PART III.4. AN APPLICATION: CRYSTALS

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INTRODUCTION
1. Crystals on prestacks and inf-schemes

In this section we will reap the fruits of the works done in [Book-III.3]. Namely, we will show how the theory of IndCoh gives rise to the theory of crystals.

1.1. The de Rham functor and crystals: recollections. The category \( \text{Crys}(X) \) of crystals on a prestack \( X \) is defined to be \( \text{IndCoh} \) on the prestack \( X_{dR} \). In this subsection we recall the functor \( X \mapsto X_{dR} \) and study its basic properties.

1.1.1. For \( Z \in \text{PreStk} \), we denote by \( Z_{dR} \) the corresponding de Rham prestack. I.e., for \( S \in \text{Sch}^{\text{aff}} \),

\[
\text{Maps}(S, Z) := \text{Maps}(\text{red } S, Z).
\]

For a morphism \( f : Z_1 \to Z_2 \), we let \( f_{dR} \) denote the corresponding morphism \( Z_{dR}^1 \to Z_{dR}^2 \).

1.1.2. Note that the functor \( dR \) commutes both with limits and colimits. Note also that \( Z_{dR} \cong (\text{red } Z)_{dR} \).

So, if a morphism \( f : Z_1 \to Z_2 \) is a nil-isomorphism (i.e., \( \text{red } Z_1 \to \text{red } Z_1 \) is an isomorphism), then \( (Z_1)_{dR} \to (Z_2)_{dR} \) is an isomorphism.

1.1.3. We claim:

**Proposition 1.1.4.** The functor \( dR \) takes \( \text{PreStk}_{laff} \) to \( \text{PreStk}_{laff} \).

**Proof.** Let \( Z \) be an object of \( \text{PreStk}_{laff} \). We need to show that \( Z_{dR} \) satisfies:

- It is convergent;
- For every \( n \), the truncation \( \leq^n Z \) belongs to \( \leq^n \text{PreStk}_{laff} \).

The convergence of \( Z_{dR} \) is obvious. To show that \( \leq^n Z \in \leq^n \text{PreStk}_{laff} \), it suffices to show that \( Z_{dR} \) takes filtered limits in \( \text{Sch}^{\text{aff}} \) to colimits in \( \text{Spc} \). However, this follows from the fact that the functor

\[
S \mapsto \text{red } S, \quad \text{Sch}^{\text{aff}} \to \text{red } \text{Sch}^{\text{aff}}
\]

preserves filtered limits, and the fact that \( \text{red } Z \in \text{red } \text{PreStk}_{laff} \). \( \square \)

1.2. Crystals. In this subsection we introduce the category of crystals.

1.2.1. Composing the functor \( dR : \text{PreStk}_{laff} \to \text{PreStk}_{laff} \) with

\[
\text{IndCoh}_{\text{PreStk}_{laff}} : (\text{PreStk}_{laff})^{\text{op}} \to \text{DGCat}_{\text{cont}},
\]

we obtain a functor that we denote

\[
\text{Crys}_{\text{PreStk}_{laff}} : (\text{PreStk}_{laff})^{\text{op}} \to \text{DGCat}_{\text{cont}}.
\]

This is the functor which is denoted \( \text{Crys}_{\text{PreStk}_{laff}} \) in [GL:Crystals, Sect 2.3.2].

1.2.2. For \( Z \in \text{PreStk}_{laff} \) we shall denote the value of \( \text{Crys}_{\text{PreStk}_{laff}} \) on \( Z \) by \( \text{Crys}(Z) \). For a morphism \( f : Z_1 \to Z_2 \) in \( \text{PreStk}_{laff} \), we shall denote by \( f_{dR} \) the resulting functor

\[
\text{Crys}(Z_2) \to \text{Crys}(Z_1).
\]

Note that if a morphism \( f : Z_1 \to Z_2 \) is a nil-isomorphism, then

\[
f_{dR} : \text{Crys}(Z_2) \to \text{Crys}(Z_1)
\]

is an equivalence.
1.2.3. For $Z \in \text{PreStk}$, we let $p_{\text{dR},Z}$ denote the tautological projection:

$$Z \rightarrow Z_{\text{dR}}.$$ 

The map $p_{\text{dR},Z}$ gives rise to a natural transformation of functors

$$\text{oblv}_{\text{dR}} : \text{Crys}^!_{\text{PreStk}^{\text{laft}}} \rightarrow \text{IndCoh}^!_{\text{PreStk}^{\text{laft}}}.$$ 

For a map $f : Z_1 \rightarrow Z_2$, we have a commutative square of functors:

$$\begin{array}{ccc}
\text{Crys}(Z_1) & \xrightarrow{\text{oblv}_{\text{dR},Z_1}} & \text{IndCoh}(Z_1) \\
\downarrow f_{\text{dR},!} & & \downarrow f^! \\
\text{Crys}(Z_2) & \xrightarrow{\text{oblv}_{\text{dR},Z_2}} & \text{IndCoh}(Z_2).
\end{array}$$

1.2.4. Finally, let us make the following observation:

**Proposition 1.2.5.** For $Z \in \text{PreStk}^{\text{laft}}$, the functor

$$\text{Crys}(Z) \rightarrow \lim_{Z \in (C/Z)^{op}} \text{Crys}(Z)$$

is an equivalence, where $C$ is any of the following categories:

$$\text{red} \text{Sch}^{\text{aff}}_{\text{ft}}, \text{cl} \text{Sch}^{\text{aff}}_{\text{ft}}, <\infty \text{Sch}^{\text{aff}}_{\text{ft}}, \text{Sch}^{\text{aff}}_{\text{ft}}, \text{red} \text{Sch}^{\text{aff}}_{\text{ft}}, \text{cl} \text{Sch}^{\text{aff}}_{\text{ft}}, <\infty \text{Sch}^{\text{aff}}_{\text{ft}}, \text{Sch}^{\text{aff}}_{\text{ft}}.$$ 

**Proof.** It is enough to show that the functor

$$\text{dR} : \text{PreStk}^{\text{laft}} \rightarrow \text{PreStk}^{\text{laft}}$$

is isomorphic to the left Kan extension of its restriction to $C \subset \text{PreStk}^{\text{laft}}$ for $C$ as above. It is sufficient to consider the case of $C = \text{redSch}^{\text{aff}}_{\text{ft}}$.

First, we note that the functor $\text{dR}$ commutes with colimits. This implies that $\text{dR}$ is isomorphic to the left Kan extension of its restriction to $<\infty \text{Sch}^{\text{aff}}_{\text{ft}}$. Hence, it suffices to show that the functor

$$\text{dR} : \text{Sch}^{\text{aff}}_{\text{ft}} \rightarrow \text{PreStk}^{\text{laft}}$$

is isomorphic to the left Kan extension of its restriction to $\text{redSch}^{\text{aff}}_{\text{ft}}$.

I.e., we have to show that for $Z \in \text{Sch}^{\text{aff}}_{\text{ft}}$, $S \in \text{Sch}^{\text{aff}}_{\text{ft}}$ and a map

$$\text{red}S \rightarrow Z,$$

the category of its factorizations as

$$\text{red}S \rightarrow Z' \rightarrow Z$$

with $Z' \in \text{redSch}^{\text{aff}}_{\text{ft}}$, is contractible.

However, the latter is obvious as the above category has a final object, namely, $Z' := \text{red}Z$.

\[\square\]

1.3. **Crystals and (ind)-nil-schemes.** In this subsection we introduce the class of prestacks that we call \textit{(ind)-nil-schemes}, and study the category of crystals on such prestacks. (Ind)-nil-schemes play the same role vis-à-vis Crys as (ind)-inf-scheme do for IndCoh.
1.3.1. Consider the full subcategories
\[ \text{indnilSch}_{\text{laft}} := \text{PreStk}_{\text{laft}}^\text{red} \times \text{red}\text{indSch} \subset \text{PreStk}_{\text{laft}} \]
and
\[ \text{nilSch}_{\text{laft}} := \text{PreStk}_{\text{laft}}^\text{red} \times \text{red}\text{Sch} \subset \text{PreStk}_{\text{laft}} , \]
where \( \text{PreStk}_{\text{laft}} \rightarrow \text{red}\text{PreStk}_{\text{laft}} \) is the functor \( Z \mapsto \text{red}Z \).

I.e., \( Z \) belongs to \( \text{indnilSch}_{\text{laft}} \) (resp., \( \text{nilSch}_{\text{laft}} \)) if and only if \( \text{red}Z \) is a reduced ind-scheme (resp., scheme).

For example, we have
\[ \text{infSch}_{\text{laft}} \subset \text{nilSch}_{\text{laft}} \text{ and } \text{indinfSch}_{\text{laft}} \subset \text{indnilSch}_{\text{laft}} . \]

We shall refer to objects of \( \text{indnilSch}_{\text{laft}} \) (resp., \( \text{nilSch}_{\text{laft}} \)) as \( \text{ind-nil-schemes} \) (resp., \( \text{nil-schemes} \)).

1.3.2. We claim:

**Lemma 1.3.3.** The functor \( \text{dR} \) takes objects of \( \text{indnilSch}_{\text{laft}} \) (resp., \( \text{nilSch}_{\text{laft}} \)) to \( \text{indinfSch}_{\text{laft}} \) (resp., \( \text{infSch}_{\text{laft}} \)).

**Proof.** We have
\[ \text{red}(Z_{\text{dR}}) = \text{red}Z. \]

Now, we claim that for any \( Z \in \text{PreStk} \), the corresponding \( Z_{\text{dR}} \) admits deformation theory. In fact, it admits an \( \infty \)-connective deformation theory: all of its cotangent spaces are zero.

1.3.4. Recall the notions of (ind)-nil-schematic and (ind)-nil-proper map of prestacks, see [Book-III.2, Definition 1.6.11(c)].

**Corollary 1.3.5.** The functor \( \text{dR} \) takes (ind)-nil-schematic maps in \( \text{PreStk}_{\text{laft}} \) to (ind)-inf-schematic maps.

