resolution).** Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. In the classical setting, the object $\omega_{\mathcal{X}}$, when equipped with the canonical structure of $\mathfrak{L}$-module, admits a canonical 'de Rham' resolution with terms induced from

$$\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])[-n].$$

In this subsection we will carry out the corresponding construction in the derived setting.

The statement will be that the unit object in the category $\mathfrak{L}$-mod(IndCoh) has a canonical filtration with subquotients $\mathbf{ind}_{\mathfrak{L}}\,(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1]))$.

Applying this to $\mathfrak{L} = \mathcal{T}(\mathcal{X})$, we recover the Hodge filtration on

$$\omega_{\mathcal{X}_{\mathrm{dR}}} \in \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}}).$$

5.3.1. Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$, corresponding to an object

$$(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}\,.$$

Let $\omega_{\mathcal{X},\mathfrak{L}}$ denote the object of $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ corresponding to $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$. Tautologically,

$$\mathbf{oblv}_{\mathfrak{L}}(\omega_{\mathcal{X},\mathfrak{L}}) = \omega_{\mathcal{X}}.$$

We will prove the following:

Proposition-Construction 5.3.2. *There exists a canonical lift of $\omega_{\mathcal{X},\mathfrak{L}}$ to an object*

$$(\omega_{\mathcal{X},\mathfrak{L}})^{\mathrm{Fil}} \in \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))^{\mathrm{Fil},\geq 0},$$

such that

$$\text{ass-gr}^n(\omega_{\mathcal{X},\mathfrak{L}}) = \mathbf{ind}_{\mathfrak{L}}\left(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right).$$

5.3.3. *Proof of Proposition 5.3.2.* Let $\mathcal{X}^{(n)}$ be the n-th infinitesimal neighborhood of $\mathcal{X}$ in $\mathcal{Y}$, see Sect. 5. Let i_n, $i_{n-1,n}$ and f_n denote the maps

$$\mathcal{X} \to \mathcal{X}^{(n)},\ \mathcal{X}^{(n-1)} \to \mathcal{X}^{(n)} \text{ and } \mathcal{X}^{(n)} \to \mathcal{Y},$$

respectively.

We let

$$(\omega_{\mathcal{X},\mathfrak{L}})^{\leq n} := (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}).$$

Recall that by Corollary 5.2.5, the canonical map

$$\underset{n}{\mathrm{colim}}\,(f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}) \to \omega_{\mathcal{Y}}$$

is an isomorphism.

Hence, it remains to construct the isomorphisms

$$\mathrm{coFib}\left((f_{n-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}})\right) \simeq \\ \simeq f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right). \tag{5.1}$$

The left-hand side in (5.1) identifies with

$$(f_n)_*^{\mathrm{IndCoh}}\left(\mathrm{coFib}\left((i_{n-1,n})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to \omega_{\mathcal{X}^{(n)}}\right)\right).$$

Let us recall that by construction, the map $i_{n-1,n} : \mathcal{X}^{(n-1)} \to \mathcal{X}^{(n)}$ has a structure of square-zero extension corresponding to

$$(i_{n-1})_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1]\right) \in \mathrm{IndCoh}(\mathcal{X}^{(n-1)}).$$

Hence, by [Chapter IV.4, Proposition 6.4.2],

$$\mathrm{coFib}\left((i_{n-1,n})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to \omega_{\mathcal{X}^{(n)}}\right) \simeq \\ \simeq (i_{n-1,n})_*^{\mathrm{IndCoh}} \circ (i_{n-1})_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right) \simeq \\ \simeq (i_n)_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right).$$

And hence, the left-hand side in (5.1) identifies with

$$(f_n)_*^{\mathrm{IndCoh}} \circ (i_n)_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right) \simeq f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right),$$

where

$$T(\mathcal{X}/\mathcal{Y}) \simeq \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}),$$

as desired.

□

5.4. **Proof of Theorem 5.1.3: reduction to the case of vector groups.** In this subsection we will reduce the assertion of Theorem 5.1.3 to the case when $\mathcal{Y}$ is of the form $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

5.4.1. Since the functor
$$\text{ass-gr} : \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq 0} \to \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}$$
is conservative, it is enough to prove that in
$$\text{ass-gr}((T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}$$
the lowest graded piece is in degree n, and is canonically isomorphic to $\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1]$.

5.4.2. By Sect. 2.5.2 and the compatibility of the functor RealSqZ with base change (see [Chapter IV.4, Proposition 5.4.3]) this reduces the assertion of the proposition to considering the case of
$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in (\mathrm{FormMod}_{/\mathcal{X}})^{\mathbb{A}^1_{\text{left-lax}}},$$
for
$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, =1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

5.5. **Proof of Theorem 5.1.3: the case of vector groups.** When dealing with vector groups we 'know' what the n-infinitesimal neighborhood must be, and this is what we will establish, along with the assertion of Theorem 5.1.3 in this case.

5.5.1. Consider the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}$. Let
$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}) \subset \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0})$$
be the full subcategory consisting of objects, for which the augmentation co-ideal belongs to $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}$.

Consider also the category
$$\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}).$$

We have a pair of adjoint functors
$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}) : \mathrm{coChev}^{\mathrm{enh}}. \tag{5.2}$$

By [FraG, Proposition 4.1.2], the adjoint functors in (5.2) are mutually inverse equivalences.

5.5.2. For
$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, =1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}$$
we consider the objects
$$\mathrm{Sym}(\mathcal{F}) \text{ and } \mathrm{Sym}^{\leq n}(\mathcal{F}) \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, >0}).$$

Note that there is a canonical isomorphism
$$\mathrm{coChev}^{\mathrm{enh}}(\mathrm{Sym}(\mathcal{F})) = \mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}[-1]).$$
(The above isomorphism is a particular case of [Chapter IV.2, Theorem 4.2.4], but is much simpler, since we are in the graded category, and the functors in (5.2) are equivalences.)

Note that
$$B_{\mathcal{X}}(\mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}[-1])) \simeq \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}).$$

Denote
$$\mathfrak{g}^{(n)} := \mathrm{coChev}^{\mathrm{enh}}(\mathrm{Sym}^{\leq n}(\mathcal{F})).$$

For example,

$$\mathfrak{g}^{(1)} = \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}[-1]).$$

(Again, this isomorphism holds because the functors in (5.2) are equivalences.)

