CHAPTER III.4. AN APPLICATION: CRYSTALS

Contents

Introduction

In this Chapter we will establish one of the goals indicated in the Introduction to Part III: we will show that inf-schemes give a common framework for ind-coherent sheaves and D-modules. In particular, we will show that the induction and forgetful functors

$$\mathbf{ind}_X : \text{IndCoh}(X) \rightleftarrows \text{D-mod}(X) : \mathbf{oblv}_X \tag{0.1}$$

interact with the direct and inverse image functors in the expected way.

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0.1. **Let's do D-modules!** The usual definition of the category of D-modules on a smooth affine scheme X is as the category

$$\mathrm{Diff}_X\text{-mod},$$

where Diff_X is the (classical) ring of Grothendieck operations.

This approach to D-modules is very explicit, and is indispensable for concrete applications (e.g. to define regular D-modules and study the notion of holonomicity). However, this approach is not particularly convenient for setting up the theory from the point of view of higher category theory.

Here are some typical issues that become painful in this approach.

0.1.1. One often encounters the question of how to define the category of D-modules on a singular scheme X? The usual answer is that we first assume that X is affine, and choose an embedding $X \hookrightarrow Y$, where Y is smooth. Now, define $\text{D-mod}(X)$ to be $\text{D-mod}(Y)_X$, i.e., the full subcategory of $\text{D-mod}(Y)$ consisting of objects with set-theoretic support on X.

Then, using Kashiwara's lemma, one shows that this construction is canonically independent of the choice of Y. For general X, one considers an affine Zariski cover and glues the corresponding categories.

Note, however, that the words 'choose an embedding $X \hookrightarrow Y$' mean that in the very definition, we appeal to resolutions. From the homotopical point of view, this exacts a substantial price and is too cumbersome to be convenient.

0.1.2. Another example is the definition of the direct image functor. For a morphism $f : X \to Y$ between smooth affine schemes, one introduces an explicit object

$$\mathrm{Diff}_{X,Y} : (\mathrm{Diff}_Y \otimes \mathrm{Diff}_X^{\mathrm{op}})\text{-mod},$$

which defines the desired functor

$$\mathrm{Diff}_X\text{-mod} \to \mathrm{Diff}_Y\text{-mod}.$$

When X and Y are not necessarily smooth, one again embeds this situation into one where X and Y are smooth. When X and Y are non-affine, this is performed locally on X and Y.

All of this can be made to work for an individual morphism: we can prove the proper adjunction between pullbacks and pushforwards, and the base change isomorphism. However, it is not clear how to establish the full functoriality of the category D-mod in this way; namely, as a functor out of the category of correspondences.

0.1.3. Another layer of complexity (=homotopical nuisance) arises when one wants to construct D-modules together with the adjoint pair (0.1).

0.2. **D-modules via crystals.** In this book, we take a different approach to the theory of D-modules. We *define* the category of D-modules as crystals, establish all the needed functorialities, and then in the case of smooth schemes and morphisms between them identify the resulting categories and functors with the classical ones from the theory of D-modules.

0.2.1. By definition, for a laft prestack $\mathcal{Z}$, the category of crystals on $\mathcal{Z}$ is

$$\mathrm{Crys}(\mathcal{Z}) := \mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}}),$$

where $\mathcal{Z}_{\mathrm{dR}}$ is the de Rham prestack of $\mathcal{Z}$.

Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map of laft prestacks. Then !-pullback on IndCoh defines a functor

$$f^!_{\mathrm{dR}} : \mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1).$$

This is the pullback functor for crystals.

0.2.2. Assume now that f is ind-nil-schematic, which means that the corresponding morphism ${}^{\mathrm{red}}\mathcal{Z}_1 \to {}^{\mathrm{red}}\mathcal{Z}_2$ is ind-schematics. Then one (easily) sees that the resulting morphism

$$(f_{\mathrm{dR}}) : (\mathcal{Z}_1)_{\mathrm{dR}} \to (\mathcal{Z}_2)_{\mathrm{dR}}$$

is ind-inf-schematic. Now, using [Chapter III.3, Sect. 4], we define the functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$$

to be the functor $(f_{\mathrm{dR}})_*$. This is the de Rham direct image functor.

0.2.3. Taking $\mathcal{Z}_1 = \mathcal{Z}$ (so that ${}^{\mathrm{red}}\mathcal{Z}$ is an ind-scheme) and $\mathcal{Z}_2 = \mathrm{pt}$, we obtain the functor of de Rham sections

$$\Gamma_{\mathrm{dR}}(\mathcal{Z}, -) : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{Vect}\,.$$

Moreover, the above constructions automatically extend to the data of a functor out of a suitable $(\infty, 2)$-category of correspondences. Namely, we consider the category $\mathrm{PreStk}_{\mathrm{laft}}$ equipped with the following classes of functors:

–'horizontal' maps are all maps in $\mathrm{PreStk}_{\mathrm{laft}}$;

–'vertical' maps are those maps f that ${}^{\mathrm{red}}f$ is ind-schematic (we call them *ind-nil-schematic*);

–'admissible' maps are those vertical maps that are also ind-proper.

One shows that the assignment $\mathcal{Z} \mapsto \mathcal{Z}_{\mathrm{dR}}$ defines a functor

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}.$$

Composing with the functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

we obtain a functor

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}\,.$$

The above functor $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}}$ is the desired expression of functoriality of the assignment

$$\mathcal{Z} \mapsto \mathrm{Crys}(\mathcal{Z}).$$

0.2.4. Now suppose that $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ admits deformation theory. One shows that in the case the tautological map

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}$$

is an inf-schematic nil-isomorphism. Hence, by [Chapter III.3, Prop. 3.1.2], the functor

$$p^!_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits the left adjoint.

Thus, we obtain the desired adjoint pair:

$$\mathbf{ind}_{\mathrm{dR},\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Crys}(\mathcal{Z}) : \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}.$$

0.2.5. *But what does this have to do with D-modules?* The basic observation, essentially due to Grothendieck[1], is that for a smooth scheme X, the category $\text{Crys}(X)$, together with the forgetful functor

$$\Psi_X \circ \mathbf{oblv}_{\text{dR},X} : \text{Crys}(X) \to \text{QCoh}(X),$$

is canonically equivalent to the category of right D-modules, together with its tautological forgetful functor to $\text{QCoh}(X)$.

We describe this identification in Sect. 4 of this Chapter. We also show that the functors on the category of crystals (direct and inverse image for a map $f : X \to Y$) described above map to the corresponding functors for D-modules under this identification.

This is thus our ansatz to the construction of the theory of D-modules: instead of developing the theory of D-modules directly, we develop the theory of crystals, and then identity it with D-modules when D-modules are conveniently defined; namely, in the case of smooth schemes.

0.3. **What else is done in this chapter?**

0.3.1. In Sect. 1 we introduce the category of crystals $\text{Crys}(\mathcal{Z})$, where $\mathcal{Z} \in \text{PreStk}_{\text{laft}}$.

The key observation here is the following: let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map between prestacks such that the induced map

$$^{\text{red}}\mathcal{Z}_1 \to {}^{\text{red}}\mathcal{Z}_2$$

is (ind)-schematic. Then we show that the resulting map

$$(\mathcal{Z}_1)_{\text{dR}} \to (\mathcal{Z}_2)_{\text{dR}}$$

is (ind)-inf-schematic.

This observation, along with the fact that pushforward is defined on IndCoh for (ind)-nil-schematic morphisms, is what makes the theory work. I.e., this is the framework that allows to treat the de Rham pushfoward (in particular, de Rham (co)homology) on the same footing as the $\mathcal{O}$-module pushforward (in its IndCoh variant).

We then establish some properties, expected from the theory of D-modules:

(i) For a closed embedding $i : \mathcal{Y} \hookrightarrow \mathcal{Z}$, the functor $i_{\text{dR},*} : \text{IndCoh}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{Z})$ is fully faithful;

(ii) If $\mathcal{Z}$ is an (ind)-nil-scheme, the category $\text{Crys}(\mathcal{Z})$ is compactly generated and has a reasonably behaved t-structure.

0.3.2. In Sect. 2 we apply the results of [Chapter III.3, Sect. 5 and 6] and construct Crys as a functor out of the category of correspondences.

We show that when evaluated on ind-nil-schemes, this gives rise to the operation of Verdier duality.

[1]We learned it from A. Beilinson.

0.3.3. In Sect. 3 we study the functor of forgetting the crystal structure:

$$\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z}),$$

which, in our framework, is just the pullback functor for the morphism

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

The key observation is that if $\mathcal{Z}$ admits deformation-theory, then the map $p_{\mathrm{dR},\mathcal{Z}}$ is inf-schematic. Hence, in this case the functor $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ admits a left adjoint, given by $(p_{\mathrm{dR},\mathcal{Z}})_*^{\mathrm{IndCoh}}$. This left adjoint, denoted $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$, is the functor of induction from ind-coherent sheaves to crystals.

When $\mathcal{Z} = X$ is a smooth affine scheme, under the identification

$$\mathrm{Crys}(X) \simeq (\mathrm{Diff}_X^{\mathrm{op}})\text{-mod},$$

the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ corresponds to

$$\mathcal{F} \mapsto \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathrm{Diff}_X .$$

We show that if $\mathcal{Z}$ is an ind-scheme, then the morphism $p_{\mathrm{dR},\mathcal{Z}}$ is *ind-schematic*. We use this fact to deduce that the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is t-exact.

0.3.4. In Sect. 3.3, we develop the theory of crystals *relative* to a given prestack $\mathcal{Y}$. Namely, for $\mathcal{Z}$ over $\mathcal{Y}$, we set

$$\mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}} := \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}$$

and we set

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) := \mathrm{IndCoh}(\mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}}).$$

When $\mathcal{Z} = X$ and $\mathcal{Y} = Y$ are smooth affine schemes, and the map $X \to Y$ is smooth, category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ identifies with

$$(\mathrm{Diff}_{X/Y})^{\mathrm{op}}\text{-mod},$$

where $\mathrm{Diff}_{X/Y}$ is the (classical) ring of *vertical* differential operators (i.e., the subring of Diff_X consisting of elements that commute with functions on Y).

If $\mathcal{Z}$ *admits deformation theory relative to* $\mathcal{Y}$, then the morphism

$$p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}}$$

is again inf-schematic, and hence the forgetful functor

$$(p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}})^! : {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint, given by $(p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}})_*^{\mathrm{IndCoh}}$.

0.3.5. In Sect. 4 we show how to identify the theory of crystals with D-modules in the case of smooth schemes. Our exposition here is not self contained: we make frequent references to [GaRo2].