**Proof.** For a map of prestacks \( f : Z_1 \rightarrow Z_2 \) and \( S \in (\text{Sch}_{\text{aff}}^\text{laft})/(Z_2)_{\text{dR}} \), the Cartesian product
\[ S \times (Z_1)_{\text{dR}} \]
identifies with
\[ S \times (\text{red}S \times Z_1)_{\text{dR}} \]
Now, we use Lemma 1.3.3 and the fact that the subactegory \( \text{indinfSch}_{\text{laft}} \) is preserved by finite limits.

1.3.6. We claim:

**Lemma 1.3.7.** Let \( f : Z_1 \rightarrow Z_2 \) be a ind-nil-proper map in \( \text{PreStk}_{\text{laft}} \). Then:
(a) The functor \( f_{\text{dR},*} : \text{Crys}(Z_1) \rightarrow \text{Crys}(Z_2) \), left adjoint to \( f_{\text{dR}!} \), is well-defined, and satisfies base change.
(b) If \( f \) is ind-nil-closed, then \( f_{\text{dR},*} \) is fully faithful.
**Proof.** Point (a) follows from Corollary 1.3.5 and [Book-III.3, Proposition 3.2.4].

To prove point (b), we need to show that the unit of the adjunction

\[ \text{Id}_{\text{Crys}(\mathbb{Z}_1)} \to f_{\text{dR},*} \]

is an isomorphism.

Consider the Cartesian square:

\[
\begin{array}{ccc}
\mathbb{Z}_1 \times \mathbb{Z}_1 & \xrightarrow{p_1} & \mathbb{Z}_1 \\
\downarrow p_2 & & \downarrow \\
\mathbb{Z}_1 & \to & \mathbb{Z}_2.
\end{array}
\]

The above unit of the adjunction equals the composed map

\[ \text{Id}_{\text{Crys}(\mathbb{Z}_1)} \simeq (p_2)_{\text{dR},*} \circ (\Delta_{\mathbb{Z}_1})_{\text{dR},*} \circ (\Delta_{\mathbb{Z}_1})_{\text{dR},!} \circ (p_1)_{\text{dR},!} \to (p_2)_{\text{dR},*} \circ (p_1)_{\text{dR},!} \to f_{\text{dR},!} \circ f_{\text{dR},*}, \]

where \( \Delta_{\mathbb{Z}_1} \) is the diagonal map

\[ \mathbb{Z}_1 \to \mathbb{Z}_1 \times \mathbb{Z}_2, \]

and second arrow is the co-unit of the \(((\Delta_{\mathbb{Z}_1})_{\text{dR},*}, (\Delta_{\mathbb{Z}_1})_{\text{dR},!})\)-adjunction.

Now, by base change,

\[ (p_2)_{\text{dR},*} \circ (p_1)_{\text{dR},!} \to f_{\text{dR},!} \circ f_{\text{dR},*}, \]

is an isomorphism. Hence, it is enough to show that

\[ (p_2)_{\text{dR},*} \circ (\Delta_{\mathbb{Z}_1})_{\text{dR},*} \circ (\Delta_{\mathbb{Z}_1})_{\text{dR},!} \circ (p_1)_{\text{dR},!} \to (p_2)_{\text{dR},*} \circ (p_1)_{\text{dR},!} \]

is an isomorphism as well. However, the map

\[ (\Delta_{\mathbb{Z}_1})_{\text{dR},*} \circ (\Delta_{\mathbb{Z}_1})_{\text{dR},!} \to \text{Id}_{\text{Crys}(\mathbb{Z}_1 \times \mathbb{Z}_1)} \]

is an isomorphism, since \((\Delta_{\mathbb{Z}_1})_{\text{dR},!}\) is equivalence (the latter is because the map \( \Delta_{\mathbb{Z}_1} \) is a nil-isomorphism).

\[ \square \]

1.4. **The functor of de Rham direct image.** In this subsection we develop the functor of de Rham direct image (a.k.a. pushforward) for crystals.

1.4.1. Recall the functor

\[ \text{IndCoh}_{\text{indnilSch}} : \text{indnilSch} \to \text{DGCat}_{\text{cont}}, \]

that sends a morphism \( f \) to the functor \( f_{\text{IndCoh}}^* \), see [Book-III.3, Sect. 4.3].

Precomposing it with the functor

\[ \text{dR} : \text{indnilSch} \to \text{indnilSch} \]

we obtain a functor

\[ \text{Crys}_{\text{indnilSch}} : \text{indnilSch} \to \text{DGCat}_{\text{cont}}. \]

1.4.2. For a morphism \( f : \mathbb{Z}_1 \to \mathbb{Z}_2 \) in nilSch we shall denote the resulting functor

\[ \text{Crys}(\mathbb{Z}_1) \to \text{Crys}(\mathbb{Z}_2) \]

by \( f_{\text{dR},*} \).

In other words,

\[ f_{\text{dR},*} = (f_{\text{dR}})^* \text{IndCoh}. \]
1.4.3. From [Book-III.3, Corollary 5.2.4(a)] we obtain:

**Corollary 1.4.4.** The restriction of the functor  \( \text{Crys}_{\text{ind-nilSch}_{\text{laft}}} \) to the 1-full subcategory \( \text{ind-nil-proper} \subset \text{indnilSch}_{\text{laft}} \) is obtained by passing to left adjoints from the restriction functor \( \text{Crys}_{\text{ind-nilSch}_{\text{laft}}} \) to 
\[
\left((\text{ind-nil-proper})^{\text{op}} \subset (\text{indnilSch}_{\text{laft}})^{\text{op}}\right).
\]

**Remark 1.4.5.** Note that we have used the notation \( f_{\text{dR}}^* \) when \( f \) is ind-nil-proper earlier, to denote the left adjoint of \( f^{\text{dR},!} \). So, the above corollary implies that the notations are consistent.

1.5. **Crystals on ind-nil-schemes as extended from schemes.** The material of this subsection will not be used in the sequel and is included for the sake of completeness. We show that theory of \( \text{Crys} \) on ind-nil-schemes can be obtained by extending the same theory on schemes.

1.5.1. Consider now the category \( \text{redSch}_{\text{ft}} \), and consider the functors

\[
\text{Crys}^!_{\text{redSch}_{\text{ft}}} : (\text{redSch}_{\text{ft}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}}
\]

and

\[
\text{Crys}_{\text{redSch}_{\text{ft}}} : \text{redSch}_{\text{ft}} \rightarrow \text{DGCat}_{\text{cont}}.
\]

From Proposition 1.2.5 we obtain:

**Corollary 1.5.2.** The natural map

\[
\text{Crys}^!_{\text{indnilSch}_{\text{laft}}} \rightarrow \text{RKE}_{(\text{redSch}_{\text{ft}})^{\text{op}} \hookrightarrow (\text{ind-nil Sch}_{\text{laft}})^{\text{op}} (\text{Crys}^!_{\text{redSch}_{\text{ft}}})}
\]

is an isomorphism.

We are going to prove the following:

**Proposition 1.5.3.** The natural map

\[
\text{LKE}_{\text{redSch}_{\text{ft}} \hookrightarrow \text{indnilSch}_{\text{laft}}} (\text{Crys}_{\text{redSch}_{\text{ft}}}) \rightarrow \text{Crys}_{\text{indnilSch}_{\text{laft}}}
\]

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

1.5.4. Consider now the 1-full subcategory of \( \text{nilSch}_{\text{laft}} \) equal to

\[
(\text{ind-nil Sch}_{\text{laft}})^{\text{nil-closed}} = \text{PreStk}_{\text{laft}} \times_{\text{redPreStk}_{\text{ft}}} (\text{red-nil Sch}_{\text{laft}})^{\text{closed}}.
\]

I.e., we restrict 1-morphisms to nil-closed maps.

It is enough to show that the map in Proposition 1.5.3 becomes an isomorphism when restricted to the above subcategory. This follows by [Book-III.3, Corollary 4.3.4] from Proposition 1.4.4 and the following statement:

**Proposition 1.5.5.**

(a) The map

\[
\left(\text{RKE}_{(\text{redSch}_{\text{ft}})^{\text{op}} \hookrightarrow (\text{ind-nil Sch}_{\text{laft}})^{\text{op}}} (\text{Crys}^!_{\text{redSch}_{\text{ft}}})\right) \mid ((\text{ind-nil Sch}_{\text{laft}})^{\text{nil-closed}})^{\text{op}} \rightarrow \\
\text{RKE}_{(\text{redSch}_{\text{ft}})^{\text{closed}} \hookrightarrow (\text{ind-nil Sch}_{\text{laft}})^{\text{nil-closed}}^{\text{op}}} (\text{Crys}^!_{\text{redSch}_{\text{ft}}}) \mid ((\text{redSch}_{\text{ft}})^{\text{closed}})^{\text{op}}
\]

is an isomorphism.
(b) The map

\[
\text{LKE}_{\text{redSch}^{\text{closed}}} \to \text{indnilSch}^{\text{nil-closed}}(\text{Crys}_{\text{redSch}^{\text{closed}}}) \to \text{LKE}_{\text{redSch}^{\text{closed}}} \to \text{indnilSch}^{\text{nil-closed}}(\text{Crys}_{\text{redSch}^{\text{closed}}}) \mid \text{indnilSch}^{\text{nil-closed}}
\]

is an isomorphism.

**Proof.** Follows from the fact that for \( Z \in \text{indnilSch}^{\text{nil-closed}} \),
the category

\[
\{ f : Z \to Z, \ Z \in \text{redSch}^{\text{nil-closed}} \}
\]

is cofinal in

\[
\{ f : Z \to Z, \ Z \in \text{redSch}^{\text{nil-closed}} \}.
\]

see [Book-III.2, Corollary 1.7.5(b)] \( \square \)

1.6. **Properties of the category of crystals.** In this subsection we study properties of the category \( \text{Crys}(Z) \) on a given object \( Z \in \text{indnilSch}^{\text{nil-closed}} \).

