CHAPTER V.1. THE $(\infty, 2)$–CATEGORY OF CORRESPONDENCES

Contents

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Introduction

This chapter contains one of the two main innovations in this book: functors out of the $(\infty,2)$-category of correspondences (the other one being the notion of inf-scheme).

The idea of the $(\infty,2)$–category of correspondences, as a way to encode bi-variant functors that satisfy base change, was explained to us by J. Lurie. So, in a sense we realize his suggestion, even though our approach to the definition of the $(\infty,2)$-category of correspondences is different from what he had originally envisaged[1].

0.1. **Why do we need it?**

0.1.1. Suppose we have an $(\infty,1)$-category $\mathbf{C}$, and let $\mathbf{S}$ be a target $(\infty,1)$-category.

We want to express, in a functorial way, what it means to have a *bi-variant* assignment

$$\mathbf{c}\in\mathbf{C} \rightsquigarrow \Phi(\mathbf{c})\in\mathbf{S}$$

that satisfies base change.

In other words, we want to have a functor

$$\Phi:\mathbf{C}\to\mathbf{S}$$

and also a functor

$$\Phi^{!}:\mathbf{C}^{\mathrm{op}}\to\mathbf{S}$$

that interact as follows:

(1) At the level of objects, for any $\mathbf{c}\in\mathbf{C}$ we are given an isomorphism $\Phi(\mathbf{c})\simeq\Phi^{!}(\mathbf{c})$.
(2) Whenever we have a Cartesian square

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0}, \end{tikzcd} \tag{0.1}$$

we want to be *given* a base change isomorphism

$$\Phi(\beta_1)\circ\Phi^{!}(\alpha_0)\simeq\Phi^{!}(\alpha_1)\circ\Phi(\beta_0). \tag{0.2}$$

We emphasize that, in general, for a morphism $\gamma:\mathbf{c}\to\mathbf{c}'$ in $\mathbf{C}$ and $\mathbf{S}=1\text{-Cat}$, the 1-morphisms

$$\Phi(\gamma):\Phi(\mathbf{c})\to\Phi(\mathbf{c}') \text{ and } \Phi^{!}(\gamma):\Phi(\mathbf{c}')\to\Phi(\mathbf{c})$$

are not adjoint on either side; therefore, in (0.2) there is no a priori defined map in either direction.

An example to keep in mind is when $\mathbf{C}=\mathrm{Sch}_{\mathrm{aft}}$ and $\mathbf{S}=1\text{-Cat}$, and we take $\Phi(S)=\mathrm{IndCoh}(S)$ with the morphism $(S_1\xrightarrow{f}S_2)$ being sent to

$$f_*^{\mathrm{IndCoh}}:\mathrm{IndCoh}(S_1)\to\mathrm{IndCoh}(S_2) \text{ and } f^{!}:\mathrm{IndCoh}(S_2)\to\mathrm{IndCoh}(S_1),$$

respectively.

[1] Lurie's idea was to construct it combinatorially in terms simplicial *sets* starting from quasi-categories, whereas our approach is independent of a particular model for $(\infty,1)$-categories.

0.1.2. A challenge is to even express what it means for the data (1) and (2) above to be functorial. Let us give a typical example of why one would want that.

Say, we want to extend the functor

$$\mathrm{IndCoh} : \mathrm{Sch}_{\mathrm{aft}} \to 1\text{-Cat}$$

to a functor

$$\mathrm{IndCoh} : (\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch}} \to 1\text{-Cat},$$

where $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch}}$ is the 1-full subcategory of $\mathrm{PreStk}_{\mathrm{laft}}$, where we restrict morphisms to maps that are schematic.

In other words, we want to assign to a schematic map $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$ between prestacks a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2),$$

and we want this assignment to be functorial in the ∞-categorical sense.

By definition, for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, we have

$$\mathrm{IndCoh}(\mathcal{Y}) \coloneqq \lim_{Y \in (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{Y}}} \mathrm{IndCoh}(Y),$$

where for a map $Y' \xrightarrow{g} Y''$ over $\mathcal{Y}$, the corresponding functor $\mathrm{IndCoh}(Y_2) \to \mathrm{IndCoh}(Y_1)$ is $g^!$.

For each $Y_1 \to \mathcal{Y}_1$, set $Y_2 \coloneqq Y_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$, which is a scheme since f was assumed to be schematic, and let $\widetilde{f}$ denote the resulting map $Y_1 \to Y_2$.

The sought-for functor f_*^{IndCoh} is given by the *compatible* family of functors

$$\widetilde{f}_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y_1) \to \mathrm{IndCoh}(Y_2),$$

where the compatibility is exactly encoded by the base change isomorphisms.

We will carry out this construction in detail in [Chapter V.2, Sect. 6].

0.1.3. Of course, if $\mathbf{S}$ is an ordinary category, one can express the required compatibility conditions on the data (1) and (2) in Sect. 0.1.1 by hand: one specifies the natural transformations (0.2) for each Cartesian square (0.1) requiring that for a Cartesian diagram

$$\begin{tikzcd}
\mathbf{c}_{0,2} \arrow[r, "\alpha'_0"] \arrow[d, "\beta_2"'] & \mathbf{c}_{0,1} \arrow[r, "\alpha''_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\
\mathbf{c}_{1,2} \arrow[r, "\alpha'_1"] & \mathbf{c}_{1,1} \arrow[r, "\alpha''_1"] & \mathbf{c}_{1,0},
\end{tikzcd}$$

the resulting two natural isomorphisms

$$\Phi(\beta_2) \circ \Phi^!(\alpha'_0) \circ \Phi^!(\alpha''_0) \rightrightarrows \Phi^!(\alpha'_1) \circ \Phi^!(\alpha''_1) \circ \Phi(\beta_0)$$

coincide, and similarly for every Cartesian diagram

$$\begin{tikzcd}
\mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta'_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta'_0"] \\
\mathbf{c}_{1,1} \arrow[r, "\alpha_1"] \arrow[d, "\beta''_1"'] & \mathbf{c}_{1,0} \arrow[d, "\beta''_0"] \\
\mathbf{c}_{2,1} \arrow[r, "\alpha_2"] & \mathbf{c}_{2,0}.
\end{tikzcd}$$

But if $\mathbf{S}$ is an ∞-category, the word 'coincides' must be replaced by 'a specified homotopy', and thus we need to specify the data of infinitely many homotopies for $(m \times n)$-diagrams for every m and n.

0.1.4. That said, a way to formulate the above data at the level of ∞-categories readily presents itself. Here is one possibility: we specify a map between *bi-simplicial spaces* (i.e., a map in the category $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$) between the following two objects:

The source is the object of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ that assigns to $([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ the full subspace in $\mathrm{Maps}([m] \times [n], \mathbf{C})$ that consists of *Cartesian diagrams*.

The target is the object of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ that assigns to $([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ simply $\mathrm{Maps}([m] \times [n], \mathbf{S})$.

This is a valid formulation, and we will prove (see Theorem 2.1.3) that it is equivalent to the one we will 'officially' take. Its disadvantage is that a datum of a map in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ does not look like a datum of a functor between ∞-categories (however, Theorem 2.1.3, mentioned above, says that it is actually equivalent to one).

0.1.5. Here is an alternative approach, which is the one we will adopt in this book as the definition. Namely, we will introduce an $(\infty, 1)$-category that we will denote $\mathrm{Corr}(\mathbf{C})$, and our datum will be simply a functor from $\mathrm{Corr}(\mathbf{C})$ to $\mathbf{S}$.

In order to define $\mathrm{Corr}(\mathbf{C})$, recall following [Chapter A.1, Sect. 1.2], that the datum of an $(\infty, 1)$-category $\mathbf{D}$ is equivalent to one of the *complete Segal space*

$$\mathrm{Seq}_\bullet(\mathbf{D}) \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}, \quad \mathrm{Seq}_n(\mathbf{D}) = \mathrm{Maps}([n], \mathbf{D}).$$

For the desired $(\infty, 1)$-category $\mathrm{Corr}(\mathbf{C})$, we take the corresponding object $\mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{C}))$ to be the complete Segal space, denoted $\mathrm{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})$, defined as follows. We let $\mathrm{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})$ be the space of functors

$$([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to \mathbf{C}$$

for which every inner square is Cartesian.

Here $([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$ is the ordinary category, equal to the full subcategory of $[n] \times [n]^{\mathrm{op}}$ spanned by the objects $\{i\}, \{j\}$ with $i \leq j$.

Note that $\mathrm{Corr}(\mathbf{C})$ receives a pair of functors

$$\mathbf{C} \to \mathrm{Corr}(\mathbf{C}) \leftarrow \mathbf{C}^{\mathrm{op}}$$

corresponding to the projections

$$[n] \leftarrow ([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to [n]^{\mathrm{op}},$$

respectively.

0.1.6. Thus, for example, the space of objects of $\mathrm{Corr}(\mathbf{C})$ equals that of $\mathbf{C}$. The space of 1-morphisms in $\mathrm{Corr}(\mathbf{C})$ is that of diagrams

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_{0,0} \\ \mathbf{c}_{1,1}. \end{tikzcd} \tag{0.3}$$

The space of two-fold compositions is that of diagrams

$$\begin{tikzcd}
\mathbf{c}_{0,2} \arrow[r] \arrow[d] & \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_{0,0} \\
\mathbf{c}_{1,2} \arrow[r] \arrow[d] & \mathbf{c}_{1,1} & \\
\mathbf{c}_{2,2}, & &
\end{tikzcd}$$

in which the square

$$\begin{tikzcd}
\mathbf{c}_{0,2} \arrow[r] \arrow[d] & \mathbf{c}_{0,1} \arrow[d] \\
\mathbf{c}_{1,2} \arrow[r] & \mathbf{c}_{1,1}
\end{tikzcd}$$

is Cartesian.

0.1.7. If the target $\mathbf{S}$ is an ordinary category, it is an easy exercise to see that the datum of a functor

$$\mathrm{Corr}(\mathbf{C}) \to \mathbf{S}$$

is equivalent to that of a bi-simplicial map as in Sect. 0.1.4.

If $\mathbf{S}$ is a general $(\infty,1)$-category such an equivalence is the content of (a particular case of) Theorem 2.1.3, mentioned above.

0.2. **The actual reason we need it.**

0.2.1. The above was meant to explain why we need something like the category of correspondences if we want to have a bi-variant functor with base change. However, we were led to consider the category $\mathrm{Corr}(\mathbf{C})$ for a different reason.

Namely, we simply wanted to define $\mathrm{IndCoh}^!$ as a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-Cat} \tag{0.4}$$

that assigns to $S \in \mathrm{IndCoh}$ the category $\mathrm{IndCoh}(S)$ and to a morphism $S_1 \xrightarrow{f} S_2$ the functor

$$f^! : \mathrm{IndCoh}(S_2) \to \mathrm{IndCoh}(S_1).$$

The problem, known since at least Hartshorne's 'Residues and duality', is that for an arbitrary morphism f, the functor $f^!$ is not adjoint to anything.

Namely, when f is proper, $f^!$ is the *right* adjoint to f_*^{IndCoh}, and when f is an open embedding, $f^!$ is the *left* adjoint to f_*^{IndCoh}. In general, one decomposes f as a composition $f_1 \circ f_2$ with f_2 an open embedding and f_1 proper, and defines

$$f^! := ((f_2)_*^{\mathrm{IndCoh}})^L \circ ((f_1)_*^{\mathrm{IndCoh}})^R.$$

0.2.2. This gives a valid definition of $f^!$ for a single morphism f, and it is not difficult to show that it is independent of the decomposition of f as $f_1 \circ f_2$. However, to make this construction functorial (i.e., to have IndCoh as a functor as in (0.4)) becomes a challenge.

For example, let us see what happens with a simple composition $f' \circ f''$. We write

$$f' = f'_1 \circ f'_2 \text{ and } f'' = f''_1 \circ f''_2.$$

Then to show that

$$(f' \circ f'')^! \simeq (f'')^! \circ (f')^!$$

we will need to show that

$$(f''_1)^! \circ (f'_2)^! \simeq g^! \circ h^!,$$

where $f'_2 \circ f''_1 = h \circ g$ with g an open embedding and h proper (while f''_1 was proper and f'_2 was open, so we need to perform a swap).

One can imagine that this becomes quite combinatorially involved in the ∞-categorical setting, where one needs to consider n-fold compositions for any n.

0.2.3. Now, it turns out that the notion of a functor out of the category of correspondences provides a convenient framework to deal with the above issues.

However, there is a caveat: the $(\infty,1)$-category $\mathrm{Corr}(\mathbf{C})$ as defined above is not sufficient. We need to enlarge it to an $(\infty,2)$-category by allowing non-invertible 2-morphisms.

This is not surprising: in the construction of $f^!$ we appeal to the notion of *adjoint functor*, and the latter is a 2-categorical notion; i.e., it is intrinsic not to the $(\infty,1)$-category 1-Cat, but to the $(\infty,2)$-category $\mathbf{1}\text{-}\mathbf{Cat}$.

0.2.4. Given a class of 1-morphisms in $\mathbf{C}$, denoted *adm*, satisfying some reasonable conditions (see Sect. 1.1.1), we will attach to it an $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}$, such that the underlying $(\infty,1)$-category is $\mathrm{Corr}(\mathbf{C})$ as was defined above.

In the example of $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$, we take *adm* to be the class of proper morphisms.

Let us explain the idea of $\mathrm{Corr}(\mathbf{C})^{adm}$. As was said above, 1-morphisms in $\mathrm{Corr}(\mathbf{C})^{adm}$ are diagrams (0.3). Given such a 1-morphism and another 1-morphism between the same two objects

$$\begin{tikzcd} \mathbf{c}'_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ \mathbf{c}_1, & \end{tikzcd}$$

a 2-morphism from the former to the latter is a commutative diagram

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[dr, "\gamma"] \arrow[drr] \arrow[ddr] & & \\ & \mathbf{c}'_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ & \mathbf{c}_1 & \end{tikzcd}$$

with $\gamma \in adm$.

0.2.5. In order to spell out this definition in the ∞-categorical world, we use the approach to $(\infty, 2)$-categories explained in [Chapter A.1, Sect. 2]. Namely, we will think of a datum of an $(\infty, 2)$-category $\mathbb{T}$ in terms of the corresponding object

$$\mathrm{Seq}_\bullet(\mathbb{T}) \in 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

For the desired $(\infty, 2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}$, we take the corresponding object of $1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ to be the simplicial $(\infty, 1)$-category $''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}$, defined as follows.

For any n, the space underlying $''\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}$ equals $\mathrm{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})$ (as it should be, since we want $(\mathrm{Corr}(\mathbf{C})^{adm})^{1\text{-}\mathrm{Cat}} = \mathrm{Corr}(\mathbf{C})$). Now, $''\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}$ is defined as a 1-full subcategory in

$$\mathrm{Funct}(([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}),$$

where as 1-morphisms we allow natural transformations $\underline{\mathbf{c}} \to \underline{\mathbf{c}}'$ such that for every i, j, the corresponding map

$$\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$$

belongs to the class adm and is an isomorphism for $i = j$.

0.2.6. Here is the theorem that one can prove regarding the functor IndCoh (this is the combination of Theorems 3.2.2, 4.1.3 and 5.2.4), see Sect. 0.3.7:

Theorem 0.2.7. *There exists a* uniquely *defined functor*

$$\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all;all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}} \to \mathbf{1\text{-}Cat} \tag{0.5}$$

whose restriction along

$$\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}$$

is identified with the functor

$$\mathrm{IndCoh} : \mathrm{Sch}_{\mathrm{aft}} \to 1\text{-}\mathrm{Cat}.$$

The sought-for functor

$$\mathrm{IndCoh}^! : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}$$

is obtained from the functor $\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all;all}}$ of (0.5) by restriction along

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}.$$

0.3. What is done in this chapter?

0.3.1. In Sect. 1 we define the $(\infty, 2)$-category of correspondences.

The setting here is slightly more general than the one described in Sects. 0.1 and 0.2 (and this generalization is necessary for what we will develop in subsequent sections). Namely, in addition to adm we choose two more classes of morphisms in $\mathbf{C}$, denoted $vert$ and $horiz$, respectively, so that $adm \subset vert \cap horiz$.

We restrict the class of one 1-morphisms in $\mathrm{Corr}(\mathbf{C})^{adm}$ by only allowing diagrams

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathbf{c}_0 \\ \mathbf{c}_1 & \end{tikzcd}$$

where $\alpha \in horiz$ and $\beta \in vert$.

We denote the resulting $(\infty,2)$-category by $\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}$. It is endowed with a pair of functors

$$\mathbf{C}_{vert} \to \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \leftarrow (\mathbf{C}_{horiz})^{\mathrm{op}}.$$

We describe explicitly the simplicial categories

$$''\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz} = \mathrm{Seq}_{\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}),$$

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz} = \mathrm{Seq}^{\mathrm{Pair}}_{\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}, \mathbf{C}_{adm})$$

and

$$\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz} = \mathrm{Seq}^{\mathrm{Pair}}_{\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}, \mathbf{C}_{vert}).$$

For every $[n] \in \Delta$, each of these categories is a 1-full subcategory inside

$$\mathrm{Funct}(([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}).$$

All three have the same underlying space, denoted $\mathrm{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathbf{C})^{adm}_{vert,horiz}$. Its objects are *half-grids* of objects of $\mathbf{C}$, where each internal square is Cartesian, all vertical arrows belong to *vert* and all horizontal arrows belong to *horiz*.

0.3.2. To specify a functor

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S}, \tag{0.6}$$

where $\mathbb{S}$ is another $(\infty,2)$-category is equivalent to specifying a map of bi-simplicial spaces in any of the following versions:

$$\mathrm{Seq}_{\bullet}(''\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}),$$

$$\mathrm{Seq}_{\bullet}('\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$$

or

$$\mathrm{Seq}_{\bullet}(\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathbf{C})^{adm}_{vert,horiz}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}).$$

In Sect. 2 we give two additional formulations of what it takes to specify a functor as in (0.6). These formulations are used in the proofs of the various results establishing the existence and uniqueness of a functor out of a given category of correspondences with specified properties, discussed in the subsequent sections.

The first of these two formulations is given in terms of maps of bi-simplicial spaces

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}).$$

The main difference between $\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ and the space $\mathrm{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathbf{C})^{adm}_{vert,horiz}$ is that objects of $\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ are functors

$$[m] \times [n]^{\mathrm{op}} \to \mathbf{C}.$$

I.e., here we are dealing with full $(m \times n)$-grids rather than half-grids. This is in the spirit of the initial idea in Sect. 0.1.4 of how to functorially account for base change.

The second description involves a bi-simplicial *category*, denoted $\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$; we refer the reader to Sect. 2.2 for the precise statement.

0.3.3. We now ask the following question: how does one ever construct a functor out of a given category of correspondences $\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}$?

In Sect. 3 we state and prove Theorem 3.2.2 that gives an answer to this question (and, quite probably, any functor out of a category of correspondences ultimately comes down to the paradigm described in this theorem).