We first consider the case of *left* D-modules, and we show that the category $\mathrm{Crys}^l(X)$ of *left* crystals on a smooth affine scheme X, defined as $\mathrm{QCoh}(X_{\mathrm{dR}})$, identifies with Diff_X-mod.

We then show that the category of right crystals (i.e., the usual category of crystals)

$$\mathrm{Crys}^r(X) := \mathrm{Crys}(X) := \mathrm{IndCoh}(X_{\mathrm{dR}})$$

identifies with $(\mathrm{Diff}_X)^{\mathrm{op}}$-mod.

Next, we show that the functor

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}})$$

that identifies $\mathrm{Crys}^l(X)$ with $\mathrm{Crys}^r(X)$ corresponds under the above equivalences

$$\mathrm{Crys}^l(X) \simeq \mathrm{Diff}_X\text{-mod} \text{ and } \mathrm{Crys}^r(X) \simeq (\mathrm{Diff}_X)^{\mathrm{op}}\text{-mod} \tag{0.2}$$

with the functor

$$\mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X), \quad \mathcal{M} \mapsto M \otimes \det(T^*(X))[\dim(X)].$$

Finally, we show that for a map between $f : X \to Y$ between smooth schemes, under the identifications (0.2), the functor

$$f^{\blacktriangle,l} : \mathrm{Diff}_Y\text{-mod} \to \mathrm{Diff}_X\text{-mod}$$

from the theory of D-modules corresponds to pullback

$$f^*_{\mathrm{dR}} : \mathrm{QCoh}(Y_{\mathrm{dR}}) \to \mathrm{QCoh}(X_{\mathrm{dR}}),$$

and the functor

$$f_{\text{D-mod},*} : \mathrm{Diff}_X\text{-mod} \to \mathrm{Diff}_Y\text{-mod}$$

from the theory of D-modules corresponds to push-forward

$$f_{\mathrm{dR},*} : \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathrm{QCoh}(Y_{\mathrm{dR}}).$$

1. Crystals on prestacks and inf-schemes

In this section we will reap the fruits of the work done in [Chapter III.3]. Namely, we will show how the theory of IndCoh gives rise to the theory of *crystals*.

1.1. **The de Rham functor and crystals: recollections.** The category $\mathrm{Crys}(\mathcal{X})$ of crystals on a prestack $\mathcal{X}$ is defined to be IndCoh on the corresponding prestack $\mathcal{X}_{\mathrm{dR}}$. In this subsection we recall the functor $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{dR}}$ and study its basic properties.

1.1.1. For $\mathcal{Z} \in \mathrm{PreStk}$, we denote by $\mathcal{Z}_{\mathrm{dR}}$ the corresponding de Rham prestack, defined as

$$\mathrm{Maps}(S, \mathcal{Z}_{\mathrm{dR}}) := \mathrm{Maps}({}^{\mathrm{red}}S, \mathcal{Z}),$$

for $S \in \mathrm{Sch}^{\mathrm{aff}}$.

For a morphism $f : \mathcal{Z}^1 \to \mathcal{Z}^2$, let $f_{\mathrm{dR}} : \mathcal{Z}^1_{\mathrm{dR}} \to \mathcal{Z}^2_{\mathrm{dR}}$ denote the corresponding morphism between deRham prestacks.

1.1.2. Note that the functor dR commutes both with limits and colimits.

Also, note that

$$\mathcal{Z}_{\mathrm{dR}} \simeq ({}^{\mathrm{red}}\mathcal{Z})_{\mathrm{dR}}.$$

So, if a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a nil-isomorphism (i.e., ${}^{\mathrm{red}}\mathcal{Z}_1 \to {}^{\mathrm{red}}\mathcal{Z}_1$ is an isomorphism), then $(\mathcal{Z}_1)_{\mathrm{dR}} \to (\mathcal{Z}_2)_{\mathrm{dR}}$ is an isomorhism.

1.1.3. We claim:

Proposition 1.1.4. *The functor* dR *takes* $\mathrm{PreStk}_{\mathrm{laft}}$ *to* $\mathrm{PreStk}_{\mathrm{laft}}$.

Proof. Let $\mathcal{Z}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$. We need to show that $\mathcal{Z}_{\mathrm{dR}}$ satisfies:

- It is convergent;
- For every n, the truncation ${}^{\leq n}\mathcal{Z}$ belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

The convergence of $\mathcal{Z}_{\mathrm{dR}}$ is obvious. To show that ${}^{\leq n}\mathcal{Z} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, it suffices to show that $\mathcal{Z}_{\mathrm{dR}}$ takes filtered limits in $\mathrm{Sch}^{\mathrm{aff}}$ to colimits in Spc. However, this follows from the fact that the functor

$$S \mapsto {}^{\mathrm{red}}S, \quad \mathrm{Sch}^{\mathrm{aff}} \to {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$$

preserves filtered limits, and the fact that ${}^{\mathrm{red}}\mathcal{Z} \in {}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}$.

□

1.2. **Crystals.** In this subsection we introduce the category of crystals.

1.2.1. Composing the functor $\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$ with

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

we obtain a functor denoted by

$$\mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

This is the functor which is denoted $\mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}$ in [GaRo2, Sect 2.3.2].

1.2.2. For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ we shall denote the value of $\mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ on $\mathcal{Z}$ by $\mathrm{Crys}(\mathcal{Z})$. For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathrm{PreStk}_{\mathrm{laft}}$, we shall denote by $f^!_{\mathrm{dR}}$ the resulting functor

$$\mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1).$$

Note that if a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a nil-isomorphism, then

$$f^!_{\mathrm{dR}} : \mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1)$$

is an equivalence.

1.2.3. For $\mathcal{Z} \in \mathrm{PreStk}$, we let $p_{\mathrm{dR},\mathcal{Z}}$ denote the tautological projection:

$$\mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

The map $p_{\mathrm{dR},\mathcal{Z}}$ gives rise to a natural transformation of functors

$$\mathbf{oblv}_{\mathrm{dR}} : \mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

For a map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, we have a commutative square of functors:

$$\begin{array}{ccc} \mathrm{Crys}(\mathcal{Z}_1) & \xrightarrow{\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}_1}} & \mathrm{IndCoh}(\mathcal{Z}_1) \\ {\scriptstyle f^!_{\mathrm{dR}}}\uparrow & & \uparrow{\scriptstyle f^!} \\ \mathrm{Crys}(\mathcal{Z}_2) & \xrightarrow{\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}_2}} & \mathrm{IndCoh}(\mathcal{Z}_2). \end{array}$$

1.2.4. Finally, we make the following observation:

Proposition 1.2.5. *For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$, the functor*

$$\mathrm{Crys}(\mathcal{Z}) \to \lim_{Z \in (\mathbf{C}_{/\mathcal{Z}})^{\mathrm{op}}} \mathrm{Crys}(Z)$$

is an equivalence, where $\mathbf{C}$ *is any of the following categories:*

$$ {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},\ \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}},\ {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}},\ {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}},\ {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}},\ \mathrm{Sch}_{\mathrm{aft}}\,.$$

Proof. It is enough to show that the functor

$$\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$$

is isomorphic to the left Kan extension of its restriction to $\mathbf{C} \subset \mathrm{PreStk}_{\mathrm{laft}}$ for $\mathbf{C}$ as above. It is sufficient to consider the case of $\mathbf{C} = {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

First, we note that the functor dR commutes with colimits. This implies that dR is isomorphic to the left Kan extension of its restriction to ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. Hence, it suffices to show that the functor

$$\mathrm{dR} : \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \to \mathrm{PreStk}_{\mathrm{laft}}$$

is isomorphic to the left Kan extension of its restriction to ${}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

In other words, we have to show that given $Z \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and a map

$${}^{\mathrm{red}}S \to Z,$$

the category of its factorizations as

$${}^{\mathrm{red}}S \to Z' \to Z$$

with $Z' \in {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$, is contractible.

However, the latter is obvious as the above category has a final object, namely, $Z' := {}^{\mathrm{red}}Z$. □

1.3. **Crystals and (ind)-nil-schemes.** In this subsection we introduce the class of prestacks that we call *(ind)-nil-schemes*, and study the category of crystals on such prestacks. (Ind)-nil-schemes play the same role vis-à-vis Crys as (ind)-inf-schemes do for IndCoh.

1.3.1. Consider the full subcategories

$$\mathrm{indnilSch}_{\mathrm{laft}} := \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} {}^{\mathrm{red}}\mathrm{indSch} \subset \mathrm{PreStk}_{\mathrm{laft}}$$

and

$$\mathrm{nilSch}_{\mathrm{laft}} := \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} {}^{\mathrm{red}}\mathrm{Sch} \subset \mathrm{PreStk}_{\mathrm{laft}},$$

where $\mathrm{PreStk}_{\mathrm{laft}} \to {}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}$ is the functor $\mathcal{Z} \mapsto {}^{\mathrm{red}}\mathcal{Z}$.

In other words, $\mathcal{Z}$ belongs to $\mathrm{indnilSch}_{\mathrm{laft}}$ (resp., $\mathrm{nilSch}_{\mathrm{laft}}$) if and only if ${}^{\mathrm{red}}\mathcal{Z}$ is a reduced ind-scheme (resp., scheme).

For example, we have

$$\mathrm{infSch}_{\mathrm{laft}} \subset \mathrm{nilSch}_{\mathrm{laft}} \text{ and } \mathrm{indinfSch}_{\mathrm{laft}} \subset \mathrm{indnilSch}_{\mathrm{laft}}\,.$$

We shall refer to objects of $\mathrm{indnilSch}_{\mathrm{laft}}$ (resp., $\mathrm{nilSch}_{\mathrm{laft}}$) as *ind-nil-schemes* (resp., *nil-schemes*).

1.3.2. We claim:

Lemma 1.3.3. *The functor* dR *takes objects of* $\text{indnilSch}_{\text{laft}}$ *(resp.,* $\text{nilSch}_{\text{laft}}$*) to* $\text{indinfSch}_{\text{laft}}$ *(resp.,* $\text{infSch}_{\text{laft}}$*).*

Proof. We have

$$^{\text{red}}(\mathcal{Z}_{\text{dR}}) = {}^{\text{red}}\mathcal{Z}.$$

Now, we claim that for *any* $\mathcal{Z} \in \text{PreStk}$, the corresponding $\mathcal{Z}_{\text{dR}}$ admits deformation theory. In fact, it admits an ∞-connective deformation theory: all of its cotangent spaces are zero.

□

1.3.4. Recall from [Chapter III.2, Definitions 1.6.5(a), 1.6.7(c) and 1.6.11(c)], the notions of *(ind)-schematic* and *(ind)-proper* maps of prestacks, as well as *(ind)-closed embeddings* of prestacks.