Denote

$$\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)} := B_{\mathcal{X}}(\mathfrak{g}^{(n)}) \in (\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}.$$

Let $\widetilde{i}_n$ denote the map $\mathcal{X} \to \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$.

5.5.3. Consider

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in (\mathrm{FormMod}_{/\mathcal{X}})^{\mathbb{A}^1_{\text{left-lax}}},$$

and the corresponding object $\mathcal{X}^{(n)} \in (\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}$.

We are going to prove that there exists a canonical isomorphism in $(\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}$.

$$\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)} \simeq \mathcal{X}^{(n)},$$

and that the assertion of Theorem 5.1.3 holds for $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$. More precisely, we will prove the following assertion:

Proposition 5.5.4.
(a) *The lowest graded terms in the objects of* $\mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}$

$$T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\,\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)})|_{\mathcal{X}} \text{ and } T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\,\mathrm{Vect}_{\mathcal{X}}(\mathcal{F}))|_{\mathcal{X}}$$

are in degree n; the map

$$T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\,\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)})|_{\mathcal{X}} \to T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\,\mathrm{Vect}_{\mathcal{X}}(\mathcal{F}))|_{\mathcal{X}}$$

induces an isomorphism of degree n terms, and both identify canonically with $\mathrm{Sym}^n(\mathcal{F})[-1]$.
(b) *The map*

$$\mathrm{RealSqZ}\left((\widetilde{i}_{n-1})^{\mathrm{IndCoh}}_*(\mathrm{Sym}^n(\mathcal{F})[-1]) \to T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)})\right) \to \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)},$$

induced by the identification in (a), is an isomorphism.

5.5.5. The assertion of Proposition 5.5.4 implies the required properties of $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$ and $\mathcal{X}^{(n)}$ by induction on n.

5.6. **Proof of Proposition 5.5.4.** The proof of Proposition 5.5.4 will involve some 'cheating': instead of performing the crucial computation, we will reduce it to the case of classical algebraic geometry, namely, the embedding of 0 into a finite-dimensional vector space.

5.6.1. Before we prove Proposition 5.5.4, let us translate its assertion into the language of Lie algebras.

Point (a) says that the objects

$$\mathrm{Fib}\left(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n)})\right) \text{ and } \mathrm{Fib}\left(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathcal{F}[-1]\right)$$

both live in degrees $\geq n$, and their degree n part is isomorphic to $\mathrm{Sym}^n(\mathcal{F})[-2]$.

Point (b) says the following. Let $\mathcal{F}_n$ denote the object

$$\mathrm{coFib}\left(\mathrm{Sym}^n(\mathcal{F})[-2] \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)})\right) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}.$$

We have a canonical map

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathcal{F}_n.$$

Let

$$\mathbf{free}_{\mathrm{LieAlg}_{\mathfrak{g}^{(n-1)}/}}(\mathcal{F}_n)$$

be the corresponding free object in the category of Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$ *under* $\mathfrak{g}^{(n-1)}$.

By point (a) of Proposition 5.5.4, we have a canonical map

$$\mathbf{free}_{\mathrm{LieAlg}_{\mathfrak{g}^{(n-1)}/}}(\mathcal{F}_n) \to \mathfrak{g}^{(n)}. \tag{5.3}$$

Now, point (b) of Proposition 5.5.4 is equivalent to the fact that the map (5.3) is an isomorphism.

5.6.2. The above reformulation of Proposition 5.5.4 makes sense when $\mathrm{IndCoh}(\mathcal{X})$ is replaced by an arbitrary symmetric monoidal DG category $\mathbf{O}$. Furthermore, the assertion of both points of Proposition 5.5.4 is about the comparison of pairs of functors $\mathbf{O} \to \mathbf{O}$, given by symmetric sequences.

Hence, we can replace $\mathrm{IndCoh}(\mathcal{X})$ by Vect, and we can assume that $\mathcal{F}$ is a finite-dimensional vector space that lives in the cohomological degree 0.

In the latter case, the assertion of Proposition 5.5.4 is manifest.

6. Filtration on the universal enveloping algebra of a Lie algebroid

Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. Recall that in [Chapter IV.4, Sect. 4.2] to $\mathfrak{L}$ we associated its universal enveloping algebra $U(\mathfrak{L})$, which was an algebra object in the monoidal DG category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})).$$

In this subsection we define a crucial piece of structure that $U(\mathfrak{L})$ possesses, namely, the canonical (a.k.a. PBW) filtration.

6.1. The statement.

In this subsection we state the main result of the present section, Theorem 6.1.2.

6.1.1. Consider the monoidal category

$$(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}.$$

We claim:

Theorem 6.1.2. *The object*

$$U(\mathfrak{L}) \in \mathrm{AssocAlg}\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))\right),$$

canonically lifts to an object

$$U(\mathfrak{L})^{\mathrm{Fil}} \in \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}\right).$$

The corresponding associated graded identifies canonically with the monad given by tensor product with $\mathbf{free}_{\mathrm{Com}} \circ \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$.

The theorem will be proved in Sects 6.2-6.4.

6.1.3. The following corollary results from the construction of the filtration and Theorem 4.1.3:

Corollary 6.1.4. *For* $\mathfrak{L} = \mathrm{diag}(\mathfrak{h})$*, the filtration on* $U(\mathfrak{L})$ *defined in Theorem 6.1.2 identifies with the one coming from the canonical filtration on* $U(\mathfrak{h})$.

6.2. **Constructing the filtration.** As a first step, we will construct $U(\mathfrak{L})^{\mathrm{Fil}}$ as an object of the category

$$\mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}\right),$$

where we identify the latter with

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \otimes \mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m}.$$

6.2.1. Consider the forgetful functor

$$\left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}} \to \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{G}_m}.$$

Let us consider the resulting prestacks

$$\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m \text{ and } \mathcal{Y}_{\mathrm{scaled}}/\mathbb{G}_m$$

over $\mathbb{A}^1/\mathbb{G}_m$.