Namely, suppose we have a category $\mathbf{C}$, with *horiz* a class of 1-morphisms (with some reasonable properties) as well as a a functor

$$\Phi : \mathbf{C} \to \mathbb{S},$$

where $\mathbb{S}$ is some $(\infty, 2)$-category. Assume that this functor has the following *property*[2]: for every arrow $\mathbf{c} \xrightarrow{\alpha} \mathbf{c}'$ in *horiz*, the 1-morphism

$$\Phi(\mathbf{c}) \xrightarrow{\Phi(\alpha)} \Phi(\mathbf{c}')$$

admits a right adjoint. Moreover, assume that these right adjoints satisfy a base change *property* against the 1-morphisms

$$\Phi(\mathbf{d}) \xrightarrow{\Phi(\beta)} \Phi(\mathbf{d}'), \quad (\mathbf{d} \xrightarrow{\beta} \mathbf{d}') \in \mathbf{C}.$$

Consider the $(\infty, 2)$-category $\mathrm{Corr}(\mathbf{C})^{horiz}_{\mathrm{all},horiz}$. The statement of Theorem 3.2.2 is that there exists a uniquely defined functor

$$\Phi^{horiz}_{\mathrm{all},horiz} : \mathrm{Corr}(\mathbf{C})^{horiz}_{\mathrm{all},horiz} \to \mathbb{S},$$

whose composition with $\mathbf{C} \to \mathrm{Corr}(\mathbf{C})^{horiz}_{\mathrm{all},horiz}$ is identified with the initial functor Φ.

For example, if we start with the functor

$$\mathrm{IndCoh} : \mathrm{Sch}_{\mathrm{aft}} \to 1\text{-}\mathrm{Cat},$$

the above theorem allows us to (uniquely) extend it to a functor

$$\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all},\mathrm{proper}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all},\mathrm{proper}} \to \mathbf{1\text{-}Cat}$$

and to a functor

$$\mathrm{IndCoh}^{\mathrm{open}}_{\mathrm{all},\mathrm{open}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all},\mathrm{open}} \to (\mathbf{1\text{-}Cat})^{2\text{-}\mathrm{op}}.$$

0.3.4. In Sect. 4 we prove Theorem 4.1.3, which is the first out of the two basic theorems of this chapter[3] that say that starting from a functor from a given $(\infty, 2)$-category of correspondences, there is a canonical way to extend it to a larger one.

Let $adm' \supset adm$ be a larger class of morphisms, satisfying the following assumption: for any $\gamma : \mathbf{c} \to \mathbf{c}'$ from adm', the diagonal morphism

$$\mathbf{c} \to \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c}$$

belongs to *adm*.

In this case, Theorem 4.1.3 says that restriction under $\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathrm{Corr}(\mathbf{C})^{adm'}_{vert,horiz}$ defines a fully faithful embedding from the space of functors

$$\mathrm{Corr}(\mathbf{C})^{adm'}_{vert,horiz} \to \mathbb{S}$$

to that of functors

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S}.$$

[2]We emphasize that this is a property and *not* an additional piece of data.

[3]A few more theorems of this kind will be given in [Chapter V.2].

Moreover, we give an explicit description of the essential image of this fully faithful embedding.

0.3.5. Here are some typical applications of Theorem 4.1.3:

(i) Take $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ with $adm' = \mathrm{open}$, and let $adm = \mathrm{isom}$, so that

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{isom}}_{\mathrm{all,all}} = \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all,all}} = \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})$$

is the $(\infty,1)$-category of correspondences from Sect. 0.1.5. We obtain that the datum of the functor

$$\mathrm{IndCoh}^{\mathrm{open}}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all,all}} \to (\mathbf{1\text{-}Cat})^{2\text{-op}}$$

is equivalent to that of its restriction under $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all,all}}$,

$$\mathrm{IndCoh}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all,all}} \to \text{1-Cat}\,.$$

(i') Same as above, but we consider the pair

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all,open}} := \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{isom}}_{\mathrm{all,open}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all,open}}$$

and the functor

$$\mathrm{IndCoh}^{\mathrm{open}}_{\mathrm{all,open}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all,open}} \to (\mathbf{1\text{-}Cat})^{2\text{-op}}.$$

(ii) Take $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ with $adm' = \mathrm{proper}$, and let $adm = \mathrm{closed}$ be the class of closed embeddings. Then Theorem 4.1.3 implies that the datum of a functor

$$\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat},$$

can be uniquely recovered from its restriction under $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{closed}}_{\mathrm{all,all}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}}$,

$$\mathrm{IndCoh}^{\mathrm{closed}}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{closed}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat}.$$

(ii') The same applies to the case when $\mathbf{C} = {}^{\mathrm{cl}}\mathrm{Sch}$ and the functor in question is

$$\text{D-mod}^{\mathrm{proper}}_{\mathrm{all,all}} : \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat}.$$

(iii) We take $\mathbf{C} = {}^{\mathrm{cl}}\mathrm{Sch}$ with $adm' = \mathrm{closed}$, while $adm = \mathrm{isom}$, so that

$$\mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})^{\mathrm{isom}}_{\mathrm{all,all}} = \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})_{\mathrm{all,all}} = \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})$$

is the $(\infty,1)$-category of correspondences. We start with a functor

$$\text{D-mod}^{\mathrm{closed}}_{\mathrm{all,all}} : \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})^{\mathrm{closed}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat},$$

and we conclude that it can be uniquely recovered from its restriction under the functor $\mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})_{\mathrm{all,all}} \to \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})^{\mathrm{closed}}_{\mathrm{all,all}}$,

$$\text{D-mod}_{\mathrm{all,all}} : \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})_{\mathrm{all,all}} \to \text{1-Cat}\,.$$

Note that combining with (ii') and (iii), we obtain that the datum of the functor

$$\text{D-mod}^{\mathrm{proper}}_{\mathrm{all,all}} : \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat}$$

can be uniquely recovered from its restriction under $\mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})_{\mathrm{all,all}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}}$,

$$\text{D-mod}_{\mathrm{all,all}} : \mathrm{Corr}({}^{\mathrm{cl}}\mathrm{Sch})_{\mathrm{all,all}} \to \text{1-Cat}\,.$$

Remark 0.3.6. The point is that in (iii) we take fiber products in the category ${}^{cl}\mathrm{Sch}$, and in this category, for a closed embedding $S \to S'$, the map

$$S \to S \underset{S'}{\times} S$$

is an isomorphism (which is of course completely false in Sch). On the other hand, if we tried to consider IndCoh out of the category ${}^{cl}\mathrm{Sch}$, it would fail to satisfy base change.

0.3.7. In Sect. 5 we prove the second of the two extension results, Theorem 5.2.4. It is this theorem that allows to construct the functor

$$\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat},$$

starting from just

$$\mathrm{IndCoh} : \mathrm{Sch}_{\mathrm{aft}} \to 1\text{-}\mathrm{Cat}\,.$$

In this theorem, we start with four classes of morphisms *vert*, *horiz*, as well as *adm* and *co-adm* with some reasonable assumptions. The example one should keep in mind is when $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ with $vert = horiz = \mathrm{all}$, $adm = \mathrm{proper}$ and $co\text{-}adm = \mathrm{open}$.

We start with a functor

$$\Phi_{vert,co\text{-}adm} : \mathrm{Corr}(\mathbf{C})_{vert,co\text{-}adm} \to \mathbb{S}^{1\text{-}\mathrm{Cat}}, \tag{0.7}$$

and we wish to extend it to a functor

$$\Phi^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S}. \tag{0.8}$$

Theorem 5.2.4 says that under a certain assumption on *horiz*, *adm* and *co-adm*, restriction along $\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathrm{Corr}(\mathbf{C})_{vert,co\text{-}adm}$ defines a fully faithful map from the space of functors (0.8) to the space of functors (0.7), whose essential image is explicitly described.

The assumption on our classes of morphisms is that for a given 1-morphism $\mathbf{c}_0 \overset{\alpha}{\to} \mathbf{c}_1$, the category of its factorizations as

$$\mathbf{c}_0 \overset{\epsilon}{\to} \overline{\mathbf{c}}_0 \overset{\gamma}{\to} \mathbf{c}_1, \quad \epsilon \in co\text{-}adm,\ \gamma \in adm$$

is *contractible.*

In our main application, we start with the functor

$$\mathrm{IndCoh}^{\mathrm{open}}_{\mathrm{all,open}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all,open}} \to (\mathbf{1\text{-}Cat})^{2\text{-}\mathrm{op}}$$

(which is uniquely constructed starting from $\mathrm{IndCoh} : \mathrm{Sch}_{\mathrm{aft}} \to 1\text{-}\mathrm{Cat}$, see Sect. 0.3.3); we restrict it to a functor

$$\mathrm{IndCoh}_{\mathrm{all,open}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all,open}} \to 1\text{-}\mathrm{Cat}$$

(this restriction does not lose information, see Example (i') in Sect. 0.3.5 above); and finally apply Theorem 5.2.4 to extend $\mathrm{IndCoh}_{\mathrm{all,open}}$ to the desired functor

$$\mathrm{IndCoh}^{\mathrm{proper}}_{\mathrm{all,all}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathbf{1\text{-}Cat}.$$

0.3.8. We should remark that if one is only interested in constructing the functor

$$\mathrm{IndCoh}^! : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-Cat},$$

then one can make do with a vastly simplified version of Theorem 5.2.4.

Namely, starting from the functor (0.7) we can first restrict it to $\mathrm{Corr}(\mathbf{C})_{adm,co\text{-}adm}$, and then apply Theorem 4.1.3 to obtain a functor

$$\mathrm{Corr}(\mathbf{C})^{adm \cap co\text{-}adm}_{adm,co\text{-}adm} \to \mathbb{S}.$$

Then a much simplified version of Steps B and C of the proof will produce from the latter functor the desired functor

$$\Phi^! : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}.$$

We note, however, that both steps of the procedure just described lose information; whereas the one in Theorem 5.2.4 does not.

1. The 2-category of correspondences

In this section, given an $(\infty, 1)$-category $\mathbf{C}$ with three distinguished classes of 1-morphisms *vert*, *horiz* and *adm*, we will construct the corresponding $(\infty, 2)$-category of correspondences $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

1.1. **The set-up.** In this subsection we will list the requirements on the classes of morphisms *vert*, *horiz* and *adm*, and explain what the desired $(\infty, 2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ is when $\mathbf{C}$ is an *ordinary* category. In this case, $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ will be an ordinary 2-category, which can specified by saying what are its objects, 1-morphisms and 2-morphisms.

1.1.1. Let $\mathbf{C}$ be an $(\infty, 1)$-category. Let $vert, horiz$ be two classes of 1-morphisms in $\mathbf{C}$, and $adm \subset vert \cap horiz$ a third class, such that:

(1) The identity maps of objects of $\mathbf{C}$ belong to all three classes;
(2) If a 1-morphism belongs to a given class, then so do all isomorphic 1-morphisms;
(3) All three classes are closed under compositions.
(4) Given a morphism $\alpha_1 : \mathbf{c}_{1,1} \to \mathbf{c}_{1,0}$ in *horiz* and a morphism $\beta_0 : \mathbf{c}_{0,0} \to \mathbf{c}_{1,0}$ *vert*, the Cartesian square

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0} \end{tikzcd} \tag{1.1}$$

exists, and $\alpha_0 \in horiz$ and $\beta_1 \in vert$. Moreover, if α_1 (resp., β_0) belongs to *adm*, then so does $\alpha_0 \in horiz$ (resp., $\beta_1 \in vert$).
(5) The class *adm* satisfies the '2 out of 3' property: if

$$\mathbf{c}_1 \xrightarrow{\alpha} \mathbf{c}_2 \xrightarrow{\beta} \mathbf{c}_3$$

are maps with β and $\beta \circ \alpha$ in *adm*, then α is also in *adm*.

For example, if $\mathbf{C}$ contains fiber products for all morphisms, we can take $vert = horiz = adm$ to be the class of all 1-morphisms.

We let

$$\mathbf{C}_{vert}, \ \mathbf{C}_{horiz} \text{ and } \mathbf{C}_{adm}$$

denote the corresponding 1-full subcategories of $\mathbf{C}$.

1.1.2. Note the following consequences of the above conditions.

First, we claim that if $\mathbf{c} \to \mathbf{c}'$ is a 1-morphism that belongs to adm, then so does the diagonal morphism $\mathbf{c} \to \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c}$.

Second, the above observation implies that for a diagram

$$\begin{tikzcd}
 & \mathbf{c}_1 \arrow[d] \arrow[dr] & \\
\mathbf{c}_2 \arrow[r] \arrow[dr] & \mathbf{c}_3 \arrow[dr] & \mathbf{c}_1' \arrow[d] \\
 & \mathbf{c}_2' \arrow[r] & \mathbf{c}_3'
\end{tikzcd}$$

with the slanted maps in adm, vertical arrows in $vert$ and horizontal arrows in $horiz$, the resulting map

$$\mathbf{c}_1 \underset{\mathbf{c}_3}{\times} \mathbf{c}_2 \to \mathbf{c}_1' \underset{\mathbf{c}_3'}{\times} \mathbf{c}_2'$$

is in adm.

1.1.3. We wish to define a $(\infty, 2)$-category, denoted $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$. We want its objects be the same as objects of $\mathbf{C}$.

For $\mathbf{c}_0, \mathbf{c}_1 \in \mathrm{Corr}(\mathbf{C})$, we want the $(\infty, 1)$-category

$$\mathrm{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c}_0, \mathbf{c}_1)$$

to have as objects *correspondences*

$$\begin{tikzcd}
\mathbf{c}_{0,1} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathbf{c}_0 \\
\mathbf{c}_1 &
\end{tikzcd} \tag{1.2}$$

where $\alpha \in horiz$ and $\beta \in vert$.

For a pair of correspondences $(\mathbf{c}_{0,1}, \alpha, \beta)$ and $(\mathbf{c}_{0,1}', \alpha', \beta')$, we want the space of maps between them to be that of commutative diagrams

$$\begin{tikzcd}
\mathbf{c}_{0,1} \arrow[dr, "\gamma"] \arrow[drr, "\alpha'"] \arrow[ddr, "\beta'"'] & & \\
 & \mathbf{c}_{0,1}' \arrow[r, "\alpha"'] \arrow[d, "\beta"] & \mathbf{c}_0 \\
 & \mathbf{c}_1 &
\end{tikzcd} \tag{1.3}$$

with $\gamma \in adm$.

Composition of 1-morphisms should be given by forming Cartesian products:

$$\begin{tikzcd}
\mathbf{c}_{0,2} \arrow[r] \arrow[d] & \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\
\mathbf{c}_{1,2} \arrow[r] \arrow[d] & \mathbf{c}_1 & \\
\mathbf{c}_2. & &
\end{tikzcd}$$

1.1.4. It is easy to check that when $\mathbf{C}$ is an ordinary category, the above construction indeed gives rise to an (ordinary) 2-category.

To give the actual definition in the ∞-categorical framework, we shall use the formalism of [Chapter A.1, Sect. 2.1]

1.2. **The Segal category of correspondences.** In this subsection we will carry out the construction of the sought-for category $\operatorname{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

The construction will be very intuitive: when we think of the datum of an $(\infty, 2)$-category in terms of its image under the functor

$$\operatorname{Seq}_\bullet : 2\text{-Cat} \to 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}},$$

it is quite clear what the simplicial $(\infty, 1)$-category corresponding to $\operatorname{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ should be. Namely, for each n, the corresponding $(\infty, 1)$-category, denoted $''\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$, is one whose objects are half-grids of size n of objects of $\mathbf{C}$, in which every square is Cartesian.

This category $''\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$ of half-grids is easy to make sense of in the ∞-categorical setting, because it is obtained from a category of functors from a (very simple) ordinary category to $\mathbf{C}$.

1.2.1. Let $(\mathbf{C}, vert, horiz, adm)$ be as in Sect. 1.1.1. We will now define the desired object

$$''\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} \in 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Consider the following co-simplicial object of 1-Cat, which in fact takes values in ordinary categories. For each $n = 0, 1, 2, ...$ we consider the full subcategory

$$([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \subset [n] \times [n]^{\mathrm{op}},$$

spanned by objects (i, j) with $i \leq j$.

Consider the $(\infty, 1)$-category

$$\mathbf{Maps}\big(([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big).$$

I.e., this is the category of commutative diagrams $\underline{\mathbf{c}}$

$$
\begin{tikzcd}
\mathbf{c}_{0,n} \arrow[r] \arrow[d] & \mathbf{c}_{0,n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_{0,0} \\
\mathbf{c}_{1,n} \arrow[r] \arrow[d] & \mathbf{c}_{1,n-1} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \mathbf{c}_{1,1} & \\
\dots \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \dots & & \\
\mathbf{c}_{n-1,n} \arrow[r] \arrow[d] & \mathbf{c}_{n-1,n-1} & & & \\
\mathbf{c}_{n,n}. & & & &
\end{tikzcd}
\tag{1.4}
$$

1.2.2. For $n = 0, 1, 2, \dots$ we define an $(\infty, 1)$-category $''\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ to be the following 1-full subcategory of $\mathbf{Maps}\big(([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big)$.

The objects of $''\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ consist of those diagrams (1.4) that:

(1) All the vertical maps belong to *vert*;
(2) All the horizontal maps belong to *horiz*;
(3) All squares are Cartesian.

We restrict 1-morphisms to those maps $\underline{\mathbf{c}} \to \underline{\mathbf{c}}'$ between diagrams (1.4), for which the corresponding map

$$\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$$

belongs to *adm* for all $0 \leq i \leq j \leq n$ and is an *isomorphism* when $i = j$.

1.2.3. It is clear that for a map $[m] \to [n]$ in $\mathbf{\Delta}$, the restriction functor

$$\mathbf{Maps}\big(([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big) \to \mathbf{Maps}\big(([m] \times [m]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big)$$

sends

$$''\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} \to {''\mathbf{Grid}_m^{\geq \mathrm{dgnl}}}(\mathbf{C})_{vert;horiz}^{adm}.$$

Hence, we obtain a well-defined object

$$''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} \in 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Proposition 1.2.4. *The object* $''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} \in 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ *lies in the essential image of the functor* $\mathrm{Seq}_\bullet$.

Proof. The fact that 0-simplices of $''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ are a space follows from the definition. The fact that $''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ is a Segal category is also clear. Thus, we only have to check that every invertible 1-simplex is degenerate.

A 1-simplex in $''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ is given by a diagram

$$
\begin{tikzcd}
\mathbf{d} \arrow[r] \arrow[d] & \mathbf{c}_0 \\
\mathbf{c}_1, &
\end{tikzcd}
$$

and the fact that it is invertible means that this diagram can be completed to a diagram

$$\begin{tikzcd}
\mathbf{d}' \arrow[r] \arrow[d] & \mathbf{c}_1 \arrow[r] \arrow[d] & \mathbf{d}' \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathbf{c}_1 \\
\mathbf{c}_0 \arrow[r] \arrow[d] & \mathbf{d} \arrow[r] \arrow[d] & \mathbf{c}_0 & \\
\mathbf{d}' \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathbf{c}_1 & & \\
\mathbf{c}_0, & & &
\end{tikzcd}$$

with all the squares being Cartesian, and both composite maps $\mathbf{c}_0 \to \mathbf{c}_0$ (resp., $\mathbf{c}_1 \to \mathbf{c}_1$) are the identity maps. However, this implies that all the arrows in this diagram are isomorphisms.