Definition 1.3.5.

(a) *We shall say that a map of prestacks is* (ind)-nil-schematic *if the map of the corresponding reduced prestacks is (ind)-schematic.*

(b) *We shall say that a map of prestacks is an* nil-closed-embedding(ind)-nil-closed embedding *if the map of the corresponding reduced prestacks is an ind-closed embedding.*

Recall the notion of an (ind)-inf-schematic map of prestacks, see [Chapter III.2, Definitions 3.1.5]. We have:

Corollary 1.3.6. *The functor* dR *takes (ind)-nil-schematic maps in* $\text{PreStk}_{\text{laft}}$ *to (ind)-inf-schematic maps.*

Proof. For a map of prestacks $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ and $S \in (\text{Sch}^{\text{aff}}_{\text{aft}})_{/(\mathcal{Z}_2)_{\text{dR}}}$, the Cartesian product

$$S \underset{(\mathcal{Z}_2)_{\text{dR}}}{\times} (\mathcal{Z}_1)_{\text{dR}}$$

identifies with

$$S \underset{S_{\text{dR}}}{\times} ({}^{\text{red}}S \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1)_{\text{dR}}.$$

Now, we use Lemma 1.3.3 and the fact that the subactegory $\text{indinfSch}_{\text{laft}}$ is preserved by finite limits.

□

1.3.7. We claim:

Lemma 1.3.8. *Let* $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ *be an ind-nil-proper map in* $\text{PreStk}_{\text{laft}}$*. Then:*

(a) *The functor* $f_{\text{dR},*} : \text{Crys}(\mathcal{Z}_1) \to \text{Crys}(\mathcal{Z}_2)$*, left adjoint to* $f^!_{\text{dR}}$*, is well-defined, and satisfies base change with respect to !-pullbacks.*

(b) *If* f *is an ind-nil-closed embedding, then* $f_{\text{dR},*}$ *is fully faithful.*

Proof. Point (a) follows from Corollary 1.3.6 and [Chapter III.3, Proposition 3.2.4].

To prove point (b), we need to show that the unit of the adjunction

$$\text{Id}_{\text{Crys}(\mathcal{Z}_1)} \to f^!_{\text{dR}} \circ f_{\text{dR},*}$$

is an isomorphism.

Consider the Cartesian square:

$$\begin{array}{ccc} \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1 & \xrightarrow{p_1} & \mathcal{Z}_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ \mathcal{Z}_1 & \longrightarrow & \mathcal{Z}_2. \end{array}$$

The above unit of the adjunction equals the composite map

$$\mathrm{Id}_{\mathrm{Crys}(\mathcal{Z}_1)} \simeq (p_2)_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \circ (p_1)^!_{\mathrm{dR}} \to (p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}} \to f^!_{\mathrm{dR}} \circ f_{\mathrm{dR},*},$$

where $\Delta_{\mathcal{Z}_1}$ is the diagonal map

$$\mathcal{Z}_1 \to \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1,$$

and second arrow is the co-unit of the $((\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*}, (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}})$-adjunction.

Now, by base change,

$$(p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}} \to f^!_{\mathrm{dR}} \circ f_{\mathrm{dR},*}$$

is an isomorphism. Hence, it is enough to show that

$$(p_2)_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \circ (p_1)^!_{\mathrm{dR}} \to (p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}}$$

is an isomorphism as well. However, the map

$$(\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \to \mathrm{Id}_{\mathrm{Crys}(\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1)}$$

is an isomorphism, since $(\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}}$ is an equivalence (because the map $\Delta_{\mathcal{Z}_1}$ is a nil-isomorphism). □

1.4. **The functor of de Rham direct image.** In this subsection we develop the functor of de Rham direct image (a.k.a., pushforward) for crystals.

1.4.1. Recall the functor

$$\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}} : \mathrm{indinfSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

that sends a morphism f to the functor f^{IndCoh}_*, see [Chapter III.3, Sect. 4.3].

Precomposing it with the functor

$$\mathrm{dR} : \mathrm{indnilSch}_{\mathrm{laft}} \to \mathrm{indinfSch}_{\mathrm{laft}}$$

we obtain a functor

$$\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}} : \mathrm{indnilSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}\,. \tag{1.1}$$

1.4.2. For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathrm{indnilSch}_{\mathrm{laft}}$ we shall denote the resulting functor

$$\mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$$

by $f_{\mathrm{dR},*}$.

In other words,

$$f_{\mathrm{dR},*} = (f_{\mathrm{dR}})^{\mathrm{IndCoh}}_*.$$

1.4.3. From [Chapter III.3, Corollary 5.2.3], we obtain:

Corollary 1.4.4. *The restriction of the functor* $\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *to the* 1*-full subcategory*

$$(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}} \subset \mathrm{indnilSch}_{\mathrm{laft}}$$

is obtained by passing to left adjoints from the restriction functor $\mathrm{Crys}^!_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *to*

$$((\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}} \subset (\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}.$$

Remark 1.4.5. Note that we have used the notation $f_{\mathrm{dR},*}$ when f is ind-proper earlier (in Lemma 1.3.8), to denote the left adjoint of $f^!_{\mathrm{dR}}$. The above corollary implies that the notations are consistent.

1.5. **Crystals on ind-nil-schemes as extended from schemes.** The material of this subsection will not be used in the sequel and is included for the sake of completeness. We show that the theory of Crys on ind-nil-schemes can be obtained by extending the same theory on schemes.

1.5.1. Consider the category ${}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$, and consider the functors

$$\mathrm{Crys}^!_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}} : ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}} : {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

From Proposition 1.2.5 we obtain:

Corollary 1.5.2. *The natural map*

$$\mathrm{Crys}^!_{\mathrm{indnilSch}_{\mathrm{laft}}} \to \mathrm{RKE}_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{Crys}^!_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})$$

is an isomorphism.

We are going to prove the following:

Proposition 1.5.3. *The natural map*

$$\mathrm{LKE}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}} \hookrightarrow \mathrm{indnilSch}_{\mathrm{laft}}}(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}) \to \mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$$

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

1.5.4. Consider the 1-full subcategory of $\mathrm{indnilSch}_{\mathrm{laft}}$ equal to

$$(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}} = \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} ({}^{\mathrm{red}}\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{closed}}.$$

I.e., we restrict 1-morphisms to be nil-closed maps.

It is enough to show that the map in Proposition 1.5.3 becomes an isomorphism when restricted to the above subcategory. This follows by [Chapter III.3, Corollary 4.1.4] from Proposition 1.4.4 and the following statement:

Proposition 1.5.5.

(a) *The map*

$$\Big(\big(\mathrm{RKE}_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}}\big)(\mathrm{Crys}^!_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})\Big)|_{((\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}})^{\mathrm{op}}} \to$$
$$\to \mathrm{RKE}_{(({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}})^{\mathrm{op}}}(\mathrm{Crys}^!_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}|_{(({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed}})^{\mathrm{op}}})$$

is an isomorphism.

(b) *The map*

$$\mathrm{LKE}_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed}}\hookrightarrow(\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}}}(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}|_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed}}}) \to$$
$$\to \left(\mathrm{LKE}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}\hookrightarrow\mathrm{indnilSch}_{\mathrm{laft}}}(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})\right)|_{(\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}}}$$

is an isomorphism.

Proof. Follows from the fact that for

$$\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}},$$

the category

$$\{f : Z \to \mathcal{Z}, \quad Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}, \quad f \text{ is nil-closed}\}$$

is cofinal in

$$\{f : Z \to \mathcal{Z}, \quad Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}\},$$

by [Chapter III.2, Corollary 1.7.5(b)]

□

1.6. **Properties of the category of crystals on (ind)-nil-schemes.** In this subsection we study properties of the category $\mathrm{Crys}(\mathcal{Z})$ on a given object $\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}}$.

1.6.1. We claim:

Proposition 1.6.2. *The functor*

$$\mathrm{Crys}(\mathcal{Z}) \to \lim_{f:Z\to\mathcal{Z}} \mathrm{Crys}(Z)$$

is an equivalence, where the limit is taken over the index ∞-category

$$\{f : Z \to \mathcal{Z},\ Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}},\ f \text{ is nil-closed}\}.$$

For every $f : Z \to \mathcal{Z}$ as above, the corresponding functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(Z) \to \mathrm{Crys}(\mathcal{Z})$$

is fully faithful.

Proof. The first assertion follows from Proposition 1.2.5 for $\mathbf{C} = {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and [Chapter III.2, Corollary 1.7.5(b)].

The second assertion follows from Lemma 1.3.8(b).

□

1.6.3. *Compact generation.* From [Chapter III.3, Corollary 3.2.2] and , we obtain:

Corollary 1.6.4. *The category* $\mathrm{Crys}(\mathcal{Z})$ *is compactly generated.*

From Proposition 1.6.2, combined with [DrGa2, Corollary 1.9.4 and Lemma 1.9.5], we have the following more explicit description of the subcategory

$$\mathrm{Crys}(\mathcal{Z})^c \subset \mathrm{Crys}(\mathcal{Z}).$$

Corollary 1.6.5. *Compact objects of* $\mathrm{Crys}(\mathcal{Z})$ *are those that can be obtained as*

$$f_{\mathrm{dR},*}(\mathcal{M}), \quad \mathcal{M} \in \mathrm{Crys}(Z)^c,\ Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}} \text{ and } f \text{ is a nil-closed map } Z \to \mathcal{Z}.$$

1.6.6. *t-structure.* According to [Chapter III.3, Sect. 3.4], the category $\mathrm{Crys}(\mathcal{Z})$ carries a canonical t-structure. It is characterized by the following property:

$$\mathcal{M} \in \mathrm{Crys}(\mathcal{Z})^{\geq 0} \Leftrightarrow \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}(\mathcal{M}) \in \mathrm{IndCoh}(\mathcal{Z})^{\geq 0}.$$

In addition, from [Chapter III.3, Corollary 3.4.4], we obtain:

Corollary 1.6.7.
(a) *An object* $\mathcal{M} \in \mathrm{Crys}(\mathcal{Z})$ *lies in* $\mathrm{Crys}(\mathcal{Z})^{\geq 0}$ *if and only if for every nil-closed map* $f : Z \to \mathcal{Z}$ *with* $Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *we have*

$$f^!_{\mathrm{dR}}(\mathcal{M}) \in \mathrm{Crys}(Z)^{\geq 0}.$$

(b) *The category* $\mathrm{Crys}(\mathcal{Z})^{\leq 0}$ *is generated under colimits by the essential images of* $\mathrm{Crys}(Z)^{\leq 0}$ *for* $f : Z \to \mathcal{Z}$ *with* $Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *and* f *nil-closed.*

2. Crystals as a functor out of the category of correspondences

In this section we extend the formalism of crystals to a functor out of the category of correspondences.

