PART II.2. THE !-PULLBACK AND BASE CHANGE

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INTRODUCTION
1. Factorizations of morphisms of DG schemes

In this section we will study what happens to the notion of the closure of the image of a morphism between schemes in derived algebraic geometry. The upshot is that there is essentially “nothing new” as compared to the classical case.

1. Colimits of closed embeddings. In this subsection we will show that colimits exist and are well-behaved in the category of closed subschemes of a given ambient scheme.

1.1. Recall that a map $X \to Y$ in Sch is called a closed embedding if the map $\text{cl}X \to \text{cl}Y$

is a closed embedding of classical schemes.

1.1.2. Let $f : X \to Y$ be a morphism in Sch. We let $\text{Sch}_{X/\text{closed in } Y}$ denote the full subcategory of $\text{Sch}_{X/Y}$ consisting of diagrams $X \to X' \xrightarrow{f'} Y$,

where the map $f'$ is a closed embedding.

We claim:

**Proposition 1.1.3.** Suppose that $f$ is quasi-compact.

(a) The category $\text{Sch}_{X/\text{closed in } Y}$ has finite colimits (including the initial object).

(b) The formation of colimits commutes with Zariski localization on $Y$.

**Proof.**

**Step 1.** Assume first that $Y$ is affine, $Y = \text{Spec}(A)$. Let

$$i \leadsto (X \to X' \xrightarrow{f'} Y),$$

be a finite diagram in $\text{Sch}_{X/\text{closed in } Y}$.

Set $B := \Gamma(X, \mathcal{O}_X)$. This is a (not necessarily connective) commutative $k$-algebra. Set also $X'_i = \text{Spec}(B'_i)$. Consider the corresponding diagram

$$i \leadsto (A \to B'_i \to B)$$

in $\text{ComAlg}_{A/\text{B}}$.

Set

$$(\overline{B'} \to B) := \lim_i (B'_i \to B),$$

where the limit taken in $\text{ComAlg}_{/B}$. Note that we have a canonical map $A \to \overline{B}'$, and

$$(A \to \overline{B'} \to B) \in \text{ComAlg}_{A/\text{B}}$$

maps isomorphically to the limit of (1.2) taken in, $\text{ComAlg}_{A/\text{B}}$.

Set

$$B' := \tau_{\leq 0}(\overline{B}') \times_{H^0(\overline{B}')} \text{Im} \left( H^0(A) \to H^0(\overline{B}') \right),$$

where the fiber product is taken in the category of **connective** commutative algebras (i.e., it is $\tau_{\leq 0}$ of the fiber product taken in the category of all commutative algebras).
We still have the canonical maps
\[ A \to B' \to B, \]
and it is easy to see that for \( X' := \text{Spec}(B') \), the object
\[ (X \to X' \to Y) \in \text{Sch}_{X/\text{closed in } Y} \]
is the colimit of (1.1).

Step 2. To treat the general case it suffices to show that the formation of colimits in the affine case commutes with Zariski localization. I.e., we need to show that if \( Y \) is affine, \( \tilde{Y} \subset Y \) is a basic open, then for \( \tilde{X} := f^{-1}(\tilde{Y}) \), \( \tilde{X}' := (\phi'_i)^{-1}(\tilde{Y}) \), \( \tilde{X} := (f')^{-1}(\tilde{Y}) \), then the map
\[ \text{colim}_i \tilde{X}'_i \to \tilde{X}', \]
is an isomorphism, where the colimit is taken in \( \text{Sch}_{X/\text{closed in } \tilde{Y}} \).

However, the required isomorphism follows from the description of the colimit in Step 1. (The assumption that \( f \) be quasi-compact is used in the commutation of the formation of \( \Gamma(X, \mathcal{O}_X) \) with localization along \( Y \).)

1.1.4. We note the following property of colimits in the situation of Proposition 1.1.3.

Let \( g : Y \to \tilde{Y} \) be a closed embedding. Set
\[ (X \to X' \to Y) = \text{colim}_i (X \to X'_i \to Y) \]
and \( (X \to \tilde{X}' \to \tilde{Y}) = \text{colim}_i (X \to X'_i \to \tilde{Y}) \),
where the colimits are taken in \( \text{Sch}_{X/\text{closed in } Y} \) and \( \text{Sch}_{\tilde{X}/\text{closed in } \tilde{Y}} \), respectively.

Consider the composition
\[ X \to X' \to Y \to \tilde{Y}, \]
and the corresponding object
\[ (X \to X' \to \tilde{Y}) \in \text{Sch}_{X/\text{closed in } \tilde{Y}} \).

It is endowed with a compatible family of maps
\[ (X \to X'_i \to \tilde{Y}) \to (X \to X' \to \tilde{Y}). \]

Hence, by the universal property of \((X \to \tilde{X}' \to \tilde{Y})\), we obtain a canonically defined map (1.3)
\[ \tilde{X}' \to X'. \]

We claim:

**Lemma 1.1.5.** The map (1.3) is an isomorphism.

**Proof.** We construct the inverse map as follows. We note that by the universal property of \((X \to \tilde{X}' \to \tilde{Y})\), we have a canonical map
\[ (X \to \tilde{X}' \to \tilde{Y}) \to (X \to Y \to \tilde{Y}). \]

This produces a compatible family of maps
\[ (X \to X'_i \to Y) \to (X \to \tilde{X}' \to Y), \]
and hence the desired map
\[ X' \to \tilde{X}'. \]

□
1.1.6. In the situation of Proposition 1.1.3 let us consider the case of \( X = \emptyset \). We shall denote the resulting category by \( \text{Sch}_{\text{closed}} \) in \( Y \). Thus, Proposition 1.1.3 guarantees the existence and compatibility with Zariski localization of finite colimits in \( \text{Sch}_{\text{closed}} \) in \( Y \).

Explicitly, if \( Y = \text{Spec}(A) \) is affine and
\[
i \rightsquigarrow Y'_i \subset Y
\]
is a diagram of closed subschemes, \( Y'_i = \text{Spec}(A'_i) \), then
\[
\colim_i Y'_i = Y',
\]
where
\[
Y' = \text{Spec}(A'), \quad A' = \tau^{\leq 0}(\bar{\mathcal{A}}') \times \text{Im} \left( A \to \bar{\mathcal{A}}' \right), \quad \bar{\mathcal{A}}' := \lim_i A'_i.
\]

1.2. The closure. In this subsection we will define the notion of closure of the image of a morphism of schemes.

1.2.1. In what follows, in the situation of Proposition 1.1.3, we shall refer to the initial object in the category \( \text{Sch}_{X/, \text{closed}} \) in \( Y \) as the closure of \( X \) and \( Y \), and denote it by \( \overline{f(X)} \).