1.6.1. We claim:

**Proposition 1.6.2.** The functor

\[
\text{Crys}(Z) \to \lim_{f:Z\to Z} \text{Crys}(Z)
\]

is an equivalence, where the limit is taken over the index category

\[
\{ f : Z \to Z, \ Z \in \text{redSch}^{\text{nil-closed}} \}
\]

For every \( f : Z \to Z \) as above, the corresponding functor

\[
f_{\text{dR},*} : \text{Crys}(Z) \to \text{Crys}(Z)
\]

is fully faithful.

**Proof.** The first assertion follows from Proposition 1.2.5 for \( C = \text{redSch}^{\text{aff}} \) and [Book-III.2, Corollary 1.7.5(b)].

The second assertion follows from Lemma 1.3.7(b). \( \square \)

1.6.3. **Compact generation.** From [Book-III.3, Corollary 3.2.2], we obtain:

**Corollary 1.6.4.** The category \( \text{Crys}(Z) \) is compactly generated.

From Proposition 1.6.2, we have the following more explicit description of the subcategory

\[
\text{Crys}(Z)^{\text{c}} \subset \text{Crys}(Z).
\]

**Corollary 1.6.5.** Compact objects of \( \text{Crys}(Z) \) are those that can be obtained as

\[
f_{\text{dR},*}(M), \ \ M \in \text{Crys}(Z)^{\text{c}}, \ Z \in \text{redSch}^{\text{nil-closed}} \text{ and } f \text{ is a nil-closed map } Z \to Z.
\]
1.6.6. \textit{t-structure}. According to [Book-III.3, Sect. 3.3], the category Crys(\mathbb{Z}) carries a canonical t-structure. It is characterized by the following property:

\[ \mathcal{M} \in \text{Crys}(\mathbb{Z})^{\geq 0} \iff \text{oblv}_{\text{dR}, \mathbb{Z}}(\mathcal{M}) \in \text{IndCoh}(\mathbb{Z})^{\geq 0}. \]

In addition, from [Book-III.3, Corollary 3.3.4] we obtain:

**Corollary 1.6.7.**

(a) An object \( \mathcal{M} \in \text{Crys}(\mathbb{Z}) \) is contained in \( \text{Crys}(\mathbb{Z})^{\geq 0} \) if and only if for every nil-closed map \( f : Z \to \mathbb{Z} \) with \( Z \in \text{red} \text{Sch}_R \) we have

\[ f^{\text{dR},!}(\mathcal{M}) \in \text{Crys}(\mathbb{Z})^{\geq 0}. \]

(b) The category \( \text{Crys}(\mathbb{Z})^{\leq 0} \) is generated under colimits by the essential images of \( \text{Crys}(\mathbb{Z})^{\leq 0} \) for \( f : Z \to \mathbb{Z} \) with \( Z \in \text{red} \text{Sch}_R \) and \( f \) nil-closed.

2. **Crystals as a functor out of the category of correspondences**

In this section we extend the formalism of crystal to a functor out of the category of correspondences.

2.1. **Correspondences and the de Rham functor.** In this subsection we show that the de Rham functor turns (ind)-nil-schematic morphisms into (ind)-inf-schematic ones.

2.1.1. Recall that the functor dR commutes with Cartesian products. Combining this observation with Lemma 1.3.5, we obtain that dR gives rise to a functor:

\[
\text{dR}^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}} : (\text{PreStk}_{\text{left}})^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}} \to (\text{PreStk}_{\text{left}})^{\text{indinfsch \& ind-nil-proper}}_{\text{corr;indinfsch;all}}.
\]

Hence, from [Book-III.3, Theorem 5.3.3 and Proposition 5.4.3], we obtain:

**Theorem 2.1.2.** There exists a canonically defined functor

\[
\text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}}} : (\text{PreStk}_{\text{left}})^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}} \to \text{DGCat}^{2-\text{Cat}}_{\text{cont}},
\]

equipped with an isomorphism

\[
\text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}}} |_{(\text{PreStk}_{\text{left}})^{\text{op}}} \simeq \text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{op}}}.
\]

The restriction

\[
\text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{nil-open}}_{\text{corr;indnilsch;all}}} := \text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{ind-nil-proper}}_{\text{corr;indnilsch;all}}} |_{(\text{PreStk}_{\text{left}})^{\text{op}}} \text{DGCat}^{2-\text{Cat}}_{\text{cont}}^{\text{op}}
\]

uniquely extends to a functor

\[
\text{Crys}^{(\text{PreStk}_{\text{left}})^{\text{nil-open}}_{\text{corr;indnilsch;all}}} : (\text{PreStk}_{\text{left}})^{\text{nil-open}}_{\text{corr;indnilsch;all}} \to \left(\text{DGCat}^{2-\text{Cat}}_{\text{cont}}\right)^{\text{op}}.
\]
2.1.3. As in the case of [Book-III.3, Theorem 5.3.3], the content of Theorem 2.1.2 is the existence of the functor

\[ f_{dR,*} : \text{Crys}(\mathcal{Z}_1) \to \text{Crys}(\mathcal{Z}_2) \]

for ind-nil-schematic morphisms of prestacks \( f : \mathcal{Z}_1 \to \mathcal{Z}_2 \), and of the base change isomorphisms. Namely, for a Cartesian diagram of inf-schemes

\[
\begin{array}{ccc}
\mathcal{Z}_1' & \xrightarrow{g_1} & \mathcal{Z}_1 \\
\downarrow f' & & \downarrow f \\
\mathcal{Z}_2' & \xrightarrow{g_2} & \mathcal{Z}_2
\end{array}
\]

we have a canonical isomorphism

\[ (2.1) \quad f'_{dR,*} \circ g_1^{dR,!} \to g_2^{dR,!} \circ f_{dR,*}. \]

Moreover, if \( f \) is dR-proper, then \( f_{dR,*} \) is the left adjoint of \( f^{dR,!} \). Furthermore, the isomorphism (2.1) is the one arising by adjunction if either \( f_X \) or \( g_2 \) is dR-proper.

2.1.4. Let is now restrict the functor \( \text{Crys}(\text{PreStk}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{indnilsch};\text{all}} \) to

\( (\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{indnilsch};\text{all}} \subset (\text{PreStk}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{indnilsch};\text{all}}. \)

We denote the resulting functor by \( \text{Crys}_{(\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{all};\text{all}}} \). From [Book-III.3, Theorems 5.1.4 and 5.3.3] we obtain:

**Corollary 2.1.5.** The restriction of \( \text{Crys}_{(\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{all};\text{all}}} \) to

\( \text{indnilSch}_{\text{left}} \subset (\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{all};\text{all}} \)

identifies canonically with the functor \( \text{Crys}_{(\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{all};\text{all}}} \) of (1.1).

2.1.6. Restricting even further under

\( (\text{nilSch}_{\text{left}})_{\text{nil-proper}}^{\text{nil-proper}} \to (\text{indnilSch}_{\text{left}})_{\text{ind-nil-proper}}^{\text{corr}:\text{all};\text{all}}, \)

we obtain a functor

\[ \text{Crys}_{(\text{nilSch}_{\text{left}})_{\text{nil-proper}}^{\text{nil-proper}}} \to (\text{DGCat}_{\text{cont}})^{2\text{-Cat}} \]

that we denote by \( \text{Crys}_{(\text{nilSch}_{\text{left}})_{\text{nil-proper}}^{\text{nil-proper}}} \).

In particular, we obtain a functor

\[ \text{Crys}_{\text{nilSch}_{\text{left}}} := \text{Crys}_{(\text{Sch}_{\text{left}})_{\text{nil-proper}}^{\text{nil-proper}}} |_{\text{nilSch}_{\text{left}}}, \]

which is also isomorphic to

\[ \text{Crys}_{(\text{indnilSch}_{\text{left}})_{\text{nilSch}_{\text{left}}}^{\text{nil-nil}}}. \]

2.2. The multiplicative structure of the functor of crystals. In this subsection we show how the formalism of crystals as a functor out of the category of correspondences gives rise to Verdier duality.
2.2.1. **Duality.** From [Book-III.3, Theorem 6.1.6], we obtain:

**Theorem 2.2.2.** We have a commutative diagram of functors

\[
\begin{array}{ccc}
((\text{nilSch}_{\text{laft}})_{\text{corr;all;all}})^{\text{op}} & \xrightarrow{\text{(Crys(\text{nilSch}_{\text{laft}})_{\text{corr;all;all}})^{\text{op}}}} & \left(D\text{GCat}_{\text{cont;all;all}}^{\text{dualizable}}\right)^{\text{op}} \\
(\text{nilSch}_{\text{laft}})_{\text{corr;all;all}} & \xrightarrow{\text{Crys(\text{nilSch}_{\text{laft}})_{\text{corr;all;all}}}} & D\text{GCat}_{\text{cont;all;all}}^{\text{dualizable}}.
\end{array}
\]

2.2.3. Concretely, this theorem says that for \(Z \in \text{nilSch}_{\text{laft}}\) there is a canonical involutive equivalence

\[
(D_{\text{Verdier}})_{\text{Z}} : \text{Crys}(Z)^{\text{v}} \simeq \text{Crys}(Z),
\]

and for a map \(f : Z_1 \to Z_2\) in \(\text{nilSch}_{\text{laft}}\) there is a canonical identification

\[fdR^! \simeq (fdR,^*)^{\text{v}}.\]

2.2.4. As in [IndCohonInf, Sect. 6.2.6], we can write the unit and counit maps

\[
\mu_Z^{\text{adR}} : \text{Vect} \to \text{Crys}(Z) \otimes \text{Crys}(Z) \quad \text{and} \quad \epsilon_Z^{\text{adR}} : \text{Crys}(Z) \otimes \text{Crys}(Z) \to \text{Vect}
\]

explicitly.