The map $f_{\mathrm{scaled}} : \mathcal{X}\times\mathbb{A}^1 \to \mathcal{Y}_{\mathrm{scaled}}$ gives rise to a $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$-linear functor

$$(f_{\mathrm{scaled}}/\mathbb{G}_m)^! : \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}}/\mathbb{G}_m) \to \mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m).$$

Since the symmetric monoidal category $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ is rigid, the left adjoint of the functor $(f_{\mathrm{scaled}}/\mathbb{G}_m)^!$, i.e., $(f_{\mathrm{scaled}}/\mathbb{G}_m)_*^{\mathrm{IndCoh}}$, is also $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$-linear.

Hence, by [Chapter I.1, Sect. 8.4.4], the composition

$$(f_{\mathrm{scaled}}/\mathbb{G}_m)^! \circ (f_{\mathrm{scaled}}/\mathbb{G}_m)_*^{\mathrm{IndCoh}}$$

has a natural structure of algebra object on the monoidal category

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m), \mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m)),$$

while the latter identifies with

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \otimes \mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m}.$$

This provides the sought-for lifting.

6.2.2. Our task is now to show that the object

$$U(\mathfrak{L})^{\mathrm{Fil}} \in \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}\right)$$

constructed above, belongs to the essential image of the (fully faithful) functor

$$\mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}\right) \to$$
$$\to \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}\right).$$

Note, however, that for any monoidal DG category $\mathbf{O}$ the following diagram is a pullback square:

$$\begin{tikzcd}
\mathrm{AssocAlg}\left(\mathbf{O}^{\mathrm{Fil},\geq 0}\right) \arrow[r] \arrow[d, "\mathbf{oblv}_{\mathrm{Assoc}}"'] & \mathrm{AssocAlg}\left(\mathbf{O}^{\mathrm{Fil}}\right) \arrow[d, "\mathbf{oblv}_{\mathrm{Assoc}}"] \\
\mathbf{O}^{\mathrm{Fil},\geq 0} \arrow[r] & \mathbf{O}^{\mathrm{Fil}}.
\end{tikzcd}$$

Hence, we obtain that it suffices to show that the object

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}}) \in (\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}$$

in fact belongs to $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$.

6.3. **The categorical setting for the non-negative filtration.** In this subsection we explain a general categorical paradigm for establishing that the filtration on

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}})$$

is non-negative.

6.3.1. Consider the following presheaf of categories

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}, \quad S \mapsto \mathrm{QCoh}(S)\text{-}\,\mathbf{mod}, \tag{6.1}$$

see Example (iii) in Sect. 1.2.5.

We denote the value of this functor on $\mathcal{Z} \in \mathrm{PreStk}$ by $\mathrm{ShvCat}(\mathcal{Z})$.

6.3.2. We regard (6.1) as equipped with the trivial action of a monoid $\mathcal{G}$.

According to Sect. 1.2.3, for a prestack $\mathcal{Z}$, equipped with an action of $\mathcal{G}$ one can talk about the category

$$\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\text{left-lax}}}.$$

6.3.3. Assume now that $\mathcal{Z} = Z \in \mathrm{Sch}^{\mathrm{aff}}$. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}'$ be three objects in $\mathrm{ShvCat}(Z)^{\mathcal{G}_{\text{right-lax}}}$, and let $\mathsf{G} : \mathbf{C} \to \mathbf{D}$ and $\mathsf{F}' : \mathbf{C} \to \mathbf{D}'$ be morphisms.

Applying the forgetful functor

$$\mathrm{ShvCat}(Z)^{\mathcal{G}_{\text{right-lax}}} \to \mathrm{ShvCat}(Z) \simeq \mathrm{QCoh}(Z)\text{-}\,\mathbf{mod},$$

the objects $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}'$ give rise to $\mathrm{QCoh}(Z)$-module categories, and G and F' to $\mathrm{QCoh}(Z)$-linear functors.

Assume that G, viewed as a functor between $\mathrm{QCoh}(Z)$-linear categories admits a left adjoint, denoted F. Since the monoidal category $\mathrm{QCoh}(Z)$ is rigid, the functor F is also naturally $\mathrm{QCoh}(Z)$-linear.

6.3.4. Assume now that $\mathbf{D}$ and $\mathbf{D}'$ are of the form

$$\mathbf{D}_0 \otimes \mathrm{QCoh}(Z) \text{ and } \mathbf{D}'_0 \otimes \mathrm{QCoh}(Z),$$

respectively, where the structure on $\mathbf{D}$ and $\mathbf{D}'$ of objects of $\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\text{right-lax}}}$ is induced by the structure on $\mathrm{QCoh}(Z)$ of an object of $\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\text{right-lax}}}$ (in fact, $\mathrm{ShvCat}(\mathcal{Z})$), arising from the $\mathcal{G}$-action on Z.

We have:

Lemma 6.3.5. *Under the above circumstances, the object*

$$\mathsf{F}' \circ \mathsf{F} \in \mathrm{Funct}_{\mathrm{QCoh}(Z)}(\mathbf{D}, \mathbf{D}') \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_0, \mathbf{D}'_0) \otimes \mathrm{QCoh}(Z)$$

corresponding to $\mathsf{G} \circ \mathsf{F}$*, admits a canonical lift to an object in the category*

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_0, \mathbf{D}'_0) \otimes \mathrm{QCoh}(Z)^{\mathcal{G}_{\text{left-lax}}}.$$

6.4. **Implementing the categorical setting.** We will now apply the setting of Sect. 6.4 to deduce the filtration on $U(\mathfrak{L})$.

6.4.1. We take the monoid $\mathcal{G}$ to be $\mathbb{A}^1$ and $Z = \mathbb{A}^1$, equipped with an action on itself by multiplication.

We take $\mathbf{D}_0, \mathbf{D}'_0 = \mathrm{IndCoh}(\mathcal{X})$.

6.4.2. We take $\mathbf{C}$ to be $\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}})$, where the structure on $\mathbf{C}$ of an object of the category $\mathrm{ShvCat}(\mathbb{A}^1)^{\mathbb{A}^1_{\text{left-lax}}}$ is given by the lift of $\mathcal{Y}_{\mathrm{scaled}}$ to the object of

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}.$$

We take G and F' to both be the functor of pullback along the map

$$\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{\mathrm{scaled}}.$$

The functor $\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))$ is then one corresponding to $\mathsf{F}' \circ \mathsf{F}$. The lifting of Lemma 6.3.5 defines the sought-for filtered structure.