Explicitly, if \( Y = \text{Spec}(A) \) is affine, we have:
\[
\overline{f(X)} = \text{Spec}(A'), \quad A' = \tau^{\leq 0}(\Gamma(X, \mathcal{O}_X)) \times \text{Im} \left( H^0(A) \to H^0(\Gamma(X, \mathcal{O}_X)) \right).
\]

A particular case of Lemma 1.1.5 says:

**Corollary 1.2.2.** If \( f : X \to Y \) is a closed embedding, then \( X \to \overline{f(X)} \) is an isomorphism.

1.2.3. The following property of the operation of taking the closure will be used in the sequel. Let us be in the situation of Proposition 1.1.3,
\[
X = X_1 \cup X_2,
\]
where \( X_1 \subset X \) are open and set \( X_{12} = X_1 \cap X_2 \). Denote \( f_i := f|_{X_i} \).

We have a canonical map
\[
\overline{f_1(X_1)} \sqcup \overline{f_2(X_2)} \to \overline{f(X)},
\]
where the colimit is taken in \( \text{Sch}_{\text{closed}} \) in \( Y \).

**Lemma 1.2.4.** The map (1.5) is an isomorphism.

*Proof.* Follows by reducing to the case when \( Y \) is affine, and in the latter case by (1.4). \( \square \)

1.2.5. We give the following definition:

**Definition 1.2.6.** A map \( f : X \to Y \) is said to be a locally closed embedding, if \( Y \) contains an open \( \tilde{Y} \subset Y \), such that \( f \) defines a closed embedding \( X \to \tilde{Y} \).

We have:

**Lemma 1.2.7.** Suppose that \( f \) is a quasi-compact locally closed embedding. Then \( f \) defines an open embedding of \( X \) into \( \overline{f(X)} \).

*Proof.* Follows by combining Corollary 1.2.2 and Proposition 1.1.3(b). \( \square \)
1.3. Transitivity of closure. The basic fact established in this subsection, Proposition 1.3.2, will be of crucial technical importance for the proof of Theorem 2.1.4.

1.3.1. Consider now a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

where $f$ and $g$ are quasi-compact.

Set $Y' := \overline{f(X)}$ and $g' := g|_{Y'}$. By the universal property of closure, we have a canonical map

$$g \circ f(X) \to g'(Y').$$

We claim:

**Proposition 1.3.2.** The map (1.6) is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

1.3.3. Step 1. As in the proof of Proposition 1.1.3, the assertion reduces to the case when $Z = \text{Spec}(A)$ is affine. Assume first that $Y$ is affine as well, $Y = \text{Spec}(B)$. Then we have the following descriptions of the two sides in (1.6).

Set $C := \Gamma(X, O_X)$. We have

$$Y' = \text{Spec}(B'),$$

where

$$B' = \tau^{\leq 0}(C) \times_{H^0(C)} \text{Im}(H^0(B) \to H^0(C)),$$

where here and below the fiber product is taken in the category of connective commutative algebras.

Furthermore, $\overline{g \circ f(X)} = \text{Spec}(A')$, where

$$A' = \tau^{\leq 0}(C) \times_{H^0(C)} \text{Im}(H^0(A) \to H^0(C)).$$

Finally, $\overline{g'(Y')}$ = Spec($A''$), where

$$A'' = B' \times_{H^0(B')} \text{Im}(H^0(A) \to H^0(B')).$$

Note that

$$H^0(B') = \text{Im}(H^0(B) \to H^0(C))$$

and

$$\text{Im}(H^0(A) \to H^0(B')) = \text{Im}(H^0(A) \to H^0(C)).$$

The map (1.6) corresponds to the homomorphism

$$A'' = B' \times_{H^0(B')} \text{Im}(H^0(A) \to H^0(B')) =$$

$$= \left( \tau^{\leq 0}(C) \times_{H^0(C)} \text{Im}(H^0(B) \to H^0(C)) \right) \times_{H^0(B')} \text{Im}(H^0(A) \to H^0(C)) \simeq$$

$$\simeq \tau^{\leq 0}(C) \times_{H^0(C)} \text{Im}(H^0(A) \to H^0(C)) \simeq A',$$

which is an isomorphism, as required.

1.3.4. Step 2. Let $Y$ be arbitrary. Choose an open affine cover $Y = \bigcup Y_i$ and set $X_i = f^{-1}(Y_i)$. Then the assertion of the proposition follows from Step 1 using Lemma 1.2.4.

\[\square\]
2. IndCoh as a functor from the category of correspondences

This section realizes one of the main goals of our project, namely, the construction of IndCoh as a functor out of the category of correspondences.

It will turn out that IndCoh, equipped with the operation of direct image, and left and right adjoints, corresponding to open embeddings and proper morphisms, respectively, will uniquely extend to the sought-for formalism of correspondences.

2.1. The category of correspondences. In this subsection we introduce the category of correspondences on schemes and state our main theorem.

2.1.1. We consider the category \( \text{Sch}_{\text{aff}} \) equipped with the following classes of morphisms:

\[
\text{vert} = \text{all, } \text{horiz} = \text{all, } \text{adm} = \text{proper},
\]

and consider the corresponding category

\[
(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}},
\]

see [Funct, Sect. 8].

Our goal in this section is to extend the functor

\[
\text{IndCoh}_{\text{Sch}_{\text{aff}}} : \text{Sch}_{\text{aff}} \to \text{DGCat}_{\text{cont}}
\]

of [Book-II.1, Sect. 2.2] to a functor

\[
\text{IndCoh}_{(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}}} : (\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}} \to \text{DGCat}_{\text{cont}}^{2\text{-Cat}}.
\]

We shall do so in several stages.

2.1.2. We start with the functor

\[
\text{IndCoh}_{\text{Sch}_{\text{aff}}} : \text{Sch}_{\text{aff}} \to \text{DGCat}_{\text{cont}}
\]

and consider the class of morphisms

\[
\text{open} \subset \text{all}.
\]

By [Book-II.1, Proposition 3.2.2], the functor \( \text{IndCoh}_{\text{Sch}_{\text{aff}}} \), viewed as a functor

\[
\text{Sch}_{\text{aff}} \to \left( \text{DGCat}_{\text{cont}}^{2\text{-Cat}} \right)^{\text{2-op}},
\]

satisfies the left base-change condition.

Applying [Funct, Theorem 12.2.2], we extend \( \text{IndCoh}_{\text{Sch}_{\text{aff}}} \) to a functor

\[
\text{IndCoh}_{(\text{Sch}_{\text{aff}})^{\text{open}}_{\text{corr:all;open}}} : (\text{Sch}_{\text{aff}})^{\text{open}}_{\text{corr:all;open}} \to \left( \text{DGCat}_{\text{cont}}^{2\text{-Cat}} \right)^{\text{2-op}}.
\]

We restrict the latter functor to

\[
(\text{Sch}_{\text{aff}})^{\text{corr:all;open}} \subset (\text{Sch}_{\text{aff}})^{\text{open}}_{\text{corr:all;open}},
\]

and denote the resulting functor by \( \text{IndCoh}_{(\text{Sch}_{\text{aff}})^{\text{corr:all;open}}} \), viewed as a functor

\[
(\text{Sch}_{\text{aff}})^{\text{corr:all;open}} \to \text{DGCat}_{\text{cont}}.
\]

2.1.3. The main result of this section reads:

**Theorem 2.1.4.** There exists a unique extension of the functor \( \text{IndCoh}_{(\text{Sch}_{\text{aff}})^{\text{corr:all;open}}} \) to a functor

\[
\text{IndCoh}_{(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}}} : (\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}} \to \text{DGCat}_{\text{cont}}^{2\text{-Cat}}.
\]
2.1.5. **Proof of Theorem 2.1.4.** We start with the following three classes of morphisms

\[ \text{vert} = \text{all}, \quad \text{horiz} = \text{all}, \quad \text{et} = \text{open}, \quad \text{adm} = \text{proper}. \]

We note that the class \( \text{open} \cap \text{proper} \) is that of embeddings of a connected component. This implies that the condition of [Funct, Sect. 15.1.2] holds.