Namely, \(\epsilon_Z^{\text{adR}}\) is the composition

\[
\text{Crys}(Z) \otimes \text{Crys}(Z) \simeq \text{Crys}(Z \times Z) \xrightarrow{\Delta_{Z,^!}} \text{Crys}(Z) \xrightarrow{\Gamma_{dR}(Z,-)} \text{Vect},
\]

where

\[
\Gamma_{dR}(Z,-) := (p_Z)_{dR,*},
\]

and \(\mu_Z^{\text{adR}}\) is the composition

\[
\text{Vect} \xrightarrow{\omega_Z^{\text{adR}}} \text{Crys}(Z) \xrightarrow{(\Delta_Z)^{dR,*}} \text{Crys}(Z \times Z) \simeq \text{Crys}(Z) \otimes \text{Crys}(Z).
\]

2.2.5. **Verdier duality.** For \(Z \in \text{nilSch}_{\text{laft}}\) we let \(D_{\text{Verdier}}^{\text{Z}}\) denote the canonical equivalence

\[
(\text{Crys}(Z)^{c})^{\text{op}} \to \text{Crys}(Z)^{c},
\]

corresponding to isomorphism (2.2).

In other words,

\[
D_{Z}^{\text{Verdier}} = D_{Z}^{\text{Serre}}.
\]

2.2.6. As a particular case of [Book-III.3, Corollary 6.2.9], we obtain:

**Corollary 2.2.7.** Let \(f : Z_1 \to Z_2\) be a proper map of objects of \(\text{nilSch}_{\text{laft}}\). Then we have a commutative diagram:

\[
\begin{array}{ccc}
(\text{Crys}(Z_1)^{c})^{\text{op}} & \xrightarrow{\text{D}_{Z_1}^{\text{Verdier}}} & \text{Crys}(Z_1)^{c} \\
(fdR,^*)^{\text{op}} & \xrightarrow{(fdR,^*)^{\text{op}}} & fdR,* \\
(\text{Crys}(Z_2)^{c})^{\text{op}} & \xrightarrow{\text{D}_{Z_2}^{\text{Verdier}}} & \text{Crys}(Z_2)^{c}.
\end{array}
\]

In view of Corollary 1.6.5, the last corollary gives an expression of the Verdier duality functor on \(Z \in \text{nilSch}_{\text{laft}}\) in terms of that on schemes.
2.2.8. **Convolution for crystals.** Returning to the entire \((\text{PreStk}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr;ind-nil-sch;all}}\) and the corresponding functor

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr;ind-nil-sch;all}}},
\]

from [Book-III.3, Sect. 6.3], we obtain that the functor

\[
\text{Crys}_{(\text{PreStk}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr;ind-nil-sch;all}}} : (\text{PreStk}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr;ind-nil-sch;all}} \rightarrow \text{DGCat}_{\text{cont}}^2
\]

carries a canonical right-lax symmetric monoidal structure.

As in [Book-III.3, Sect. 6.3.2], we have:

(i) Given a category-object \(X^*\) of \(\text{PreStk}_{\text{laft}}\), with the target and composition maps ind-nil-schematic, the category \(\text{Crys}(\mathcal{R})\) acquires a monoidal structure given by convolution, and as such it acts on \(\text{Crys}(X)\) (here, as in [Book-II.2, Sect. 5.1.1], \(X = X^0\) and \(\mathcal{R} = X^1\)).

(ii) If the composition map is ind-nil-proper, then \(\omega_{\mathcal{R}} \in \text{Crys}(\mathcal{R})\) acquires a structure of algebra in \(\text{Crys}(\mathcal{R})\). The action of this algebra on \(\text{IndCoh}(X)\), viewed as a plain endo-functor, is

\[
(p_t)_{d\mathcal{R}^*} \circ (p_s)^{d\mathcal{R},!}.
\]

3. **Inducing crystals**

In this section we study the interaction between the functors \(\text{IndCoh}\) and \(\text{Crys}\).

3.1. **The functor of induction.** In this subsection we show that the forgetful functor

\[
\text{Crys}(X) \rightarrow \text{IndCoh}(X)
\]

admits a left adjoint, provided that \(X\) is prestack that admits deformation theory.

3.1.1. For an object \(Z \in \text{PreStk}_{\text{laft}}\) consider the canonical map

\[
p_{d\mathcal{R},Z} : Z \rightarrow Z_{d\mathcal{R}}.
\]

We are going to prove:

**Proposition 3.1.2.** Suppose that \(Z\) admits deformation theory. Then the map \(p_{d\mathcal{R},Z}\) is an inf-schematic nil-isomorphism.

**Proof.** We need to show that for \(S \in (\text{Sch}_{\text{aff}})^{d\mathcal{R}}_{/Z_{d\mathcal{R}}}\), the Cartesian product

\[
S \times_{Z_{d\mathcal{R}}} Z
\]

is an inf-scheme.

Clearly, the above Cartesian product belongs to \(\text{PreStk}_{\text{laft}}\), and its underlying reduced prestack identifies with \(\text{red}S\). Hence, it remains to show that (3.1) admits deformation theory. This holds because the category \(\text{PreStk}_{\text{def-laft}}\) is closed under finite limits. \(\square\)

3.1.3. From Proposition 3.1.2 and [Book-III.3, Proposition 3.1.2(a)] we obtain:

**Corollary 3.1.4.** Let \(Z\) be an object of \(\text{PreStk}_{\text{def-laft}}\). Then the functor

\[
\text{oblv}_{d\mathcal{R},Z} : \text{Crys}(Z) \rightarrow \text{IndCoh}(Z)
\]

admits a left adjoint, denoted \(\ind_{d\mathcal{R},Z}\).
3.1.5. Thus, for \( X \in \text{PreStk}_{\text{def-laft}} \), we obtain an adjoint pair

\[
\text{ind}_{\text{dR},X} : \text{IndCoh}(X) \rightleftarrows \text{Crys}(X) : \text{oblv}_{\text{dR},X}.
\]

We claim:

**Lemma 3.1.6.** The pair (3.2) is monadic.

**Proof.** Since \( \text{oblv}_{\text{dR},X} \) is continuous, we only need to check that it is conservative. However, this follows from [Book-III.3, Proposition 3.1.2(b)]. \( \square \)

3.1.7. The next assertion expresses the functoriality of the operation of induction:

**Corollary 3.1.8.** There is a canonically defined natural transformation

\[
\text{ind}_{\text{dR}} : \text{IndCoh}(\text{PreStk}_{\text{def-laft}}) \longrightarrow \text{IndCoh}(\text{PreStk}_{\text{inf-sch}}) \Rightarrow \text{Crys}(\text{PreStk}_{\text{def-laft}}) \Rightarrow \text{Crys}(\text{PreStk}_{\text{inf-sch}}),
\]

as functors

\[
(\text{PreStk}_{\text{def-laft}})_{\text{inf-sch}} \rightarrow \text{DGCat}_{\text{cont}}.
\]

In particular, the above corollary says that for an ind-inf-schematic morphism \( f : Z_1 \rightarrow Z_2 \) of objects of \( \text{PreStk}_{\text{def-laft}} \), the following diagram of functors commutes:

\[
\begin{array}{ccc}
\text{IndCoh}(Z_1) & \xrightarrow{\text{ind}_{\text{dR}},Z_1} & \text{Crys}(Z_1) \\
\downarrow f^{\text{IndCoh}} & & \downarrow f_{\text{dR}} \\
\text{IndCoh}(Z_2) & \xrightarrow{\text{ind}_{\text{dR}},Z_2} & \text{Crys}(Z_2).
\end{array}
\]

3.2. **Induction on ind-inf-schemes.** In this subsection we let \( Z \) be an object of \( \text{indinfSch}_{\text{laft}} \). We study the interaction with the functor of induction with that of Serre and Verdier dualities.

3.2.1. We have:

**Lemma 3.2.2.** The functor \( \text{ind}_{\text{dR},Z} \) sends \( \text{IndCoh}(Z)^c \) to \( \text{Crys}(Z)^c \).

**Proof.** Follows from the fact that the functor \( \text{oblv}_{\text{dR},Z} \) is continuous and conservative. \( \square \)

3.2.3. **Induction and duality.** Let us apply isomorphism [Book-III.3, Equation 6.2] to the map

\[
p_{\text{dR},Z} : Z \rightarrow Z_{\text{dR}}.
\]

We obtain:

**Corollary 3.2.4.** Under the isomorphisms

\[
D^\text{Serre}_Z : \text{IndCoh}(Z)^\circ \simeq \text{IndCoh}(Z) \quad \text{and} \quad D^\text{Verdier}_Z : \text{Crys}(Z)^\circ \simeq \text{Crys}(Z),
\]

we have a canonical identification

\[
(\text{oblv}_{\text{dR},Z})^\circ \simeq \text{ind}_{\text{dR},Z}.
\]

In addition, by [Book-III.3, Corollary 6.2.7]

**Corollary 3.2.5.** The following diagram of functors commutes:

\[
\begin{array}{ccc}
(\text{IndCoh}(Z)^c)^{\text{op}} & \xrightarrow{D^\text{Serre}_Z} & \text{IndCoh}(Z)^c \\
\downarrow (\text{ind}_{\text{dR},Z})^{\text{op}} & & \downarrow \text{ind}_{\text{dR},Z} \\
(\text{Crys}(Z)^c)^{\text{op}} & \xrightarrow{D^\text{Verdier}_Z} & \text{Crys}(Z)^c.
\end{array}
\]
3.2.6. **Induction and t-structure.** Recall that by the definition of the t-structure on $\text{Crys}(\mathbb{Z})$, the functor $\text{obl}_{dR, \mathbb{Z}}$ is left t-exact. We claim:

**Corollary 3.2.7.** Assume that $\mathbb{Z}$ is an ind-scheme. Then the functor $\text{ind}_{dR, \mathbb{Z}}$ is t-exact.

**Proof.** The fact that $\text{ind}_{dR, \mathbb{Z}}$ is right t-exact follows by adjunction. To show that it is left t-exact we use [Book-III.3, Lemma 3.3.6]. Thus, we have to show that $p_{dR, \mathbb{Z}}$ is ind-schematic.