2.1. **Correspondences and the de Rham functor.** In this subsection we show that the de Rham functor turns (ind)-nil-schematic morphisms into (ind)-inf-schematic ones.

2.1.1. Recall that the functor dR commutes with Cartesian products. Combining this observation with Lemma 1.3.6, we obtain that dR gives rise to a functor of $(\infty, 2)$-categories:

$$\mathrm{Corr}(\mathrm{dR})^{\text{ind-proper}}_{\text{indnilsch;all}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indinfsch \& ind-proper}}_{\text{indinfsch;all}}.$$

Hence, from [Chapter III.3, Theorem 5.4.3 and Proposition 5.5.3], we obtain:

Theorem 2.1.2. *There exists a canonically defined functor*

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

equipped with an isomorphism

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}}|_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}} \simeq \mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

Furthermore, the restriction

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\text{indnilsch;all}}} := \mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}}|_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\text{indnilsch;all}}}$$

uniquely extends to a functor

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{nil-open}}_{\text{indnilsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\text{nil-open}}_{\text{indnilsch;all}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}.$$

2.1.3. As in the case of [Chapter III.3, Theorem 5.5.3], the content of Theorem 2.1.2 is the existence of the functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$$

for ind-nil-schematic morphisms of prestacks $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, and of the base change isomorphisms compatible with proper and nil-open adjunctions. Namely, for a Cartesian diagram of prestacks

$$\begin{array}{ccc} \mathcal{Z}'_1 & \xrightarrow{g_1} & \mathcal{Z}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z}'_2 & \xrightarrow{g_2} & \mathcal{Z}_2, \end{array}$$

with f ind-nil-schematic, we have *a canonical isomorphism*

$$f'_{\mathrm{dR},*} \circ g^!_{1,\mathrm{dR}} \xrightarrow{\sim} g^!_{2,\mathrm{dR}} \circ f_{\mathrm{dR},*}. \tag{2.1}$$

Moreover, if f is ind-proper, then $f_{\mathrm{dR},*}$ is the left adjoint of $f^!_{\mathrm{dR}}$. Furthermore, the isomorphism (2.1) is the one arising by adjunction if either f_X or g_2 is ind-proper.

If f is a nil-open embedding (i.e., the map of the corresponding reduced prestacks is an open embedding), then $f_{\mathrm{dR},*}$ is the right adjoint of $f^!_{\mathrm{dR}}$. Furthermore, the isomorphism (2.1) is the one arising by adjunction if either f_X or g_2 is a nil-open embedding.

2.1.4. Now, let us restrict the functor $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}}$ to

$$\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}.$$

We denote the resulting functor by $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$. From [Chapter III.3, Theorems 5.2.2 and 5.4.3] we obtain:

Corollary 2.1.5. *The restriction of* $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$ *to*

$$\mathrm{indnilSch}_{\mathrm{laft}} \subset \mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}$$

identifies canonically with the functor $\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *of* (1.1).

2.1.6. Further restricting along

$$\mathrm{Corr}(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}},$$

we obtain a functor

$$\mathrm{Crys}_{(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} \to (\mathrm{DGCat}_{\mathrm{cont}})^{2\text{-}\mathrm{Cat}}$$

denoted by $\mathrm{Crys}_{(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$.

In particular, we obtain a functor

$$\mathrm{Crys}_{\mathrm{nilSch}_{\mathrm{aft}}} := \mathrm{Crys}_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}|_{\mathrm{nilSch}_{\mathrm{aft}}},$$

which is also isomorphic to

$$\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}|_{\mathrm{nilSch}_{\mathrm{aft}}}.$$

2.2. **The multiplicative structure of the functor of crystals.** In this subsection we show how the formalism of crystals as a functor out of the category of correspondences gives rise to Verdier duality.

2.2.1. *Duality.* From [Chapter III.3, Theorem 6.2.2], we obtain:

Theorem 2.2.2. *We have a commutative diagram of functors*

$$\begin{array}{ccc} (\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}})^{\mathrm{op}} & \xrightarrow{\left(\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}}}\right)^{\mathrm{op}}} & \left(\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\right)^{\mathrm{op}} \\ \varpi\downarrow & & \downarrow \text{dualization} \\ \mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}} & \xrightarrow{\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}}}} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}. \end{array}$$

2.2.3. Concretely, this theorem says that for $\mathcal{Z} \in \text{indnilSch}_{\text{laft}}$ there is a canonical involutive equivalence

$$\mathbf{D}^{\text{Verdier}}_{\mathcal{Z}} : \text{Crys}(\mathcal{Z})^{\vee} \simeq \text{Crys}(\mathcal{Z}), \tag{2.2}$$

and for a map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\text{indnilSch}_{\text{laft}}$ there is a canonical identification

$$f^!_{\text{dR}} \simeq (f_{\text{dR},*})^{\vee}.$$

2.2.4. As in [Chapter III.3, Sect. 6.2.6], we can write the unit and counit maps

$$\mu_{\mathcal{Z}_{\text{dR}}} : \text{Vect} \to \text{Crys}(\mathcal{Z}) \otimes \text{Crys}(\mathcal{Z}) \text{ and } \epsilon_{\mathcal{Z}_{\text{dR}}} : \text{Crys}(\mathcal{Z}) \otimes \text{Crys}(\mathcal{Z}) \to \text{Vect}$$

explicitly.

Namely, $\epsilon_{\mathcal{Z}_{\text{dR}}}$ is the composition

$$\text{Crys}(\mathcal{Z}) \otimes \text{Crys}(\mathcal{Z}) \simeq \text{Crys}(\mathcal{Z} \times \mathcal{Z}) \overset{\Delta^!_{\mathcal{Z},\text{dR}}}{\longrightarrow} \text{Crys}(\mathcal{Z}) \overset{\Gamma_{\text{dR}}(\mathcal{Z},-)}{\longrightarrow} \text{Vect},$$

where

$$\Gamma_{\text{dR}}(\mathcal{Z}, -) := (p_{\mathcal{Z}})_{\text{dR},*},$$

and $\mu_{\mathcal{Z}_{\text{dR}}}$ is the composition

$$\text{Vect} \overset{\omega_{\mathcal{Z}_{\text{dR}}}}{\longrightarrow} \text{Crys}(\mathcal{Z}) \overset{(\Delta_{\mathcal{Z}})_{\text{dR},*}}{\longrightarrow} \text{Crys}(\mathcal{Z} \times \mathcal{Z}) \simeq \text{Crys}(\mathcal{Z}) \otimes \text{Crys}(\mathcal{Z}).$$

2.2.5. *Verdier duality.* For $\mathcal{Z} \in \text{indnilSch}_{\text{laft}}$, let $\mathbb{D}^{\text{Verdier}}_{\mathcal{Z}}$ denote the canonical equivalence

$$(\text{Crys}(\mathcal{Z})^c)^{\text{op}} \to \text{Crys}(\mathcal{Z})^c,$$

corresponding to the isomorphism (2.2).

In other words,

$$\mathbb{D}^{\text{Verdier}}_{\mathcal{Z}} = \mathbb{D}^{\text{Serre}}_{\mathcal{Z}_{\text{dR}}}.$$

2.2.6. As a particular case of [Chapter III.3, Corollary 6.2.9], we obtain:

Corollary 2.2.7. *Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be an ind-proper map in* $\text{indnilSch}_{\text{laft}}$. *Then we have a commutative diagram:*

$$\begin{array}{ccc} (\text{Crys}(\mathcal{Z}_1)^c)^{\text{op}} & \xrightarrow{\mathbb{D}^{\text{Verdier}}_{\mathcal{Z}_1}} & \text{Crys}(\mathcal{Z}_1)^c \\ {\scriptstyle (f_{\text{dR},*})^{\text{op}}}\downarrow & & \downarrow{\scriptstyle f_{\text{dR},*}} \\ (\text{Crys}(\mathcal{Z}_2)^c)^{\text{op}} & \xrightarrow{\mathbb{D}^{\text{Verdier}}_{\mathcal{Z}_2}} & \text{Crys}(\mathcal{Z}_2)^c. \end{array}$$

In view of Corollary 1.6.5, the above corollary gives an expression of the Verdier duality functor on $\mathcal{Z} \in \text{indnilSch}_{\text{laft}}$ in terms of that on schemes.

2.2.8. *Convolution for crystals.*

Returning to the entire $(\infty, 2)$-category $\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch}\,\&\,\text{ind-proper}}_{\text{indnilsch;all}}$ and the corresponding functor

$$\text{IndCoh}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch}\,\&\,\text{ind-proper}}_{\text{indnilsch;all}}},$$

we obtain, from [Chapter III.3, Sect. 6.3], that the functor

$$\text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch}\,\&\,\text{ind-proper}}_{\text{indnilsch;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch}\,\&\,\text{ind-proper}}_{\text{indnilsch;all}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}}$$

carries a canonical right-lax symmetric monoidal structure.

As in [Chapter III.3, Sect. 6.3.2], we have:

(i) Given a Segal object $\mathcal{R}^\bullet$ of $\mathrm{PreStk}_{\mathrm{laft}}$, with the target and composition maps ind-nil-schematic, the category $\mathrm{Crys}(\mathcal{R})$ acquires a monoidal structure given by convolution, and as such it acts on $\mathrm{Crys}(\mathcal{X})$ (here, as in [Chapter II.2, Sect. 5.1.1], $\mathcal{X} = \mathcal{R}^0$ and $\mathcal{R} = \mathcal{R}^1$).

(ii) If the composition map is ind-proper, then $\omega_{\mathcal{R}} \in \mathrm{Crys}(\mathcal{R})$ acquires the structure of an algebra in $\mathrm{Crys}(\mathcal{R})$. The action of this algebra on $\mathrm{IndCoh}(\mathcal{X})$, viewed as a plain endo-functor, is given by

$$(p_t)_{\mathrm{dR},*} \circ (p_s)^!_{\mathrm{dR}}.$$

3. Inducing crystals

In this section we study the interaction between the functors IndCoh and Crys.

3.1. **The functor of induction.** In this subsection we show that the forgetful functor

$$\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint, provided that $\mathcal{Z}$ is a prestack that admits deformation theory.

3.1.1. For an object $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ consider the canonical map

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

We claim:

Proposition 3.1.2. *Suppose that $\mathcal{Z}$ admits deformation theory. Then the map $p_{\mathrm{dR},\mathcal{Z}}$ is an inf-schematic nil-isomorphism.*

Proof. We need to show that for $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}$, the Cartesian product

$$S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z} \tag{3.1}$$

is an inf-scheme.