The fact that

\[ \text{IndCoh}(\text{Sch}_{\text{all}})_{\text{corr}:\text{all};\text{open}} | \text{Sch}_{\text{all}} = \text{IndCoh}_{\text{Sch}_{\text{all}}} | \text{Sch}_{\text{all}} \]

satisfies the left base change condition with respect to the class of proper maps is the content of [Book-II.1, Proposition 5.2.1].

Finally, the fact that the condition of [Funct, Sect. 15.2.2] holds is the content of [Book-II.1, Proposition 5.3.4].

Hence, in order to deduce Theorem 2.1.4 from [Funct, Theorem 15.2.5], it remains to verify that the condition of [Funct, Sect. 15.1.3]. I.e. we need to prove the following:

**Proposition 2.1.6.** For a morphism \( f : X \to Y \) in \( \text{Sch}_{\text{all}} \), the category Factor\((f)\) of factorizations of \( f \) as

\[ (2.1) \quad X \xrightarrow{\ j \ } Z \xrightarrow{\ g \ } Y, \]

where \( j \) is an open embedding, and \( g \) is proper, is contractible.

\[ \square \]

2.2. **Proof of Proposition 2.1.6.** Modulo the classical Nagata theorem, the proof will be a simple manipulation with the notion of closure, developed in the previous section.

2.2.1. **Step 1.** First we show that Factor\((f)\) is non-empty. By Nagata’s theorem, we can factor the morphism

\[ \text{red}X \to \text{red}Y \]

as

\[ \text{red}X \to Z_{\text{red}} \to \text{red}Y, \]

where \( \text{red}X \to Z_{\text{red}} \) is an open embedding and \( Z_{\text{red}} \to \text{red}Y \) is proper.

We define an object of Factor\((f)\) by setting

\[ Z := X \sqcup_{\text{red}X} Z_{\text{red}}, \]

where we use [Book-III.1, Corollary 1.4.4(a)] for the existence and the properties of push-out in this situation.

2.2.2. **Step 2.** Let Factor\((f)_{\text{dense}} \subset \text{Factor}(f)\) be the full subcategory consisting of those objects

\[ X \xrightarrow{\ j \ } Z \xrightarrow{\ g \ } Y, \]

for which the map

\[ j(X) \to Z \]

is an isomorphism.

We claim that the tautological embedding

\[ \text{Factor}(f)_{\text{dense}} \hookrightarrow \text{Factor}(f) \]

admits a right adjoint that sends a given object (2.1) to

\[ X \to j(X) \to Y. \]
Indeed, the fact that the map $X \to j(X)$ is an open embedding follows from Proposition 1.1.3(b). The fact that the above operation indeed produces a right adjoint follows from Proposition 1.3.2.

Hence, it suffices to show that the category $\text{Factor}(f)_{\text{dense}}$ is contractible. Since it is non-empty (by Step 1), it suffices to show that it contains products.

2.2.3. Step 3. Given two objects $(X \to Z_1 \to Y)$ and $(X \to Z_2 \to Y)$ of $\text{Factor}(f)_{\text{dense}}$ consider

$$W := Z_1 \times_{Y} Z_2,$$

and let $h$ denote the resulting map $X \to W$.

Set $Z := h(X)$. We claim that the map $X \to Z$ is an open embedding. Indeed, consider the open subscheme of $\overset{\circ}{W} \subset W$ equal to $X \times Y$. By Proposition 1.1.3(b),

$$\overset{\circ}{Z} := Z \cap \overset{\circ}{W}$$

is the closure of the map

$$\Delta_{X/Y} : X \to X \times_{Y} X.$$  

However, $X \to \overline{\Delta_{X/Y}(X)}$ is an isomorphism by Corollary 1.2.2.

2.2.4. Step 4. Finally, we claim that the resulting object

$$X \to Z \to Y$$

is the product of $X \to Z_1 \to Y$ and $X \to Z_2 \to Y$ in $\text{Factor}(f)_{\text{dense}}$.

Indeed, let

$$X \to Z' \to Y$$

be another object of $\text{Factor}(f)_{\text{dense}}$, endowed with maps to $X \to Z_1 \to Y$ and to $X \to Z_2 \to Y$. Let $i$ denote the resulting morphism

$$Z' \to Z_1 \times_{Y} Z_2 = W.$$

We have a canonical map in $\text{Factor}(f)$

$$(X \to Z' \to Y) \to (X \to i(Z') \to Y).$$

However, from Proposition 1.3.2 we obtain that the natural map

$$Z \to i(Z')$$

is an isomorphism. This gives rise to the desired map

$$(X \to Z' \to Y) \to (X \to Z \to Y).$$

3. The functor of $!$-pullback

Having defined IndCoh as a functor out of the category of correspondences, restricting to "horizontal morphisms", we in particular obtain the functor of $!$-pullback, which is now defined on all morphisms.

In this section we study the basic properties of this functor.
3.1. Definition of the functor. In this subsection we summarize the basic properties of the \( ! \)-pullback that follow formally from Theorem 2.1.4.

3.1.1. We let \( \text{IndCoh}^{!}_{\text{Sch}^{\text{aff}}} \) denote the restriction of the functor \( \text{IndCoh}_{(\text{Sch}^{\text{aff}})^{\text{proper}};\text{all}\rightarrow\text{all}} \)

In particular, for a morphism \( f : X \to Y \), we let \( f^{!} : \text{IndCoh}(Y) \to \text{IndCoh}(X) \) the resulting morphism.

The functor \( \text{IndCoh}^{!}_{\text{Sch}^{\text{aff}}} \) is essentially defined by the following two properties:

- The restriction \( \text{IndCoh}^{!}_{\text{Sch}^{\text{aff}}}|_{(\text{Sch}^{\text{aff}})^{\text{proper}}} \) identifies with \( \text{IndCoh}^{!}_{(\text{Sch}^{\text{aff}})^{\text{proper}}} \).
- The restriction \( \text{IndCoh}^{!}_{\text{Sch}^{\text{aff}}}|_{(\text{Sch}^{\text{aff}})^{\text{open}}} \) identifies with \( \text{IndCoh}^{!}_{(\text{Sch}^{\text{aff}})^{\text{open}}} \).