□

1.2.5. We define the $(\infty,2)$-category

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \in 2\text{-Cat},$$

to be such that

$$\mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}) = {}''\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}.$$

The $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ can thus be recovered as

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \simeq \mathfrak{L}({}''\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}),$$

see [Chapter A.1, Sect. 4.4.1] where the functor $\mathfrak{L}$ is introduced.

Note that we have a canonical identification

$$(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz})^{1\text{-op}} \simeq \mathrm{Corr}(\mathbf{C})^{adm}_{horiz;vert}.$$

1.3. **Changing the class of 2-morphisms.** In this subsection we show that if we replace the class *adm* (which gives rise to 2-morphisms) by a smaller one, things work as they should.

In particular, we will see that the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ contains $\mathbf{C}_{vert}$ and $\mathbf{C}^{\mathrm{op}}_{horiz}$ as 1-full subcategories.

1.3.1. Let us now be given two classes adm' and adm as in Sect. 1.1.1 with $adm' \subset adm$. On the one hand, we can consider the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm'}_{vert;horiz}$. On the other hand, we can consider the 2-full subcategory

$$\mathrm{Corr}(\mathbf{C})^{adm' \subset adm}_{vert;horiz} \subset \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},$$

obtained by leaving by keeping objects and 1-morphisms the same, but restricting 2-morphisms to those diagrams (1.3), where $\gamma \in adm'$.

The tautological functor

$$\mathrm{Corr}(\mathbf{C})^{adm'}_{vert;horiz} \to \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

factors through a canonical functor

$$\mathrm{Corr}(\mathbf{C})^{adm'}_{vert;horiz} \to \mathrm{Corr}(\mathbf{C})^{adm' \subset adm}_{vert;horiz}. \tag{1.5}$$

The following is tautological:

Lemma 1.3.2. *The functor* (1.5) *is an equivalence.*

1.3.3. Let us consider a particular case of the above construction when $adm = \text{isom}$, i.e., is the class of all isomorphisms.

In this case, $''\mathbf{Grid}^{\geq \text{dgnl}}_\bullet(\mathbf{C})^{\text{isom}}_{vert;horiz}$ belongs to $\text{Spc}^{\mathbf{\Delta}^{\text{op}}}$. Therefore, $\text{Corr}(\mathbf{C})^{\text{isom}}_{vert;horiz}$ is an $(\infty,1)$-category. We shall denote it simply by $\text{Corr}(\mathbf{C})_{vert;horiz}$.

By Lemma 1.3.2, for any adm, we have

$$(\text{Corr}(\mathbf{C})^{adm}_{vert;horiz})^{\text{1-Cat}} \simeq \text{Corr}(\mathbf{C})_{vert;horiz}.$$

1.3.4. Let us now take both classes *horiz* and *adm* to be isom. In this case the corresponding $(\infty,1)$-category

$$\text{Corr}(\mathbf{C})_{vert;\text{isom}} = \text{Corr}(\mathbf{C})^{\text{isom}}_{vert;\text{isom}}$$

is canonically equivalent to $\mathbf{C}_{vert}$. Indeed, it is easy to see that for every n, the corresponding category $''\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{\text{isom}}_{vert;\text{isom}}$ is canonically equivalent to $\mathbf{Maps}([n],\mathbf{C}_{vert})^{\text{Spc}}$.

Similarly, the $(\infty,1)$-category

$$\text{Corr}(\mathbf{C})_{\text{isom};horiz} = \text{Corr}(\mathbf{C})^{\text{isom}}_{\text{isom},vert}$$

is canonically equivalent to $(\mathbf{C}_{horiz})^{\text{op}}$.

In particular, we obtain the functors:

$$\mathbf{C}_{vert} \simeq \text{Corr}(\mathbf{C})^{\text{isom}}_{vert;\text{isom}} \to \text{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

and

$$(\mathbf{C}_{horiz})^{\text{op}} \simeq \text{Corr}(\mathbf{C})^{\text{isom}}_{\text{isom},vert} \to \text{Corr}(\mathbf{C})^{adm}_{vert;horiz}.$$

1.4. **Distinguishing a class of 1-morphisms.** Recall from [Chapter A.1, Sect. 4.3.7] that in addition to the functor

$$\text{Seq}_\bullet : \text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\text{op}}}$$

there is also a functor $\text{Seq}^{\text{Pair}}_\bullet$ that takes as an input an $(\infty,2)$-category and a class of its 1-morphisms.

In this subsection we will describe the result applying $\text{Seq}^{\text{Pair}}_\bullet$ to $\text{Corr}(\mathbf{C})^{adm}_{vert;horiz}$, with respect to the following two 1-full subcategories

$$\mathbf{C}_{adm} \subset \mathbf{C}_{vert} \subset (\text{Corr}(\mathbf{C})^{adm}_{vert;horiz})^{\text{1-Cat}}.$$

As a result we will see two versions of the simplicial $(\infty,1)$-category $''\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$, denoted $'\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$ and $\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$, respectively.

Both these versions are useful in the applications.

1.4.1. For a natural number n, we define the $(\infty,1)$-categories

$$\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} \text{ and } {}'\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$$

to be 1-full subcategories $\mathbf{Maps}\big(([n]\times[n]^{\text{op}})^{\geq \text{dgnl}},\mathbf{C}\big)$ having the same objects as the category $''\mathbf{Grid}^{\geq \text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$, but a larger class of 1-morphisms:

For $\mathbf{Grid}^{\geq \text{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$ we allow those maps $\underline{\mathbf{c}} \to \underline{\mathbf{c}}'$ between diagrams (1.4), such that the corresponding maps

$$\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$$

belong to *vert* for all $0 \le i \le j \le n$, and such that for every $0 \le j-1 < j \le n$, the map

$$\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j} \underset{\mathbf{c}'_{i,j-1}}{\times} \mathbf{c}_{i,j-1}$$

belongs to *adm* (i.e., the defect of the corresponding square to be Cartesian is a 1-morphism from *adm*).

For $'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$ we allow those maps $\underline{\mathbf{c}} \to \underline{\mathbf{c}}'$ between diagrams (1.4), such that the corresponding maps

$$\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$$

belong to *adm* for all $0 \le i \le j \le n$.

Denote also

$$\mathrm{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} := \mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{\mathrm{isom}}_{vert;horiz} \simeq (\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}} \simeq$$
$$\simeq ('\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}} \simeq (''\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}}.$$

1.4.2. As in the case of $''\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}$, the restriction functor

$$\mathbf{Maps}\big(([n]\times[n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big) \to \mathbf{Maps}\big(([m]\times[m]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C}\big)$$

sends

$$\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} \to \mathbf{Grid}^{\geq \mathrm{dgnl}}_m(\mathbf{C})^{adm}_{vert;horiz}$$

and

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} \to {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_m(\mathbf{C})^{adm}_{vert;horiz}$$

Thus, we obtain well-defined objects

$$\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} \text{ and } {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$$

in $1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Similarly, we obtain an object

$$\mathrm{Grid}^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz} \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

The following results from the definitions:

Lemma 1.4.3. *The objects*

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} \text{ and } \mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$$

of $1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ *are both Segal categories.*

1.4.4. We now claim:

Proposition 1.4.5. *We have:*

(a) $\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} = \mathrm{Seq}^{\mathrm{Pair}}_\bullet(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}, \mathbf{C}_{vert})$.

(b) $'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} = \mathrm{Seq}^{\mathrm{Pair}}_\bullet(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}, \mathbf{C}_{adm})$.

Proof. We will give the proof for $\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$; the case of $'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$ is similar.

First, we claim that $\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}$ lies in the essential image of the functor $\mathrm{Seq}^{\mathrm{Pair}}_\bullet$, i.e., that it is a half-symmetric Segal category. We can check this at the level of ordinary Segal categories. Thus, we can replace

$$\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} \text{ by } (\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})^{1\text{-}\mathrm{ordn}},$$

while

$$(\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})^{1\text{-ordn}} \simeq \mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C}^{1\text{-ordn}})_{vert;horiz}^{adm}.$$

Now, for $\mathbf{C}^{1\text{-ordn}}$, its is easy to see that $'\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C}^{1\text{-ordn}})_{vert;horiz}^{adm}$ is equivalent to

$$\mathrm{Seq}_\bullet^{\mathrm{Pair}}(\mathrm{Corr}(\mathbf{C}^{1\text{-ordn}})_{vert;horiz}^{adm}, (\mathbf{C}^{1\text{-ordn}})_{adm}).$$

Since

$$''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} \simeq \mathfrak{r}(\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}),$$

we conclude that

$$\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} = \mathrm{Seq}_\bullet^{\mathrm{Pair}}(\mathrm{Corr}(\mathbf{C})_{vert;horiz}^{adm}, \mathbf{D}),$$

where $\mathbf{D}$ is a 1-full subcategory of $(\mathrm{Corr}(\mathbf{C})_{vert;horiz}^{adm})^{1\text{-Cat}}$, with the same class of objects.

It remains to show that $\mathbf{D} = \mathbf{C}_{vert}$. However, this also follows from the corresponding fact for the underlying ordinary categories.

□

Corollary 1.4.6. *We have canonical identifications*

$$\mathfrak{L}^{\mathrm{ext}}(\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}) \simeq \mathrm{Corr}(\mathbf{C})_{vert;horiz}^{adm},$$

and

$$\mathfrak{L}^{\mathrm{ext}}('\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}) \simeq \mathrm{Corr}(\mathbf{C})_{vert;horiz}^{adm},$$

where $\mathfrak{L}^{\mathrm{ext}}$ *is as in* [Chapter A.1, Sect. 4.4.1].

2. The category of correspondences via grids

According to the previous section, for an $(\infty,2)$-category $\mathbb{S}$ the data of a functor

$$\mathrm{Corr}(\mathbf{C})_{vert;horiz}^{adm} \to \mathbb{S} \tag{2.1}$$

amounts to any of the following:

(i) A map of bi-simplicial spaces $\mathrm{Seq}_\bullet(''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$;

(ii) A map of bi-simplicial spaces $\mathrm{Seq}_\bullet('\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$;

(iii) A map of bi-simplicial spaces $\mathrm{Seq}_\bullet(\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$.

In this section we will give yet two more interpretations of what it takes to define a functor (2.1).

One (given by Theorem 2.1.3) will still have the form of maps of from a certain bi-simplicial space to $\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$, but the flavor of this new bi-simplicial space will be different: instead of half-grids we will have $m \times n$-grids of objects of $\mathbf{C}$. The other one (given by Theorem 2.2.7) involves three-dimensional grids.

We should say right away that the contents of this section are pure combinatorics, i.e., manipulating diagrams. The reader may do well by absorbing the statements of the two main results, Theorems 2.1.3 and 2.2.7, and skipping the proofs on the first pass.

2.1. **The bi-simplicial space of grids with defect.** In this subsection we will introduce the bi-simplicial space of grids with defect and state Theorem 2.1.3.

2.1.1. Consider the following object

$$\text{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \in \text{Spc}^{\mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta}^{\text{op}}}.$$

Namely, the space $\text{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ is the full subspace in

$$\text{Maps}([m] \times [n]^{\text{op}}, \mathbf{C})$$

that consists of objects $\underline{\mathbf{c}}$ with the following properties:

(1) For any $0 \le i < i+1 \le m$, the map $\mathbf{c}_{i,j} \to \mathbf{c}_{i+1,j}$ belongs to *vert*;
(2) For any $0 \le j-1 < j \le n$, the map $\mathbf{c}_{i,j} \to \mathbf{c}_{i,j-1}$ belongs to *horiz*;
(3) For any $0 \le i < i+1 \le m$ and $0 \le j-1 < j \le n$ in the commutative square

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i,j-1} \arrow[d] \\ \mathbf{c}_{i+1,j} \arrow[r] & \mathbf{c}_{i+1,j-1}, \end{tikzcd} \tag{2.2}$$

its *defect of Cartesianness*, i.e., the map

$$\mathbf{c}_{i,j} \to \mathbf{c}_{i+1,j} \underset{\mathbf{c}_{i+1,j-1}}{\times} \mathbf{c}_{i,j-1},$$

belongs to *adm*.

So, the objects of $\text{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ are grids of objects of $\mathbf{C}$ of height m and width n, with vertical arrows in *vert*, horizontal arrows in *horiz*, and the defect of Cartesianness of each square in *adm*.

2.1.2. There exists a canonical map in $\text{Spc}^{\mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta}^{\text{op}}}$

$$\text{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \text{Sq}_{\bullet,\bullet}(\text{Corr}(\mathbf{C})^{adm}_{vert;horiz}), \tag{2.3}$$

explained in Sect. 2.3.

We will prove:

Theorem 2.1.3. *For* $\mathbb{S} \in 2\text{-Cat}$*, the map* (2.3) *defines an isomorphism between the space of functors* $\text{Maps}(\text{Corr}(\mathbf{C})^{adm}_{vert;horiz}, \mathbb{S})$ *and the subspace of maps in* $\text{Spc}^{\mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta}^{\text{op}}}$

$$\text{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \text{Sq}_{\bullet,\bullet}(\mathbb{S})$$

with the following property:
For every object in $\underline{\mathbf{c}} \in \text{defGrid}_{1,1}(\mathbf{C})^{adm}_{vert;horiz}$*, for which the diagram*

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_{0,0} \arrow[d] \\ \mathbf{c}_{1,1} \arrow[r] & \mathbf{c}_{1,0} \end{tikzcd} \tag{2.4}$$

is Cartesian*, the corresponding object*

$$\begin{tikzcd} s_{0,1} \arrow[d] \arrow[dr, Rightarrow] & s_{0,0} \arrow[l] \arrow[d] \\ s_{1,1} & s_{1,0} \arrow[l] \end{tikzcd} \tag{2.5}$$

in $\text{Sq}_{1,1}(\mathbb{S})$ *should represent an* invertible *2-morphism.*

Remark 2.1.4. When $\mathbb{S}$ is an ordinary category, the statement of Theorem 2.1.3 is easy to verify directly (and we recommend to anyone who wants to study its proof to do this exercise).

In general, the intuition behind Theorem 2.1.3 should also be rather clear: both sides describe the data of a pair of functors

$$\mathbf{C}_{vert} \to \mathbb{S} \text{ and } (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}$$

and an assignment to a every commutative square (2.4) in $\mathbf{C}$ (with vertical arrows in *vert*, horizontal arrows in *horiz* and the defect of Cartesianness in *adm*) of a datum of a natural transformation (2.5) in $\mathbb{S}$, such that if (2.4) is Cartesian, then the natural transformation in (2.5) is an isomorphism.

2.2. **The bi-simplicial category of grids.** In this subsection we will formulate Theorem 2.2.7 which gives yet another description of what it takes to define a functor out of the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

This description uses bi-simplicial $(\infty,1)$-*categories* (rather than spaces), and as a result also *tri-simplicial* spaces.

Having to deal with tri-simplicial spaces may appear as a grueling task, but unfortunately it seems that one has no choice: we will need Theorem 2.2.7 in order to prove Theorem 4.1.3, which is one of the key results of this chapter[4].

2.2.1. We consider the following object of $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$, denoted $\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$.

For $m,n=0,1,...$ we let

$$\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$$

be the 1-full subcategory of

$$\mathbf{Maps}([m]\times[n]^{\mathrm{op}},\mathbf{C}),$$

where we restrict objects to diagrams $\underline{\mathbf{c}}$ satisfying:

(1) For every $0\le i<i+1\le m$, the map $\mathbf{c}_{i,j}\to\mathbf{c}_{i+1,j}$ belongs to *vert*;
(2) For every $0\le j-1<j\le n$, the map $\mathbf{c}_{i,j}\to\mathbf{c}_{i,j-1}$ belongs to *horiz*;
(3) For every $0\le i<i+1\le m$ and $0\le j-1<j\le n$, the square

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i,j-1} \arrow[d] \\ \mathbf{c}_{i+1,j} \arrow[r] & \mathbf{c}_{i+1,j-1}, \end{tikzcd}$$

is Cartesian.

We restrict 1-morphisms to those maps of diagrams $\underline{\mathbf{c}}\to\underline{\mathbf{c}}'$ such that for every $0\le i\le m$ and $0\le j\le n$, the map

$$\mathbf{c}_{i,j}\to\mathbf{c}'_{i,j}$$

belongs to *vert*, and for every $0\le i\le m$ and $0\le j-1<j\le n$, the defect of Cartesianness of the square

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i,j-1} \arrow[d] \\ \mathbf{c}'_{i,j} \arrow[r] & \mathbf{c}'_{i,j-1} \end{tikzcd}$$

belongs to *adm*.

[4] As a dubious comfort to the reader, let us point out that the proof of Theorem 4.1.3 uses quadri-simplicial spaces, so tri-simplicial ones are not yet the worst.

Remark 2.2.2. Note that we impose no condition on the defect of the commutative diagrams

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i+1,j} \arrow[d] \\ \mathbf{c}'_{i,j} \arrow[r] & \mathbf{c}'_{i+1,j}. \end{tikzcd}$$

2.2.3. Consider also the following object of $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$, denoted $'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$.

For $m,n=0,1,...$ we let

$$'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$$

be the 1-full subcategory of $\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$, which has the same objects, but where we restrict 1-morphisms to those maps of diagrams $\underline{\mathbf{c}}\to\underline{\mathbf{c}}'$ such that for every $0\le i\le m$ and $0\le j\le n$, the map $\mathbf{c}_{i,j}\to\mathbf{c}'_{i,j}$ is in adm.

Denote also

$$\mathrm{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}\in\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}},$$

where

$$\mathrm{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}:=(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}}.$$

Remark 2.2.4. The difference between $\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ (resp., $'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$) and $\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$ is that the former has fewer objects, but we allow non-invertible morphisms.

2.2.5. Applying the functor $\mathrm{Seq}_\bullet:1\text{-Cat}\to\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ term-wise to

$$\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}\text{ and }{}'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$$

we obtain objects

$$\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})\text{ and }\mathrm{Seq}_\bullet({}'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})$$

in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$, so that the corresponding spaces of (l,m,n)-simplices are

$$\mathrm{Seq}_l(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz})\text{ and }\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}),$$

respectively.

2.2.6. Recall now (see [Chapter A.1, Sect. 4.6.1]) that to an $(\infty,2)$-category $\mathbb{S}$ we can canonically attach an object

$$\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S})\in\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}.$$

There are canonically defined maps

$$\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})\to\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}),\tag{2.6}$$

$$\mathrm{Seq}_\bullet({}'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})\to\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}),\tag{2.7}$$

explained in Sect. 2.4 below.

We will prove:

Theorem 2.2.7.
(a) *For an* $(\infty,2)$*-category* $\mathbb{S}$*, the map* (2.6) *induces an isomorphism from*

$$\mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbb{S})$$

to the subspace of maps in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$

$$\mathrm{Maps}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S})),$$

consisting of those maps, for which for every $(0,1,1)$*-simplex in* $\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})$*, the corresponding* $(0,1,1)$*-simplex in* $\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S})$*, thought of as a* $(1,1)$*-simplex in* $\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$*, represents an* invertible *2-morphism in* $\mathbb{S}$*.*
(b) *Ditto for* $'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$ *instead of* $\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}$.