6.4.3. The associated graded of $U(\mathfrak{L})^{\mathrm{Fil}}$ has the prescribed shape by Sect. 2.5.2.

6.5. **The filtration via infinitesimal neighborhoods.** In this subsection we realize the following (intuitively clear) idea: the canonical filtration on $U(\mathfrak{L})$ stated in Theorem 6.1.2, can be realized by considering n-th infinitesimal neighborhoods of the diagonal in the corresponding groupoid.

6.5.1. For $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$, consider the corresponding groupoid

$$\mathcal{R} := \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X},$$

and the algebroid $\mathfrak{L}$.

Let p_s, p_t denote the two projections $\mathcal{R} \rightrightarrows \mathcal{X}$. Let $\Delta_{\mathcal{X}/\mathcal{Y}}$ denote the diagonal (i.e., unit) map $\mathcal{X} \to \mathcal{R}$.

Let $'\mathcal{X}^{(n)}$ denote the n-th infinitesimal neighborhood of $\mathcal{X}$ in $\mathcal{R}$, defined as in Sect. 5.1. Let $p_i^{(n)}$ denote the restriction of p_i to $'\mathcal{X}^{(n)}$, $i = s, t$.

6.5.2. Consider the object of $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$ given by

$$n \mapsto (p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^!, \quad \mathbb{Z}^{\geq 0} \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \tag{6.2}$$

Let as assume Theorem 6.1.2. From it we will deduce:

Theorem 6.5.3. *There exists a canonical isomorphism between the object* (6.2) *and*

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}})$$

in the category $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$.

The rest of this subsection is devoted to the proof of this theorem.

6.5.4. *Proof of Theorem 6.5.3, Step 1.* To prove the proposition, we need to construct a compatible family of maps

$$(p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n},$$

so that the induced maps

$$\mathrm{coFib}\left((p_t^{(n-1)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n-1)})^! \to (p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^!\right) \to$$
$$\to \mathrm{coFib}\left(\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n-1} \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n}\right)$$

are isomorphisms.

First, the base change isomorphism

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})) := f^! \circ f_*^{\mathrm{IndCoh}} \simeq (p_t)_*^{\mathrm{IndCoh}} \circ p_s^!$$

of [Chapter III.3, Proposition 2.1.2] defines a compatible system of maps

$$(p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})). \tag{6.3}$$

6.5.5. *Proof of Theorem 6.5.3, Step 2.* As in Sects. 6.3 and 6.4, the prestack

$$'\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}},$$

and the corresponding maps

$$(p_s^{(n),\mathrm{Fil}}), (p_t^{(n),\mathrm{Fil}}) : \mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \to \mathcal{X}\times\mathbb{A}^1,$$

lift the system (6.2) to an assignment

$$n \mapsto (p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!, \quad \mathbb{Z}^{\geq 0} \to (\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}.$$

In addition, the system of maps (6.3) lifts to a system of maps

$$(p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}}). \tag{6.4}$$

Hence, taking into account (the filtered version of) Corollary 5.2.5, to prove the proposition, it suffices to show the following:

Lemma 6.5.6.
(a) *For every n, the filtration on $(p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!$ stabilizes at n, i.e., the maps*

$$\left((p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!\right)^{\leq m} \to \left((p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!\right)^{\leq m+1}$$

are isomorphisms for $m \geq n$.
(b) *For every n, the map* (6.4) *induces an isomorphism of the n-th associated graded quotients.*

6.5.7. *Proof of Theorem 6.5.3, Step 3.* In order to prove Lemma 6.5.6, since the functor ass-gr is conservative on $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$, it is enough to prove the corresponding assertion at the associated graded level.

By Sect. 2.5.2, this reduces are to the situation when $\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

However, in the latter case, the assertion of Lemma 6.5.6 is manifest from Corollary 6.1.4 and Sect. 5.5.3.

7. The case of a regular embedding

Recall that if $f : X \to Y$ is a regular closed embedding of classical schemes, then we have Grothendieck's formula that says that the functors f^* and $f^!$ are related by tensoring by the determinant of the normal bundle.

In this section we will establish an analog of this assertion in the derived setting.

7.1. **The notion of regular embedding.** In this subsection we will introduce the notion of regular embedding in the context of formal moduli problems.

7.1.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\text{laft-def}}$, and let $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$.

We shall say that f is a *regular embedding of relative codimension n* if

$$T^*(\mathcal{X}/\mathcal{Y})[-1] \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)$$

belongs to $\mathrm{QCoh}(\mathcal{X})^-$ and is a *vector bundle of rank n* (i.e., its pullback to any affine scheme S is Zariski-locally isomorphic to $\mathcal{O}_S^{\oplus n}$). Throughout this subsection we will assume that f has this property.

7.1.2. Denote
$$\det(T^*(\mathcal{X}/\mathcal{Y})) := \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y}));$$
this is a cohomologically shifted (by $[n]$) line bundle.

Consider the objects $\mathrm{Sym}^m(T^*(\mathcal{X}/\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{X})$. Note that they all are also vector bundles. Moreover, $\mathrm{Sym}^m(T^*(\mathcal{X}/\mathcal{Y}))$ vanishes for $m > n$.

7.1.3. Note also that
$$T(\mathcal{X}/\mathcal{Y}) \simeq \Upsilon_X((T^*(\mathcal{X}/\mathcal{Y}))^\vee),$$
where $(T^*(\mathcal{X}/\mathcal{Y}))^\vee \in \mathrm{QCoh}(\mathcal{X})$ is the tensor dual of $T^*(\mathcal{X}/\mathcal{Y})$ in $\mathrm{QCoh}(\mathcal{X})$. In particular, $T(\mathcal{X}/\mathcal{Y})$ is dualizable as an object of the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$.

Furthermore, $\mathrm{Sym}^m(T(\mathcal{X}/\mathcal{Y})) \in \mathrm{IndCoh}(\mathcal{X})$ is dualizable for any m, and vanishes for $m > n$.