In the above formula, \( \text{IndCoh}^{!}_{(\text{Sch}^{\text{aff}})^{\text{open}}} := \text{IndCoh}^{!}_{(\text{Sch}^{\text{aff}})^{\text{event-coconn}}} \), see [Book-II.1, Corollary 3.1.10], where the functor \( \text{IndCoh}^{*}_{(\text{Sch}^{\text{aff}})^{\text{event-coconn}}} : ((\text{Sch}^{\text{aff}})^{\text{event-coconn}})^{\text{op}} \to \text{DGCat}_{\text{cont}} \) is introduced.

3.1.2. In what follows we shall denote by \( \omega_X \in \text{IndCoh}(X) \) the canonical object equal to \( p^!_X(k) \), where \( p_X : X \to \text{pt} \).

3.1.3. Base change. Let

be a Cartesian diagram in \( \text{Sch}^{\text{aff}} \). As the main corollary of Theorem 2.1.4, using [Funct, Theorem 12.2.2], we obtain:

**Corollary 3.1.4.** There exists a canonical isomorphism of functors

\[
\text{IndCoh}^{!}_{\text{Sch}^{\text{aff}}} : ((\text{Sch}^{\text{aff}})^{\text{open}})^{\text{op}} \to \text{DGCat}_{\text{cont}}
\]

compatible with compositions of vertical and horizontal morphisms in the natural sense. Furthermore:

(a) Suppose that \( g_Y \) (and hence \( g_X \)) is proper. Then the morphism \( \leftarrow \) in (3.1) comes by the \( (g^!_{\text{IndCoh}}, g^!_Y) \)-adjunction from the isomorphism

\[
(f_2)^!_{\text{IndCoh}} \circ (g_X)^!_{\text{IndCoh}} \simeq (g_Y)^!_{\text{IndCoh}} \circ (f_1)^!_{\text{IndCoh}}.
\]

(b) Suppose that \( f_2 \) (and hence \( f_1 \)) is proper. Then the morphism \( \rightarrow \) in (3.1) comes by the \( (f^!_{\text{IndCoh}}, f^!_Y) \)-adjunction from the isomorphism

\[
f_2^! \circ g_Y^! \simeq g_X^! \circ f_2^!.
\]

(c) Suppose that \( g_Y \) (and hence \( g_X \)) is an open embedding. Then the morphism \( \rightarrow \) in (3.1) comes by the \( (g^!_{\text{IndCoh}}, g^!_Y) \)-adjunction from the isomorphism

\[
(f_2)^!_{\text{IndCoh}} \circ (g_X)^!_{\text{IndCoh}} \simeq (g_Y)^!_{\text{IndCoh}} \circ (f_1)^!_{\text{IndCoh}}.
\]
Suppose that \( f_2 \) (and hence \( f_1 \)) is an open embedding. Then the morphism \( \to \) in (3.1) comes by the \( (f^1, f^\text{IndCoh}) \)-adjunction from the isomorphism
\[
f^1_1 \circ g^1_Y \simeq g^1_X \circ f^1_2.
\]

**Remark 3.1.5.** The real content of Theorem 2.1.4 is that there exists a uniquely defined family of functors \( f^l \), that satisfies the properties listed in Corollary 3.1.4 and those of Sect. 3.1.1.

### 3.2. Some properties.

#### 3.2.1. Let \( \text{IndCoh}^1_{\text{Sch}} \) denote the restriction
\[
\text{IndCoh}_{\text{Sch}}^1 |_{(\text{Sch}_{\text{aff}})^{op}}.
\]
We claim:

**Lemma 3.2.2.** The functor
\[
\text{IndCoh}^1_{\text{Sch}} \to \text{RKE}_{(\text{Sch}_{\text{aff}})^{op} \to (\text{Sch}_{\text{aff}})^{op}}(\text{IndCoh}_{\text{Sch}_{\text{aff}}}^1) \to \text{IndCoh}^1_{\text{Sch}}
\]
is an isomorphism.

**Proof.** Follows from the fact that \( \text{IndCoh}^1_{\text{Sch}} \) satisfies Zariski descent ([Book-II.1, Proposition 4.2.2]), combined with the fact that affine schemes form a basis for the Zariski topology. \(\square\)

#### 3.2.3. Convergence.
Let \( \text{IndCoh}^1_{\text{Sch}} \) denote the restrictions of \( \text{IndCoh}^1_{\text{Sch}} \) to the corresponding subcategories.

We claim:

**Lemma 3.2.4.** The functors
\[
\text{IndCoh}^1_{\text{Sch}} \to \text{RKE}_{(\leq \text{Sch}_{\text{aff}})^{op} \to (\text{Sch}_{\text{aff}})^{op}}(\text{IndCoh}_{\leq \text{Sch}_{\text{aff}}}^1) \to \text{IndCoh}^1_{\text{Sch}}
\]
and
\[
\text{IndCoh}^1_{\text{Sch}_{\text{aff}}} \to \text{RKE}_{(\leq \text{Sch}_{\text{aff}})^{op} \to (\text{Sch}_{\text{aff}})^{op}}(\text{IndCoh}_{\leq \text{Sch}_{\text{aff}}}^1)
\]
are isomorphisms.

**Proof.** Both statements are equivalent to the assertion that for \( X \in \text{Sch}_{\text{aff}} \), the functor
\[
\text{IndCoh}(X) \to \lim_n \text{IndCoh}(\leq^n X)
\]
is an equivalence.

The latter assertion is the content of [Book-II.1, Proposition 6.4.3]. \(\square\)

### 3.3. h-descent.
We will now use proper descent for \( \text{IndCoh} \) to show that it in fact has h-descent.
3.3.1. Let $C$ be a category with Cartesian products, and let $\alpha$ be an isomorphism class of 1-morphisms, closed under base change.

We define the Grothendieck topology generated by $\alpha$ to be the minimal Grothendieck topology that contains all morphisms from $\alpha$ and has the following “2-out-of-3” property:

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps in $C$ such that $f$ and $g \circ f$ are coverings, then so is $g$.

The following is well-known:

**Lemma 3.3.2.** Let $F$ be a presheaf on $C$ that satisfies descent with respect to morphisms from the class $\alpha$. Then $F$ is a sheaf with respect to the Grothendieck topology generated by $\alpha$.

3.3.3. We recall that the $h$-topology on $\text{Sch}_{\text{aff}}$ is the one generated by the class of proper surjective maps and Zariski covers.

From Lemma 3.3.2, combined with [Book-II.1, Propositions 4.2.2 and 7.2.2] we obtain:

**Corollary 3.3.4.** The functor

$$\text{IndCoh}_{\text{Sch}_{\text{aff}}}^! : (\text{Sch}_{\text{aff}})^{\text{op}} \to \text{DGCat}_{\text{op}}$$

satisfies $h$-descent.

3.3.5. We have:

**Lemma 3.3.6.** Any fpqc covering is an $h$-covering.

**Proof.** Let $f : X \to Y$ be an fpqc covering. Consider the Cartesian square

$$
\begin{array}{ccc}
\cl Y \times_X Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
\cl Y & \longrightarrow & Y
\end{array}
$$

It suffices to show that $\cl Y \times_X Y \to \cl Y$ is an $h$-covering. By flatness, $\cl Y \times_X Y$ is classical. Hence, we are reduced to an assertion at the classical level, in which case it is well-known.