Remark 2.2.8. The content of Remark 2.1.4 applies also to Theorem 2.2.7: it is rather clear why this kind of statement should be true (convince yourself for $\mathbb{S}$ ordinary).

2.3. **Construction of the map-I.** In this subsection we will construct the map (2.3).

The reader who prefers to take Theorem 2.1.3 on faith, may choose to skip this subsection (or better tinker with the relevant objects and invent what the map (2.3) should be).

2.3.1. The map (2.3) is constructed as a composition of a map

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}\to\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}),\tag{2.8}$$

followed by the map

$$\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz})\simeq\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbf{C}_{vert})\to\mathrm{Sq}_{\bullet,\bullet}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}),$$

where the first arrow is the isomorphism of Proposition 1.4.5(a).

2.3.2. In its turn, the map (2.8) is constructed as follows. For every n, let

$$\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}$$

be the 1-full subcategory of $\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})$, where we restrict objects to those n-strings $\underline{\mathbf{c}}$, for which for every $0\leq j-1<j\leq n$, the corresponding map $\mathbf{c}_j\to\mathbf{c}_{j-1}$ belongs to *horiz*, and where we restrict 1-morphisms to those maps $\underline{\mathbf{c}}\to\underline{\mathbf{c}}'$ that satisfy:

1. For every $0\leq j\leq n$, the map $\mathbf{c}_j\to\mathbf{c}'_j$ belongs to *vert*;
2. For every $0\leq j-1<j\leq n$, the defect of Cartesianness of the square

$$\begin{tikzcd}
\mathbf{c}_j \arrow[r] \arrow[d] & \mathbf{c}_{j-1} \arrow[d] \\
\mathbf{c}'_j \arrow[r] & \mathbf{c}'_{j-1}
\end{tikzcd}$$

belongs to *adm*.

It is easy to see that

$$\mathrm{defGrid}_{\bullet,n}(\mathbf{C})^{adm}_{vert;horiz}\simeq\mathrm{Seq}_\bullet(\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}),$$

as simplicial spaces.

2.3.3. Now, the sought-for map (2.8) comes from a canonically defined functor

$$\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}\to\mathbf{Grid}^{\geq\mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}.\tag{2.9}$$

In fact, the above functor is a fully faithful embedding, whose image consists of those half-grids, in which the vertical maps are isomorphisms.

2.3.4. In what follows, we will use the following notation. Let $\mathbf{I}$ and $\mathbf{J}$ be $(\infty,1)$-categories. Let

$$\mathrm{Maps}(\mathbf{I}\times\mathbf{J}^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}\subset\mathrm{Maps}(\mathbf{I}\times\mathbf{J}^{\mathrm{op}},\mathbf{C})$$

be the subspace consisting of those functors such that:

(1) for every morphism $(i_0\to i_1)\in\mathbf{I}$ and every object $j\in\mathbf{J}$, the image of the morphism $(i_0,j)\to(i_1,j)$ lies in $vert$;
(2) for every object $i\in\mathbf{I}$ and every morphism $(j_0\to j_1)\in\mathbf{J}$, the image of the morphism $(i,j_0)\to(i,j_1)$ lies in $horiz$; and
(3) for every pair of morphisms $(i_0\to i_1)\in\mathbf{I}$ and $(j_0\to j_1)\in\mathbf{J}$, the defect of Cartesianness of the resulting diagram in $\mathbf{C}$

$$\begin{tikzcd}\mathbf{c}_{i_0,j_1} \arrow[r] \arrow[d] & \mathbf{c}_{i_0,j_0} \arrow[d] \\ \mathbf{c}_{i_1,j_1} \arrow[r] & \mathbf{c}_{i_1,j_0}\end{tikzcd} \tag{2.10}$$

lies in adm.

2.3.5. We claim that the datum of a map of bi-simplicial spaces

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}\to\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}) \tag{2.11}$$

gives rise to a map

$$\mathrm{Maps}(\mathbf{I}\times\mathbf{J}^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}\to\mathrm{Maps}(\mathbf{I}\circledast\mathbf{J},\mathbb{S}) \tag{2.12}$$

that behaves functorially in $\mathbf{I}$ and $\mathbf{J}$.

In the above formula, $\circledast$ is the Gray tensor product, see [Chapter A.1, Sect. 3.2].

Moreover, if the map (2.11) satisfies the additional condition of Theorem 2.1.3, then the map (2.12) has the property that it sends every square (2.10) that is Cartesian to a diagram in $\mathbb{S}$ representing an invertible 2-morphism.

Indeed, each side in (2.12) is the limit over the index category

$$([m]\to\mathbf{I})\times([n]\to\mathbf{J})$$

of terms equal to

$$\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}\text{ and }\mathrm{Sq}_{m,n}(\mathbb{S}),$$

respectively.

2.4. **Construction of the map-II.** In this subsection we will carry out the construction of the map (2.6). The case of (2.7) is similar.

The reader who prefers to take Theorem 2.2.7 on faith may choose to skip this subsection.

2.4.1. Recall (see [Chapter A.1, Proposition 3.2.9]) that for an $(\infty,2)$-category $\mathbb{S}$ and n, there exists a canonical monomorphism of bi-simplicial spaces

$$\mathrm{Sq}_{\bullet,\bullet}(\mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{S}))\to\mathrm{Cu}_{\bullet,\bullet,n}(\mathbb{S}), \tag{2.13}$$

where $\mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{S})\in 1\text{-Cat}$ is regarded as an $(\infty,2)$-category. Explicitly,

$$\mathrm{Sq}_{l,m}(\mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{S}))=\mathrm{Maps}(([l]\times[m])\circledast[n],\mathbb{S})\text{ and }\mathrm{Cu}_{\bullet,\bullet,n}(\mathbb{S})=\mathrm{Maps}([l]\circledast[m]\circledast[n],\mathbb{S}),$$

and the map (2.13) comes from

$$[l]\circledast[m]\circledast[n]\simeq([l]\circledast[m])\circledast[n]\to([l]\times[m])\circledast[n].$$

2.4.2. Recall the category

$$\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}$$

(see Sect. 2.3.2).

The functor (2.6) will be defined as the composition of a map

$$\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,n}(\mathbf{C})^{adm}_{vert;horiz}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}) \tag{2.14}$$

(where $\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C}) \in$ 1-Cat is regarded as an $(\infty,2)$-category), followed by

$$\begin{aligned}\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Maps}([n]^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz}) \overset{\text{Proposition 1.4.5}}{\simeq} \\ \simeq \mathrm{Sq}_{\bullet,\bullet}(\mathrm{Seq}_n^{\mathrm{Pair}}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbf{C}_{vert})) \to \mathrm{Sq}_{\bullet,\bullet}(\mathrm{Seq}_n^{\mathrm{ext}}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz})) \to \\ \to \mathrm{Cu}_{\bullet,\bullet,n}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}),\end{aligned}$$

where the first arrow is induced by the functor (2.9) and the last arrow from (2.13).

2.4.3. To define (2.14), let us write both sides out explicitly. The space of (l,m)-simplices in the right-hand side is the full subspace of

$$\mathrm{Maps}([l]\times[m]\times[n]^{\mathrm{op}},\mathbf{C}),$$

consisting of diagrams $\underline{\mathbf{c}}$ satisfying the following conditions:

(1) For every i,j and k the map $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i,j,k-1}$ belongs to *horiz*;
(2) For every i,j and k the map $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i,j+1,k}$ belongs to *vert*;
(3) For every i,j and k the map $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i+1,j,k}$ belongs to *vert*;
(4) For every i,j and k, the defect of Cartesianness of the diagram

$$\begin{tikzcd} \mathbf{c}_{i,j,k} \arrow[r] \arrow[d] & \mathbf{c}_{i,j,k-1} \arrow[d] \\ \mathbf{c}_{i+1,j,k} \arrow[r] & \mathbf{c}_{i+1,j,k-1} \end{tikzcd}$$

belongs to *adm*.

(5) For every i,j and k, the defect of Cartesianness of the diagram

$$\begin{tikzcd} \mathbf{c}_{i,j,k} \arrow[r] \arrow[d] & \mathbf{c}_{i,j,k-1} \arrow[d] \\ \mathbf{c}_{i,j+1,k} \arrow[r] & \mathbf{c}_{i,j+1,k-1} \end{tikzcd}$$

belongs to *adm*.

Now, the left-hand side in (2.14) is the subspace of the space of diagrams as above, where we strengthen condition (5) as follows:

We require that the square

$$\begin{tikzcd} \mathbf{c}_{i,j,k} \arrow[r] \arrow[d] & \mathbf{c}_{i,j,k-1} \arrow[d] \\ \mathbf{c}_{i,j+1,k} \arrow[r] & \mathbf{c}_{i,j+1,k-1} \end{tikzcd}$$

be Cartesian.

2.4.4. One can view the resulting map

$$\mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}, \mathbb{S}) \to \mathrm{Maps}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}), \mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))$$

in Theorem 2.2.7 (which we have just defined by completing the construction of (2.6)) also as follows.

It is the composition of the map

$$\mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}, \mathbb{S}) \to \mathrm{Maps}(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}, \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))$$

in Theorem 2.1.3, followed by a map

$$\begin{aligned} \text{(2.15)} \quad & \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}, \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})) \to \\ & \to \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}), \mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S})), \end{aligned}$$

constructed as follows.

2.4.5. Given a map

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}), \tag{2.16}$$

we need to construct maps

$$\mathrm{Seq}_l(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}) \to \mathrm{Cu}_{l,m,n}(\mathbb{S}) \tag{2.17}$$

that depend functorially on $[l], [m], [n] \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$.

By Sect. 2.3.5, a map in (2.16) gives rise to a map

$$\mathrm{Maps}(([l] \times [m]) \times [n]^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Maps}(([l] \times [m]) \circledast [n]), \mathbb{S}),$$

while

$$\mathrm{Seq}_l(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}) \subset \mathrm{Maps}(([l] \times [m]) \times [n]^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz}$$

and

$$\mathrm{Maps}(([l] \times [m]) \circledast [n]), \mathbb{S}) \subset \mathrm{Cu}_{l,m,n}(\mathbb{S}).$$

Now, we obtain the desired map in (2.17) as the composite:

$$\begin{aligned} & \mathrm{Seq}_l(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}) \to \mathrm{Maps}(([l] \times [m]) \times [n]^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \to \\ & \to \mathrm{Maps}(([l] \times [m]) \circledast [n]), \mathbb{S}) \to \mathrm{Cu}_{l,m,n}(\mathbb{S}). \end{aligned}$$

2.5. **Proof of Theorems 2.1.3 and 2.2.7: initial remarks.** We shall only consider point (a) of Theorem 2.2.7, point (b) being similar.

The proof will consist of constructing the inverse maps. Its key idea (which is completely straightforward once you understand what should map where) is given in Sect. 2.5.4.

2.5.1. It it is easy to see that the essential image of the map in Theorem 2.1.3 belongs to

$$\mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))\subset \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})),$$

where $\mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))$ is the subspace, singled out by the condition in Theorem 2.1.3.

Furthermore, the map in Theorem 2.2.7 is the composition of the above map

$$\mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbb{S})\to \mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))$$

and a map

$$\begin{aligned}(2.18)\quad &\mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))\to\\ &\to \mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S})),\end{aligned}$$

where

$$\begin{aligned}&\mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\subset\\ &\subset \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\end{aligned}$$

is the subspace singled out by the condition in Theorem 2.2.7, and the map (2.18) is induced by (2.15).

2.5.2. We will construct a map

$$\begin{aligned}(2.19)\quad &\mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\to\\ &\to \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S})),\end{aligned}$$

where

$$\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}))\simeq \mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbb{S})$$

by Corollary 1.4.6.

It will be a tedious but straightforward verification that each of the three compositions

$$\mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbb{S})\to \mathrm{Maps}(\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz},\mathbb{S}),$$

$$\mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))\to \mathrm{Maps}^0(\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))$$

and

$$\begin{aligned}&\mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\to\\ &\to \mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\end{aligned}$$

is canonically isomorphic to the identity map. We will leave this verification to the reader.

2.5.3. Note that the left-hand side in (2.19) is a subspace in the space of maps of tri-simplicial spaces, from $\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})$ to $\mathrm{Maps}([\bullet]\circledast([\bullet]\times[\bullet]),\mathbb{S})$, where the latter assigns to i,j,k the space

$$\mathrm{Maps}([i]\circledast([j]\times[k]),\mathbb{S}).$$

By definition, the right-hand side in (2.19) is the space of maps of bi-simplicial spaces, from $\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz})$ to $\mathrm{Maps}([\bullet]\circledast[\bullet],\mathbb{S})$, where the latter assigns to m,n the space

$$\mathrm{Maps}([m]\circledast[n],\mathbb{S}).$$

Note that the diagonal map $[n]\to([n]\times[n])^{\geq\mathrm{dgnl}}$ defines a map of bi-simplicial spaces

$$\mathrm{Maps}([\bullet]\circledast([\bullet]\overset{=}{\times}[\bullet])^{\geq\mathrm{dgnl}},\mathbb{S})\to \mathrm{Maps}([\bullet]\circledast[\bullet],\mathbb{S}),$$

where the former bi-simplicial space assigns to m,n the space

$$\mathrm{Maps}([m]\circledast([n]\times[n])^{\geq \mathrm{dgnl}},\mathbb{S}).$$

We will construct a map

$$\begin{aligned}(2.20)\quad &\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Maps}([\bullet]\circledast([\bullet]\times[\bullet]),\mathbb{S}))\to\\ &\to\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Maps}([\bullet]\circledast([\bullet]\overset{=}{\times}[\bullet])^{\geq\mathrm{dgnl}},\mathbb{S})).\end{aligned}$$

Then, by composing from (2.20) we obtain a map

$$\begin{aligned}&\mathrm{Maps}^0(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}))\hookrightarrow\\ &\to\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Maps}([\bullet]\circledast([\bullet]\times[\bullet]),\mathbb{S}))\overset{(2.20)}{\to}\\ &\to\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Maps}([\bullet]\circledast([\bullet]\overset{=}{\times}[\bullet])^{\geq\mathrm{dgnl}},\mathbb{S}))\to\\ &\to\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz}),\mathrm{Maps}([\bullet]\circledast[\bullet],\mathbb{S}))=\\ &=\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathbf{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S})).\end{aligned}$$

giving rise to the desired map (2.19).

2.5.4. *The key idea.* Let us explain where the map (2.20) will come from. For simplicity, let us consider the corresponding map

$$\begin{aligned}(2.21)\quad &\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Maps}([\bullet]\times[\bullet],\mathbb{S}))\to\\ &\to\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Grid}^{\geq\mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz},\mathrm{Maps}(([\bullet]\overset{=}{\times}[\bullet])^{\geq\mathrm{dgnl}},\mathbb{S})),\end{aligned}$$

where we recall that

$$\mathrm{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}:=(\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}}$$

and

$$\mathrm{Grid}^{\geq\mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz}:=(\mathbf{Grid}^{\geq\mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})^{\mathrm{Spc}}.$$

The idea behind the existence of the map (2.21) is the following motto 'if we know how to map grids of objects in $\mathbf{C}$ to diagrams on $\mathbb{S}$, then we can extend this to half-grids'.

We will turn this motto into an actual construction in the next few subsections.

2.6. **Digression: clusters.** By a *cluster* we mean a category that 'looks like' $[m]\times[n]$ or $([n]\times[n])^{\geq\mathrm{dgnl}}$. This class of categories will come handy for the construction of the map (2.20).

2.6.1. Let $\mathbf{Q}$ be an $(\infty,1)$-category, equipped with a pair of 1-full subcategories $\mathbf{Q}_{vert}$ and $\mathbf{Q}_{horiz}$. To it we associate a bi-simplicial space $\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})$ as follows.

For every m,n, the space $\mathrm{Sq}_{m,n}(\mathbf{Q}_{vert,horiz})$ is a subspace of $\mathrm{Sq}_{m,n}(\mathbf{Q})$ consisting of objects such that for every $[1]\times[0]\to[m]\times[n]$ (resp., $[0]\times[1]\to[m]\times[n]$) the resulting object of $\mathrm{Sq}_{0,1}(\mathbf{Q})$ (resp., $\mathrm{Sq}_{1,0}(\mathbf{Q})$) belongs to *vert* (resp., *horiz*).

For any $(\infty,1)$-category $\mathbf{D}$ we have a tautologically defined map

$$(2.22)\qquad \mathrm{Maps}(\mathbf{Q},\mathbf{D})\to\mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),\mathrm{Sq}_{\bullet,\bullet}(\mathbf{D}))$$

The map (2.22) does not have to be an isomorphism in general. Note, however, that if (2.22) is an isomorphism for any $\mathbf{D}$, then the triple $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ is uniquely determined by the bi-simplicial space $\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})$.

Below we will describe a class of triples $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ for which (2.22) is an isomorphism.

2.6.2. Let Q be a convex subset of $\{0,...,m\}\times\{0,...,n\}$ for some m and n. To such Q we attach a triple $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ by letting $\mathbf{Q}$ be the full subcategory of $[m]\times[n]$ spanned by Q, and $\mathbf{Q}_{vert}$ (resp., $\mathbf{Q}_{horiz}$) be given by vertical (resp., horizontal) arrows (where we are thinking of the first coordinate as the vertical direction, and the second coordinate as the horizontal diction).

By a *cluster* we shall mean a triple $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ which is equivalent to one coming from a convex subset $Q\subset\{0,...,m\}\times\{0,...,n\}$ as above.

In Sect. 2.8 we will prove:

Proposition 2.6.3. *If* $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ *is a cluster, then the map* (2.22) *is an isomorphism for any* $\mathbf{D}$.

2.6.4. For a bi-simplicial space B, let $B^{\text{horiz-op}}$ be the bi-simplicial space obtained by applying the involution $\text{rev}:\mathbf{\Delta}\to\mathbf{\Delta}$ along the second copy of $\mathbf{\Delta}^{\text{op}}$ in $\mathbf{\Delta}^{\text{op}}\times\mathbf{\Delta}^{\text{op}}$.

Note that if $(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})$ is a cluster, there exists a canonically defined cluster

$$(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})^{\text{horiz-op}},$$

characterized by the property that

$$\text{Sq}_{\bullet,\bullet}((\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz})^{\text{horiz-op}})=(\text{Sq}_{\bullet,\bullet}(\mathbf{Q},\mathbf{Q}_{vert},\mathbf{Q}_{horiz}))^{\text{horiz-op}}.$$

We let $Q(n)$ denote the cluster given by the subset

$$(\{0,...n\}\times\{0,...,n\})^{\geq\text{dgnl}}\subset\{0,...n\}\times\{0,...,n\}.$$

Note that the underlying category is $([n]\times[n])^{\geq\text{dgnl}}$.

Consider the cluster $Q(n)^{\text{horiz-op}}$. Note that its underlying category is $([n]\times[n]^{\text{op}})^{\geq\text{dgnl}}$.

2.7. **Proof of Theorems 2.1.3 and 2.2.7: continuation.** We recall that we need to construct the map (2.20). The idea is that we can construct something a lot more general: namely, a map from the left-hand side in (2.20) to

$$\text{Maps}_{\text{Spc}^{\mathbf{\Delta}^{\text{op}}}}(\text{Seq}_\bullet(\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz}),\text{Maps}([\bullet]\circledast\mathbf{Q},\mathbb{S}))$$

for any cluster Q (see below for the notation $\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz}$).