Indeed, for $S \in \text{Sch}^{\text{aff}}$ and $S \to \mathbb{Z}_{dR}$, the Cartesian product $S \times \mathbb{Z}_{dR}$ identifies with the formal completion of $S \times \mathbb{Z}$ along the graph of the map $\text{red}S \to \mathbb{Z}$. □

3.3. **Relative crystals.** In this subsection we comment how the discussion of crystals generalizes to the relative situation.

3.3.1. Let now $\mathfrak{y}$ be a fixed object of $\text{PreStk}^{\text{aff}}$. We consider the category $$(\text{PreStk}^{\text{aff}})_{/\mathfrak{y}}$$ and the corresponding category $$((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}}.$$ Restricting the functor $\text{IndCoh}_{(\text{PreStk}^{\text{aff}})_{/\text{ind-nil-proper}}}^{\text{ind-nil-sch}}$ along the forgetful functor $$((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}} \to ((\text{PreStk}^{\text{aff}})_{/\text{corr}})^{\text{inf-proper}}_{/\text{inf-sch};\text{all}},$$ we obtain a functor, denoted

$$\text{IndCoh}_{(\text{PreStk}^{\text{aff}})_{/\mathfrak{y}}}^{\text{ind-nil-proper}} : ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}} \to \text{DGCat}_{\text{cont}}^{2 \cdot \text{Cat}},$$

with properties specified by [Book-III.3, Theorem 5.3.3]. In particular, we shall denote by

$$\text{IndCoh}^{\text{op}}_{(\text{PreStk}^{\text{aff}})_{/\mathfrak{y}}} : ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{op}} \to \text{DGCat}_{\text{cont}},$$

the resulting functor.

3.3.2. The category $((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})$ carries an endo-functor, denoted $^{/\mathfrak{y}}_{dR}$:

$$\mathbb{Z} \mapsto \mathbb{Z}_{dR} \times ^{\mathfrak{y}}_{dR}.$$ 

Corollary 1.3.5 implies that the functor $^{/\mathfrak{y}}_{dR}$ carries the class of morphisms $\text{ind-nilSch} \to \text{ind-sch}$. Hence, $^{/\mathfrak{y}}_{dR}$ induces a functor

$$(3.3) \quad ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}} \to ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}}.$$ 

Thus, precomposing $\text{IndCoh}_{(\text{PreStk}^{\text{aff}})_{/\text{ind-nil-proper}}}^{\text{ind-nil-sch}}$ with $^{/\mathfrak{y}}_{dR}$, we obtain a functor that we denote by

$$^{/\mathfrak{y}}_{\text{Crys}} ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}} : ((\text{PreStk}^{\text{aff}})_{/\mathfrak{y}})^{\text{ind-nil-proper}}_{/\text{ind-nil-sch};\text{all}} \to \text{DGCat}_{\text{cont}}^{2 \cdot \text{Cat}}.$$
3.3.3. We let 
\[ \mathcal{C}_{\text{Crys}} \left( \text{PreSt}_\text{laft} / \mathcal{Y} \right) \rightarrow \text{DGCat}_{\text{cont}} \]
and
\[ \mathcal{C}_{\text{Crys}} \left( \text{PreSt}_\text{laft} / \mathcal{Y} \right)_{\text{ind-nil}} \rightarrow \text{DGCat}_{\text{cont}} \]
denote the corresponding functors obtained by restriction.

For a map \( \mathcal{Z}_1 \rightarrow \mathcal{Z}_2 \) in \( \text{PreSt}_\text{laft} / \mathcal{Y} \), we shall denote by \( f_!^{\mathcal{Y} \text{dR}}, f_*^{\mathcal{Y} \text{dR}} \) the corresponding functors
\[ \mathcal{C}_{\text{Crys}}(\mathcal{Z}_1) \leftrightarrow \mathcal{C}_{\text{Crys}}(\mathcal{Z}_2) \].

These functors satisfy an adjointness property for \( f \) being ind-nil-proper/nil-open.

3.3.4. Let \( \mathcal{C}_{\text{ind-nilSch}} \) denote the full subcategory of \( \text{PreSt}_\text{laft} / \mathcal{Y} \) equal to the preimage of \( \text{indinfSch} / \mathcal{Y} \) under the functor \( \mathcal{Y} \text{dR} \).

Restricting the functor \( \mathcal{C}_{\text{Crys}}(\text{PreSt}_\text{laft} / \mathcal{Y}) \) to \( \mathcal{C}_{\text{ind-nilSch}} \), we obtain a functor that we denote by
\[ \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \rightarrow \text{DGCat}^{2 \text{-Cat}}_{\text{cont}}. \]
Furthermore, we have the following properties of the category \( \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \) for \( \mathcal{Z} \in \mathcal{C}_{\text{ind-nilSch}} / \mathcal{Y} \):

1. The category \( \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \) is compactly generated, and is self-dual in the same sense as in Theorem 2.2.2.

2. The category \( \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \) carries a t-structure in which an object \( \mathcal{F} \) is coconnective if and only if its image under the forgetful functor \( \text{obl}_{\mathcal{Y} \text{dR}, \mathcal{Z}} : \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \rightarrow \text{IndCoh}(\mathcal{Z}) \) is coconnective, for \( \text{obl}_{\mathcal{Y} \text{dR}, \mathcal{Z}} := (p_{\mathcal{Y} \text{dR}, \mathcal{Z}})_! \), where \( p_{\mathcal{Y} \text{dR}, \mathcal{Z}} \) denotes the canonical morphism \( \mathcal{Z} \rightarrow \mathcal{Y} \text{dR}(\mathcal{Z}) \).

3. If \( \mathcal{Z} \) admits deformation theory over \( \mathcal{Y} \) (see [Book-III.1, Sect. 7.1.6] for what this means), then the morphism \( p_{\mathcal{Y} \text{dR}, \mathcal{Z}} \) is an inf-schematic nil-isomorphism, and hence the functor \( \text{obl}_{\mathcal{Y} \text{dR}, \mathcal{Z}} \) admits a left adjoint, denoted \( \text{ind}_{\mathcal{Y} \text{dR}, \mathcal{Z}} \).

Remark 3.3.5. The essential difference between Crys and \( \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \) is that for \( \mathcal{Z} \in \text{PreSt}_\text{laft} / \mathcal{Y} \), the category \( \mathcal{C}_{\text{Crys}}(\mathcal{Z}) \) depends not just on the underlying reduced prestack. E.g., for \( \mathcal{Z} = \mathcal{Y} \),
\[ \mathcal{C}_{\text{Crys}}(\mathcal{Y}) = \text{IndCoh}(\mathcal{Y}). \]

3.3.6. Assume for a moment that \( \mathcal{Y} = Y \) is a smooth classical scheme, and \( \mathcal{Z} = Z \) is also a classical scheme smooth over \( Y \). Then, as in [GL:Crystals, Sect. 4.7], one shows that the category
\[ \mathcal{C}_{\text{Crys}}(Z) \]
identifies with the DG category associated with the abelian category of quasi-coherent sheaves of modules on \( Y \) with respect to the algebra of “vertical” differential operators, i.e., the subalgebra of \( \text{Diff}(Z) \) that consists of differential operators that commute with \( \mathcal{O}_Y \).

4. Comparison with the classical theory of D-modules

In this section we will identify the theory of crystals as developed in the previous sections with the theory of D-modules.
4.1. **Left D-modules and left crystals.** In this subsection we will recollect (and rephrase) the contents of [GL:Crystals, Sect. 5].

4.1.1. Let $X$ be a classical scheme of finite type. Consider the category $\text{QCoh}(X \times X)^\vee$ and its full subcategory $(\text{QCoh}(X \times X)_{\Delta_X})^\vee$, consisting of objects that are set-theoretically supported on the diagonal. Let 

$$(\text{QCoh}(X \times X)_{\Delta_X})^\vee_{\text{rel,flat}} \subset \text{QCoh}(X \times X)_{\Delta_X}^\vee$$

be the full subcategory, consisting of objects that are $X$-flat with respect to both projections $p_s, p_t : X \times X \to X$.

The category $(\text{QCoh}(X \times X)_{\Delta_X})^\vee_{\text{rel,flat}}$ has a naturally defined monoidal structure, given by convolution.

Moreover, we have a canonically defined fully faithful monoidal (!) functor

$$(4.1) \quad (\text{QCoh}(X \times X)_{\Delta_X})^\vee_{\text{rel,flat}} \to \text{QCoh}(X \times X),$$

where $\text{QCoh}(X \times X)$ is a monoidal category via [Book-II.2, Sect. 5.1.4].

4.1.2. Assume now that $X$ is smooth. In this case, we have a canonically defined object $\text{Diff}_X \in \text{AssocAlg}( (\text{QCoh}(X \times X)_{\Delta_X})^\vee_{\text{rel,flat}})$, namely, the Grothendieck algebra of differential operators.

Composing (4.1) with the monoidal equivalence

$$\text{QCoh}(X \times X) \to \text{Funct}_{\text{cont}}(\text{QCoh}(X), \text{QCoh}(X)),$$

we obtain that $\text{Diff}_X$ gives rise to a monad on $\text{QCoh}(X)$.

We consider the category $\text{Diff}_X \text{-mod}(\text{QCoh}(X))$. It is equipped with a t-structure, characterized by the property that the tautological forgetful functor $\text{Diff}_X \text{-mod}(\text{QCoh}(X)) \to \text{QCoh}(X)$ is t-exact.

The category $\text{D-mod}^l(X)$ of left D-modules on $X$ is defined as the canonical DG model of the derived category of the abelian category $(\text{Diff}_X \text{-mod}(\text{QCoh}(X)))^\vee$.

As in [GL:Crystals, Proposition 4.7.3] one shows that the canonical functor

$$(4.2) \quad \text{D-mod}^l(X) \to \text{Diff}_X \text{-mod}(\text{QCoh}(X))$$

is an equivalence.