7.1.4. We now claim:

Proposition 7.1.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a regular embedding. Then the functor*
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$
admits a left *adjoint (to be denoted $f^{\mathrm{IndCoh},*}$).*

Proof. In order to show that the functor f_*^{IndCoh} admits a left adjoint, it suffices to show that it commutes with limits. Since the functor $f^!$ is conservative and commutes with limits (being a right adjoint), it suffices to show that the composition
$$f^! \circ f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$
commutes with limits.

Let $\mathfrak{L}$ denote the Lie algebroid $\mathcal{T}(\mathcal{X}/\mathcal{Y})$. We have to show that $U(\mathfrak{L})$, viewed as an endo-functor of $\mathrm{IndCoh}(\mathcal{X})$, commutes with limits.

Note now that Sect. 7.1.3 implies that the canonical filtration on $U(\mathfrak{L})$ has the property that $\text{ass-gr}^m(U(\mathfrak{L}))$ vanishes for $m > n$. I.e., the filtration is finite. Hence, it is enough to see that each graded term, viewed as endo-functor of $\mathrm{IndCoh}(\mathcal{X})$, commutes with limits.

However,
$$\text{ass-gr}^m(U(\mathfrak{L})) \simeq \Upsilon_{\mathcal{X}}(\mathrm{Sym}^m(\mathcal{T}(\mathcal{X}/\mathcal{Y}))) \otimes -,$$
and the assertion follows.

□

7.2. **Grothendieck's formula.** In this subsection we state the main result of this section: Grothendieck's formula that relates $f^{\mathrm{IndCoh},*}$ and $f^!$.

7.2.1. The goal of this section is to prove the following result:

Theorem 7.2.2. *Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\text{laft-def}}$, and let $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$ be a regular embedding. Then:*

(a) *The natural transformation*
$$f^{\mathrm{IndCoh},*}(-) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}) \overset{!}{\otimes} f^!(-)$$
is an isomorphism.

(b) *There exists a canonical isomorphism*
$$f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}}) \simeq \Upsilon_{\mathcal{X}}\left(\det(T^*(\mathcal{X}/\mathcal{Y}))\right).$$

7.2.3. Combining points (a) and (b) of the theorem, we obtain:

Corollary 7.2.4. *There exists a canonical isomorphism of functors* $\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})$

$$f^{\mathrm{IndCoh},*}(-) \simeq \det(T^*(\mathcal{X}/\mathcal{Y})) \otimes f^!(-),$$

where $\otimes$ *is understood in the sense of the action of* $\mathrm{QCoh}(-)$ *on* $\mathrm{IndCoh}(-)$.

7.3. **Applications.** In this subsection we give some applications of Theorem 7.2.2.

7.3.1. *Schematic regular embeddings.* Let $f : \mathcal{X} \to \mathcal{Y}$ be a schematic map between objects of $\mathrm{PreStk}_{\mathrm{laft}}$. Assume that f is a closed embedding, and that the map, denoted

$$f^{\wedge} : \mathcal{X} \to \mathcal{Y}^{\wedge}_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}$$

is a regular embedding of relative codimension n in the sense of Sect. 7.1.1.

From Theorem 7.2.2 we shall now deduce:

Corollary 7.3.2. *The functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to f_*^{IndCoh}*, is defined, and we have a canonical isomorphism*

$$f^{\mathrm{IndCoh},*}(-) \simeq \det(T^*(\mathcal{X}/\mathcal{Y})) \otimes f^!(-).$$

Proof. By base change, we can assume that $X = \mathcal{X}$ and $Y = \mathcal{Y}$ are schemes. The condition on f implies that it is *quasi-smooth*, and hence eventually coconnective (see [AG, Corollary 1.2.5]). Hence, existence of the functor $f^{\mathrm{IndCoh},*}$ follows from [Ga1, Proposition 7.1.6].

Let $U \overset{j}{\hookrightarrow} Y$ be the open embedding of the complement of image of f. Let i denote the map $Y^{\wedge}_X \to Y$.

It is easy to see that $f^{\mathrm{IndCoh},*} \circ j_* = 0$. Hence, by [GaRo1, Proposition 7.4.5], the functor $f^{\mathrm{IndCoh},*}$ factors through the co-localization $i^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y^{\wedge}_X)$, i.e.,

$$f^{\mathrm{IndCoh},*} \simeq (f^{\wedge})^{\mathrm{IndCoh},*} \circ i^!.$$

Similarly, $f^! \simeq (f^{\wedge})^! \circ i^!$. Now, the required result follows from the isomorphism of Corollary 7.2.4 for the morphism $f^{\wedge}$.

□

7.3.3. *Smooth maps.* Let now $g : \mathcal{X} \to \mathcal{Z}$ be a schematic map between objects of $\mathrm{PreStk}_{\mathrm{laft}}$. Assume that g is *smooth* of relative dimension n.

Note that in this case $T^*(\mathcal{X}/\mathcal{Z}) \in \mathrm{QCoh}(\mathcal{X})$ is a vector bundle of rank n. Denote

$$\det(T^*(\mathcal{X}/\mathcal{Z})) := \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Z})[1]).$$

This is a cohomologically shifted (by $[n]$) line bundle.

We claim:

Proposition 7.3.4. *The functor*

$$g^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to g_*^{IndCoh}*, is defined, and we have a canonical isomorphism*

$$g^{\mathrm{IndCoh},*}(-) \simeq \det(T^*(\mathcal{X}/\mathcal{Z}))^{\vee} \otimes g^!(-).$$

7.3.5. *Step 1.* By [Ga1, Propositions 7.1.6 and 7.3.8], for any map $f : \mathcal{X} \to \mathcal{X}'$, whose base change by an affine scheme is schematic and Gorenstein, the functor $f^{\mathrm{IndCoh},*}$ exists, and we have:

$$\mathcal{K}_{\mathcal{X}/\mathcal{X}'} \otimes f^{\mathrm{IndCoh},*}(-) \simeq f^!(-)$$

for a canonically defined line bundle on $\mathcal{K}_{\mathcal{X}/\mathcal{X}'}$ on $\mathcal{X}$.