Such a map is more or less tautological, modulo Proposition 2.6.3.

2.7.1. Let Q be a cluster with the underlying category $\mathbf{Q}$ and the attached bi-simplicial space $\text{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})$. We will also consider the cluster $Q^{\text{horiz-op}}$ and the corresponding category, denoted $\mathbf{Q}^{\text{horiz-op}}$.

Analogous to the definition of the category

$$\mathbf{Grid}^{\geq\text{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz},$$

we have the category $\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz}$, which is a 1-full subcategory in $\mathbf{Maps}(\mathbf{Q}^{\text{horiz-op}},\mathbf{C})$.

We will construct maps

$$\text{Maps}_{\text{Spc}^{\mathbf{\Delta}^{\text{op}}\times\mathbf{\Delta}^{\text{op}}\times\mathbf{\Delta}^{\text{op}}}}(\text{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}),\text{Maps}([\bullet]\circledast([\bullet]\times[\bullet]),\mathbb{S}))\to \tag{2.23}$$
$$\to\text{Maps}(\text{Seq}_m(\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz}),\text{Maps}([m]\circledast\mathbf{Q},\mathbb{S})),$$

functorial in Q and $[m]\in\mathbf{\Delta}^{\text{op}}$. By taking $Q=Q(n)$, we will obtain the desired map (2.20).

2.7.2. Consider the tri-simplicial space

$$\mathrm{Seq}_\bullet([m]) \boxtimes \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),$$

where $\boxtimes$ denotes the product operation

$$\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}} \times \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}.$$

We have a naturally defined map from the space

$$\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}), \mathrm{Maps}([\bullet] \circledast ([\bullet] \times [\bullet]), \mathbb{S}))$$

to the space of maps

$$\begin{multline}\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet([m]) \boxtimes \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})) \to \\ \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet([m]) \boxtimes \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([\bullet] \circledast ([\bullet] \times [\bullet]), \mathbb{S})).\end{multline} \tag{2.24}$$

We will how that the left-hand side in (2.24) identifies canonically with

$$\mathrm{Seq}_m(\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz})$$

and the right-hand side with

$$\mathrm{Maps}([m] \circledast \mathbf{Q}, \mathbb{S}),$$

as required.

2.7.3. We shall first analyze the right-hand side in (2.24). We have:

$$\begin{multline}\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet([m]) \boxtimes \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([\bullet] \circledast ([\bullet] \times [\bullet]), \mathbb{S})) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([m] \circledast ([\bullet] \times [\bullet]), \mathbb{S})) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([\bullet] \times [\bullet], \mathrm{Funct}([m], \mathbb{S})_{\text{lef-lax}}) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Sq}_{\bullet,\bullet}(\mathrm{Funct}([m], \mathbb{S})_{\text{left-lax}})),\end{multline}$$

which, using Proposition 2.6.3, we identify with

$$\mathrm{Maps}(\mathbf{Q}, \mathrm{Sq}_{\bullet,\bullet}(\mathrm{Funct}([m], \mathbb{S})_{\text{left-lax}})) \simeq \mathrm{Maps}([m] \circledast \mathbf{Q}, \mathbb{S}),$$

as required.

In the above formula, the notation $\mathrm{Funct}(-,-)_{\text{left-lax}}$ is stands for the $(\infty, 2)$-category, whose objects are functors, but whose 1-morphisms are *left-lax* natural transformations, see [Chapter A.1, Sect. 3.2.7].

2.7.4. Let us now analyze the left-hand side in (2.24). We have:

$$\begin{multline}\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Seq}_\bullet([m]) \boxtimes \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Seq}_\bullet(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Seq}_m(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz})).\end{multline}$$

Note that the latter expression is a subspace in

$$\begin{multline}\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([m] \times ([\bullet] \times [\bullet]^{\mathrm{op}}), \mathbf{C})) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}), \mathrm{Maps}([\bullet] \times [\bullet]^{\mathrm{op}}, \mathbf{Maps}([m], \mathbf{C}))) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})^{\text{horiz-op}}, \mathrm{Maps}([\bullet] \times [\bullet], \mathbf{Maps}([m], \mathbf{C}))) \simeq \\ \simeq \mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})^{\text{horiz-op}}, \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Maps}([m], \mathbf{C}))),\end{multline}$$

which, using Proposition 2.6.3, we identify

$$\mathrm{Maps}(\mathbf{Q}^{\text{horiz-op}}, \mathbf{Maps}([m], \mathbf{C})) \simeq \mathrm{Maps}([m] \times \mathbf{Q}^{\text{horiz-op}}, \mathbf{C}).$$

Similarly, $\mathrm{Seq}_m(\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz})$ is a subspace in

$$\mathrm{Maps}([m]\times\mathbf{Q}^{\text{horiz-op}},\mathbf{C}).$$

Now, under the identification obtained in this way, we have:

$$\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),\mathrm{Maps}([m]\times([\bullet]\times[\bullet]^{\mathrm{op}}),\mathbf{C}))\simeq\mathrm{Maps}([m]\times\mathbf{Q}^{\text{horiz-op}},\mathbf{C}),$$

the subspaces

$$\mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),\mathrm{Seq}_m(\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}))\subset \mathrm{Maps}_{\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),\mathrm{Maps}([m]\times([\bullet]\times[\bullet]^{\mathrm{op}}),\mathbf{C}))$$

and

$$\mathrm{Seq}_m(\mathbf{Q}^{\text{horiz-op}}(\mathbf{C})^{adm}_{vert;horiz})\subset\mathrm{Maps}([m]\times\mathbf{Q}^{\text{horiz-op}},\mathbf{C})$$

correspond to one another, as required.

2.8. **Double Segal spaces and the proof of Proposition 2.6.3.** The idea of the proof is that any cluster can be assembled from pieces for which the assertion is manifestly true.

2.8.1. By a double Segal space we shall mean a bi-simplicial space B such that for every fixed m (resp., n), the simplicial space $B_{m,\bullet}$ (resp., $B_{\bullet,n}$) is a Segal space.

By [Chapter A.1, Sect. 4.1.7], the essential image of the functor

$$\mathrm{Sq}_{\bullet,\bullet}:2\text{-Cat}\to\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$$

is a double Segal space.

2.8.2. Let Q be a cluster, realized as a convex subset of some $\{0,...,m\}\times\{0,...,n\}$. For a horizontal line

$$m_1\times\{0,...,n\}\subset\{0,...,m\}\times\{0,...,n\}$$

let $Q^{\leq m_1}$, $Q^{\geq m_1}$ and $Q^{=m_1}$ be the parts of Q that lie below, above or on that line, respectively.

Consider the corresponding bi-simplicial spaces

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz})\ \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\geq m_1}_{vert,horiz}),\ \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{=m_1}_{vert,horiz})\text{ and }\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})$$

and the categories

$$\mathbf{Q}^{\leq m_1},\ \mathbf{Q}^{\geq m_1},\ \mathbf{Q}^{=m_1}\text{ and }\mathbf{Q}.$$

We have the natural maps

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz})\underset{\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz})}{\sqcup}\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz})\to\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})\tag{2.25}$$

and

$$\mathbf{Q}^{\leq m_1}\underset{\mathbf{Q}^{=m_1}}{\sqcup}\mathbf{Q}^{\geq m_1}\to\mathbf{Q}\tag{2.26}$$

We will prove:

Lemma 2.8.3.

(a) *The map*

$$\mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}),B)\to \to\mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz}),B)\underset{\mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{=m_1}_{vert,horiz}),B)}{\times}\mathrm{Maps}(\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\geq m_1}_{vert,horiz}),B),$$

induced by (2.25)*, is an isomorphism whenever B is a double Segal space.*

(b) *The map* (2.26) *is an isomorphism in* 1-Cat.

2.8.4. Let us show how this lemma implies Proposition 2.6.3.

First, by symmetry, we have an analog of Lemma 2.8.3 when instead of horizontal lines we consider vertical ones. This reduces the verification of Proposition 2.6.3 to the following four cases: (i) $Q = \{0\} \times \{0\}$, (ii) $Q = \{0,1\} \times \{0\}$, (iii) $Q = \{0\} \times \{0,1\}$ and (iv) $Q = \{0,1\} \times \{0,1\}$.

In each of these cases, the assertion of Proposition 2.6.3 is manifest.

2.8.5. *Proof of Lemma 2.8.3.* We will prove point (a) of the lemma; point (b) is similar but simpler.

We have

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}) = \underset{[i]\to[m],[j]\to[n],\mathrm{Im}([i]\times[j])\subset Q}{\mathrm{colim}} \mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j]).$$

Cofinal in the above index category is the full subcategory, denoted E, that consists of those maps for which one of the following three scenarios happens:

(1) The image of $\{0,...,i\}$ in $\{0,...,m\}$ is $< m_1$;
(2) The image of $\{0,...,i\}$ in $\{0,...,m\}$ is $> m_1$;
(3) The element $m_1 \in \{0,...,m\}$ has a *unique* preimage in $\{0,...,i\}$.

For an object

$$([i] \to [m], [j] \to [n], \mathrm{Im}([i] \times [j]) \subset Q) =: e \in E$$

consider the fiber product

$$B_e := (\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz}) \underset{\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{=m_1}_{vert,horiz})}{\sqcup} \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\geq m_1}_{vert,horiz})) \underset{\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz})}{\times} (\mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j])),$$

taken in the category $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$.

We have:

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}_{vert,horiz}) \simeq \underset{e\in E}{\mathrm{colim}}\, \mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j]),$$

and since fiber products in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ commute with colimits, we also have

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\leq m_1}_{vert,horiz}) \underset{\mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{=m_1}_{vert,horiz})}{\sqcup} \mathrm{Sq}_{\bullet,\bullet}(\mathbf{Q}^{\geq m_1}_{vert,horiz}) \simeq \underset{e\in E}{\mathrm{colim}}\, B_e.$$

It remains to show that for every $e \in E$ and a double Segal space B, the map

$$B_e \to \mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j])$$

induces an isomorphism

$$B_{i,j} = \mathrm{Maps}(\mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j]), B) \to \mathrm{Maps}(B_e, B). \tag{2.27}$$

However, this follows from the fact that B_e has the form

(1) $\mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j])$;
(2) $\mathrm{Seq}_\bullet([i]) \boxtimes \mathrm{Seq}_\bullet([j])$;
(3) $(\mathrm{Seq}_\bullet([i_1]) \boxtimes \mathrm{Seq}_\bullet([j])) \underset{\mathrm{Seq}_\bullet([0])\boxtimes\mathrm{Seq}_\bullet([j])}{\sqcup} (\mathrm{Seq}_\bullet([i_2])) \boxtimes \mathrm{Seq}_\bullet([j]))$ for $[i_1] \underset{[0]}{\sqcup} [i_2] \simeq [i]$

in each of the three scenarios above. In the first two cases, the map (2.27) is an isomorphism for any bi-simplicial category B. In the third case, it follows from the definition of double Segal spaces:

$$B_{i,j} \to B_{i_1,j} \underset{B_{0,j}}{\times} B_{i_2,j}$$

is an isomorphism.

$\square$

3. The universal property of the category of correspondences

One of the main themes of this book is the construction of functors out of given $(\infty,2)$-category of correspondences. How does one construct such a functor?

In turns out that there is one case, where a datum of a functor

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S}$$

is equivalent to a datum of a functor $\Phi : \mathbf{C}_{vert} \to (\mathbb{S})^{1\text{-Cat}}$, having a particular property.

We emphasize that this a property, and not an additional piece of data. Moreover, this property essentially occurs at the level of the underlying ordinary 2-categories. It is called the *left Beck-Chevalley condition*, and it says that for every 1-morphism α in $\mathbf{C}_{horiz}$, the corresponding 1-morphism $\Phi(\alpha)$ in $\mathbb{S}$ *admits a right adjoint*, and that these right adjoints satisfy *base change* against $\Phi(\beta)$ for $\beta \in \mathbf{C}_{vert}$.

All other instances of a functor out of a $(\infty,2)$-category of correspondences, considered in this book, will be obtained from this case, by various extension procedures, considered in the subsequent sections in this chapter and the next.

3.1. **The Beck-Chevalley conditions.** In this subsection we will give the definition of the left and right Beck-Chevalley conditions.

3.1.1. Assume that in the context of Sect. 1.1.1, we have $horiz \subset vert$, and $adm = horiz$. Let $\mathbb{S}$ be an $(\infty,2)$-category, and let

$$\Phi : \mathbf{C}_{vert} \to \mathbb{S}$$

be a functor.

Definition 3.1.2. *We shall say that Φ satisfies the* left *Beck-Chevalley condition with respect to horiz, if for every 1-morphism $\alpha : \mathbf{c} \to \mathbf{c}'$ with $\alpha \in horiz$, the corresponding 1-morphism*

$$\Phi(\alpha) : \Phi(\mathbf{c}) \to \Phi(\mathbf{c}')$$

admits a right adjoint, to be denoted $\Phi^!(\alpha)$, such that for every Cartesian diagram

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0} \end{tikzcd} \tag{3.1}$$

with $\alpha_0, \alpha_1 \in horiz$ and $\beta_0, \beta_1 \in vert$, the 2-morphism

$$\Phi(\beta_1) \circ \Phi^!(\alpha_0) \to \Phi^!(\alpha_1) \circ \Phi(\beta_0),$$

arising by adjunction from the isomorphism

$$\Phi(\alpha_1) \circ \Phi(\beta_1) \simeq \Phi(\beta_0) \circ \Phi(\alpha_0),$$

is an isomorphism.

In particular, from the existence of the right adjoints $\Phi^!(\alpha)$, we obtain a well-defined functor

$$\Phi^! : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S},$$

see [Chapter A.3, Sect. 1.3].

Remark 3.1.3. Note that a functor $\Phi : \mathbf{C}_{vert} \to \mathbb{S}$ satisfies the (left) Beck-Chevalley condition if and only if its composition with $\mathbb{S} \to \mathbb{S}^{2\text{-ordn}}$ does. So, the Beck-Chevalley condition is something that can be checked at the level of ordinary 2-categories.

3.1.4. Let us now assume that $vert \subset horiz$ and $adm = vert$. Let $\mathbb{S}$ be a 2-category, and let

$$\Phi^! : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}$$

be a functor.

Definition 3.1.5. *We shall say that $\Phi^!$ satisfies the* right *Beck-Chevalley condition with respect to vert, if for every 1-morphism $\beta : \mathbf{c} \to \mathbf{c}'$ with $\beta \in vert$, the corresponding 1-morphism*

$$\Phi^!(\beta) : \Phi^!(\mathbf{c}') \to \Phi^!(\mathbf{c})$$

admits a left adjoint, to be denoted $\Phi(\beta)$, such that for every Cartesian diagram

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0} \end{tikzcd} \tag{3.2}$$

with $\alpha_0, \alpha_1 \in horiz$ and $\beta_0, \beta_1 \in vert$, the 2-morphism

$$\Phi(\beta_1) \circ \Phi^!(\alpha_0) \to \Phi^!(\alpha_1) \circ \Phi(\beta_0), \tag{3.3}$$

arising by adjunction from the isomorphism

$$\Phi^!(\alpha_0) \circ \Phi^!(\beta_0) \simeq \Phi^!(\beta_1) \circ \Phi^!(\alpha_1),$$

is an isomorphism.

In particular, if $\Phi^!$ satisfies the right Beck-Chevalley condition with respect to *vert*, we obtain a well-defined functor

$$\Phi : \mathbf{C}_{vert} \to \mathbb{S},$$

see [Chapter A.3, Sect. 1.3].

3.2. **Statement of the universal property.** In this subsection we state the main result of this section: it describes functors out of a $(\infty, 2)$-category of correspondences in terms of a *1-categorical datum.*

3.2.1. The goal of this section is to prove the following theorem:

Theorem 3.2.2.
(a) *Suppose that $horiz \subset vert$ and $adm = horiz$ satisfies condition (5) from Sect. 1.1.1. Then restriction along $\mathbf{C}_{vert} \to \mathrm{Corr}(\mathbf{C})^{horiz}_{vert;horiz}$ defines an equivalence between the space of functors*

$$\Phi^{horiz}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{horiz}_{vert;horiz} \to \mathbb{S}$$

and the subspace of functors

$$\Phi : \mathbf{C}_{vert} \to \mathbb{S}$$

that satisfy the left Beck-Chevalley condition with respect to horiz. For $\Phi^{horiz}_{vert;horiz}$ as above, the resulting functor $\Phi^! := \Phi^{horiz}_{vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}}$ is obtained from $\Phi|_{\mathbf{C}_{horiz}}$ by passing to right adjoints.
(b) *Suppose that $vert \subset horiz$ and $adm = vert$ satisfies condition (5) from Sect. 1.1.1. Then restriction along $(\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz}$ defines an equivalence between the space of functors*

$$\Phi^{vert}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz} \to \mathbb{S}$$

and the subspace of functors

$$\Phi^! : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}$$

that satisfy the right Beck-Chevalley condition with respect to vert. For $\Phi^{vert}_{vert;horiz}$ as above, the resulting functor $\Phi := \Phi^{vert}_{vert;horiz}|_{\mathbf{C}_{vert}}$ is obtained from $\Phi^!|_{(\mathbf{C}_{vert})^{\mathrm{op}}}$ by passing to left adjoints.

The rest of this section is devoted to the proof of Theorem 3.2.2. By symmetry, it suffices to treat case (b) of the theorem.

3.2.3. We will first establish the easy direction. Namely, we will start with a functor

$$\Phi^{vert}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz} \to \mathbb{S},$$

and we will show that its restriction

$$\Phi^! := \Phi^{vert}_{vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}} : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}$$

satisfies the left Beck-Chevalley condition and that $\Phi := \Phi^{vert}_{vert;horiz}|_{\mathbf{C}_{vert}}$ is obtained from $\Phi^!|_{(\mathbf{C}_{vert})^{\mathrm{op}}}$ by passing to left adjoints.

Remark 3.2.4. We note, however, that this step is logically unnecessary: in Sect. 3.3 we will be able to establish the desired isomorphism directly.

3.2.5. Note that according to Remark 3.1.3, we can assume that $\mathbb{S}$ is an ordinary 2-category. Since

$$(\mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz})^{2\text{-ordn}} \simeq \mathrm{Corr}(\mathbf{C}^{1\text{-ordn}})^{vert}_{vert;horiz},$$

we can assume that $\mathbf{C}$ is an ordinary 1-category. Hence, we can use a hands-on description of the 2-category $\mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz}$ given in Sect. 1.1.3.