4.1.3. Let $X$ be any scheme almost of finite type. Recall the category

$\text{Crys}^l(X)$,

see [GL:Crystals, Sect. 2.1]. It is equipped with a forgetful functor

$$\text{oblv}_{\text{dr},X}^l : \text{Crys}^l(X) \to \text{QCoh}(X).$$

Assume now that $X$ is eventually coconnective. According to [GL:Crystals, Proposition 3.4.11], in this case the functor $\text{oblv}_{\text{dr},X}^l$ admits a left adjoint, denoted $\text{ind}_{\text{dr},X}^l$, and the resulting adjoint pair of functors

$$\text{ind}_{\text{dr},X}^l : \text{QCoh}(X) \rightleftarrows \text{Crys}^l(X) : \text{oblv}_{\text{dr},X}^l,$$

is monadic.
The corresponding monad is given by an object
\[ \mathcal{D}_X^l \in \text{AssocAlg}(\text{QCoh}(X \times X)_{\Delta_X}). \]
I.e., we have an equivalence
\[ \text{Crys}^l(X) \simeq \mathcal{D}_X^l\text{-mod}(\text{QCoh}(X)). \]

4.1.4. Assume again that \( X \) is smooth. In this case one easily shows (see, e.g., [GL:Crystals, Proposition 5.3.6]) that
\[ \mathcal{D}_X^l \in (\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel,flat}}. \]
Moreover, it is a classical fact (reproved for completeness in [GL:Crystals, Lemma 5.4.3]) that there is a canonical isomorphism in \( \text{AssocAlg} \)
\[ (\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel,flat}}: \]
\[ \mathcal{D}_X^l \simeq \text{Diff}_X. \]
In particular, we obtain a canonical equivalence of categories
\[ \text{D-mod}^l(X) \simeq \text{Diff}_X\text{-mod}(\text{QCoh}(X)) \simeq \text{Crys}^l(X), \]
compatible with the forgetful functors to \( \text{QCoh}(X) \).

We denote the resulting equivalence \( \text{D-mod}^l(X) \to \text{Crys}^l(X) \) by \( F_X^l \).

4.1.5. Let \( f : X \to Y \) be a morphism between smooth classical schemes. In this the classical theory of D-modules defines a functor
\[ f^{\star,l} : \text{D-mod}^l(Y) \to \text{D-mod}^l(X) \]
that makes the diagram
\[
\begin{array}{ccc}
\text{D-mod}^l(Y) & \xrightarrow{f^{\star,l}} & \text{D-mod}^l(X) \\
\downarrow & & \downarrow \\
\text{QCoh}(Y) & \xrightarrow{f^{\star}} & \text{QCoh}(X)
\end{array}
\]
commute.

Recall also (see [GL:Crystals, Sect. 2.1.2]) that for a map \( f : X \to Y \) between arbitrary schemes almost of finite type, we have a functor
\[ f^{\dag,l} : \text{Crys}^l(Y) \to \text{Crys}^l(X), \]
the makes the diagram
\[
\begin{array}{ccc}
\text{Crys}^l(Y) & \xrightarrow{f^{\dag,l}} & \text{Crys}^l(X) \\
\downarrow & & \downarrow \\
\text{QCoh}(Y) & \xrightarrow{f^{\star}} & \text{QCoh}(X)
\end{array}
\]
commute.

The following can be established by a direct calculation:
Lemma 4.1.6. The following diagram of functors naturally commutes

\[
\begin{array}{ccc}
\text{D-mod}^l(Y) & \xrightarrow{f^*} & \text{D-mod}^l(X) \\
F_Y & & F_X \\
\downarrow & & \downarrow \\
\text{Crys}^l(Y) & \xrightarrow{f^*} & \text{Crys}^l(X),
\end{array}
\]

in a way compatible with the forgetful functors to \(\text{QCoh}(\cdot)\).

4.2. Right D-modules and right crystals.

4.2.1. Note that for any scheme \(X\), the monoidal category \(\text{QCoh}(X \times X)\) carries a canonical anti-involution, denoted \(\sigma\), corresponding to the transposition acting on \(X \times X\).

In terms of the identification

\[
\text{QCoh}(X \times X) \simeq \text{Funct}_{\text{cont}}(\text{QCoh}(X), \text{QCoh}(X)),
\]

we have

\[
\sigma(F) \simeq F^\vee,
\]

where we use the canonical identification

\[
\text{D}^{\text{naive}}_X : \text{QCoh}(X)^\vee \simeq \text{QCoh}(X),
\]

of [Book-II.3, Equation (4.2)].

4.2.2. Let \(X\) be a smooth classical scheme. Consider the object

\[
\sigma(\text{Diff}^{\text{op}}_X) \in \text{AssocAlg} \left( \left( \text{QCoh}(X) \right)^{\Delta_X} \right)_{\text{rel, flat}},
\]

and the corresponding category

\[
\sigma(\text{Diff}^{\text{op}}_X)-\text{mod}(\text{QCoh}(X)).
\]

The category \(\text{D-mod}^r(X)\) of right D-modules on \(X\) is defined as the canonical DG model of the derived category of the abelian category \(\sigma(\text{Diff}^{\text{op}}_X)-\text{mod}(\text{QCoh}(X)))^{\text{op}}\).

As in [GL:Cryystals, Proposition 4.7.3] one shows that the canonical functor

\[
(4.5) \quad \text{D-mod}^r(X) \to \sigma(\text{Diff}^{\text{op}}_X)-\text{mod}(\text{QCoh}(X))
\]

is an equivalence.

4.2.3. For an arbitrary scheme \(X\) locally almost of finite type, for the duration of this section we will denote

\[
\text{Crys}^r(X) := \text{Crys}(X),
\]

where the latter is defined as in Sect. 1.2.2, and by

\[
\text{ind}_{\text{dr}, X}^r : \text{IndCoh}(X) \rightleftharpoons \text{Crys}^r(X) : \text{oblv}_{\text{dr}, X}^r
\]

the corresponding pair of adjoint functors from (3.2). I.e., we are adding the superscript “r” (for “right”) to the notation from Sect. 1.2.2.

Let \(\mathcal{D}^r_X\) be the object of \(\text{AssocAlg}(\text{IndCoh}(X \times X)_{\Delta_X})\), corresponding to the monad

\[
\text{oblv}_{\text{dr}, X}^r \circ \text{ind}_{\text{dr}, X}^r
\]

via the equivalence of monoidal categories

\[
\text{IndCoh}(X \times X) \to \text{Funct}_{\text{cont}}(\text{IndCoh}(X), \text{IndCoh}(X)).
\]
By Lemma 3.1.6, we have:
\[
\mathcal{D}_X^{\text{r}} \text{-mod} (\text{IndCoh}(X)) \simeq \text{Crys}^r(X).
\]

4.2.4. Assume again that \(X\) is smooth. Recall that in this case the adjoint pairs
\[
\Xi_X : \text{QCoh}(X) \rightleftarrows \text{IndCoh}(X) : \Psi_X
\]
and
\[
\Xi_X^\vee : \text{QCoh}(X) \rightleftarrows \text{IndCoh}(X) : \Psi_X^\vee = \Upsilon_X
\]
are both equivalences.

In this case it is easy to show as in [GL:Crystals, Sect. 5.5] that there is a canonical
equivalence of categories
\[
F_X^r : \text{D-mod}^r(X) \simeq \text{Crys}^r(X),
\]
that makes the diagram
\[
\begin{array}{ccc}
\text{D-mod}^r(X) & \xrightarrow{F_X^r} & \text{Crys}^r(X) \\
\downarrow & & \downarrow \\
\text{QCoh}(X) & \xrightarrow{\Xi_X} & \text{IndCoh}(X)
\end{array}
\]
commute.

Remark 4.2.5. One can obtain the equivalence of (4.6) without resorting to computations.
Namely, taking into account the equivalences
\[
(\text{obl}v_{\text{fr}}^{l} \circ \text{ind}_{\text{fr}}^{l})^\vee \text{-mod}(\text{QCoh}(X)) \simeq \sigma(\text{Diff}_{X}^{\text{op}})^\vee \text{-mod}(\text{QCoh}(X)) \simeq \text{D-mod}^r(X)
\]
and
\[
(\text{obl}v_{\text{fr}}^{r} \circ \text{ind}_{\text{fr}}^{r}) \text{-mod}(\text{IndCoh}(X)) \simeq \text{Crys}^r(X),
\]
it suffices to construct an isomorphism of the monads
\[
(\Psi_X \circ (\text{obl}v_{\text{fr}}^{r} \circ \text{ind}_{\text{fr}}^{r}) \circ \Xi_X) \text{ and } (\text{obl}v_{\text{fr}}^{l} \circ \text{ind}_{\text{fr}}^{l})^\vee,
\]
acting on \(\text{QCoh}(X)\).

The latter isomorphism holds for any eventually coconnective \(X\), and follows from the iso-

\[
\text{obl}v_{\text{fr}}^{l} \circ \text{ind}_{\text{fr}}^{l} \simeq \Xi_X^\vee \circ (\text{obl}v_{\text{fr}}^{r} \circ \text{ind}_{\text{fr}}^{r}) \circ \Psi_X^\vee,
\]
see [GL:Crystals, Lemma 3.4.9].

Remark 4.2.6. The monad(s) in (4.7) correspond to the pair of adjoint functors
\[
(\text{ind}_{\text{fr}}^{r} : \text{QCoh}(X) \rightleftarrows \text{Crys}(X) : \text{obl}v_{\text{fr}}^{r})
\]
of [GL:Crystals, Sect. 4.6]. Note also the isomorphisms
\[
(\text{ind}_{\text{fr}}^{r})^\vee \simeq \text{obl}v_{\text{fr}}^{l} \text{ and } (\text{obl}v_{\text{fr}}^{r})^\vee \simeq \text{ind}_{\text{fr}}^{l},
\]
where we use the identifications
\[
\mathbf{D}_X^{\text{naive}} : \text{QCoh}(X)^\vee \simeq \text{QCoh}(X) \text{ and } \mathbf{D}_X^{\text{Serre}} : \text{IndCoh}(X)^\vee \simeq \text{IndCoh}(X).
\]

4.3. Passage between left and right D-modules/crystals.
4.3.1. According to [GL:Crystals, Proposition 2.2.4], for any scheme $X$ almost of finite type we have a canonically defined equivalence

$$\Upsilon_{X_{\text{dr}}} : \text{Crys}^l(X) \to \text{Crys}^r(X),$$

that makes the diagram

$$\begin{array}{ccc}
\text{Crys}^l(X) & \xrightarrow{\Upsilon_{X_{\text{dr}}}} & \text{Crys}^r(X) \\
\text{oblv}^l_{\text{dr}, X} \downarrow & & \downarrow \text{oblv}^r_{\text{dr}, X} \\
\text{Qcoh}(X) & \xrightarrow{\Upsilon_X} & \text{IndCoh}(X)
\end{array}$$

commute.