Clearly, the above Cartesian product belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, and its underlying reduced prestack identifies with ${}^{\mathrm{red}}S$. Hence, it remains to show that (3.1) admits deformation theory. This holds because the category $\mathrm{PreStk}_{\text{def-laft}}$ is closed under finite limits.

□

3.1.3. From Proposition 3.1.2 and [Chapter III.3, Proposition 3.1.2(a)] we obtain:

Corollary 3.1.4. *Let $\mathcal{Z}$ be an object of* $\mathrm{PreStk}_{\text{def-laft}}$*. Then the functor*

$$\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint.

We denote the left adjoint to $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$, whose existence is given by the above corollary, by $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$.

3.1.5. Thus, for $\mathcal{Z} \in \mathrm{PreStk}_{\text{def-laft}}$, we obtain an adjoint pair

$$\mathbf{ind}_{\mathrm{dR},\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Crys}(\mathcal{Z}) : \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}. \tag{3.2}$$

We claim:

Lemma 3.1.6. *The pair* (3.2) *is monadic.*

Proof. Since $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ is continuous, we only need to check that it is conservative. However, this follows from [Chapter III.3, Proposition 3.1.2(b)]. □

3.1.7. The next corollary of Proposition 3.1.2 expresses the functoriality of the operation of induction:

Corollary 3.1.8. *There is a canonically defined natural transformation*

$$\mathbf{ind}_{\mathrm{dR}} : \mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch}}}|_{(\mathrm{PreStk}_{\text{def-laft}})_{\mathrm{indinfsch}}} \Rightarrow \mathrm{Crys}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch}}}|_{(\mathrm{PreStk}_{\text{def-laft}})_{\mathrm{indinfsch}}},$$

as functors

$$(\mathrm{PreStk}_{\text{def-laft}})_{\text{indinf-sch}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

In particular, the above corollary says that for an ind-inf-schematic morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ of objects of $\mathrm{PreStk}_{\text{def-laft}}$, the following diagram of functors commutes:

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Z}_1) & \xrightarrow{\mathbf{ind}_{\mathrm{dR},\mathcal{Z}_1}} & \mathrm{Crys}(\mathcal{Z}_1) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{dR},*}} \\ \mathrm{IndCoh}(\mathcal{Z}_2) & \xrightarrow{\mathbf{ind}_{\mathrm{dR},\mathcal{Z}_2}} & \mathrm{Crys}(\mathcal{Z}_2). \end{array}$$

3.2. **Induction on ind-inf-schemes.** In this subsection, let $\mathcal{Z}$ be an object of $\mathrm{indinfSch}_{\mathrm{laft}}$. We study the interaction of the induction functor with that of Serre and Verdier dualities.

3.2.1. We have:

Lemma 3.2.2. *The functor* $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ *sends* $\mathrm{IndCoh}(\mathcal{Z})^c$ *to* $\mathrm{Crys}(\mathcal{Z})^c$*.*

Proof. Follows from the fact that the functor $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ is continuous and conservative. □

3.2.3. *Induction and duality.* Let us apply isomorphism [Chapter III.3, Equation (6.2)] to the map

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

We obtain:

Corollary 3.2.4. *Under the isomorphisms*

$$\mathbf{D}_{\mathcal{Z}}^{\mathrm{Serre}} : \mathrm{IndCoh}(\mathcal{Z})^\vee \simeq \mathrm{IndCoh}(\mathcal{Z}) \text{ and } \mathbf{D}_{\mathcal{Z}}^{\mathrm{Verdier}} : \mathrm{Crys}(\mathcal{Z})^\vee \simeq \mathrm{Crys}(\mathcal{Z}),$$

we have a canonical identification

$$(\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}})^\vee \simeq \mathbf{ind}_{\mathrm{dR},\mathcal{Z}}.$$

In addition, by [Chapter III.3, Corollary 6.2.9]

Corollary 3.2.5. *The following diagram of functors commutes:*

$$\begin{array}{ccc} (\mathrm{IndCoh}(\mathcal{Z})^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}_{\mathcal{Z}}^{\mathrm{Serre}}} & \mathrm{IndCoh}(\mathcal{Z})^c \\ {\scriptstyle (\mathbf{ind}_{\mathrm{dR},\mathcal{Z}})^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}_{\mathrm{dR},\mathcal{Z}}} \\ (\mathrm{Crys}(\mathcal{Z})^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}_{\mathcal{Z}}^{\mathrm{Verdier}}} & \mathrm{Crys}(\mathcal{Z})^c. \end{array}$$

3.2.6. *Induction and t-structure.* Recall that by the definition of the t-structure on $\mathrm{Crys}(\mathcal{Z})$, the functor $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ is left t-exact. We claim:

Corollary 3.2.7. *Assume that $\mathcal{Z}$ is an ind-scheme. Then the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is t-exact.*

Proof. The fact that $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is right t-exact follows by adjunction. To show that it is left t-exact we use [Chapter III.3, Lemma 3.4.6]: we have to show that the $p_{\mathrm{dR},\mathcal{Z}}$ is ind-schematic.

Indeed, for $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and $S \to \mathcal{Z}_{\mathrm{dR}}$, the Cartesian product

$$S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}$$

identifies with the formal completion of $S \times \mathcal{Z}$ along the graph of the map ${}^{\mathrm{red}}S \to \mathcal{Z}$. $\square$

3.3. **Relative crystals.** In this subsection we describe how the discussion of crystals generalizes to the relative situation.

3.3.1. Let $\mathcal{Y}$ be a fixed object of $\mathrm{PreStk}_{\mathrm{laft}}$. Consider the ∞-category

$$(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$$

and the corresponding $(\infty, 2)$-category

$$\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}.$$

Restricting the functor $\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}$ along the forgetful functor

$$\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{inf\text{-}proper}}_{\mathrm{inf\text{-}sch;all}},$$

we obtain the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

with properties specified by [Chapter III.3, Theorem 5.4.3].

In particular, let

$$\mathrm{IndCoh}^!_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

be the resulting functor.

3.3.2. The category $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$ has an endo-functor, denoted by ${}^{/\mathcal{Y}}\mathrm{dR}$:

$$\mathcal{Z} \mapsto \mathcal{Z}_{/\mathcal{Y}\mathrm{dR}} := \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}.$$

Corollary 1.3.6 implies that the functor ${}^{/\mathcal{Y}}\mathrm{dR}$ gives rise to a functor

$$((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilSch}} \to ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch}}.$$

Hence, ${}^{/\mathcal{Y}}\mathrm{dR}$ induces a functor

$$\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}. \tag{3.3}$$

Thus, precomposing $\mathrm{IndCoh}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}$ with ${}^{/\mathcal{Y}}\mathrm{dR}$, we obtain the functor

$${}^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}} : \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

3.3.3. Let

$$^{/\mathcal{Y}}\mathrm{Crys}^!_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$^{/\mathcal{Y}}\mathrm{Crys}_{((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

denote the corresponding functors obtained by restriction.

For a map $\mathcal{Z}_1 \to \mathcal{Z}_2$ in $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indnilsch}}$, we shall denote by $f^!_{\mathcal{Y}\mathrm{dR}}$ and $f_{\mathcal{Y}\mathrm{dR},*}$ the corresponding functors

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}_1) \leftrightarrows {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}_2).$$

These functors are adjoint if f is ind-proper/nil-open.

3.3.4. Let $^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}}$ denote the full subcategory of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$ given by the preimage of $(\mathrm{indinfSch}_{\mathrm{laft}})_{/\mathcal{Y}}$ under the functor $^{/\mathcal{Y}}\mathrm{dR}$.

Restricting the functor $^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}}$ to

$$\mathrm{Corr}({}^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \subset \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}},$$

we obtain the functor

$$^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}({}^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}({}^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

Furthermore, for an object $\mathcal{Z}$ of $^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}}$, the category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ satisfies the following properties:

(1) The category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ is compactly generated, and is self-dual in the sense of Theorem 2.2.2.

(2) The category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ carries a t-structure in which an object $\mathcal{F}$ is coconnective if and only if its image under the forgetful functor $\mathbf{oblv}_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}} : {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$ is coconnective, for $\mathbf{oblv}_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}} := (p_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}})^!$, where $p_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}}$ denotes the canonical morphism

$$\mathcal{Z} \to \mathcal{Z}_{/\mathcal{Y}\mathrm{dR}}.$$

(3) If $\mathcal{Z}$ admits deformation theory *over* $\mathcal{Y}$ (see [Chapter III.1, Sect. 7.1.6] for what this means), then the morphism $p_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}}$ is an inf-schematic nil-isomorphism, and hence the functor $\mathbf{oblv}_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}}$ admits a left adjoint, denoted $\mathbf{ind}_{/\mathcal{Y}\mathrm{dR},\mathcal{Z}}$.

Remark 3.3.5. The essential difference between Crys and $^{/\mathcal{Y}}\mathrm{Crys}$ is that for $\mathcal{Z} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$, the category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ depends *not just* on the underlying reduced prestack. E.g., for $\mathcal{Z} = \mathcal{Y}$,

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) = \mathrm{IndCoh}(\mathcal{Y}).$$

3.3.6. Assume for a moment that $\mathcal{Y} = Y$ is a smooth classical scheme, and $\mathcal{Z} = Z$ is also a classical scheme smooth over Y. Then, as in [GaRo2, Sect. 4.7], one shows that the category

$$^{/Y}\mathrm{Crys}(Z)$$

identifies with the DG category associated with the abelian category of quasi-coherent sheaves of modules on Y with respect to the algebra of 'vertical' differential operators, i.e., the subalgebra of $\mathrm{Diff}(Z)$ that consists of differential operators that commute with $\mathcal{O}_Y$.

4. Comparison with the classical theory of D-modules

In this section we will identify the theory of crystals as developed in the previous sections with the theory of D-modules.

This section can be regarded as a companion to [GaRo2, Sects. 6 and 7], and we shall assume the reader's familiarity with the contents of *loc.cit.*

4.1. **Left D-modules and left crystals.** In this subsection we will recollect (and rephrase) the contents of [GaRo2, Sect. 5]. Specifically, we will discuss the equivalence between the category of *left D-modules* on a smooth scheme X and the category of *left crystals* on X.

4.1.1. Let X be a classical scheme of finite type. Consider the category $\operatorname{QCoh}(X\times X)^\heartsuit$ and its full subcategory $(\operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit$, consisting of objects that are set-theoretically supported on the diagonal. Let

$$(\operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit_{\text{rel.flat}}\subset \operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit$$

be the full subcategory, consisting of objects that are X-flat with respect to both projections

$$p_s,p_t:X\times X\to X.$$

The category $(\operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit_{\text{rel.flat}}$ has a naturally defined monoidal structure, given by convolution.