Furthermore, it suffices to consider the universal case, namely when $\mathbb{S} = \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz}$ and $\Phi^!$ is the tautological inclusion

$$(\mathbf{C}_{\mathrm{horiz}})^{\mathrm{op}} \to \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz}.$$

3.2.6. Given $\beta : \mathbf{c} \to \mathbf{c}'$ in *vert*, we need to show that the corresponding 1-morphism $\mathbf{c}' \to \mathbf{c}$ in $\mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz}$, i.e.,

$$\begin{tikzcd} \mathbf{c} \arrow[r, "\beta"] \arrow[d, "\mathrm{id}_{\mathbf{c}}"'] & \mathbf{c}' \\ \mathbf{c}, \end{tikzcd} \tag{3.4}$$

admits a left adjoint. We claim that the left adjoint in question is given by the diagram

$$\begin{tikzcd} \mathbf{c} \arrow[r, "\mathrm{id}_{\mathbf{c}}"] \arrow[d, "\beta"'] & \mathbf{c} \\ \mathbf{c}'. \end{tikzcd} \tag{3.5}$$

Let us construct the corresponding unit and co-unit of the adjunction.

The composition (3.4)$\circ$(3.5) is given by the diagram

$$\begin{tikzcd} \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c} \arrow[r] \arrow[d] & \mathbf{c} \\ \mathbf{c} \end{tikzcd}$$

The unit of the adjunction is given by the diagram:

$$\begin{tikzcd}
\mathbf{c} \arrow[dr] \arrow[drr, "\mathrm{id}_{\mathbf{c}}"] \arrow[ddr, "\mathrm{id}_{\mathbf{c}}"'] & & \\
& \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c} \arrow[r] \arrow[d] & \mathbf{c} \\
& \mathbf{c}. &
\end{tikzcd}$$

The composition (3.5)∘(3.4) is given by the diagram

$$\begin{tikzcd}
\mathbf{c} \arrow[r, "\beta"] \arrow[d, "\beta"'] & \mathbf{c}' \\
\mathbf{c}'. &
\end{tikzcd}$$

The co-unit of the adjunction is given by the diagram:

$$\begin{tikzcd}
\mathbf{c} \arrow[dr, "\beta"'] \arrow[drr, "\beta"] \arrow[ddr, "\beta"'] & & \\
& \mathbf{c}' \arrow[r, "\mathrm{id}_{\mathbf{c}'}"'] \arrow[d, "\mathrm{id}_{\mathbf{c}'}"] & \mathbf{c}' \\
& \mathbf{c}'. &
\end{tikzcd}$$

The fact that unit and co-unit maps thus constructed satisfy the adjunction identities is a straightforward verification.

3.3. **Construction of a functor out of the category of correspondences.** In this subsection we will prove Theorem 3.2.2.

If $\mathbb{S}$ is an ordinary 2-category, the description of the adjoints given in Sect. 3.2.6 above gives an elementary proof.

However, to give a proof in the ∞-categorical setting, we need a more explicit description of the notion of adjoint functor. Such a description is furnished by [Chapter A.3, Theorem 1.2.4].

3.3.1. Let $\Phi^! : (\mathbf{C}_{horiz})^{op} \to \mathbb{S}$ be a functor, such that for every 1-morphism $\beta : \mathbf{c} \to \mathbf{c}'$ with $\beta \in vert$, the corresponding 1-morphism

$$\Phi^!(\beta) : \Phi^!(\mathbf{c}') \to \Phi^!(\mathbf{c})$$

admits a left adjoint.

According to [Chapter A.3, Theorem 1.2.4], the datum of such $\Phi^!$ is equivalent to the datum of a map of bi-simplicial spaces

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}((\mathbf{C}_{horiz})^{\mathrm{op}}, (\mathbf{C}_{vert})^{\mathrm{op}})^{\text{vert-op}} \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}).$$

By construction, for a commutative square

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0}, \end{tikzcd} \tag{3.6}$$

the resulting 2-morphism

$$\Phi(\beta_1) \circ \Phi^!(\alpha_0) \to \Phi^!(\alpha_1) \circ \Phi(\beta_0)$$

is obtained by adjunction from the tautological isomorphism

$$\Phi^!(\alpha_0) \circ \Phi^!(\beta_0) \simeq \Phi^!(\beta_1) \circ \Phi^!(\alpha_1).$$

3.3.2. According to Theorem 2.1.3, the datum of a functor $\Phi^{vert}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{vert}_{vert;horiz} \to \mathbb{S}$ is equivalent to the datum of a map of bi-simplicial spaces

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{vert}_{vert;horiz} \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$$

such that for every object in $\underline{\mathbf{c}} \in \mathrm{defGrid}_{1,1}(\mathbf{C})^{vert}_{vert;horiz}$ corresponding to a diagram (3.6) that is Cartesian, the 2-morphism in the corresponding object in $\mathrm{Sq}_{1,1}(\mathbb{S})$ is an isomorphism.

3.3.3. Note, however, that we have a natural monomorphism of bi-simplicial spaces

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{vert}_{vert;horiz} \to \mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}((\mathbf{C}_{horiz})^{\mathrm{op}}, (\mathbf{C}_{vert})^{\mathrm{op}})^{\text{vert-op}} \tag{3.7}$$

Therefore, the assertion of Theorem 3.2.2(b) manifestly follows from:

Proposition 3.3.4. *The map* (3.7) *is an isomorphism of bi-simplicial spaces.*

Proof. We need to show that for any commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ y_1 \arrow[r] & y_2 \end{tikzcd}$$

with vertical maps in *vert* and horizontal maps in *horiz*, the corresponding map

$$x_1 \to x_2 \underset{y_2}{\times} y_1$$

lies in *vert*. However, this map is given by the composite

$$x_1 \to x_2 \underset{y_2}{\times} x_1 \to x_2 \underset{y_2}{\times} y_1$$

where the first map is a base change of $x_2 \to x_2 \underset{y_2}{\times} x_2$ and the second is a base change of $x_1 \to y_1$, both of which lie in *vert*, by assumption. □

4. Enlarging the class of 2-morphisms at no cost

In this section we will prove the first of the two results of the type that given a functor from one $(\infty, 2)$-category of correspondences, we can canonically extend it to a functor from another $(\infty, 2)$-category of correspondences that has a larger class of 2-morphisms.

4.1. **The setting.** In this subsection we explain the setting for the main result of this section, Theorem 4.1.3.

4.1.1. Let $(\mathbf{C}, vert, horiz, adm)$ be as in Sect. 1.1.1. Let $(\mathbf{C}, vert, horiz, adm')$ be another such data with

$$adm \subset adm'.$$

We shall also assume that the following condition holds:

For a 1-morphism

$$\gamma' : \mathbf{c} \to \mathbf{c}'$$

with $\gamma \in adm'$, the diagonal map

$$\mathbf{c} \to \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c}$$

belongs to adm.

4.1.2. Let $\Phi^{adm}_{vert;horiz}$ be a functor

$$\Phi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S}.$$

For a morphism $\gamma : \mathbf{c}_0 \to \mathbf{c}_1$ in adm', consider the commutative (but not necessarily) Cartesian) square

$$\begin{CD}
\mathbf{c} @>{\mathrm{id}}>> \mathbf{c} \\
@V{\mathrm{id}}VV @VV{\gamma}V \\
\mathbf{c} @>{\gamma}>> \mathbf{c}'
\end{CD}$$

which gives rise to a (not necessarily invertible) 2-morphism

$$\mathrm{id} \to \Phi^!(\gamma) \circ \Phi(\gamma).$$

We impose the condition that the above 2-morphism define the unit of an adjunction.

Under the above circumstances, we claim:

Theorem 4.1.3. *The functor $\Phi^{adm}_{vert;horiz}$ admits a unique extension to a functor*

$$\Phi^{adm'}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm'}_{vert;horiz} \to \mathbb{S}.$$

The rest of this section is devoted to the proof of this theorem.

Remark 4.1.4. The proof of Theorem 4.1.3 is essentially combinatorics, i.e., playing with various diagrams. The reader may find it useful to first prove Theorem 4.1.3 in the case when $\mathbb{S}$ is an ordinary 2-category; in this case, this is simple exercise.

4.1.5. Starting from $\Phi^{adm}_{vert;horiz}$, we shall construct the data of the functor $\Phi^{adm'}_{vert;horiz}$ as a map of tri-simplicial spaces

$$\mathrm{Seq}_\bullet({}'\mathbf{Grid}_{\bullet,\bullet}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Cu}_{\bullet,\bullet,\bullet}(\mathbb{S}), \tag{4.1}$$

satisfying the additional condition of Theorem 2.2.7. I.e., we need to construct a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}([l] \circledast [m] \circledast [n], \mathbb{S}), \tag{4.2}$$

functorial in

$$[l] \times [m] \times [n] \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}.$$

The fact that this extension is uniquely defined will follow from the construction.

4.2. **Idea of the construction.**

4.2.1. As was said above, starting from the data of $\Phi^{adm}_{vert;horiz}$, we need to construct the map (4.2). The problem that we will have to confront already occurs when $l = 1$, $m = 0$ and $n = 1$. I.e., given a square

$$\begin{tikzcd} \mathbf{c}^0_1 \arrow[r, "\alpha_0"] \arrow[d, "\gamma_1"'] & \mathbf{c}^0_0 \arrow[d, "\gamma_0"] \\ \mathbf{c}^1_1 \arrow[r, "\alpha_1"] & \mathbf{c}^1_0 \end{tikzcd}$$

and $\alpha_0, \alpha_1 \in horiz$ and $\gamma_0, \gamma_1 \in adm'$, whose defect of Cartesianness belongs to adm', we want to construct a diagram

(4.3)
$$\begin{tikzcd} \Phi(\mathbf{c}^0_1) \arrow[d, "\Phi(\gamma_1)"'] \arrow[dr, Rightarrow] & \Phi(\mathbf{c}^0_0) \arrow[l, "\Phi^!(\alpha_0)"'] \arrow[d, "\Phi(\gamma_0)"] \\ \Phi(\mathbf{c}^1_1) & \Phi(\mathbf{c}^1_0) \arrow[l, "\Phi^!(\alpha_1)"] \end{tikzcd}$$

The problem is that the data of $\Phi^{adm}_{vert;horiz}$ does not produce such diagrams: according to Theorem 2.1.3, we can a priori only construct such diagrams for squares in which the defect of Cartesianness belongs to adm.

4.2.2. The trick is the following. Consider the 3-dimensional diagram

(4.4)
$$\begin{tikzcd} & \mathbf{c}^{0,0}_1 := \mathbf{c}^0_1 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{c}^{0,0}_0 := \mathbf{c}^0_0 \arrow[dl] \arrow[dd] \\ \mathbf{c}^{0,1}_1 := \mathbf{c}^0_0 \underset{\mathbf{c}^1_0}{\times} \mathbf{c}^1_1 \arrow[rr] \arrow[dd] & & \mathbf{c}^{0,1}_0 := \mathbf{c}^0_0 \arrow[dd] & \\ & \mathbf{c}^{1,0}_1 := \mathbf{c}^0_1 \arrow[rr] \arrow[dl] & & \mathbf{c}^{1,0}_0 := \mathbf{c}^0_0 \arrow[dl] \\ \mathbf{c}^{1,1}_1 := \mathbf{c}^1_1 \arrow[rr] & & \mathbf{c}^{1,1}_0 := \mathbf{c}^1_0 & \end{tikzcd}$$

We will construct the diagram (4.3) by first constructing the diagram

(4.5)
$$\begin{tikzcd}
& \Phi(\mathbf{c}_1^{0,0}) \arrow[dl, "\Phi"'] \arrow[dd, "\Phi"] & & \Phi(\mathbf{c}_0^{0,0}) \arrow[ll, "\Phi^!"'] \arrow[dl, "\Phi"] \arrow[dd, "\Phi"] \\
\Phi(\mathbf{c}_1^{0,1}) \arrow[dd, "\Phi"] & & \Phi(\mathbf{c}_0^{0,1}) \arrow[ll, "\Phi^!"', crossing over] & \\
& \Phi(\mathbf{c}_1^{1,0}) \arrow[dl, "\Phi"'] & & \Phi(\mathbf{c}_0^{1,0}) \arrow[ll, "\Phi^!"] \arrow[dl, "\Phi"] \\
\Phi(\mathbf{c}_1^{1,1}) & & \Phi(\mathbf{c}_0^{1,1}) \arrow[ll, "\Phi^!"'] \arrow[from=uu, "\Phi"', crossing over] &
\end{tikzcd}$$

(with appropriate 2-morphisms), and then restricting to the diagonal

$$\begin{tikzcd}
\Phi(\mathbf{c}_1^{0,0}) \arrow[d] & \Phi(\mathbf{c}_0^{0,0}) \arrow[l] \arrow[d] \\
\Phi(\mathbf{c}_1^{1,1}) & \Phi(\mathbf{c}_0^{1,1}). \arrow[l]
\end{tikzcd}$$

4.2.3. We will obtain the diagram (4.5) from the diagram

(4.6)
$$\begin{tikzcd}
& \Phi(\mathbf{c}_1^{0,0}) \arrow[dd, "\Phi"] & & \Phi(\mathbf{c}_0^{0,0}) \arrow[ll, "\Phi^!"'] \arrow[dd, "\Phi"] \\
\Phi(\mathbf{c}_1^{0,1}) \arrow[ur, "\Phi^!"] \arrow[dd, "\Phi"] & & \Phi(\mathbf{c}_0^{0,1}) \arrow[ll, "\Phi^!"', crossing over] \arrow[ur, "\Phi^!"] & \\
& \Phi(\mathbf{c}_1^{1,0}) & & \Phi(\mathbf{c}_0^{1,0}) \arrow[ll, "\Phi^!"] \\
\Phi(\mathbf{c}_1^{1,1}) \arrow[ur, "\Phi^!"] & & \Phi(\mathbf{c}_0^{1,1}) \arrow[ll, "\Phi^!"'] \arrow[ur, "\Phi^!"'] \arrow[from=uu, "\Phi"', crossing over] &
\end{tikzcd}$$

by passing to right adjoints along the slanted arrows.

4.2.4. Now, the point is that the data of the latter diagram, i.e., diagram (4.6), *is* contained in the datum of $\Phi^{adm}_{vert;horiz}$ in its guise as a map of bi-simplicial spaces

$$\text{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \text{Sq}_{\bullet,\bullet}(\mathbb{S}). \tag{4.7}$$

Namely, let us recall from Sect. 2.3.5, that the data of a map (4.7) assigns to a functor $\mathbf{I} \times \mathbf{J}^{\text{op}} \to \mathbf{C}$ (that sends arrows along $\mathbf{I}$ to *vert*, arrows along $\mathbf{J}^{\text{op}}$ to *horiz* and where defect of

Cartesianness of squares belongs to adm) a functor

$$\mathbf{I} \circledast \mathbf{J} \to \mathbb{S}.$$

We take $\mathbf{I} = [1]$ and $\mathbf{J} = [1] \times [1]$, and we take the functor

$$\mathbf{I} \times \mathbf{J}^{\mathrm{op}} \to \mathbf{C}$$

to be given by the diagram (4.4), where $\mathbf{I}$ corresponds to the first upper index (i.e., the "k" in $\mathbf{c}_j^{k,l}$). In other words, $\mathbf{I}$ is the direction depicted as vertical in the diagram (4.4).

The key observation is that that the condition on $adm \subset adm'$ from Sect. 4.1.1 implies that the defect of the Cartesianness of the relevant squares in (4.4) (i.e., the squares where one side is vertical), does belong to adm.

The resulting functor

$$[1] \circledast ([1] \times [1]) \to \mathbb{S}$$

exactly produces the desired diagram (4.6).

4.3. **Proof of Theorem 4.1.3, the key construction.** In this subsection we will formally implement the idea explained above.

The map (4.2) will be constructed as a composition of several maps.

4.3.1. Recall the notation

$$\mathrm{Maps}(\mathbf{I} \times \mathbf{J}^{op}, \mathbf{C})^{adm}_{vert;horiz} \subset \mathrm{Maps}(\mathbf{I} \times \mathbf{J}^{\mathrm{op}}, \mathbf{C})$$

from Sect. 2.3.4.

As a first step in constructing the map (4.2), we will produce a functor

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}(([l] \times [m]) \times ([n] \times [l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz}. \tag{4.8}$$

The functor (4.8) is given by the following explicit procedure. To an object of the space $\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz})$, given by

$$\mathbf{c}^{l'}_{m',n'}, \quad 0 \le l' \le l, \ 0 \le m' \le m, \ 0 \le n' \le n$$

we assign a map

$$[l] \times [m] \times [n]^{\mathrm{op}} \times [l] \to \mathbf{C} \tag{4.9}$$

that sends

$$(k', m', n', l') \mapsto \mathbf{c}^{k',l'}_{m',n'} := \begin{cases} \mathbf{c}^{k'}_{m',0} \underset{\mathbf{c}^{l'}_{m,0}}{\times} \mathbf{c}^{l'}_{m,n'} \text{ for } k' \le l'; \\ \mathbf{c}^{k'}_{m',0} \underset{\mathbf{c}^{k'}_{m,0}}{\times} \mathbf{c}^{l'}_{m,n'} \text{ for } k' \ge l'. \end{cases}$$

It is easy to see that the map (4.9) thus constructed above has the following properties:

(1) For fixed k', l', each square

$$\begin{tikzcd} \mathbf{c}^{k',l'}_{m',n'} \arrow[r] \arrow[d] & \mathbf{c}^{k',l'}_{m',n'-1} \arrow[d] \\ \mathbf{c}^{k',l'}_{m'+1,n'} \arrow[r] & \mathbf{c}^{k',l'}_{m'+1,n'-1} \end{tikzcd}$$

is Cartesian.

(2) For fixed n' and k', the square

$$\begin{tikzcd} \mathbf{c}^{k',l'}_{m',n'} \arrow[r] \arrow[d] & \mathbf{c}^{k',l'}_{m'+1,n'} \arrow[d] \\ \mathbf{c}^{k',l'+1}_{m',n'} \arrow[r] & \mathbf{c}^{k',l'+1}_{m'+1,n'} \end{tikzcd}$$

is Cartesian.

(3) For fixed m' and l', the square

$$\begin{tikzcd} \mathbf{c}^{k',l'}_{m',n'} \arrow[r] \arrow[d] & \mathbf{c}^{k',l'}_{m',n'-1} \arrow[d] \\ \mathbf{c}^{k'+1,l'}_{m',n'} \arrow[r] & \mathbf{c}^{k'+1,l'}_{m',n'-1} \end{tikzcd}$$

is Cartesian.

(4) For fixed m' and n', the defect of Cartesianness of the square

$$\begin{tikzcd} \mathbf{c}^{k',l'}_{m',n'} \arrow[r] \arrow[d] & \mathbf{c}^{k',l'+1}_{m',n'} \arrow[d] \\ \mathbf{c}^{k'+1,l'}_{m',n'} \arrow[r] & \mathbf{c}^{k'+1,l'+1}_{m',n'} \end{tikzcd}$$

belongs to adm.

(5) For fixed k', m', n', the map $\mathbf{c}^{k',l'}_{m',n'} \to \mathbf{c}^{k',l'+1}_{m',n'}$ belongs to adm'.

(6) For fixed l', m', n', the map $\mathbf{c}^{k',l'}_{m',n'} \to \mathbf{c}^{k+1',l'}_{m',n'}$ belongs to adm'.

In particular, we obtain that the map (4.9) indeed belongs to

$$\mathrm{Maps}(([l] \times [m]) \times ([n] \times [l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz}.$$

Remark 4.3.2. For an object $\underline{\mathbf{c}} \in \mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz})$, given by

$$[l] \times [m] \times [n]^{op} \to \mathbf{C},$$

the corresponding map (4.9) is uniquely characterized by properties (2) and (3) above and the following.