□

Hence, combining, we obtain:

**Corollary 3.3.7.** The functor

$$\text{IndCoh}_{\text{Sch}_{\text{aff}}}^! : (\text{Sch}_{\text{aff}})^{\text{op}} \to \text{DGCat}_{\text{op}}$$

satisfies fpqc-descent.

3.4. **Extension to prestacks.** The functor of $!$-pullback for arbitrary morphisms of schemes allows to define the category IndCoh on arbitrary prestacks (locally almost of finite type).

3.4.1. We consider the category $\text{PreStk}_{\text{aff}}$ and define the functor

$$\text{IndCoh}_{\text{PreStk}_{\text{aff}}}^! : (\text{PreStk}_{\text{aff}})^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

as the right Kan extension of $\text{IndCoh}_{\text{Sch}_{\text{aff}}}^!$ along the Yoneda functor

$$(\text{Sch}_{\text{aff}})^{\text{op}} \hookrightarrow (\text{PreStk}_{\text{aff}})^{\text{op}}.$$

According to Lemmas 3.2.2 and 3.2.4, we can equivalently define $\text{IndCoh}_{\text{PreStk}_{\text{aff}}}^!$ as the right Kan extension of

$$\text{IndCoh}_{\text{Sch}_{\text{aff}}}^!, \text{IndCoh}_{\leq \text{Sch}_{\text{aff}}}^!, \text{or IndCoh}_{< \text{Sch}_{\text{aff}}}^!$$
from the corresponding subcategories.

For \( X \in \text{PreStk}_{\text{laft}} \) we let \( \text{IndCoh}(X) \) denote the value of \( \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}} \) on it. For a morphism \( f : X_1 \to X_2 \) we let

\[
f^! : \text{IndCoh}(X_2) \to \text{IndCoh}(X_1)
\]
denote the corresponding functor.

For \( X \in \text{PreStk}_{\text{laft}} \), we let \( \omega_X \in \text{IndCoh}(X) \) denote the canonical object equal to \( p_X^!(k) \), where \( p_X : X \to \text{pt} \).

3.4.2. We now consider the category

\[
(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}
\]
where

\[
\text{sch \& qc \ and \ sch \& proper}
\]
signify the classes of schematic and quasi-compact (resp., schematic and proper) morphisms between prestacks.

We claim:

**Theorem 3.4.3.** There exists a uniquely defined functor

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}} : (\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}}
\]
equipped with isomorphisms

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}} |_{(\text{PreStk}_{\text{laft}})^{\text{op}}} \simeq \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}}
\]
and

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}} |_{(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}}} \simeq \text{IndCoh}^!_{(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}}}
\]
where the latter two isomorphisms are compatible in a natural sense.

**Proof.** We consider the functor

\[
(\text{Sch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}} \to (\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}
\]
and define

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}}
\]
to be the right Kan extension of \( \text{IndCoh}_{(\text{infSch}_{\text{aff}})^{\text{proper}}_{\text{corr:all;all}}} \) (see [Funct, 1-categorical RKE]) for what this means).

To prove the theorem, it suffices to show that for \( X \in \text{PreStk}_{\text{laft}} \), the natural map

\[
\text{IndCoh}_{(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{corr:sch \& qc;all}}} (X) \to \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}}(X)
\]
is an equivalence.

However, the latter is given by [Funct, Horizontal Extension].
3.4.4. The actual content of Theorem 3.4.3 can be summarized as follows:

First, for any schematic morphism \( f : X \to Y \) we have a well-defined functor

\[ f_\bullet^\text{IndCoh} : \text{IndCoh}(X) \to \text{IndCoh}(Y). \]

Furthermore, if \( f : X \to Y \) is schematic and proper, the functor \( f_\bullet^\text{IndCoh} \) is the left adjoint of \( f^! \).

When \( Y \) is a scheme (and hence \( X \) is one as well), the above functor \( f_\bullet^\text{IndCoh} \) is the usual \( f_\bullet^\text{IndCoh} \) defined in this case, and similarly for the \((f_\bullet^\text{IndCoh}, f^!)\)-adjunction.

Second, let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{g_X} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{g_Y} & Y_2
\end{array}
\]

be a canonical isomorphism of functors

\[
g_1^! \circ (f_2)_\bullet^\text{IndCoh} \simeq (f_1)_\bullet^\text{IndCoh} \circ g_X^!,
\]

compatible with compositions. Furthermore, if the vertical (resp., horizontal) morphisms are proper (resp., schematic and proper), the map \( \leftarrow \) in (3.3) comes by adjunction in a way similar to Corollary 3.1.4(a) (resp., Corollary 3.1.4(b)).

3.4.5. In [Book-III.3, Proposition 5.3.6], we will show that for a morphism \( f : X \to Y \), which is an open embedding, the functor \( f_\bullet^\text{IndCoh} \) is the right adjoint of \( f^! \).

Furthermore, if in the Cartesian diagram (3.2) the vertical (resp., horizontal) morphisms are open embeddings, the map \( \rightarrow \) in (3.3) comes by adjunction in a way similar to Corollary 3.1.4(c) (resp., Corollary 3.1.4(d)).

3.4.6. For future use, we note that the statement and proof of [Book-II.1, Proposition 7.2.2] for groupoid objects in \((\text{PreStk}_{\text{laft}}, \text{sch & proper})\).

4. The multiplicative structure and duality

In this section we will show that the functor IndCoh, when viewed as a functor out of the category of correspondences, and equipped with a natural symmetric monoidal structure, encodes Serre duality.

4.1. IndCoh as a symmetric monoidal functor. In this subsection we show that the functor IndCoh possesses a natural symmetric monoidal structure.

4.1.1. We recall that by [Book-II.1, Proposition 6.3.6], the functor

\[ \text{IndCoh}_{\text{Sch}_{\text{laft}}} : \text{Sch}_{\text{laft}} \to \text{DGCat}_{\text{cont}} \]

carries a natural symmetric monoidal structure.

Applying [Funct, Multiplicative structure on correspondences], we obtain:

**Theorem 4.1.2.** The functor

\[ \text{IndCoh}_{\text{Sch}_{\text{laft}}}^{\text{proper}} : (\text{Sch}_{\text{laft}})^{\text{proper}} \to \text{DGCat}_{\text{cont}}^{2\text{-Cat}} \]

carries a canonical symmetric monoidal structure that extends one on \( \text{IndCoh}_{\text{Sch}_{\text{laft}}} \).
In particular, we obtain that the functors
\[ \text{IndCoh}_{(\text{Sch}_{\text{saft}})_{\text{corr:all;all}}} : (\text{Sch}_{\text{saft}})_{\text{corr:all;all}} \to \text{DGCat}_{\text{cont}} \]
and
\[ \text{IndCoh}^!_{\text{Sch}_{\text{saft}}} : (\text{Sch}_{\text{saft}})^{\text{op}} \to \text{DGCat}_{\text{cont}} \]
both carry natural symmetric monoidal structures.