Moreover, we have a canonically defined fully faithful *monoidal* (!) functor

$$(\operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit_{\text{rel.flat}}\to \operatorname{QCoh}(X\times X), \tag{4.1}$$

where $\operatorname{QCoh}(X\times X)$ is a monoidal category as in [Chapter II.2, Sects. 5.2.3 and 5.3.3].

4.1.2. Now suppose that X is smooth. In this case, we have a canonically defined object

$$\operatorname{Diff}_X\in \operatorname{AssocAlg}\left((\operatorname{QCoh}(X\times X)_{\Delta_X})^\heartsuit_{\text{rel.flat}})\right),$$

namely, the Grothendieck algebra of differential operators.

Composing (4.1) with the monoidal equivalence

$$\operatorname{QCoh}(X\times X)\to \operatorname{Funct}_{\text{cont}}(\operatorname{QCoh}(X),\operatorname{QCoh}(X)),$$

we obtain that Diff_X gives rise to a monad on $\operatorname{QCoh}(X)$.

We consider the category $\operatorname{Diff}_X\text{-mod}(\operatorname{QCoh}(X))$. It is equipped with a t-structure, characterized by the property that the tautological forgetful functor $\operatorname{Diff}_X\text{-mod}(\operatorname{QCoh}(X))\to\operatorname{QCoh}(X)$ is t-exact.

The category $\text{D-mod}^l(X)$ of *left* D-modules on X is defined as the canonical DG model of the derived category of the abelian category $(\operatorname{Diff}_X\text{-mod}(\operatorname{QCoh}(X)))^\heartsuit$.

As in [GaRo2, Proposition 4.7.3] one shows that the canonical functor

$$\text{D-mod}^l(X)\to \operatorname{Diff}_X\text{-mod}(\operatorname{QCoh}(X)) \tag{4.2}$$

is an equivalence.

4.1.3. Let X be any scheme almost of finite type. Recall the category

$$\mathrm{Crys}^l(X) := \mathrm{QCoh}(X_{\mathrm{dR}}),$$

see [GaRo2, Sect. 2.1]. It is equipped with a forgetful functor

$$\mathbf{oblv}^l_{\mathrm{dR},X} : \mathrm{Crys}^l(X) \to \mathrm{QCoh}(X).$$

Assume now that X is eventually coconnective. According to [GaRo2, Proposition 3.4.11], in this case the functor $\mathbf{oblv}^l_{\mathrm{dR},X}$ admits a left adjoint, denoted $\mathbf{ind}^l_{\mathrm{dR},X}$, and the resulting adjoint pair of functors

$$\mathbf{ind}^l_{\mathrm{dR},X} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}^l(X) : \mathbf{oblv}^l_{\mathrm{dR},X},$$

is monadic.

The corresponding monad is given by an object

$$\mathcal{D}^l_X \in \mathrm{AssocAlg}(\mathrm{QCoh}(X \times X)_{\Delta_X}).$$

I.e., we have an equivalence

$$\mathrm{Crys}^l(X) \simeq \mathcal{D}^l_X\text{-mod}(\mathrm{QCoh}(X)).$$

4.1.4. Again, assume that X is smooth. In this case one easily shows (see, e.g., [GaRo2, Proposition 5.3.6]) that

$$\mathcal{D}^l_X \in (\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}.$$

Moreover, it is a classical fact (reproved for completeness in [GaRo2, Lemma 5.4.3]) that there is a canonical isomorphism in $\mathrm{AssocAlg}\Big((\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}\Big)$:

$$\mathcal{D}^l_X \simeq \mathrm{Diff}_X\,. \tag{4.3}$$

In particular, we obtain a canonical equivalence of categories

$$\text{D-mod}^l(X) \simeq \mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X)) \simeq \mathcal{D}^l_X\text{-mod}(\mathrm{QCoh}(X)) \simeq \mathrm{Crys}^l(X),$$

compatible with the forgetful functors to $\mathrm{QCoh}(X)$.

We denote the resulting equivalence $\text{D-mod}^l(X) \to \mathrm{Crys}^l(X)$ by F^l_X.

4.1.5. Let $f : X \to Y$ be a morphism between smooth classical schemes. In the classical theory of D-modules, one defines a functor

$$f^{\blacktriangle,l} : \text{D-mod}^l(Y) \to \text{D-mod}^l(X)$$

that makes the diagram

$$\begin{array}{ccc} \text{D-mod}^l(Y) & \xrightarrow{f^{\blacktriangle,l}} & \text{D-mod}^l(X) \\ \downarrow & & \downarrow \mathbf{oblv}^l_{\mathrm{dR},X} \\ \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \end{array} \tag{4.4}$$

commute.

Also recall (see [GaRo2, Sect. 2.1.2]) that for a map $f : X \to Y$ between arbitrary schemes almost of finite type, we have a functor

$$f^{\dagger,l} : \mathrm{Crys}^l(Y) \to \mathrm{Crys}^l(X),$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{Crys}^l(Y) & \xrightarrow{f^{\dagger,l}} & \mathrm{Crys}^l(X) \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X)
\end{array}
\tag{4.5}
$$

commute.

The following can be established by a direct calculation:

Lemma 4.1.6. *The following diagram of functors naturally commutes*

$$
\begin{array}{ccc}
\text{D-mod}^l(Y) & \xrightarrow{f^{\blacktriangle,l}} & \text{D-mod}^l(X) \\
{\scriptstyle F^l_Y}\downarrow & & \downarrow{\scriptstyle F^l_X} \\
\mathrm{Crys}^l(X) & \xrightarrow{f^{\dagger,l}} & \mathrm{Crys}^l(X),
\end{array}
$$

in a way compatible with the forgeful functors to $\mathrm{QCoh}(-)$.

4.2. **Right D-modules and right crystals.** In this subsection we will discuss the equivalence between the category of right D-modules on a smooth scheme X, and the category $\mathrm{Crys}(X)$, considered in the earlier sections of this Chapter.

4.2.1. Note that for any scheme X, the monoidal category $\mathrm{QCoh}(X \times X)$ carries a canonical anti-involution, denoted σ, corresponding to the transposition acting on $X \times X$.

In terms of the identification

$$\mathrm{QCoh}(X \times X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

we have

$$\sigma(F) \simeq F^\vee, \quad F \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

where we use the canonical identification

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X),$$

of [Chapter II.3, Equation (4.2)].

In particular for an algebra object A in $\mathrm{QCoh}(X \times X)$, we can regard $\sigma(A^{\mathrm{op}})$ again as an algebra object in $\mathrm{QCoh}(X \times X)$, and we have

$$(M_A)^\vee \simeq M_{\sigma(A^{\mathrm{op}})}, \tag{4.6}$$

where M_B denotes the monad on $\mathrm{QCoh}(X)$, corresponding to an algebra object $B \in \mathrm{QCoh}(X)$.

4.2.2. Let X be smooth. Consider the object

$$\sigma(\mathrm{Diff}_X^{\mathrm{op}}) \in \mathrm{AssocAlg}\left((\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}})\right),$$

and the corresponding category

$$\sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X)).$$

The category $\text{D-mod}^r(X)$ of *right* D-modules on X is defined as the canonical DG model of the derived category of the abelian category $(\sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X)))^{\heartsuit}$.

As in [GaRo2, Proposition 4.7.3] one shows that the canonical functor

$$\text{D-mod}^r(X) \to \sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X)) \tag{4.7}$$

is an equivalence.

4.2.3. Let X be a scheme almost of finite type. For the duration of this section, we will denote by

$$\mathrm{Crys}^r(X) := \mathrm{Crys}(X) := \mathrm{IndCoh}(X_{\mathrm{dR}}),$$

where the latter is defined as in Sect. 1.2.2, and by

$$\mathbf{ind}^r_{\mathrm{dR},X} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{Crys}^r(X) : \mathbf{oblv}^r_{\mathrm{dR},X}$$

the corresponding pair of adjoint functors from (3.2). I.e., we are adding the superscript 'r' (for 'right') to the notation from Sect. 1.2.2 to emphasize the comparison with right D-modules.

Let $\mathcal{D}^r_X$ be the object of $\mathrm{AssocAlg}(\mathrm{IndCoh}(X \times X)_{\Delta_X})$, corresponding to the monad

$$\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X}$$

via the equivalence of monoidal categories

$$\mathrm{IndCoh}(X \times X) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{IndCoh}(X)).$$

By Lemma 3.1.6, we have:

$$\mathcal{D}^r_X\text{-mod}(\mathrm{IndCoh}(X)) \simeq \mathrm{Crys}^r(X).$$

4.2.4. Suppose that X is smooth. Recall that in this case the adjoint pairs

$$\Xi_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi_X$$

and

$$\Xi^\vee_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi^\vee_X = \Upsilon_X$$

are both equivalences.

In this case one shows as in [GaRo2, Sect. 5.5] that there is a canonical equivalence of categories

$$F^r_X : \text{D-mod}^r(X) \simeq \mathrm{Crys}^r(X), \tag{4.8}$$

that makes the diagram

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \mathrm{Crys}^r(X) \\ \downarrow & & \downarrow \mathbf{oblv}^r_{\mathrm{dR},X} \\ \mathrm{QCoh}(X) & \xrightarrow{\Xi_X} & \mathrm{IndCoh}(X) \end{array}$$

commute.

Remark 4.2.5. One can obtain the equivalence of (4.8) formally from the corresponding computation for left D-modules.

Namely, taking into account the equivalences

$$(\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X})^\vee\text{-mod}(\mathrm{QCoh}(X)) \simeq \sigma(\mathrm{Diff}^{\mathrm{op}}_X)\text{-mod}(\mathrm{QCoh}(X)) \simeq \text{D-mod}^r(X)$$

(where the first equivalence comes from (4.3) and (4.6)), and

$$(\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X})\text{-mod}(\mathrm{IndCoh}(X)) \simeq \mathrm{Crys}^r(X),$$

it suffices to construct an isomorphism of the monads

$$\Psi_X \circ (\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X}) \circ \Xi_X \text{ and } (\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X})^\vee, \tag{4.9}$$

acting on $\mathrm{QCoh}(X)$.

The latter isomorphism holds for any eventually coconnective X, and follows from the isomorphism of monads

$$\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X} \simeq \Xi^\vee_X \circ (\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X}) \circ \Psi^\vee_X,$$

see [GaRo2, Lemma 3.4.9].