(1) The composite map with the diagonal

$$[l] \times [m] \times [n]^{op} \to [l] \times [m] \times [n]^{op} \times [l] \to \mathbf{C}$$

is isomorphic to $\underline{\mathbf{c}}$;

(2) For $k' \le l'$ and for all m', the map

$$\mathbf{c}^{k'}_{m',0} \simeq \mathbf{c}^{k',k'}_{m',0} \to \mathbf{c}^{k',l'}_{m',0}$$

is an isomorphism;

(3) For $l' \le k'$ and for all n', the map

$$\mathbf{c}^{l'}_{m,n'} \simeq \mathbf{c}^{l',l'}_{m,n'} \to \mathbf{c}^{k',l'}_{m,n'}$$

is an isomorphism.

4.3.3. By Theorem 2.1.3 and Sect. 2.3.5, the functor $\Phi^{adm}_{vert;horiz}$ gives rise to a map

$$\mathrm{Maps}(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Maps}(([l]\times[m])\circledast([n]\times[l]^{\mathrm{op}}), \mathbb{S}). \tag{4.10}$$

Let

$$\mathrm{Maps}^0(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \subset \mathrm{Maps}(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz}$$

be the subspace consisting of maps satisfying properties (1) and (3) in Sect. 4.3.1.

Then the map (4.10) has the property that the image of the composition

$$\begin{aligned}\mathrm{Maps}^0(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} &\to \mathrm{Maps}(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz}\\ \to \mathrm{Maps}(([l]\times[m])\circledast([n]\times[l]^{\mathrm{op}}), \mathbb{S}) &\to \mathrm{Maps}(([l]\times[m])\circledast[n]\circledast[l]^{\mathrm{op}}, \mathbb{S})\end{aligned}$$

belongs to

$$\mathrm{Maps}(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S}) \subset \mathrm{Maps}(([l]\times[m])\circledast[n]\circledast[l]^{\mathrm{op}}, \mathbb{S}).$$

I.e., we have a well-defined map

$$\mathrm{Maps}^0(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Maps}(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S}). \tag{4.11}$$

Composing with (4.8), we obtain a map

$$\begin{aligned}\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}^0(([l]\times[m])\times([n]\times[l]^{\mathrm{op}})^{\mathrm{op}}, \mathbf{C})^{adm}_{vert;horiz} \to\\ \to \mathrm{Maps}(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S}).\end{aligned} \tag{4.12}$$

4.3.4. Let

$$\mathrm{Maps}^0(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S}) \subset \mathrm{Maps}(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S})$$

be the subspace of maps that for every fixed k', m', n', the resulting 1-morphism

$$s^{k',l'+1}_{m',n'} \to s^{k',l'}_{m',n'}$$

admits a left adjoint for every $0 \le l' < l'+1 \le l$.

We shall now use the fact that for a map γ in $\mathbf{C}$ that belongs to adm', the 1-morphsim $\Phi^!(\gamma)$ in $\mathbb{S}$ admits a left adjoint (see the assumption on $\Phi^{adm}_{vert;horiz}$ in Sect. 4.1.2).

Using property (5) in Sect. 4.3.1, this implies that the image of the map (4.12) belongs to $\mathrm{Maps}^0(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S})$.

By [Chapter A.3, Corollary 3.1.7], we have a canonically defined map

$$\mathrm{Maps}^0(([l]\times[m]\times[n])\circledast[l]^{\mathrm{op}}, \mathbb{S}) \to \mathrm{Maps}([l]\circledast([l]\times[m]\times[n]), \mathbb{S}), \tag{4.13}$$

giving by passing to left adjoints along the $[l]^{\mathrm{op}}$-direction.

Thus, composing (4.12) and (4.13), we obtain a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}([l]\circledast([l]\times[m]\times[n]), \mathbb{S}). \tag{4.14}$$

4.3.5. Consider the composition of (4.14) with the embedding

$$\mathrm{Maps}([l] \circledast ([l] \times [m] \times [n]), \mathbb{S}) \to \mathrm{Maps}([l] \circledast ([l] \times [m]) \circledast [n], \mathbb{S}).$$

We thus obtain a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}([l] \circledast ([l] \times [m]) \circledast [n], \mathbb{S}). \tag{4.15}$$

We claim that its image belongs to the subspace

$$\mathrm{Maps}(([l] \times [l] \times [m]) \circledast [n], \mathbb{S}) \subset \mathrm{Maps}([l] \circledast ([l] \times [m]) \circledast [n], \mathbb{S}).$$

This follows from properties (2) and (4) in Sect. 4.3.1, and the next lemma:

Lemma 4.3.6. *Let*

$$\begin{tikzcd} \mathbf{c}_0 \arrow[r, "\alpha"] \arrow[d, "\gamma_0"'] & \mathbf{c}_1 \arrow[d, "\gamma_1"] \\ \mathbf{c}'_0 \arrow[r, "\alpha'"] & \mathbf{c}'_1 \end{tikzcd}$$

be a commutative diagram in $\mathbf{C}$ *with* $\gamma_0, \gamma_1 \in adm'$ *and* $\alpha, \alpha' \in horiz$, *and whose defect of Cartesianness belongs to adm. Then the 2-morphism*

$$(\Phi^!(\gamma_0))^L \circ \Phi^!(\alpha) \to \Phi^!(\alpha') \circ (\Phi^!(\gamma_1))^L,$$

arising by adjunction from the isomorphism

$$\Phi^!(\alpha) \circ \Phi^!(\gamma_1) \simeq \Phi^!(\gamma_0) \circ \Phi^!(\alpha)$$

identifies with the 2-morphism

$$\Phi(\gamma_0) \circ \Phi^!(\alpha) \to \Phi^!(\alpha') \circ \Phi^!(\gamma_1),$$

obtained from the functor $\Phi^{adm}_{vert;horiz}$.

Proof. Follows from the assumption in Sect. 4.1.2 by diagram chase. □

Remark 4.3.7. It is easy to see the the conclusion of Lemma 4.3.6 is in fact equivalent to the assumption in Sect. 4.1.2.

4.3.8. Thus, we obtain a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}(([l] \times [l] \times [m]) \circledast [n], \mathbb{S}). \tag{4.16}$$

Finally, using the diagonal map

$$[l] \to [l] \times [l],$$

we obtain a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}(([l] \times [m]) \circledast [n], \mathbb{S}). \tag{4.17}$$

The composition of (4.17) with the embedding

$$\mathrm{Maps}(([l] \times [m]) \circledast [n], \mathbb{S}) \hookrightarrow \mathrm{Maps}([l] \circledast [m] \circledast [n], \mathbb{S})$$

is a map

$$\mathrm{Seq}_l({}'\mathbf{Grid}_{m,n}(\mathbf{C})^{adm'}_{vert;horiz}) \to \mathrm{Maps}([l] \circledast [m] \circledast [n], \mathbb{S}). \tag{4.18}$$

This is the desired map of (4.2).

By constriction, its image belongs to

$$\mathrm{Maps}([l] \circledast ([m] \times [n]), \mathbb{S}) \subset \mathrm{Maps}([l] \circledast [m] \circledast [n], \mathbb{S}).$$

4.4. **Verification of the tri-simplicial functoriality.** In order to be more explicit, we will describe the situation for an individual map in $\mathbf{\Delta}\times\mathbf{\Delta}\times\mathbf{\Delta}$

$$[l_1]\to[l_2],\ [m_1]\to[m_2],\ [n_1]\to[n_2]. \tag{4.19}$$

4.4.1. Let $\underline{\mathbf{c}}_2$ be an object of $\mathrm{Seq}_{l_2}({}'\mathbf{Grid}_{m_2,n_2}(\mathbf{C})^{adm_2}_{vert;horiz})$. We let $\underline{\mathbf{c}}_1$ denote the object of $\mathrm{Seq}_{l_1}({}'\mathbf{Grid}_{m_1,n_1}(\mathbf{C})^{adm'}_{vert;horiz})$, obtained from $\underline{\mathbf{c}}_2$ by restricting along (4.19).

Let $\underline{s}_2$ be the point of $\mathrm{Maps}([l_2]\circledast[m_2]\circledast[n_2],\mathbb{S})$ corresponding to $\underline{\mathbf{c}}_2$ via the map (4.2). Let $\underline{s}_1$ be the point of $\mathrm{Maps}([l_1]\circledast[m_1]\circledast[n_1],\mathbb{S})$ corresponding to $\underline{\mathbf{c}}_1$ via the map (4.2).

Let $\widetilde{\underline{s}}_1$ be the point of $\mathrm{Maps}([l_1]\circledast[m_1]\circledast[n_1],\mathbb{S})$ obtained from $\underline{s}_2$ by restricting along (4.19).

We need to establish a canonical isomorphism

$$\underline{s}_1\simeq\widetilde{\underline{s}}_1. \tag{4.20}$$

4.4.2. Let $\overline{\mathbf{c}}_2$ denote the object of

$$\mathrm{Maps}^0(([l_2]\times[m_2])\times([n_2]\times[l_2]^{\mathrm{op}})^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}$$

obtained from $\underline{\mathbf{c}}_2$ by the map (4.8), see Sect. 4.3.3 for the notation $\mathrm{Maps}^0(-,\mathbf{C})^{adm}_{vert;horiz}$.

Let $\overline{\mathbf{c}}_1$ denote the object of

$$\mathrm{Maps}^0(([l_1]\times[m_1])\times([n_1]\times[l_1]^{\mathrm{op}})^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}$$

obtained from $\underline{\mathbf{c}}_1$ by the map (4.8).

Let $\widetilde{\overline{\mathbf{c}}}_1$ denote the object of

$$\mathrm{Maps}^0(([l_1]\times[m_1])\times([n_1]\times[l_1]^{\mathrm{op}})^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}$$

obtained from $\overline{\mathbf{c}}_2$ by restricting along (4.19).

Let $\overline{s}_2$ be the object of

$$\mathrm{Maps}(([l_2]\times[l_2])\circledast[m_2]\circledast[n_2],\mathbb{S}),$$

obtained from $\overline{\mathbf{c}}_2$ via the map (4.11) and passing to left adjoints along the last variable.

Let $\overline{s}_1$ and $\widetilde{\overline{s}}_1$ be the objects of

$$\mathrm{Maps}(([l_1]\times[l_1])\circledast[m_1]\circledast[n_1],\mathbb{S}),$$

obtained from $\overline{\mathbf{c}}_1$ and $\widetilde{\overline{\mathbf{c}}}_1$, respectively, by the same procedure.

We shall now construct a natural transformation

$$\overline{s}\in\mathrm{Maps}([1]\times(([l_1]\times[l_1])\circledast[m_1]\circledast[n_1]),\mathbb{S}), \tag{4.21}$$

whose restriction to $\{0\}\in[1]$ and $\{1\}\in[1]$ identifies with $\overline{s}_1$ and $\widetilde{\overline{s}}_1$, respectively.

Remark 4.4.3. Note, however, that this natural transformation will *not* be an isomorphism.

4.4.4. We note that there is a canonically defined map

$$\overline{\mathbf{c}}_1 \to \widetilde{\overline{\mathbf{c}}}_1,$$

which can be regarded as an object $\overline{\mathbf{c}}$ of

$$\operatorname{Maps}([1]\times([l_1]\times[m_1])\times([n_1]\times[l_1]^{\mathrm{op}})^{\mathrm{op}},\mathbf{C}),$$

and that this object in fact belongs to

$$\operatorname{Maps}^0(([1]\times[l_1]\times[m_1])\times([n_1]\times[l_1]^{\mathrm{op}})^{\mathrm{op}},\mathbf{C})^{adm}_{vert;horiz}.$$

Consider the object of

$$\operatorname{Maps}(([1]\times[l_1]\times[m_1])\circledast([n_1]\times[l_1]^{\mathrm{op}}),\mathbb{S})$$

attached to $\overline{\mathbf{c}}$ by means of the functor $\Phi^{adm}_{vert;horiz}$.

The image of the above object under

$$\operatorname{Maps}(([1]\times[l_1]\times[m_1])\circledast([n_1]\times[l_1]^{\mathrm{op}}),\mathbb{S})\to\operatorname{Maps}(([1]\times[l_1]\times[m_1])\circledast[n_1]\circledast[l_1]^{\mathrm{op}},\mathbb{S})$$

belongs

$$\operatorname{Maps}(([1]\times[l_1]\times[m_1]\times[n_1])\circledast[l_1]^{\mathrm{op}},\mathbb{S})\subset\operatorname{Maps}(([1]\times[l_1]\times[m_1])\circledast[n_1]\circledast[l_1]^{\mathrm{op}},\mathbb{S}),$$

and furthermore to

$$\operatorname{Maps}^0(([1]\times[l_1]\times[m_1]\times[n_1])\circledast[l_1]^{\mathrm{op}},\mathbb{S})\subset\operatorname{Maps}(([1]\times[l_1]\times[m_1]\times[n_1])\circledast[l_1]^{\mathrm{op}},\mathbb{S})$$

(see Sect. 4.3.4 for the notation $\operatorname{Maps}^0(-,\mathbb{S})$).

4.4.5. Passing to left adjoints along the last variable, we obtain an object of

$$\operatorname{Maps}([l_1]\circledast([1]\times[l_1]\times[m_1]\times[n_1]),\mathbb{S}).$$

The image of the latter object under

$$\operatorname{Maps}([l_1]\circledast([1]\times[l_1]\times[m_1]\times[n_1]),\mathbb{S})\to\operatorname{Maps}([l_1]\circledast([1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S})$$

belongs to

$$\operatorname{Maps}(([l_1]\times[1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S})\subset\operatorname{Maps}([l_1]\circledast([1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S}).$$

Further, the image of the latter object under

$$\operatorname{Maps}(([l_1]\times[1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S})=\operatorname{Maps}(([1]\times[l_1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S})\to$$
$$\to\operatorname{Maps}([1]\circledast([l_1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S})$$

belongs to

$$\operatorname{Maps}([1]\times(([l_1]\times[l_1]\times[m_1])\circledast[n_1]),\mathbb{S})\subset\operatorname{Maps}([1]\circledast([l_1]\times[l_1]\times[m_1])\circledast[n_1],\mathbb{S}).$$

Let us map

$$\operatorname{Maps}([1]\times(([l_1]\times[l_1]\times[m_1])\circledast[n_1]),\mathbb{S})\to\operatorname{Maps}([1]\times(([l_1]\times[l_1])\circledast[m_1]\circledast[n_1]),\mathbb{S}),$$

and denote the resulting object of

$$\operatorname{Maps}([1]\times(([l_1]\times[l_1])\circledast[m_1]\circledast[n_1]),\mathbb{S})$$

by $\overline{s}$. respectively.

This $\overline{s}$ is the desired natural transformation in (4.21).

4.4.6. Restricting $\overline{s}$ under $[l_1]\to[l_1]\times[l_1]$, we obtain an object of

$$\mathrm{Maps}([1]\times([l_1]\circledast[m_1]\circledast[n_1]),\mathbb{S}),$$

denoted $\underline{s}$.

By construction, the restrictions of $\underline{s}$ to $\{0\}\in[1]$ and $\{1\}\in[1]$ identify with $\underline{s}_1$ and $\underline{\widetilde{s}}_1$, respectively.

This provides the natural transformation from $\underline{s}_1$ to $\underline{\widetilde{s}}_1$. However, we claim that this natural transformation *is* an isomorphism, thereby providing the isomorphism in (4.20).

Indeed, this follows from the fact that the object $\overline{\mathbf{c}}$, regarded as a natural transformation from $\overline{\mathbf{c}}_1$ to $\widetilde{\overline{\mathbf{c}}}_1$, viewed as functors

$$[l_1]\times[m_1]\times[n_1]^{\mathrm{op}}\times[l_1]\to\mathbf{C},$$

becomes an isomorphism when restricted to the diagonal copy of $[l_1]\times[m_1]\times[n_1]^{\mathrm{op}}$.

5. Functors constructed by factorization

In the section we will describe the second result that allows to start from a functor out of an $(\infty,2)$-category of correspondences and (canonically) extend it to a larger $(\infty,2)$-category of correspondences.

This setting arises in practice when we want to construct the !-pullback as a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\to\mathrm{DGCat}_{\mathrm{cont}}, \tag{5.1}$$

starting from the functor

$$\mathrm{Sch}_{\mathrm{aft}}\to\mathrm{DGCat}_{\mathrm{cont}},\quad X\rightsquigarrow\mathrm{IndCoh}(X),\quad (X\xrightarrow{f}Y)\rightsquigarrow f_*^{\mathrm{IndCoh}}.$$

The idea is that $f^!$ should be the *right* adjoint of f_*^{IndCoh} if f is proper and the *left* adjoint of f_*^{IndCoh} if f is an open embedding. For a general f we want to decompose it as $f_1\circ f_2$, where f_2 is an open embedding and f_1 is proper. The challenge is to establish the independence of $f^!$ of such a factorization in a functorial way (i.e., in the context of ∞-categories).

It appears, however, that even if the original problem of defining functor (5.1) does not involve the 2-category of correspondences, the construction does necessarily use one (in particular, at a crucial stage we will invoke the already established Theorem 4.1.3).

5.1. **Set-up for the source.** In this subsection we will describe what categories of correspondences are involved in the extension procedure that is that goal of this section.

5.1.1. We start with a datum of an $(\infty,1)$-category $\mathbf{C}$ equipped with three classes of 1-morphisms $(vert,horiz,adm)$ as in Sect. 1.1.1.

We fix yet one more class $co\text{-}adm\subset horiz$, such that the triple $(vert,co\text{-}adm,\mathrm{isom})$ also satisfies the conditions of Sect. 1.1.1.

We shall assume that *horiz* and *co-adm* also satisfy the '2 out of 3' property. I.e., for a pair of composable morphisms β_1, β_2, if both β_1 and $\beta_1\circ\beta_2$ belong to *horiz* (resp., *co-adm*), then so does β_2.

5.1.2. We now impose the following additional condition on the pair $(co\text{-}adm, adm)$. Namely, we require that for a Cartesian diagram

$$
\begin{tikzcd}
\mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c} \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & \mathbf{c} \arrow[d, "\delta"] \\
\mathbf{c} \arrow[r, "\delta"] & \mathbf{c}'
\end{tikzcd}
\tag{5.2}
$$

with $\delta \in adm \cap co\text{-}adm$, the maps δ_1 and δ_2 both be *isomorphisms*. In other words, we require that the diagram

$$
\begin{tikzcd}
\mathbf{c} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathbf{c} \arrow[d, "\delta"] \\
\mathbf{c} \arrow[r, "\delta"] & \mathbf{c}'
\end{tikzcd}
$$

be Cartesian. Equivalently, we require that every map $\delta \in adm \cap co\text{-}adm$ be a *monomorphism.*

5.1.3. The example to keep in mind is that of $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$, with $vert = horiz$ being all morphisms, adm being proper morphisms and $co\text{-}adm$ being open embeddings.

In this case, the class $adm \cap co\text{-}adm$ consists of embeddings of unions of connected components.

5.1.4. We now impose the following crucial condition on the relationship between the classes adm, $co\text{-}adm$ and $horiz$.