4.1.3. Note that the symmetric monoidal structure on \(\text{IndCoh}^!_{\text{Sch}_{\text{saft}}}\) automatically upgrades the functor \(\text{IndCoh}^!_{\text{Sch}_{\text{saft}}}\) to a functor
\[ (\text{Sch}_{\text{saft}})^{\text{op}} \to \text{DGCat}^{\text{SymMon}}_{\text{cont}}, \]
due to the fact that the identity functor on \((\text{Sch}_{\text{saft}})^{\text{op}}\) naturally lifts to a functor
\[ (\text{Sch}_{\text{saft}})^{\text{op}} \to \text{ComAlg} ((\text{Sch}_{\text{saft}})^{\text{op}}) \]
via the diagonal map.

Explicitly, the monoidal operation on \(\text{IndCoh}(X)\) is given by
\[ \text{IndCoh}(X) \otimes \text{IndCoh}(X) \xrightarrow{\otimes} \text{IndCoh}(X \times X) \xrightarrow{\Delta_X} \text{IndCoh}(X). \]

We shall denote the above monoidal operation by \(\otimes):\)
\[ \mathcal{F}_1, \mathcal{F}_2 \in \text{IndCoh}(X) \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2 \in \text{IndCoh}(X). \]

The unit for this symmetric monoidal structure is given by \(\omega_X \in \text{IndCoh}(X)\).

4.1.4. Applying the functor of right Kan extension along
\[ (\text{Sch}_{\text{saft}})^{\text{op}} \to (\text{PreStk}_{\text{laft}})^{\text{op}} \]
of the functor (4.1), we obtain that the functor
\[ \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}} : (\text{PreStk}_{\text{laft}})^{\text{op}} \to \text{DGCat}_{\text{cont}} \]
naturally upgrades to a functor
\[ \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}} : (\text{PreStk}_{\text{laft}})^{\text{op}} \to \text{DGCat}^{\text{SymMon}}_{\text{cont}}. \]

As in [Book-II.1, Proposition 6.3.6], the datum of the functor (4.3) is equivalent to that of right-lax symmetric monoidal structure on the functor (4.2).

4.1.5. The above right-lax symmetric monoidal structure on \(\text{IndCoh}^!_{\text{PreStk}_{\text{laft}}}\) can be enhanced:

Indeed, applying [Funct, Horizontal extension monoidal], we obtain that the functor
\[ \text{IndCoh}^!_{(\text{PreStk}_{\text{laft}})_{\text{sch & proper}}} : (\text{PreStk}_{\text{laft}})_{\text{sch & proper}}^{\text{op}} \to \text{DGCat}^{\text{2-Cat}}_{\text{cont}} \]
carries a canonical right-lax symmetric monoidal structure.

4.2. Duality. In this subsection we will formally deduce Serre duality for schemes from the symmetric monoidal structure on \(\text{IndCoh}\).
4.2.1. Let \( \mathcal{O} \) be a symmetric monoidal category, and let \( \mathcal{O}^{\text{dualizable}} \subset \mathcal{O} \) be the full subcategory spanned by dualizable objects. This subcategory carries a canonical symmetric monoidal anti-involution
\[
(\mathcal{O}^{\text{dualizable}})^{\text{op}} \xrightarrow{\text{dualization}} \mathcal{O}^{\text{dualizable}},
\]
given by the passage to the dual object.
\[
o \mapsto o^\vee.
\]
Recall now that if
\[
F : \mathcal{O}_1 \to \mathcal{O}_2
\]
is a symmetric monoidal functor between symmetric monoidal categories, then it maps
\[
\mathcal{O}_1^{\text{dualizable}} \to \mathcal{O}_2^{\text{dualizable}},
\]
and the following diagram commutes
\[
\begin{array}{ccc}
(\mathcal{O}_1^{\text{dualizable}})^{\text{op}} & \xrightarrow{F^{\text{op}}} & (\mathcal{O}_2^{\text{dualizable}})^{\text{op}} \\
\text{dualization} & & \text{dualization} \\
\mathcal{O}_1^{\text{dualizable}} & \xrightarrow{F} & \mathcal{O}_2^{\text{dualizable}}.
\end{array}
\]

4.2.2. Recall now that by [Funct, Anti-involution on Corr] the category \((\text{Sch}_{\text{aft}})_{\text{corr:all;all}}\) carries a canonical anti-involution \(\varpi\), which is the identity on objects, and at the level of 1-morphisms is maps a 1-morphism
\[
X_{12} \xrightarrow{f} X_1 \\
g \downarrow \\
X_2
\]
to
\[
X_{12} \xrightarrow{g} X_2 \\
f \downarrow \\
X_1.
\]
Moreover, by [Funct, Anti-involution on Corr], we have:

**Theorem 4.2.3.** The inclusion
\[
((\text{Sch}_{\text{aft}})_{\text{corr:all;all}})^{\text{dualizable}} \subset (\text{Sch}_{\text{aft}})_{\text{corr:all;all}}
\]
is an isomorphism. The anti-involution \(\varpi\) identifies canonically with the dualization functor
\[
((\text{Sch}_{\text{aft}})_{\text{corr:all;all}})^{\text{dualizable}})^{\text{op}} \to ((\text{Sch}_{\text{aft}})_{\text{corr:all;all}})^{\text{dualizable}}.
\]

4.2.4. Combining Theorem 4.2.3 with Theorem 4.1.2 we obtain:

**Theorem 4.2.5.** We have the following commutative diagram of functors
\[
\begin{array}{ccc}
((\text{Sch}_{\text{aft}})_{\text{corr:all;all}})^{\text{op}} & \xrightarrow{\text{IndCoh}(\text{Sch}_{\text{aft}})_{\text{corr:all;all}}^{\text{op}}} & \left(D\text{GCat}^{\text{dualizable}}\right)^{\text{cont}} \\
\cong & & \text{dualization} \\
(\text{Sch}_{\text{aft}})_{\text{corr:all;all}} & \xrightarrow{\text{IndCoh}(\text{Sch}_{\text{aft}})_{\text{corr:all;all}}} & D\text{GCat}^{\text{dualizable}}^{\text{cont}}.
\end{array}
\]
4.2.6. Let us explain the concrete meaning of this theorem. It says that for \( X \in \text{Sch}_{\text{aff}} \) there is a canonical equivalence
\[
\mathbf{D}^\text{Serre}_X : \text{IndCoh}(X)^{\vee} \simeq \text{IndCoh}(X),
\]
and for a map \( f : X \to Y \) an isomorphism
\[
(f^!)^{\vee} \simeq f_*^{\text{IndCoh}},
\]
where \((f^!)^{\vee}\) is viewed as a functor
\[
\text{IndCoh}(X) \xrightarrow{\mathbf{D}^\text{Serre}_X^{-1}} \text{IndCoh}(Y)^{\vee} \xrightarrow{(f^!)} \text{IndCoh}(Y) \xrightarrow{\mathbf{D}^\text{Serre}_Y} \text{IndCoh}(Y).
\]

4.2.7. Let us write down explicitly the unit and co-unit for the identification \(\mathbf{D}^\text{Serre}_X\):

The co-unit, denoted \(\epsilon_X\) is given by
\[
\text{IndCoh}(X) \otimes \text{IndCoh}(X) \xrightarrow{\cong} \text{IndCoh}(X \times X) \xrightarrow{\Delta_X} \text{IndCoh}(X) \xrightarrow{(p_X)^{\text{IndCoh}}} \text{IndCoh}(\text{pt}) = \text{Vect},
\]
where \(p_X : X \to \text{pt}\).