Remark 4.2.6. The monads in (4.9) correspond to the pair of adjoint functors

$$'\mathbf{ind}^r_{\mathrm{dR},X} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}(X) : {}'\mathbf{oblv}^r_{\mathrm{dR},X}$$

of [GaRo2, Sect. 4.6]. Furthermore, in terms of the

$$\mathbf{D}^{\mathrm{naive}}_X : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathbf{D}^{\mathrm{Verdier}}_X : \mathrm{Crys}(X)^\vee \simeq \mathrm{Crys}(X),$$

we have the isomorphisms

$$('\mathbf{ind}^r_{\mathrm{dR},X})^\vee \simeq \mathbf{oblv}^l_{\mathrm{dR},X} \text{ and } ('\mathbf{oblv}^r_{\mathrm{dR},X})^\vee \simeq \mathbf{ind}^l_{\mathrm{dR},X}.$$

4.3. **Passage between left and right D-modules/crystals.** In this subsection we will compare the abstractly defined functor

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X)$$

from [Chapter II.3] and the 'hands-on' functor

$$\text{D-mod}^l(X) \to \text{D-mod}^r(X),$$

given by tensoring a given left D-module with the right D-module

$$\det(T^*(X))[\dim(X)].$$

4.3.1. According to [GaRo2, Proposition 2.2.4], for any scheme X almost of finite type we have a canonically defined equivalence

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X),$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{Crys}^l(X) & \xrightarrow{\Upsilon_{X_{\mathrm{dR}}}} & \mathrm{Crys}^r(X) \\ \Big\downarrow{\scriptstyle \mathbf{oblv}^l_{\mathrm{dR},X}} & & \Big\downarrow{\scriptstyle \mathbf{oblv}^r_{\mathrm{dR},X}} \\ \mathrm{QCoh}(X) & \xrightarrow{\Upsilon_X} & \mathrm{IndCoh}(X) \end{array}$$

commute.

Concretely, the functor $\Upsilon_{X_{\mathrm{dR}}}$ is the functor from [Chapter II.3, Sect. 3.3.4] applied to X_{dR}, and it is given by

$$\mathcal{M} \mapsto \mathcal{M} \otimes \omega_{X_{\mathrm{dR}}},$$

where $\otimes$ is the action of $\mathrm{QCoh}(X_{\mathrm{dR}})$ ($= \mathrm{Crys}^l(X)$) on $\mathrm{IndCoh}(X_{\mathrm{dR}})$ ($= \mathrm{Crys}^r(X)$).

4.3.2. Now, suppose that X is a smooth classical scheme. Recall that in this case there is a canonical equivalence

$$\text{D-mod}^l(X) \to \text{D-mod}^r(X), \tag{4.10}$$

given by tensoring a given left D-module with the right D-module

$$\omega_{\text{D-mod},X} := \det(T^*(X))[\dim(X)].$$

We denote the above functor by $\Upsilon_{\text{D-mod},X}$.

4.3.3. We will prove:

Theorem 4.3.4. *The following diagram of functors canonically commutes:*

$$\begin{array}{ccc} \text{D-mod}^l(X) & \xrightarrow{F^l_X} & \text{Crys}^l(X) \\ \Upsilon_{\text{D-mod},X}\downarrow & & \downarrow\Upsilon_{X_{\text{dR}}} \\ \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \text{Crys}^r(X) \end{array} \tag{4.11}$$

By applying Theorem 4.3.4 to $\mathcal{O}_X \in \text{D-mod}^l(X)$, we obtain:

Corollary 4.3.5. *There exists a canonical isomorphism in* $\text{Crys}^r(X)$*:*

$$F^r_X(\omega_{\text{D-mod},X}) \simeq \omega_{X_{\text{dR}}}. \tag{4.12}$$

4.3.6. Applying the forgetful functor

$$\mathbf{oblv}^r_{\text{dR},X} : \text{Crys}^r(X) \to \text{IndCoh}(X),$$

to the isomorphism of (4.12), we obtain:

Corollary 4.3.7. *There exists a canonical isomorphism in* $\text{IndCoh}(X)$*:*

$$\Xi_X(\det(T^*(X))[\dim(X)] \simeq \omega_X. \tag{4.13}$$

Note that latter corollary is the well-known identification of the abstractly defined dualizing sheaf with the shifted line bundle of top forms.

Remark 4.3.8. The isomorphism (4.13) can be proved without involving D-modules or crystals, by an argument along the same lines as that proving the isomorphism (4.12) in Sect. 4.4 below. This argument is given in a more general context in [Chapter IV.4, Proposition 7.3.4].

4.4. Proof of Theorem 4.3.4.

4.4.1. First, we make the following observation, which follows from the constructions:

Lemma 4.4.2. *For* $\mathcal{M} \in \text{D-mod}^l(X)$ *and* $\mathcal{N} \in \text{D-mod}^r(X)$ *we have a canonical isomorphism*

$$F^r_X(\mathcal{M} \otimes \mathcal{N}) \simeq F^l_X(\mathcal{M}) \otimes F^r_X(\mathcal{N}).$$

This lemma reduces the assertion of Theorem 4.3.4 to establishing the isomorphism (4.12).

4.4.3. Recall that for a map $f : X \to Y$ between smooth schemes, one *defines* the functor

$$f^{\blacktriangle,r} : \text{D-mod}^r(Y) \to \text{D-mod}^r(X)$$

by requiring that the diagram

$$\begin{array}{ccc} \text{D-mod}^l(Y) & \xrightarrow{f^{\blacktriangle,l}} & \text{D-mod}^l(X) \\ \Upsilon_{\text{D-mod},Y}\downarrow & & \downarrow\Upsilon_{\text{D-mod},X} \\ \text{D-mod}^r(Y) & \xrightarrow{f^{\blacktriangle,r}} & \text{D-mod}^r(X) \end{array} \tag{4.14}$$

commute.

Assume for the moment that f is a closed embedding of smooth schemes. Let

$$\text{D-mod}^r(Y)_X \subset \text{D-mod}^r(Y)$$

be the full subcategory consisting of objects with set-theoretic support on X.

Recall that in this case we have Kashiwara's lemma which says that the functor $f^{\blacktriangle,r}$ induces an equivalence $\text{D-mod}^r(Y)_X \to \text{D-mod}^r(X)$.

4.4.4. For a morphism $f : X \to Y$ between schemes almost of finite type, let

$$f^{\dagger,r} : \mathrm{Crys}^r(Y) \to \mathrm{Crys}^r(X)$$

be the corresponding pullback functor, see [GaRo2, Sect. 2.3.4], i.e., $f^{\dagger,r} = f^!_{\mathrm{dR}}$.

Assume for the moment that f is a closed embedding. Let $\mathrm{Crys}^r(Y)_X \subset \mathrm{Crys}^r(Y)$ be the full subcategory consisting of objects with set-theoretic support on X.

Recall (see [GaRo2, Proposition 2.5.6]) that in this case the functor $f^{\dagger,r}$ induces an equivalence $\mathrm{Crys}^r(Y)_X \to \mathrm{Crys}^r(X)$.

4.4.5. Let $f : X \to Y$ be a closed embedding of smooth classical schemes. The next assertion also follows from the constructions:

Lemma 4.4.6. *Under the equivalences*

$$f^{\blacktriangle,r} : \text{D-mod}^r(Y)_X \to \text{D-mod}^r(X) \text{ and } f^{\dagger,r} : \mathrm{Crys}^r(Y)_X \to \mathrm{Crys}(X),$$

the diagram

$$\begin{array}{ccc} \text{D-mod}^r(Y)_X & \xrightarrow{F^r_Y} & \mathrm{Crys}^r(Y)_X \\ {\scriptstyle f^{\blacktriangle,r}}\downarrow & & \downarrow{\scriptstyle f^{\dagger,r}} \\ \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \mathrm{Crys}^r(X) \end{array}$$

commutes.

As a corollary, we obtain:

Corollary 4.4.7. *For a closed embedding of smooth schemes $f : X \to Y$, the diagram*

$$\begin{array}{ccc} \text{D-mod}^r(Y) & \xrightarrow{F^r_Y} & \mathrm{Crys}^r(Y) \\ {\scriptstyle f^{\blacktriangle,r}}\downarrow & & \downarrow{\scriptstyle f^{\dagger,r}} \\ \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \mathrm{Crys}^r(X) \end{array}$$

commutes.

Proof. Follows from the fact that the functor $f^{\blacktriangle,r}$ (resp., $f^{\dagger,r}$) factors through the co-localization $\text{D-mod}^r(Y) \to \text{D-mod}^r(Y)_X$ (resp., $\mathrm{Crys}^r(Y) \to \mathrm{Crys}^r(Y)_X$). □

4.4.8. We are finally ready to construct the isomorphism (4.12) and thereby prove Theorem 4.3.4.

Consider the object

$$\omega_{\text{D-mod},X} \boxtimes \omega_{\text{D-mod},X} = \omega_{\text{D-mod},X\times X} \in \text{D-mod}^r(X \times X).$$

Consider the isomorphism

$$F^r_X \circ \Delta^{\blacktriangle,r}_X(\omega_{\text{D-mod},X\times X}) \simeq \Delta^{\dagger,r} \circ F^r_{X\times X}(\omega_{\text{D-mod},X} \boxtimes \omega_{\text{D-mod},X}) \tag{4.15}$$

of Corollary 4.4.7.

On the one hand,

$$F^r_X \circ \Delta^{\blacktriangle,r}_X(\omega_{\text{D-mod},X\times X}) \simeq F^r_X \circ \Delta^{\blacktriangle,r}_X \circ \Upsilon_{\text{D-mod},X\times X}(\mathcal{O}_{X\times X}) \simeq$$
$$\simeq F^r_X \circ \Upsilon_{\text{D-mod},X} \circ \Delta^{\blacktriangle,l}_X(\mathcal{O}_{X\times X}) \simeq F^r_X \circ \Upsilon_{\text{D-mod},X}(\mathcal{O}_X) \simeq F^r_X(\omega_{\text{D-mod},X}).$$

On the other hand,

$$\Delta^{\dagger,r} \circ F^r_{X\times X}(\omega_{\text{D-mod},X} \boxtimes \omega_{\text{D-mod},X}) \simeq \Delta^{\dagger,r}(F^r_X(\omega_{\text{D-mod},X}) \boxtimes F^r_X(\omega_{\text{D-mod},X})) \simeq$$
$$\simeq F^r_X(\omega_{\text{D-mod},X}) \overset{!}{\otimes} F^r_X(\omega_{\text{D-mod},X}),$$

where $\overset{!}{\otimes}$ denotes the symmetric monoidal operation on $\text{Crys}^r(X)$, i.e., the $\overset{!}{\otimes}$ tensor product on $\text{IndCoh}(X_{\text{dR}})$.

Thus, from (4.15) we obtain an isomorphism

$$F^r_X(\omega_{\text{D-mod},X}) \simeq F^r_X(\omega_{\text{D-mod},X}) \overset{!}{\otimes} F^r_X(\omega_{\text{D-mod},X})$$

in $\text{Crys}^r(X)$.