It follows formally that for a Cartesian diagram with vertical arrows Gorenstein

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "h"] \arrow[d, "f_1"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{X}_1' \arrow[r, "h'"] & \mathcal{X}' \end{tikzcd}$$

we have a canonical isomorphism in $\mathrm{QCoh}(\mathcal{X}_1)$

$$h^*(\mathcal{K}_{\mathcal{X}/\mathcal{X}'}) \simeq \mathcal{K}_{\mathcal{X}_1/\mathcal{X}_1'}, \tag{7.1}$$

Furthermore, it follows that for a composition of Gorenstein maps

$$\mathcal{X} \overset{f}{\to} \mathcal{X}' \overset{h}{\to} \mathcal{X}'',$$

we have a canonical isomorphism in $\mathrm{QCoh}(\mathcal{X})$

$$f^*(\mathcal{K}_{\mathcal{X}'/\mathcal{X}''}) \otimes \mathcal{K}_{\mathcal{X}/\mathcal{X}'} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{X}''}. \tag{7.2}$$

Let $g : \mathcal{X} \to \mathcal{Z}$ be a Gorenstein map. Consider the diagram

$$\begin{tikzcd} \mathcal{Y} := \mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{X} \arrow[r, "p_t"] \arrow[d, "p_s"'] & \mathcal{X} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "g"] & \mathcal{Z}. \end{tikzcd}$$

By (7.1) and (7.2), we have:

$$\mathcal{K}_{\mathcal{Y}/\mathcal{Z}} \simeq (p_s)^*(\mathcal{K}_{\mathcal{X}/\mathcal{Z}}) \otimes (p_t)^*(\mathcal{K}_{\mathcal{X}/\mathcal{Z}}). \tag{7.3}$$

7.3.6. *Step 2.* Assume now that $g : \mathcal{X} \to \mathcal{Z}$ is smooth. We need to show that

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \det(T^*(\mathcal{X}/\mathcal{Z})).$$

Let f denote the map

$$\mathcal{X} \to \mathcal{Y} := \mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{X}.$$

Since g is smooth, the map f is a regular closed embedding, i.e., satisfies the assumptions of Sect. 7.3.1. In particular, it is Gorenstein, and by Corollary 7.3.2

$$\mathcal{K}_{\mathcal{X}/\mathcal{Y}} \simeq \det(T^*(\mathcal{X}/\mathcal{Y}))^{\otimes -1}.$$

Combining (7.3) and (7.2), we obtain:

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{Z}} \otimes \mathcal{K}_{\mathcal{X}/\mathcal{Z}} \otimes \mathcal{K}_{\mathcal{X}/\mathcal{Y}}.$$

Hence,

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{Y}}^{\otimes -1} \simeq \det(T^*(\mathcal{X}/\mathcal{Y})).$$

I.e., it remains to show that

$$\det(T^*(\mathcal{X}/\mathcal{Z})) \simeq \det(T^*(\mathcal{X}/\mathcal{Y})).$$

However, this follows from the canonical identification

$$T^*(\mathcal{X}/\mathcal{Y}) \simeq T^*(\mathcal{X}/\mathcal{Z})[1].$$

7.4. **Introducing the filtration.** The rest of this section is devoted to the proof of Theorem 7.2.2. The idea is to upgrade the required isomorphism to one between filtered objects, using the deformation to the normal cone of Sect. 2.

7.4.1. Consider again the object

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}},$$

and we will regard it as an object of $\left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{G}_m}$ via the forgetful functor

$$\left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\text{left-lax}}} \to \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{G}_m}.$$

7.4.2. The construction of Sect. 6.2.1 upgrades the endo-functor $f^{\mathrm{IndCoh},*}\circ f^{\mathrm{IndCoh}}_*$ of $\mathrm{IndCoh}(\mathcal{X})$ to an object

$$\left(f^{\mathrm{IndCoh},*}\circ f^{\mathrm{IndCoh}}_*(\mathcal{F})\right)^{\mathrm{Fil}} \in \left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil}}. \tag{7.4}$$

By construction, the object (7.4) is the left-dual of $\mathbf{oblv}_{\mathrm{assoc}}(U(\mathfrak{L})^{\mathrm{Fil}})$, when both are viewed as objects in the monoidal category $\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil}}$.

Since,

$$\mathbf{oblv}_{\mathrm{assoc}}(U(\mathfrak{L})^{\mathrm{Fil}}) \in \left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil},\geq 0,\leq n},$$

we obtain that

$$\left(f^{\mathrm{IndCoh},*}\circ f^{\mathrm{IndCoh}}_*(\mathcal{F})\right)^{\mathrm{Fil}} \in \left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil},\geq -n,\leq 0}. \tag{7.5}$$

7.4.3. Similarly, the object $f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})$ naturally upgrades to an object

$$(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}))^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}.$$

Finally, the natural transformation

$$f^{\mathrm{IndCoh},*}(-) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})\otimes f^!(-)$$

also lifts to a natural transformation of functors

$$\mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}.$$

7.4.4. We claim:

Lemma 7.4.5.

$$(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}))^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n}.$$

Proof. Recall the construction in Proposition 5.3.2. Note that the assignment

$$k \mapsto f^{\mathrm{IndCoh},*}((f_k)^{\mathrm{IndCoh}}_*(\omega_{\mathcal{X}^{(k)}})),$$

as well as the maps

$$f^{\mathrm{IndCoh},*}\circ (f_k)^{\mathrm{IndCoh}}_*(\omega_{\mathcal{X}^{(k)}}) \to f^{\mathrm{IndCoh},*}\circ (f_{k+1})^{\mathrm{IndCoh}}_*(\omega_{\mathcal{X}^{(k+1)}})$$

and

$$f^{\mathrm{IndCoh},*}\circ (f_k)^{\mathrm{IndCoh}}_*(\omega_{\mathcal{X}^{(k)}}) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})$$

all lift to the category $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}$.