The unit, denoted \(\mu_X\) is given by
\[
\text{Vect} = \text{IndCoh}(\text{pt}) \xrightarrow{\mu_X} \text{IndCoh}(X) \xrightarrow{\Delta_X} \text{IndCoh}(X \times X) \xrightarrow{\cong} \text{IndCoh}(X) \otimes \text{IndCoh}(X).
\]

**Remark 4.2.8.** One does not need to rely on Theorems 4.2.3 and 4.1.2 in order to show that the maps \(\mu_X\) and \(\epsilon_X\), defined above, give rise to an identification
\[
\text{IndCoh}(X)^{\vee} \simeq \text{IndCoh}(X).
\]
Indeed, the fact that the composition
\[
\text{IndCoh}(X) \xrightarrow{\text{Id}_{\text{IndCoh}(X)} \otimes \mu_X} \text{IndCoh}(X) \otimes \text{IndCoh}(X) \xrightarrow{\epsilon_X \otimes \text{Id}_{\text{IndCoh}(X)}} \text{IndCoh}(X)
\]
is isomorphic to the identity functor follows by base change from the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times X \\
\downarrow{\Delta_X} & & \downarrow{\text{id}_X \times \Delta_X} \\
X \times X & \xrightarrow{\Delta_X \times \text{id}_X} & X \times X \times X \\
\downarrow{p_X \times \text{id}_X} & & \\
X, & &
\end{array}
\]
and similarly for the other composition. A similar diagram chase implies the isomorphism
\[
(f^!)^{\vee} \simeq f_*^{\text{IndCoh}}.
\]

**Remark 4.2.9.** Let us also note that one does not need the (difficult) Theorem 2.1.4 either in order to construct the pairing \(\epsilon_X\):

Indeed, both functors involved in \(\epsilon_X\), namely, \(\Delta_X\) and \((p_X)^{\text{IndCoh}}_*\) are “elementary”.

If one believes that the functor \(\epsilon_X\) defined in the above way is the co-unit of a duality (which is a property, and not an extra structure), then one recover the object \(\omega_X \in \text{IndCoh}(X)\). Namely,
\[
\omega_X := (p_X \times \text{id}_X)^{\text{IndCoh}}_*(\mu_X(k)).
\]
4.2.10. **Relation to the usual Serre duality.** By passage to compact objects, the equivalence
\[ D^\text{Serre}_X : \text{IndCoh}(X)^{\vee} \simeq \text{IndCoh}(X) \]
gives rise to an equivalence
\[ D^\text{Serre}_X : (\text{Coh}(X))^{\text{op}} \simeq \text{Coh}(X). \]

It is shown in [GL:IndCoh, Proposition 8.3.5] that \( D^\text{Serre}_X \) is the usual Serre duality anti-equivalence of \( \text{Coh}(X) \), given by internal Hom into \( \omega_X \).

5. **Convolution monoidal categories and algebras**

In this section we will apply the the formalism of \( \text{IndCoh} \) as a functor out of the category of correspondences to carry out the following constructions and its generalizations:

Let \( \mathcal{R} \rightarrow X \) be a groupoid-object in the category of schemes. Then the category \( \text{IndCoh}(\mathcal{R}) \) has a natural monoidal structure, given by convolution, and as such it acts on the category \( \text{IndCoh}(X) \).

The contents of this section were suggested to us by S. Raskin.

5.1. **Convolution monoidal categories.** In this sections we will show that \( \text{IndCoh} \) of a groupoid-object (and, more generally, category-object) carries a natural monoidal structure.

We note that this construction only uses the underlying 1-category of the category of correspondences.

5.1.1. Let \( X^\bullet \) be a category-object in \( \text{PreStk}_{\text{laft}} \). I.e., this is a simplicial object in \( \text{PreStk}_{\text{laft}} \), such that for any \( S \in ^{<\infty}\text{Sch}_{\text{aff}} \), the object
\[ \text{Maps}(S, X^\bullet) \in \text{Spc}^\Delta^{\text{op}} \]
is a complete Segal space.

We denote \( X := X^0 \) and \( \mathcal{R} := X^1 \). Let us assume that the “target” map \( p_t : \mathcal{R} \rightarrow X \) and the “composition” map
\[ (t, \mathcal{R}, s) : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R} \]
are schematic and quasi-compact.

5.1.2. Note that by [Funct, Algebras in corr], the object \( \mathcal{R} \) acquires a natural structure of algebra in the (symmetric) monoidal category
\[ (\text{PreStk}_{\text{laft}})_{\text{corr}, \text{sch} \& \text{qc}, \text{all}}, \]
and as such it acts on \( X \).

Applying the (symmetric) monoidal functor
\[ \text{IndCoh}(\text{PreStk}_{\text{laft}})_{\text{corr}, \text{sch} \& \text{qc}, \text{all}} : (\text{PreStk}_{\text{laft}})_{\text{corr}, \text{sch} \& \text{qc}, \text{all}} \rightarrow \text{DGCat}_{\text{cont}} \]
we obtain that the DG category \( \text{IndCoh}(\mathcal{R}) \) acquires a structure of monoidal DG category, and \( \text{IndCoh}(X) \) acquires a structure of \( \text{IndCoh}(\mathcal{R}) \).

Unwinding the definitions, we obtain that the binary operation on \( \text{IndCoh}(\mathcal{R}) \) is given by the *convolution* product, i.e., pull-push along the diagram
\[
\begin{array}{c}
\mathcal{R} \times \mathcal{R} \\
\downarrow \\
\mathcal{R}.
\end{array}
\]
5.1.3. Consider a particular case when $X = X \in \text{Sch}_aft$, and $X^n = X \times^n$. So $R = X \times X$.

We obtain that $\text{IndCoh}(X \times X)$ acquires a structure of monoidal category, and as such it acts on $\text{IndCoh}(X)$.

I.e., we obtain a monoidal functor

$$\text{IndCoh}(X \times X) \to \text{Funct}_{\text{cont}}(\text{IndCoh}(X), \text{IndCoh}(X)).$$

It is easy to see that as a functor of plain categories, (5.1) identifies with

$$\text{IndCoh}(X \times X) \simeq \text{IndCoh}(X) \otimes \text{IndCoh}(X) \xrightarrow{(D_{\text{Serre}} X)^{-1} \otimes \text{Id}} \text{IndCoh}(X)^{\vee} \otimes \text{IndCoh}(X) \simeq \text{Funct}_{\text{cont}}(\text{IndCoh}(X), \text{IndCoh}(X)).$$

In particular, the functor (5.1) is an equivalence of monoidal categories.