Now, it is easy to see that $F^r_X(\omega_{\text{D-mod},X})$ is *invertible* as an object of the symmetric monoidal category $\text{Crys}^r(X)$.

This implies that $F^r_X(\omega_{\text{D-mod},X})$ is canonically isomorphic to the unit object, i.e., $\omega_{X_{\text{dR}}}$, as required.

4.5. **Identification of functors.** In this subsection we will show that the pullback and push-forward functors on crystals correspond to the pullback and push-forward functors defined classically for D-modules.

4.5.1. We now show:

Proposition 4.5.2. *Let $f : X \to Y$ be a morphism between smooth schemes. Then the diagram of functors*

$$\begin{array}{ccc} \text{D-mod}^r(Y) & \xrightarrow{f^{\blacktriangle,r}} & \text{D-mod}^r(X) \\ {\scriptstyle F^r_Y}\downarrow & & \downarrow{\scriptstyle F^r_X} \\ \text{Crys}^r(Y) & \xrightarrow{f^{\dagger,r}} & \text{Crys}^r(X) \end{array}$$

canonically commutes.

Remark 4.5.3. It follows from the construction given below that when f is a closed embedding, the isomorphism of functors of Proposition 4.5.2 identifies canonically with one in Corollary 4.4.7.

Proof. Follows by combining the following five commutative diagrams:

$$\begin{array}{ccc} \text{D-mod}^l(Y) & \xrightarrow{f^{\blacktriangle,l}} & \text{D-mod}^l(X) \\ {\scriptstyle F^l_Y}\downarrow & & \downarrow{\scriptstyle F^l_X} \\ \text{Crys}^l(Y) & \xrightarrow{f^{\dagger,l}} & \text{Crys}^l(X) \end{array}$$

(of Lemma 4.1.6);

$$\begin{array}{ccc} \text{D-mod}^l(Y) & \xrightarrow{f^{\blacktriangle,l}} & \text{D-mod}^l(X) \\ {\scriptstyle \Upsilon_{\text{D-mod},Y}}\downarrow & & \downarrow{\scriptstyle \Upsilon_{\text{D-mod},X}} \\ \text{D-mod}^r(Y) & \xrightarrow{f^{\blacktriangle,r}} & \text{D-mod}^r(X) \end{array}$$

(of diagram (4.14));

$$\begin{array}{ccc} \mathrm{Crys}^l(Y) & \xrightarrow{f^{\dagger,l}} & \mathrm{Crys}^l(X) \\ {\scriptstyle \Upsilon_{Y_{\mathrm{dR}}}}\big\downarrow & & \big\downarrow{\scriptstyle \Upsilon_{X_{\mathrm{dR}}}} \\ \mathrm{Crys}^r(Y) & \xrightarrow{f^{\dagger,r}} & \mathrm{Crys}^r(X), \end{array}$$

(of [Chapter II.3, Sect. 3.3]) and finally the diagrams (4.11) for X and Y, respectively.

□

4.5.4. Recall that for a map $f : X \to Y$ between smooth schemes, we have a canonically defined functor

$$f_{\text{D-mod},*} : \text{D-mod}^r(X) \to \text{D-mod}^r(Y).$$

For a smooth scheme X we let $\Gamma_{\text{D-mod}}(X, -)$ denote the functor

$$\text{D-mod}^r(X) \to \mathrm{Vect}$$

equal to $(p_X)_{\text{D-mod},*}$.

Note that Verdier duality defines an equivalence

$$\mathbf{D}_X^{\mathrm{Verdier}} : \text{D-mod}^r(X)^\vee \to \text{D-mod}^r(X),$$

characterized by the fact that its unit and counit maps are

$$\mu_{\text{D-mod},X} : \mathrm{Vect} \overset{\omega_{\text{D-mod},X}}{\longrightarrow} \text{D-mod}^r(X) \overset{(\Delta_X)_{\text{D-mod},*}}{\longrightarrow} \text{D-mod}^r(X \times X) \simeq \text{D-mod}^r(X) \otimes \text{D-mod}^r(X),$$

and

$$\epsilon_{\text{D-mod},X} : \text{D-mod}^r(X) \otimes \text{D-mod}^r(X) \simeq \text{D-mod}^r(X \times X) \overset{(\Delta_X)^{\blacktriangle,r}}{\longrightarrow} \text{D-mod}^r(X) \overset{\Gamma_{\text{D-mod}}(X,-)}{\longrightarrow} \mathrm{Vect},$$

respectively.

4.5.5. We claim:

Proposition 4.5.6. *The diagram of functors*

$$\begin{array}{ccc} \text{D-mod}^r(X)^\vee & \xrightarrow{\mathbf{D}_X^{\mathrm{Verdier}}} & \text{D-mod}^r(X) \\ {\scriptstyle (F_X^r)^\vee}\big\uparrow & & \big\downarrow{\scriptstyle F_X^r} \\ \mathrm{Crys}^r(X)^\vee & \xrightarrow{\mathbf{D}_X^{\mathrm{Verdier}}} & \mathrm{Crys}^r(X) \end{array}$$

canonically commutes.

Proof. It is enough to establish the commutation of the following diagram:

$$\begin{array}{ccc} \mathrm{Vect} & \xrightarrow{\mu_{\text{D-mod},X}} & \text{D-mod}^r(X) \otimes \text{D-mod}^r(X) \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle F_X^r \otimes F_X^r} \\ \mathrm{Vect} & \xrightarrow{\mu_{X_{\mathrm{dR}}}} & \mathrm{Crys}^r(X) \otimes \mathrm{Crys}^r(X). \end{array}$$

Recall the description of the functor $\epsilon_{X_{\mathrm{dR}}}$ is Sect. 2.2.4. Thus, taking into account the isomorphism (4.12), it suffices to show that the diagram

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{(\Delta_X)_{\text{D-mod},*}} & \text{D-mod}^r(X \times X) \\ {\scriptstyle F_X^r}\big\downarrow & & \big\downarrow{\scriptstyle F_{X\times X}^r} \\ \mathrm{Crys}^r(X) & \xrightarrow{(\Delta_X)_{\mathrm{dR},*}} & \mathrm{Crys}^r(X \times X) \end{array}$$

commutes.

However, this follows by adjunction from the commutation of the diagram

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xleftarrow{(\Delta_X)^{\blacktriangle,r}} & \text{D-mod}^r(X \times X) \\ {\scriptstyle F^r_X}\downarrow & & \downarrow{\scriptstyle F^r_{X\times X}} \\ \text{Crys}^r(X) & \xleftarrow{(\Delta_X)^{\dagger,r}} & \text{Crys}^r(X \times X), \end{array}$$

while the latter commutes by Proposition 4.5.2.

□

As a consequence of Proposition 4.5.6, we obtain:

Corollary 4.5.7. *For a smooth scheme X, the following diagram of functors canonically commutes*

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \text{Crys}^r(X) \\ {\scriptstyle \Gamma_{\text{D-mod}}(X,-)}\downarrow & & \downarrow{\scriptstyle \Gamma_{\text{dR}}(X,-)} \\ \text{Vect} & \xrightarrow{\text{Id}} & \text{Vect}\,. \end{array}$$

Proof. Obtained by passing to the dual functors in the commutative diagram

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{F^r_X} & \text{Crys}^r(X) \\ {\scriptstyle \omega_{\text{D-mod},X}}\uparrow & & \uparrow{\scriptstyle \omega_{X_{\text{dR}}}} \\ \text{Vect} & \xrightarrow{\text{Id}} & \text{Vect}\,. \end{array}$$

□

4.5.8. Finally, we claim:

Proposition 4.5.9. *For a map $f : X \to Y$ between smooth schemes, the following diagram of functors canonically commutes:*

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{f_{\text{D-mod},*}} & \text{D-mod}^r(Y) \\ {\scriptstyle F^r_X}\downarrow & & \downarrow{\scriptstyle F^r_Y} \\ \text{Crys}^r(X) & \xrightarrow{f_{\text{dR},*}} & \text{Crys}^r(Y) \end{array}$$

Proof. We factor the map f as

$$X \overset{f_1}{\to} X \times Y \overset{f_2}{\to} Y,$$

where f_1 is the graph of f, and f_2 is the projection to the second factor.

Hence, it is enough to establish the commutativity of the diagrams

$$\begin{array}{ccc} \text{D-mod}^r(X) & \xrightarrow{(f_1)_{\text{D-mod},*}} & \text{D-mod}^r(X \times Y) \\ {\scriptstyle F^r_X}\downarrow & & \downarrow{\scriptstyle F^r_{X\times Y}} \\ \text{Crys}^r(X) & \xrightarrow{(f_1)_{\text{dR},*}} & \text{Crys}^r(X \times Y) \end{array} \tag{4.16}$$

and

$$
\begin{array}{ccc}
\text{D-mod}^r(X\times Y) & \xrightarrow{(f_2)_{\text{D-mod},*}} & \text{D-mod}^r(Y) \\
{\scriptstyle F^r_{X\times Y}}\big\downarrow & & \big\downarrow{\scriptstyle F^r_Y} \\
\text{Crys}^r(X\times Y) & \xrightarrow{(f_2)_{\text{dR},*}} & \text{Crys}^r(Y),
\end{array}
\tag{4.17}
$$

respectively.

Now, the commutation of (4.16) follows by adjunction from the commutation of

$$
\begin{array}{ccc}
\text{D-mod}^r(X) & \xleftarrow{(f_1)^{\blacktriangle,r}} & \text{D-mod}^r(X\times Y) \\
{\scriptstyle F^r_X}\big\downarrow & & \big\downarrow{\scriptstyle F^r_{X\times Y}} \\
\text{Crys}^r(X) & \xleftarrow{(f_1)^{\dagger,r}} & \text{Crys}^r(X\times Y),
\end{array}
$$

given by Proposition 4.5.2.

To establish the commutatition of (4.17) we rewrite it as

$$
\begin{array}{ccc}
\text{D-mod}^r(X)\otimes\text{D-mod}^r(Y) & \xrightarrow{\Gamma_{\text{D-mod}}(X,-)\otimes\text{Id}} & \text{D-mod}^r(Y) \\
{\scriptstyle F^r_X\otimes F^r_Y}\big\downarrow & & \big\downarrow{\scriptstyle F^r_Y} \\
\text{Crys}^r(X)\otimes\text{D-mod}^r(Y) & \xrightarrow{\Gamma_{\text{dR}}(X,-)\otimes\text{Id}} & \text{Crys}^r(Y),
\end{array}
$$

and the result follows from Corollary 4.5.7.

□