Hence, it is enough to show that for every k, we have

$$\left(f^{\mathrm{IndCoh},*}\circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}})\right)^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n}.$$

We note that the identification

$$\mathrm{coFib}(f^{\mathrm{IndCoh},*}\circ (f_{k-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k-1)}}) \to f^{\mathrm{IndCoh},*}\circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}}) \simeq$$
$$\simeq f^{\mathrm{IndCoh},*}\circ f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$$

lifts to an isomorphism

$$\mathrm{coFib}\left(\left(f^{\mathrm{IndCoh},*}\circ (f_{k-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k-1)}})\right)^{\mathrm{Fil}} \to \left(f^{\mathrm{IndCoh},*}\circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}})\right)^{\mathrm{Fil}}\right) \simeq$$
$$\simeq (f^{\mathrm{IndCoh},*}\circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}(\omega_{\mathcal{X}})\otimes\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right),$$

where $(f^{\mathrm{IndCoh},*}\circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}$ is as in Sect. 7.4.1, and where $\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$ is in degree k.

Now,

$$(f^{\mathrm{IndCoh},*}\circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}(\omega_{\mathcal{X}}) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n},$$

while $\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$ is non-negatively filtered.

□

7.5. **Reduction to the case of vector groups.** In this subsection we will reduce the assertion of Theorem 7.2.2 to the case when $\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F}\in\mathrm{IndCoh}(\mathcal{X})$.

7.5.1. Since the essential image of $\mathrm{IndCoh}(\mathcal{X})$ under f_*^{IndCoh} generates $\mathrm{IndCoh}(\mathcal{Y})$, in order to prove the isomorphism of Theorem 7.2.2(a), it suffices to show that the natural transformation

$$f^{\mathrm{IndCoh},*}\circ f_*^{\mathrm{IndCoh}}(\mathcal{F}) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})\otimes\mathcal{F} \tag{7.6}$$

is an isomorphism for any $\mathcal{F}\in\mathrm{IndCoh}(\mathcal{X})$.

7.5.2. Taking into account Lemma 7.4.5 and (7.5), we obtain that in order to show that (7.6) is an isomorphism, it is enough to prove that the isomorphism holds at the associated graded level.

Using Sect. 2.5.2, this reduces point (a) of Theorem 7.2.2 to the verification of the isomorphism (7.6) in the case when

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]).$$

7.5.3. We will prove the following assertion:

Proposition 7.5.4. *With respect to the canonical filtration on $f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}})$, we have*

$$\text{ass-gr}^k(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}})) \simeq \begin{cases} 0 \text{ if } k\neq -n \\ \Upsilon_{\mathcal{X}}\left(\det(T^*(\mathcal{X}/\mathcal{Y}))\right) \text{ if } k=-n. \end{cases}$$

Note that by Lemma 7.4.5, the assertion of Proposition 7.5.4 implies also Theorem 7.2.2(b).

In order to prove Proposition 7.5.4, it is also enough to do so in the case when

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]) \in \left(\mathrm{FormMod}_{\mathcal{X}/}\right)^{\mathbb{G}_m},$$

where $T(\mathcal{X}/\mathcal{Y})$ is given grading 1.

7.6. **The case of vector groups.** In this subsection we will explicitly perform the calculation stated in Theorem 7.2.2 in the case of vector groups.

7.6.1. Let
$$\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in \left(\mathrm{FormMod}_{\mathcal{X}/}\right)^{\mathbb{G}_m},$$
where $\mathcal{F} = \Upsilon_{\mathcal{X}}(\mathcal{E})$, where $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ is a vector bundle of rank n, and where we regard $\mathcal{F}$ as an object of
$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},=1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}}.$$

We note that in this case
$$\Upsilon_{\mathcal{X}}\left(\det(T^*(\mathcal{X}/\mathcal{Y})\right) \simeq \mathrm{Sym}^n(\mathcal{F}^\vee[1]),$$
where $\mathcal{F}^\vee$ is the monoidal dual of $\mathcal{F}$ in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$.

7.6.2. We have:
$$\mathrm{IndCoh}(\mathcal{Y})^{\mathbb{G}_m} \simeq \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])\text{-mod}(\mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m}),$$
where $f^!$ is the tautological forgetful functor
$$\mathbf{oblv}_{\mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])} : \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])\text{-mod}(\mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m},$$
and f_*^{IndCoh} is the functor
$$\mathbf{ind}_{\mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])} := \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1]) \otimes -.$$

7.6.3. Consider the following abstract situation: let $\mathbf{O}$ be a symmetric monoidal DG category, and let $\mathcal{F} \in \mathbf{O}$ be an object *of dimension* n, i.e., $\mathcal{F}$ is dualizable and $\mathrm{Sym}^{n+1}(\mathcal{F}[1]) = 0$. Set
$$\mathfrak{l} := \mathrm{Sym}^n(\mathcal{F}[-1]).$$

Consider the commutative algebra $A := \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])$ and the corresponding adjunction
$$\mathbf{ind}_A : \mathbf{O} \rightleftarrows A\text{-mod} : \mathbf{oblv}_A.$$

The assumption on $\mathcal{F}$ implies that $\mathbf{oblv}_{\mathrm{Com}}(A)$ is dualizable as an object of $\mathbf{O}$. Hence, the functor $\mathbf{ind}_A$ commutes with limits, and thus admits a *left* adjoint.

In this case, it is easy to see that the natural transformation
$$(\mathbf{ind}_A)^L(-) \to (\mathbf{ind}_A)^L(\mathbf{1_O}) \otimes \mathbf{oblv}_A(-)$$
is an isomorphism and that
$$(\mathbf{ind}_A)^L(\mathbf{1_O}) \simeq \mathfrak{l}^{\otimes -1}.$$

□

Appendix A. Weil restriction of scalars

In this section we will establish several facts of how the operation of Weil restriction behaves with respect to deformation theory.

In particular, we show that Weil restriction along an affine map of a prestack with deformation theory is a prestack with deformation theory, and we describe the pro-cotangent complex of the Weil restriction. Furthermore, we show that Weil restriction of formal groups can be computed using Lie algebras.

A.1. **The operation of Weil restriction of scalars.** In this subsection we recall the operation of Weil restriction of scalars of a prestack.

A.1.1. Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map of prestacks. Let $\mathcal{X}_1$ be a prestack over $\mathcal{Z}_1$. Let

$$\mathcal{X}_2 := \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}(\mathcal{X}_1) \in \mathrm{PreStk}_{/\mathcal{Z}_2}$$

be the Weil restriction of $\mathcal{X}_1$ along f.