5.1.4. We note that the above discussion goes through “as is” with $\text{IndCoh}$ replaced by $\text{Qcoh}$.

We note, however, that for the subsequent discussion that involves a 2-categorical structure, if one considers $\text{Qcoh}$ instead of $\text{IndCoh}$, the direction of the 2-morphisms must be reversed. In particular, what are algebras and monads for $\text{IndCoh}$ become co-algebras and co-monads for $\text{Qcoh}$.

5.2. Convolution algebras. In this subsection we will show how certain maps between category-objects give rise to lax monoidal functors between the corresponding convolution categories.

5.2.1. Let now $f^\bullet : \mathcal{X}^\bullet_1 \to \mathcal{X}^\bullet_2$ be a map between category-objects in $\text{PreStk}_{\text{aft}}$.

Assume that the map

$$\mathcal{R}_1 \times _{x_1} \mathcal{R}_1 \to \mathcal{R}_1 \times _{x_2} (\mathcal{R}_2 \times _{x_2} \mathcal{R}_2)$$

is schematic and proper.

5.2.2. For example, let $\mathcal{X}^\bullet_1$ (resp., $\mathcal{X}^\bullet_2$) be the Čech nerve of a map $X_1 \to Y_1$ (resp., $X_2 \to Y_2$). Let us be given a commutative (but not necessarily Cartesian) diagram

$$
\begin{array}{c}
\mathcal{X}_1 \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{X}_2 \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{Y}_1 \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{Y}_2
\end{array}
$$

Then the above condition is satisfied if the map

$$x_1 \to y_1 \times y_2$$

is schematic and proper.
5.2.3. In the above situation, by \([\text{Funct, Maps of algs in corr}]\), the map \(R_1 \to R_2\) defines a right-lax map of algebra objects
\[
R_2 \to R_1
\]
(note the direction of the arrow!) in
\[(\text{PreStk}_{\text{laft}})_{\text{sch & proper}}^{\text{corr:sch & qc;all}},\]
given by the diagram
\[
\begin{array}{ccc}
R_1 & \longrightarrow & R_2 \\
\text{id} & & \\
\downarrow & & \\
R_1 & & \\
\end{array}
\]

Applying the (symmetric) monoidal functor
\[\text{IndCoh}((\text{PreStk}_{\text{laft}})_{\text{corr:sch & qc;all}}): (\text{PreStk}_{\text{laft}})_{\text{corr:sch & qc;all}} \to \text{DGCat}_{\text{cont}},\]
we obtain a right-lax monoidal functor
\[f^! : \text{IndCoh}(R_2) \to \text{IndCoh}(R_1).\]

5.2.4. In particular, the image of the unit
\[f^! \circ (\text{unit}_{R_2}^\text{IndCoh}(\omega_{X_2}) \in \text{IndCoh}(R_1)\]
acquires a structure of algebra in the monoidal category \(\text{IndCoh}(R_1)\).

5.3. The pull-push monad. In this subsection we will show that for a category-object with a proper multiplication, its dualizing complex has a natural algebra structure in the convolution category.

5.3.1. Let us consider a particular case when \(X_2 = \text{pt}\). Denote \(X_1 = \ast\), so our condition is that the composition map
\[
\mathcal{R} \times \mathcal{R} \to \mathcal{R}
\]
be proper.

Let \(M_R\) denote the monad acting on \(X\) (where \(X\) is viewed as an object of the 2-category \((\text{PreStk}_{\text{laft}})_{\text{corr:sch & qc;all}}\), obtained from the action on \(R\) on \(X\) and the right-lax monoidal map \(\text{pt} \to \mathcal{R}\).

5.3.2. By Sect. 5.2.4, \(\omega_{\mathcal{R}} \in \text{IndCoh}(\mathcal{R})\) acquires a structure of algebra. Hence, the action of \(\omega_{\mathcal{R}}\) defines a monad on \(X\). It is easy to see that this monad, viewed as a plain endo-functor, identifies with
\[
(p_t)_s^\text{IndCoh} \circ (p_s)^!.
\]
By definition, the above monad on \(\text{IndCoh}(X)\) can also be viewed as one obtained by applying the functor \(\text{IndCoh}((\text{PreStk}_{\text{laft}})_{\text{corr:sch & qc;all}})\) to the monad \(M_R\).

5.3.3. Assume now that \(X\) is a groupoid object of \(\text{PreStk}_{\text{laft}}\), with the maps \(p_s, p_t : \mathcal{R} \to X\) being proper.

In then according to Sect. 3.4.6, the endo-functor \((p_t)_s^\text{IndCoh} \circ (p_s)^!\) acquires a (a priori different) structure of monad.

We claim, however that the above two ways of introducing a structure of monad on \((p_t)_s^\text{IndCoh} \circ (p_s)^!\) coincide. Indeed, this follows from Sect. 5.4.2 for
\[Y := |X^\ast|\].
5.4. **Action on a map.** In this subsection we will show that the pull-push monad from the previous subsection is canonically isomorphic to one from [Book-II.1, Proposition 7.2.2(a)].

5.4.1. We retain the assumptions of Sect. 5.3.

Let $Y$ be an object of $\text{PreStk}_{\text{laft}}$, and let us assume that $X$ is a category-object over $Y$. Let $g$ denote the map $X \to Y$.

Assume now that the map $p_t : \mathcal{R} \to X$ is schematic and proper. In this case, by [Funct, Modules of algs in corr], the monad $M_\mathcal{R}$, introduced above acts on the 1-morphism

$$(Y \to X) \in (\text{PreStk}_{\text{laft}})_{\text{sch} \& \text{proper}}^{\text{corr} : \text{sch} \& \text{qc}; \text{all}}$$

(not the direction of the arrow!), given by the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow^{\text{id}} & & \downarrow \\
X & & X
\end{array}
$$

Applying the functor $\text{IndCoh}(\text{PreStk}_{\text{laft}})_{\text{corr} : \text{sch} \& \text{qc}; \text{all}}$, we obtain that the functor

$$g' : \text{IndCoh}(Y) \to \text{IndCoh}(X)$$

upgrades to a functor

$$\text{(5.2)} \quad (g')^{\text{enh}} : \text{IndCoh}(Y) \to ((p_t)^{\text{IndCoh}} \circ (p_s)^{\text{IndCoh}})_{-\text{mod}}(\text{IndCoh}(X)).$$

5.4.2. Assume now that $g : X \to Y$ is schematic and proper, and that $X^\bullet$ equals the Čech nerve of $g$.

In this case, the functor (5.2) defines a map of monads

$$\text{(5.3)} \quad ((p_t)^{\text{IndCoh}} \circ (p_s)^{\text{IndCoh}}) \to (g' \circ g)^{\text{IndCoh}},$$

and it is easy to see that at the level of plain endo-functors, the map (5.3) is given by base change along

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{p_t} & X \\
p_s & & \downarrow^{g} \\
X & \xrightarrow{g} & Y
\end{array}
$$

In particular, the map (5.3) is an isomorphism of monads.
References


[Book-II.1]
[Book-III.1]
[Book-III.2]
[Book-III.3]
[Funct]
[IndSch]