Concretely, the functor $\Upsilon_{X_{\text{dr}}}$ is given by

$$M \mapsto M \otimes \omega_{X_{\text{dr}}},$$

where $\otimes$ is the action of $\text{Qcoh}(X_{\text{dr}}) (= \text{Crys}^l(X))$ on $\text{IndCoh}(X_{\text{dr}}) (= \text{Crys}^r(X))$.

4.3.2. Recall also that there is a canonical equivalence

$$(4.8) \quad \text{D-mod}^l(X) \to \text{D-mod}^r(X),$$

given by tensoring a given left D-module with the right D-module

$$\omega_{\text{D-mod}, X} := \det(T^*(X)[\dim(X)].$$

We denote the above functor by $\Upsilon_{\text{D-mod}, X}$.

4.3.3. We will prove:

**Theorem 4.3.4.** The following diagram of functors canonically commutes:

$$\begin{array}{ccc}
\text{D-mod}^l(X) & \xrightarrow{F^l_X} & \text{Crys}^l(X) \\
\Upsilon_{\text{D-mod}, X} \downarrow & & \downarrow \Upsilon_{X_{\text{dr}}} \\
\text{D-mod}^r(X) & \xrightarrow{F^r_X} & \text{Crys}^r(X)
\end{array}$$

By applying Theorem 4.3.4 to $O_X \in \text{D-mod}^l(X)$, we obtain:

**Corollary 4.3.5.** There exists a canonical isomorphism in $\text{Crys}^r(X)$:

$$(4.10) \quad F^r_X(\omega_{\text{D-mod}, X}) \simeq \omega_{X_{\text{dr}}}.$$  

4.3.6. Applying the forgetful functor

$$\text{oblv}^r_{\text{dr}, X} : \text{Crys}^r(X) \to \text{IndCoh}(X),$$

to the isomorphism of (4.10). We obtain:

**Corollary 4.3.7.** There exists a canonical isomorphism in $\text{IndCoh}(X)$:

$$(4.11) \quad \Upsilon_X(\det(T^*(X)[\dim(X)]) \simeq \omega_X.$$  

**Remark 4.3.8.** We note that the isomorphism (4.11) can be proved without involving D-modules or crystals, by an argument along the same lines as that proving the isomorphism (4.10) in Sect. 4.4 below.

This argument is given in a more general context in [Book-IV.4, Proposition 14.3.4].

4.4. **Proof of Theorem 4.3.4.**
4.4.1. The first observation is given by the following lemma that follows from the constructions:

**Lemma 4.4.2.** For $M \in \text{D-mod}^l(X)$ and $N \in \text{D-mod}^r(X)$ we have a canonical isomorphism

$$F^r_X(M \otimes N) \simeq F^l_X(M) \otimes F^r_X(N).$$

This lemma reduces the assertion of Theorem 4.3.4 to establishing the isomorphism (4.10).

4.4.3. Recall that for a map $f : X \to Y$ between smooth schemes, one defines the functor

$$f^!: \text{D-mod}^r(Y) \to \text{D-mod}^r(X)$$

by requiring that the diagram

$$
\begin{array}{ccc}
\text{D-mod}^l(Y) & \xrightarrow{f^!} & \text{D-mod}^l(X) \\
\downarrow \tau_{\text{D-mod}, Y} & & \downarrow \tau_{\text{D-mod}, X} \\
\text{D-mod}^r(Y) & \xrightarrow{f^!} & \text{D-mod}^r(X)
\end{array}
$$

(4.12)

commute.

Assume for a moment that $f$ is a closed embedding of smooth schemes. Let

$$\text{D-mod}^r(Y)_X \subset \text{D-mod}^r(Y)$$

be the full subcategory consisting of objects with set-theoretic support on $X$.

Recall that in this case we have the Kashiwara lemma that says that the functor $f^!$ induces an equivalence $\text{D-mod}^r(Y)_X \to \text{D-mod}^r(X)$.

4.4.4. For a morphism $f : X \to Y$ between schemes almost of finite type, let

$$f^\ddagger : \text{Crys}^r(Y) \to \text{Crys}^l(X)$$

be the corresponding pullback functor, see [GL:Crystals, Sect. 2.3.4].

Assume for a moment that $f$ is a closed embedding. Let $\text{Crys}^r(Y)_X \subset \text{Crys}^r(Y)$ be the full subcategory consisting of objects with set-theoretic support on $X$.

Recall (see [GL:Crystals, Proposition 2.5.6]) that in this case the functor $f^\ddagger$ induces an equivalence $\text{Crys}^r(Y)_X \to \text{Crys}^r(X)$.

4.4.5. Let $f : X \to Y$ be again a closed embedding of smooth classical schemes. The next assertion also follows from the constructions:

**Lemma 4.4.6.** Under the equivalences

$$f^!: \text{D-mod}^r(Y)_X \to \text{D-mod}^r(X) \quad \text{and} \quad f^\ddagger : \text{Crys}^r(Y)_X \to \text{Crys}^r(X),$$

the diagram

$$
\begin{array}{ccc}
\text{D-mod}^r(Y)_X & \xrightarrow{F^r_Y} & \text{Crys}^r(Y)_X \\
\downarrow f^! & & \downarrow f^\ddagger \\
\text{D-mod}^r(X) & \xrightarrow{F^r_X} & \text{Crys}^r(X)
\end{array}
$$

commutes.

As a corollary, we obtain:
Corollary 4.4.7. For a closed embedding of smooth schemes \( f : X \to Y \), the diagram
\[
\begin{array}{ccc}
\text{D-mod}^r(Y) & \xrightarrow{F_Y} & \text{Crys}^r(Y) \\
\downarrow f^\ast \cdot r & & \downarrow f^{\dagger, r} \\
\text{D-mod}^r(X) & \xrightarrow{F_X} & \text{Crys}^r(X)
\end{array}
\]
commutes.

Proof. Follows from the fact that the functor \( f^\ast \cdot r \) (resp., \( f^{\dagger, r} \)) factors through the co-localization \( \text{D-mod}^r(Y) \to \text{D-mod}^r(Y)_X \) (resp., \( \text{Crys}^r(Y) \to \text{Crys}^r(Y)_X \)). \( \square \)

4.4.8. We are finally ready to construct the isomorphism (4.10) and thereby prove Theorem 4.3.4.

Consider the object
\[
\omega_{\text{D-mod}, X} \boxtimes \omega_{\text{D-mod}, X} = \omega_{\text{D-mod}, X \times X} \in \text{D-mod}^r(X \times X).
\]

Consider the isomorphism
(4.13) \[
F_X \circ \Delta_X^{\cdot r} (\omega_{\text{D-mod}, X \times X}) \simeq \Delta_X^{\dagger, r} \circ F_X \times X (\omega_{\text{D-mod}, X} \boxtimes \omega_{\text{D-mod}, X})
\]
of Corollary 4.4.7.

On the one hand,
\[
F_X \circ \Delta_X^{\cdot r} (\omega_{\text{D-mod}, X \times X}) \simeq F_X \circ \Delta_X^{\cdot r} \circ \Upsilon_{\text{D-mod}, X \times X} (\mathcal{O}_{X \times X}) \simeq
\]
\[
F_X \circ \Upsilon_{\text{D-mod}, X} \circ \Delta_X^{\dagger, l} (\mathcal{O}_{X \times X}) \simeq F_X \circ \Upsilon_{\text{D-mod}, X} (\mathcal{O}_{X}) \simeq F_X^r (\omega_{\text{D-mod}, X}).
\]

On the other hand,
\[
\Delta_X^{\dagger, r} \circ F_X \times X (\omega_{\text{D-mod}, X} \boxtimes \omega_{\text{D-mod}, X}) \simeq \Delta_X^{\dagger, r} (F_X^r (\omega_{\text{D-mod}, X}) \boxtimes F_X^r (\omega_{\text{D-mod}, X})) \simeq
\]
\[
\simeq F_X^r (\omega_{\text{D-mod}, X}) \otimes F_X^r (\omega_{\text{D-mod}, X}).
\]

Thus, from (4.13) we obtain an isomorphism
\[
F_X^r (\omega_{\text{D-mod}, X}) \simeq F_X^r (\omega_{\text{D-mod}, X}) \otimes F_X^r (\omega_{\text{D-mod}, X})
\]
in \( \text{Crys}^r(X) \).

Now, it is easy to see that \( F_X^r (\omega_{\text{D-mod}, X}) \) is invertible as an object of the symmetric monoidal category \( \text{Crys}^r(X) \).

This implies that \( F_X^r (\omega_{\text{D-mod}, X}) \) is canonically isomorphic to the unit object, i.e., \( \omega_{X, \text{an}} \), as required.

4.5. Identification of functors.

4.5.1. We will now show:

Proposition 4.5.2. Let \( f : X \to Y \) be a morphism between smooth schemes. Then the diagram of functors
\[
\begin{array}{ccc}
\text{D-mod}^r(Y) & \xrightarrow{f^\ast \cdot r} & \text{D-mod}^r(X) \\
\downarrow F_Y & & \downarrow F_X \\
\text{Crys}^r(Y) & \xrightarrow{f^{\dagger, r}} & \text{Crys}^r(X)
\end{array}
\]
canonically commutes.
Remark 4.5.3. It follows from the construction given below that when $f$ is a closed embedding, the isomorphism of functors of Proposition 4.5.2 is canonically homotopic to one in Corollary 4.4.7.