By definition, for $S_2 \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Z}_2}$, we have

$$\mathrm{Maps}_{/\mathcal{Z}_2}(S_2, \mathcal{X}_2) := \mathrm{Maps}_{/\mathcal{Z}_1}(S_1, \mathcal{X}_1), \quad S_1 := \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} S_2.$$

A.1.2. *Assumption.* From now on we will assume that the morphism f is *affine* (i.e., its base change by an affine scheme yields an affine scheme).

A.2. **Weil restriction of scalars and deformation theory.** In this subsection we will study the deformation theory of objects obtained as by the operation of Weil restriction of scalars.

A.2.1. Assume now that $\mathcal{X}_1$ *admits deformation theory relative to* $\mathcal{Z}_1$.

It follows from the definitions that in these circumstance, $\mathcal{X}_2$ will admit deformation theory relative to $\mathcal{Z}_2$. Moreover, its cotangent complex can be described as follows.

For an S_2-point x_2 of $\mathcal{X}_2$, let x_1 be the corresponding S_1-point of $\mathcal{X}_1$, where

$$S_1 := \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} S_2.$$

Denote by f_S the corresponding map $S_1 \to S_2$.

Let $\mathrm{Pro}((f_S)_*)$ be the corresponding functor

$$\mathrm{Pro}(\mathrm{QCoh}(S_1)^-) \to \mathrm{Pro}(\mathrm{QCoh}(S_2)^-).$$

Then we have:

$$T^*_{x_2}(\mathcal{X}_2/\mathcal{Z}_2) \simeq \mathrm{Pro}((f_S)_*)(T^*_{x_1}(\mathcal{X}_1/\mathcal{Z}_1)). \tag{A.1}$$

A.2.2. Note that if in the above circumstances, $\mathcal{X}_1$ is itself of the form

$$\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{X}'_2, \quad \mathcal{X}'_2 \in \mathrm{PreStk}_{/\mathcal{Z}_2},$$

and the point x_1 comes from an S_2-point x'_2 of $\mathcal{X}'_2$, then we have

$$T^*_{x_2}(\mathcal{X}_2/\mathcal{Z}_2) \simeq T^*_{x_2}(\mathcal{X}'_2/\mathcal{Z}_2) \underset{\mathcal{O}_{S_2}}{\otimes} (f_S)_*(\mathcal{O}_{S_1}). \tag{A.2}$$

A.2.3. Assume that $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{X}_1 \in \mathrm{PreStk}_{\mathrm{laft}}$. It follows from the definitions that in this case $\mathcal{X}_2$ also belongs to $\mathrm{PreStk}_{\mathrm{laft}}$.

In this case we can talk about

$$T(\mathcal{X}_1) \in \mathrm{IndCoh}(\mathcal{X}_1) \text{ and } T(\mathcal{X}_2) \in \mathrm{IndCoh}(\mathcal{X}_2).$$

Let $(f^!)^R : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$ be the functor *right adjoint* to

$$f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1).$$

Note that $(f^!)^R$ is *not* continuous, unless f is eventually coconnective (i.e., of finite Tor dimension).

From (A.1) we obtain:

$$T(\mathcal{X}_2/\mathcal{Z}_2) \simeq (f^!)^R(T(\mathcal{X}_1/\mathcal{Z}_1)). \tag{A.3}$$

Similarly, in the circumstances of Sect. A.2.2, we have

$$T(\mathcal{X}_2/\mathcal{Z}_2) \simeq T(\mathcal{X}'_2/\mathcal{Z}_2) \underset{\mathcal{O}_{\mathcal{X}_2}}{\otimes} f_*(\mathcal{O}_{\mathcal{X}_1}), \tag{A.4}$$

where $\otimes$ understood in the sense of the action of QCoh on IndCoh.

A.3. **Weil restriction of formal groups.**

A.3.1. Let $\mathbf{O}$ be a symmetric monoidal DG category and $\mathcal{P}$ an operad (see [Chapter IV.2, Sect. 1.1] for our conventions regarding operads). For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ as above, pullback defines a functor

$$f^* : \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) \to \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_1)).$$

This functor admits a right adjoint, denoted also $\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}$ that makes the diagram

$$\begin{tikzcd} \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) \arrow[r, "\mathbf{oblv}_{\mathcal{P}}"] \arrow[d, "\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}"'] & \mathrm{QCoh}(\mathcal{Z}_1) \arrow[d, "f_*"] \\ \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) \arrow[r, "\mathbf{oblv}_{\mathcal{P}}"] & \mathrm{QCoh}(\mathcal{Z}_2) \end{tikzcd} \tag{A.5}$$

commutative.

A.3.2. Assume now that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ belong to $\mathrm{PreStk}_{\mathrm{laft}}$. Then in the above discussion we can replace QCoh and IndCoh, and the diagram (A.5) by

$$\begin{tikzcd} \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{IndCoh}(\mathcal{Z}_1)) \arrow[r, "\mathbf{oblv}_{\mathcal{P}}"] \arrow[d, "\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}"'] & \mathrm{IndCoh}(\mathcal{Z}_1) \arrow[d, "(f^!)^R"] \\ \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{IndCoh}(\mathcal{Z}_2)) \arrow[r, "\mathbf{oblv}_{\mathcal{P}}"] & \mathrm{IndCoh}(\mathcal{Z}_2) \end{tikzcd} \tag{A.6}$$

A.3.3. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ again belong to $\mathrm{PreStk}_{\mathrm{laft}}$. Note that by [Chapter IV.3, Sect. 3.5], we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_1}) \arrow[r, "\mathrm{Lie}_{\mathcal{Z}_1}"] & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_1)) \\ \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_2}) \arrow[r, "\mathrm{Lie}_{\mathcal{Z}_2}"] \arrow[u, "\mathcal{H} \mapsto \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{H}"] & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_2)). \arrow[u, "f^*"'] \end{tikzcd}$$

By passing to right adjoints along vertical arrows, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_1}) \arrow[r, "\mathrm{Lie}_{\mathcal{Z}_1}"] \arrow[d, "\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}"'] & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_1)) \arrow[d, "\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}"'] \\ \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_2}) \arrow[r, "\mathrm{Lie}_{\mathcal{Z}_2}"] & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_2)). \end{tikzcd} \tag{A.7}$$