Proof. Follows by juxtaposing the following five commutative diagrams:

\[
\begin{array}{ccc}
D\text{-mod}\hat{\mathcal{C}}\hat{\mathcal{H}}(Y) & \xrightarrow{f_{X,Y}^{\mathcal{C}}\mathcal{H}} & D\text{-mod}\hat{\mathcal{C}}\hat{\mathcal{H}}(X) \\
\downarrow F^\mathcal{C} & & \downarrow F^\mathcal{C} \\
\text{Crys}\hat{\mathcal{C}}\hat{\mathcal{H}}(Y) & \xrightarrow{f_{X,Y}^{\mathcal{C}}\mathcal{H}} & \text{Crys}\hat{\mathcal{C}}\hat{\mathcal{H}}(X)
\end{array}
\]
(of Lemma 4.1.6);

\[
\begin{array}{ccc}
D\text{-mod}\hat{\mathcal{C}}\hat{\mathcal{H}}(Y) & \xrightarrow{f_{X,Y}^{\mathcal{C}}\mathcal{H}} & D\text{-mod}\hat{\mathcal{C}}\hat{\mathcal{H}}(X) \\
\downarrow \tau_{D\text{-mod},Y} & & \downarrow \tau_{D\text{-mod},X} \\
D\text{-mod}\mathcal{D}(Y) & \xrightarrow{f^{\mathcal{D}}_{X,Y}} & D\text{-mod}\mathcal{D}(X)
\end{array}
\]
(of diagram (4.12));

\[
\begin{array}{ccc}
\text{Crys}\hat{\mathcal{C}}\hat{\mathcal{H}}(Y) & \xrightarrow{f_{X,Y}^{\mathcal{C}}\mathcal{H}} & \text{Crys}\hat{\mathcal{C}}\hat{\mathcal{H}}(X) \\
\downarrow \tau_{\text{dR},Y} & & \downarrow \tau_{\text{dR},X} \\
\text{Crys}\mathcal{D}(Y) & \xrightarrow{f^{\mathcal{D}}_{X,Y}} & \text{Crys}\mathcal{D}(X)
\end{array}
\]
(of [GL:Crystals, Sect. 2.4.2]); and finally the diagrams (4.9) for $X$ and $Y$, respectively.

4.5.4. Recall that for a map $f : X \to Y$ between smooth schemes, we have a canonically defined functor

\[ f_{D\text{-mod},*} : D\text{-mod}\mathcal{D}(X) \to D\text{-mod}\mathcal{D}(Y). \]

For a smooth scheme $X$ we let $\Gamma_{D\text{-mod}}(X,-)$ denote the functor

\[ D\text{-mod}\mathcal{D}(X) \to \text{Vect} \]

equal to $(p_X)_{D\text{-mod},X}$.

Note that Verdier duality defines an equivalence

\[ D_X^{\text{Verdier}} : D\text{-mod}\mathcal{D}(X)^\vee \to D\text{-mod}\mathcal{D}(X), \]

characterized by the fact that its unit and counit maps are

\[ \mu_{D\text{-mod},X} : \text{Vect}^{\omega_{D\text{-mod},X}} \Rightarrow D\text{-mod}\mathcal{D}(X)^{(\Delta X)_{D\text{-mod},*}} \Rightarrow D\text{-mod}\mathcal{D}(X \times X) \simeq D\text{-mod}\mathcal{D}(X) \otimes D\text{-mod}\mathcal{D}(X), \]

and

\[ \epsilon_{D\text{-mod},X} : D\text{-mod}\mathcal{D}(X) \otimes D\text{-mod}\mathcal{D}(X) \simeq D\text{-mod}\mathcal{D}(X \times X) \Rightarrow D\text{-mod}\mathcal{D}(X) \Rightarrow \text{Vect}, \]

respectively.
4.5.5. We claim:

**Proposition 4.5.6.** The diagram of functors

\[
\begin{array}{c}
\text{D-mod}^r(X) \quad \xrightarrow{F_X^r} \quad \text{D-mod}^r(X) \\
(\ell_X^r) & \quad \downarrow & \quad F_X^r \\
\text{Crys}^r(X) \quad \xrightarrow{F_X^r} \quad \text{Crys}^r(X)
\end{array}
\]

canonically commutes.

**Proof.** It is enough to establish the commutation of the following diagram:

\[
\begin{array}{c}
\text{Vect} \quad \xrightarrow{\mu_{\text{D-mod},X}} \quad \text{D-mod}^r(X) \otimes \text{D-mod}^r(X) \\
\text{Id} & \quad \downarrow & \quad F_X^r \otimes F_X^r \\
\text{Vect} \quad \xrightarrow{\mu_X^\text{dR}} \quad \text{Crys}^r(X) \otimes \text{Crys}^r(X).
\end{array}
\]

Recall the description of the functor $\epsilon_X^\text{dR}$ is Sect. 2.2.4. Thus, taking into account the isomorphism (4.10), it suffices to show that the diagram

\[
\begin{array}{c}
\text{D-mod}^r(X) \quad \xrightarrow{(\Delta_X)^{\text{D-mod},*}} \quad \text{D-mod}^r(X \times X) \\
F_X^r & \quad \downarrow & \quad F_{X \times X}^r \\
\text{Crys}^r(X) \quad \xrightarrow{(\Delta_X)^{\text{dR},*}} \quad \text{Crys}^r(X \times X)
\end{array}
\]

commutes.

However, this follows by adjunction from the commutation of the diagram

\[
\begin{array}{c}
\text{D-mod}^r(X) \quad \xleftarrow{(\Delta_X)^{\ell,r}} \quad \text{D-mod}^r(X \times X) \\
F_X^r & \quad \downarrow & \quad F_{X \times X}^r \\
\text{Crys}^r(X) \quad \xleftarrow{(\Delta_X)^{\ell,r}} \quad \text{Crys}^r(X \times X),
\end{array}
\]

while the latter commutes by Proposition 4.5.2. □

As a consequence of Proposition 4.5.6, we obtain:

**Corollary 4.5.7.** For a smooth scheme $X$, the following diagram of functors canonically commutes

\[
\begin{array}{c}
\text{D-mod}^r(X) \quad \xrightarrow{F_X^r} \quad \text{Crys}^r(X) \\
\Gamma_{\text{D-mod}(X,-)} & \quad \downarrow & \quad \Gamma_{\text{dR}(X,-)} \\
\text{Vect} \quad \xrightarrow{\text{Id}} \quad \text{Vect}.
\end{array}
\]

**Proof.** Obtained by passing to the dual functors in the commutative diagram

\[
\begin{array}{c}
\text{D-mod}^r(X) \quad \xrightarrow{F_X^\omega} \quad \text{Crys}^r(X) \\
\omega_{\text{D-mod},X} & \quad \downarrow & \quad \omega_{\text{dR}} \\
\text{Vect} \quad \xrightarrow{\text{Id}} \quad \text{Vect}.
\end{array}
\]

□
4.5.8. Finally, we claim:

**Proposition 4.5.9.** For a map \( f : X \to Y \) between smooth schemes, the following diagram of functors canonically commutes:

\[
\begin{array}{ccc}
\text{D-mod}^r(X) & \xrightarrow{\text{D-mod}^r} & \text{D-mod}^r(Y) \\
F_X & \downarrow & F_Y \\
\text{Crys}^r(X) & \xrightarrow{f_{\text{dR},r}} & \text{Crys}^r(Y)
\end{array}
\]

**Proof.** We factr the map \( f \) as

\[
X \xrightarrow{f_1} X \times Y \xrightarrow{f_2} Y,
\]

where \( f_1 \) is the graph of \( f \), and \( f_2 \) is the projector on the second factor.

Hence, it is enough to establish the commutativity of the diagrams

\[
\begin{array}{ccc}
\text{D-mod}^r(X) & \xrightarrow{(f_1)_{\text{D-mod}^r}} & \text{D-mod}^r(X \times Y) \\
F_X & \downarrow & F_{X \times Y} \\
\text{Crys}^r(X) & \xrightarrow{(f_1)_{\text{dR},r}} & \text{Crys}^r(X \times Y)
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{D-mod}^r(X \times Y) & \xrightarrow{(f_2)_{\text{D-mod}^r}} & \text{D-mod}^r(Y) \\
F_{X \times Y} & \downarrow & F_Y \\
\text{Crys}^r(X \times Y) & \xrightarrow{(f_2)_{\text{dR},r}} & \text{Crys}^r(Y),
\end{array}
\]

respectively.

Now, the commutation of (4.14) follows by adjunction from the commutation of

\[
\begin{array}{ccc}
\text{D-mod}^r(X) & \xrightarrow{(f_1)^*} & \text{D-mod}^r(X \times Y) \\
F_X & \downarrow & F_{X \times Y} \\
\text{Crys}^r(X) & \xrightarrow{(f_1)^{\text{dR}}} & \text{Crys}^r(X \times Y),
\end{array}
\]

see Proposition 4.5.2.

To establish the commutatition of (4.15) we rewrite it as

\[
\begin{array}{ccc}
\text{D-mod}^r(X) \otimes \text{D-mod}^r(Y) & \xrightarrow{\Gamma_{\text{D-mod}(X,-) \otimes \text{Id}}} & \text{D-mod}^r(Y) \\
F_X \otimes F_Y & \downarrow & F_Y \\
\text{Crys}^r(X) \otimes \text{D-mod}^r(Y) & \xrightarrow{\Gamma_{\text{dR}(X,-) \otimes \text{Id}}} & \text{Crys}^r(Y),
\end{array}
\]

and the result follows from Corollary 4.5.7.

\[\square\]
AN APPLICATION: CRYSTALS

References


[Book-II.1]
[Book-II.2]
[Book-II.3]
[Book-III.1]
[Book-III.2]
[Book-III.3]
[Book-IV.4]
[Funct]
[InfSch]
[IndCohonInf]