For a 1-morphism $\alpha : \mathbf{c}_0 \to \mathbf{c}_1$ in $horiz$, consider the $(\infty, 1)$-category $\mathrm{Factor}(\alpha)$, whose objects are

$$\mathbf{c}_0 \xrightarrow{\epsilon} \overline{\mathbf{c}}_0 \xrightarrow{\gamma} \mathbf{c}_1,$$

where $\epsilon \in co\text{-}adm$ and $\gamma \in adm$, and whose morphisms are commutative diagrams

$$
\begin{tikzcd}
 & \overline{\mathbf{c}}'_0 \arrow[dd, "\beta"] \arrow[dr, "\gamma'"] & \\
\mathbf{c}_0 \arrow[ur, "\epsilon'"] \arrow[dr, "\epsilon''"'] & & \mathbf{c}_1. \\
 & \overline{\mathbf{c}}''_0 \arrow[ur, "\gamma''"'] &
\end{tikzcd}
$$

Note that by the '2-out-of 3' property, the morphism β automatically belongs to adm.

We impose the condition that for any $\alpha : \mathbf{c}_0 \to \mathbf{c}_1$ in $horiz$, the category $\mathrm{Factor}(\alpha)$ be *contractible.*

5.2. **Set-up for the functor.** In this subsection we will describe what kind of functors out of our categories of correspondences we will consider, and formulate Theorem 5.2.4.

5.2.1. Let $\mathbb{S}$ be an $(\infty, 2)$-category. We start with functors

$$\Phi^{adm}_{vert;adm} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;adm} \to \mathbb{S}$$

and

$$\Phi^{\mathrm{isom}}_{vert;co\text{-}adm} : \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert;co\text{-}adm} \to \mathbb{S}$$

together with an identification of the corresponding functors

$$\Phi^{adm}_{vert;adm}|_{\mathbf{C}_{vert}} \simeq \Phi^{\mathrm{isom}}_{vert;co\text{-}adm}|_{\mathbf{C}_{vert}}.$$

Denote

$$\Phi := \Phi^{adm}_{vert;adm}|_{\mathbf{C}_{vert}}.$$

Note that by Theorem 3.2.2, the functor $\Phi^{adm}_{vert;adm}$ is uniquely reconstructed from that of Φ, and it exists if and only if Φ satisfies the left Beck-Chevalley condition with respect to $adm \subset vert$. So, the above data is uniquely recovered from that of $\Phi^{\mathrm{isom}}_{vert;co\text{-}adm}$.

In what follows, we shall denote by $\Phi^*_{co\text{-}adm}$ the restriction

$$\Phi_{vert;co\text{-}adm}|_{(\mathbf{C}_{co\text{-}adm})^{\mathrm{op}}},$$

and by $\Phi^!_{adm}$ the restriction

$$\Phi^{adm}_{vert;adm}|_{(\mathbf{C}_{adm})^{\mathrm{op}}}.$$

5.2.2. We now impose the following additional condition on $\Phi^{\mathrm{isom}}_{vert;co\text{-}adm}$:

Let

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\epsilon_0"] \arrow[d, "\gamma_1"'] & \mathbf{c}_{0,0} \arrow[d, "\gamma_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\epsilon_1"] & \mathbf{c}_{1,0} \end{tikzcd} \tag{5.3}$$

be a Cartesian diagram with $\epsilon_i \in co\text{-}adm$ and $\gamma_i \in adm$. Consider the 2-morphism

$$\Phi^*_{co\text{-}adm}(\epsilon_0) \circ \Phi^!_{adm}(\gamma_0) \to \Phi^!_{adm}(\gamma_1) \circ \Phi^*_{co\text{-}adm}(\epsilon_1) \tag{5.4}$$

arising by adjunction from the isomorphism

$$\Phi(\gamma_1) \circ \Phi^*_{co\text{-}adm}(\epsilon_0) \simeq \Phi^*_{co\text{-}adm}(\epsilon_1) \circ \Phi(\gamma_0),$$

the latter being a part of the data of $\Phi^{\mathrm{isom}}_{vert;co\text{-}adm}$.

We need that (5.4) be an isomorphism for all Cartesian diagrams as above.

5.2.3. We claim:

Theorem 5.2.4. *Restriction along*

$$\mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert;co\text{-}adm} \to \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

defines an isomorphism between the space of functors

$$\Phi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S}$$

and that of functors

$$\Phi^{\mathrm{isom}}_{vert;co\text{-}adm} : \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert;co\text{-}adm} \to \mathbb{S},$$

for which

$$\Phi := \Phi^{\mathrm{isom}}_{vert;co\text{-}adm}|_{\mathbf{C}_{vert}}$$

satisfies the left Beck-Chevalley condition with respect to $adm \subset vert$, and such that the condition from Sect. 5.2.2 holds.

5.3. **Proof of Theorem 5.2.4, initial remarks.** In this subsection we will explain the strategy of the proof of Theorem 5.2.4.

5.3.1. First, we shall carry out the easy direction. Namely, we will show that if we start with a functor

$$\Phi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S},$$

then the functor

$$\Phi^{\mathrm{isom}}_{vert;co\text{-}adm} : \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert;co\text{-}adm} \to \mathbb{S},$$

obtained by restriction, satisfies left Beck-Chevalley condition with respect to $adm \subset vert$, and such that the condition from Sect. 5.2.2 holds.

First, the fact that Φ satisfies the left Beck-Chevalley condition with respect to $adm \subset vert$ follows from (the easy direction of) Theorem 3.2.2.

For $\Phi^{adm}_{vert;horiz}$ as above, let $\Phi^!$ denote the restriction

$$\Phi^{adm}_{vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}}.$$

By assumption,

$$\Phi^!|_{(\mathbf{C}_{co\text{-}adm})^{\mathrm{op}}} \simeq \Phi^*_{co\text{-}adm} \text{ and } \Phi^!|_{(\mathbf{C}_{adm})^{\mathrm{op}}} \simeq \Phi^!_{adm}.$$

Furthermore, for a Cartesian diagram (5.3), the (iso)morphism

$$\Phi(\gamma_1) \circ \Phi^!(\epsilon_0) \to \Phi^!(\epsilon_1) \circ \Phi(\gamma_0),$$

is one arising by adjunction from the isomorphism

$$\Phi^!(\epsilon_0) \circ \Phi^!(\gamma_0) \simeq \Phi^!(\gamma_0 \circ \epsilon_0) \simeq \Phi^!(\epsilon_1 \circ \gamma_1) \simeq \Phi^!(\gamma_1) \circ \Phi^!(\epsilon_1).$$

In particular, the latter equals the 2-morphism (5.4), which is therefore also an isomorphism.

5.3.2. We are now going to tackle the difficult direction in Theorem 5.2.4. By Theorem 2.1.3, the datum of a functor $\Phi^{adm}_{vert;horiz}$ is equivalent to that of a map of bi-simplicial spaces

$$\mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}), \tag{5.5}$$

satisfying the additional condition from Theorem 2.1.3.

Given $\Phi^{\mathrm{isom}}_{vert;co\text{-}adm}$, we shall produce (5.5) in three steps:

Step A. We first extend the initial functor

$$\Phi^{\mathrm{isom}}_{vert;co\text{-}adm} : \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert;co\text{-}adm} \to \mathbb{S}$$

to a functor

$$\Phi^{adm\cap co\text{-}adm}_{vert;co\text{-}adm} : \mathrm{Corr}(\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm} \to \mathbb{S}. \tag{5.6}$$

The existence and uniqueness of the functor $\Phi^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$ in (5.6) follows immediately from the assumptions of Theorem 5.2.4 and Theorem 4.1.3.

Step B. We will introduce a bi-simplicial category $\mathrm{Factor}_{\bullet,\bullet}(\mathbf{C})$, equipped with a bi-simplicial functor

$$\mathrm{Factor}_{\bullet,\bullet}(\mathbf{C}) \to \mathrm{defGrid}_{\bullet,\bullet}(\mathbf{C})^{adm}_{vert;horiz}. \tag{5.7}$$

We will use $\Phi^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$ to construct a bi-simplicial functor $\Phi_{\mathrm{Factor}_{\bullet,\bullet}}$

$$\Phi_{\mathrm{Factor}_{m,n}} : \mathrm{Factor}_{m,n}(\mathbf{C}) \to \mathrm{Sq}_{m,n}(\mathbb{S}). \tag{5.8}$$

Step C. Finally, we shall use the 'contractibility of the space of factorizations' condition from Sect. 5.1.4, to show that each of the functors

$$\mathrm{Factor}_{m,n}(\mathbf{C}) \to \mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz} \tag{5.9}$$

has contractible fibers[5]. This will imply that the bi-simplicial functor $\Phi_{\mathrm{Factor}_{\bullet,\bullet}}$ uniquely factors through the sought-for bi-simplicial map (5.5) via the projection (5.7).

5.4. **Step B: introduction.**

5.4.1. We define the category $\mathrm{Factor}_{m,n}(\mathbf{C})$ as a 1-full subcategory of

$$\mathbf{Maps}\big([m]\times([n]^{\mathrm{op}}\times[n]^{\mathrm{op}})^{\geq \mathrm{dgnl}},\mathbf{C}\big),$$

as follows.

At the level of objects we take those diagrams $\underline{\mathbf{c}}$ that satisfy:

(1) The maps $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i+1,j,k}$ belong to *vert*;
(2) The maps $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i,j-1,k}$ belong to *co-adm*;
(3) The maps $\mathbf{c}_{i,j,k}\to\mathbf{c}_{i,j,k-1}$ belong to *adm*;
(4) The defect of Cartesianness of the squares

$$\begin{tikzcd} \mathbf{c}_{i,j,k} \arrow[r] \arrow[d] & \mathbf{c}_{i,j-1,k} \arrow[d] \\ \mathbf{c}_{i+1,j,k} \arrow[r] & \mathbf{c}_{i+1,j-1,k} \end{tikzcd}$$

belongs to $adm\cap co\text{-}adm$.

As 1-morphisms we allow those maps between diagrams $\underline{\mathbf{c}}\to\underline{\mathbf{c}}'$ for which the maps

$$\mathbf{c}_{i,j,k}\to\mathbf{c}'_{i,j,k}$$

belong to *adm* and are isomorphisms for $j=k$.

5.4.2. For example, when $m=1$ and $n=2$, objects of $\mathrm{Factor}_{m,n}(\mathbf{C})$ are the diagrams

(5.10)
$$\begin{tikzcd}
 & & & \mathbf{c}_{1,0,2} \arrow[dr] \arrow[dddl] & & \\
 & & \mathbf{c}_{0,1,2} \arrow[ur] \arrow[dr] \arrow[dddl] & & \mathbf{c}_{0,0,1} \arrow[dr] \arrow[dddl] & \\
 & \mathbf{c}_{0,2,2} \arrow[ur] \arrow[dddl] & & \mathbf{c}_{0,1,1} \arrow[ur] \arrow[dddl] & & \mathbf{c}_{0,0,0} \arrow[dddl] \\
 & & \mathbf{c}_{1,0,2} \arrow[dr] & & & \\
 & \mathbf{c}_{1,1,2} \arrow[ur] \arrow[dr] & & \mathbf{c}_{1,0,1} \arrow[dr] & & \\
 \mathbf{c}_{1,2,2} \arrow[ur] & & \mathbf{c}_{1,1,1} \arrow[ur] & & \mathbf{c}_{1,0,0} &
\end{tikzcd}$$

with the long slanted arrows in *vert*, northeast pointing arrows in *co-adm*, and southeast pointing arrows in *adm*.

5.4.3. The functor

$$\mathrm{Factor}_{m,n}(\mathbf{C})\to\mathbf{Grid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$$

is obtained from the diagonal embedding

$$[n]^{\mathrm{op}}\to[n]^{\mathrm{op}}\times[n]^{\mathrm{op}}.$$

[5]In particular, $|\mathrm{Factor}_{m,n}(\mathbf{C})|\simeq\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert;horiz}$, where the notation $|-|$ is as in [Chapter I.1, Sect. 2.1.5].

5.4.4. In order to perform Step B we need to construct the functor

$$\Phi_{\mathrm{Factor}_{m,n}} : \mathrm{Factor}_{m,n}(\mathbf{C}) \to \mathrm{Sq}_{m,n}(\mathbb{S}),$$

functorially in $([n],[m]) \in \boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}$.

5.4.5. Let explain the idea of this construction for $m = 2$ and $n = 1$. Namely, to a diagram as in (5.10) we want to attach a diagram

$$\begin{tikzcd}
s_{0,2,2} \arrow[d] \arrow[dr, Rightarrow] & s_{0,1,1} \arrow[l] \arrow[d] \arrow[dr, Rightarrow] & s_{0,0,0} \arrow[l] \arrow[d] \\
s_{1,2,2} & s_{1,1,1} \arrow[l] & s_{1,0,0} \arrow[l]
\end{tikzcd} \tag{5.11}$$

We will do it in several steps. First, starting from (5.10) we will use the functor $\Phi^{adm \cap co\text{-}adm}_{vert;co\text{-}adm}$ to produce a diagram

$$\begin{tikzcd}
 & & s_{0,0,2} \arrow[dl] \arrow[dr] \arrow[ddd] & & \\
 & s_{0,1,2} \arrow[dl] \arrow[dr] \arrow[ddd] & & s_{0,0,1} \arrow[dl] \arrow[dr] \arrow[ddd] & \\
s_{0,2,2} \arrow[ddd] & & s_{0,1,1} \arrow[ddd] & & s_{0,0,0}. \arrow[ddd] \\
 & & s_{1,0,2} \arrow[dl] \arrow[dr] & & \\
 & s_{1,1,2} \arrow[dl] \arrow[dr] & & s_{1,0,1} \arrow[dl] \arrow[dr] & \\
s_{1,2,2} & & s_{1,1,1} & & s_{1,0,0},
\end{tikzcd} \tag{5.12}$$

(here the squares are not necessarily commutative, but have 2-morphisms along appropriate faces), where along the long slanted arrows we take the 1-morphisms Φ, along the southwest pointing arrows we take the 1-morphisms $\Phi^*_{co\text{-}adm}$, and along the southeast pointing arrows we take the 1-morphisms Φ.

From the diagram (5.12), by taking right adjoints along the southeast pointing arrows, we obtain the diagram

$$\begin{tikzcd}
 & & s_{0,0,2} \arrow[dl] \arrow[ddd] & & \\
 & s_{0,1,2} \arrow[dl] \arrow[ddd] & & s_{0,0,1} \arrow[ul] \arrow[dl] \arrow[ddd] & \\
s_{0,2,2} \arrow[ddd] & & s_{0,1,1}, \arrow[ul] \arrow[ddd] & & s_{0,0,0} \arrow[ul] \arrow[ddd] \\
 & & s_{1,0,2} \arrow[dl] & & \\
 & s_{1,1,2} \arrow[dl] & & s_{1,0,1} \arrow[ul] \arrow[dl] & \\
s_{1,2,2} & & s_{1,1,1} \arrow[ul] & & s_{1,0,0}. \arrow[ul]
\end{tikzcd} \tag{5.13}$$

Finally, the desired diagram (5.11) is obtained from the diagram (5.13) by passing to the diagonal (by letting the 2nd and the 3rd indices be equal).

5.5. **Step B: preparations.** In order to prepare for Step B, we need to perform a certain manipulation with the functor $\Phi^{adm \cap co\text{-}adm}_{vert;co\text{-}adm}$ constructed in Step A.

5.5.1. Let us explain what we want to construct.

Let us be given a functor

$$\underline{\mathbf{c}} : [l]^{\mathrm{op}} \times [m] \times [n]^{\mathrm{op}} \to \mathbf{C},$$

such that:

(1) For every l', m', the map $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l',m',n'-1}$ belongs to $co\text{-}adm$;
(2) For every l', n', the map $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l',m'+1,n'}$ belongs to $vert$;
(3) For every m', n', the map $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l'-1,m',n'}$ belongs to adm;
(4) For every n', the square

$$\begin{CD} \mathbf{c}_{l',m',n'} @>>> \mathbf{c}_{l'-1,m',n'} \\ @VVV @VVV \\ \mathbf{c}_{l',m'+1,n'} @>>> \mathbf{c}_{l'-1,m'+1,n'} \end{CD}$$

is Cartesian;
(5) For every m', the defect of Cartesianness of the square

$$\begin{CD} \mathbf{c}_{l',m',n'} @>>> \mathbf{c}_{l'-1,m',n'} \\ @VVV @VVV \\ \mathbf{c}_{l',m',n'-1} @>>> \mathbf{c}_{l'-1,m',n'-1} \end{CD}$$

belongs to $adm \cap co\text{-}adm$;
(6) For every l', the defect of Cartesianness of the square

$$\begin{CD} \mathbf{c}_{l',m',n'} @>>> \mathbf{c}_{l',m',n'-1} \\ @VVV @VVV \\ \mathbf{c}_{l',m'+1,n'} @>>> \mathbf{c}_{l',m'+1,n'-1} \end{CD}$$

belongs to $adm \cap co\text{-}adm$.

We claim that to any such $\underline{\mathbf{c}}$ we can attach a functor

$$\underline{s} : [m] \circledast [n] \circledast [l] \to \mathbb{S},$$

such that:

(1) For every l', m', n', we have $s_{m',n',l'} = \Phi(\mathbf{c}_{l',m',n'})$;
(2) For every l', m', the 1-morphism $s_{m',n'-1,l'} \to s_{m',n',l'}$ is obtained by applying $\Phi^*_{co\text{-}adm}$ to the arrow $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l',m',n'-1}$;
(3) For every m', n', the morphism $s_{m',n',l'-1} \to s_{m',n',l'}$ is obtained by applying $\Phi^!_{adm}$ to the arrow $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l'-1,m',n'}$;
(4) For every l', n', the morphism $s_{m',n',l'} \to s_{m'+1,n',l'}$ is obtained by applying Φ to the arrow $\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l',m'+1,n'}$;

(5) For every m', the 2-morphism in the diagram

$$\begin{tikzcd}
s_{m',n',l'} & s_{m',n'-1,l'} \arrow[l] \arrow[dl, Rightarrow] \\
s_{m',n',l'-1} \arrow[u] & s_{m',n'-1,l'-1} \arrow[l] \arrow[u]
\end{tikzcd}$$

is an isomorphism.

We now explain how this is done.

5.5.2. First, by Theorem 2.1.3 and Sect. 2.3.5, the functor $\Phi^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$ gives rise to a map in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$

$$\begin{aligned}(5.14)\quad \mathrm{Maps}(([l]^{\mathrm{op}}\times[m])\times[n]^{\mathrm{op}},\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm} \to \mathrm{Maps}(([l]^{\mathrm{op}}\times[m])\circledast[n],\mathbb{S}) \to \\ \to \mathrm{Maps}([l]^{\mathrm{op}}\circledast[m]\circledast[n],\mathbb{S}).\end{aligned}$$

Let

$$\mathrm{Maps}^{adm}(([l]^{\mathrm{op}}\times[m])\times[n]^{\mathrm{op}},\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm} \subset \mathrm{Maps}(([l]^{\mathrm{op}}\times[m])\times[n]^{\mathrm{op}},\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$$

be the subspace consisting of those diagrams such that for every fixed $0\le m'\le m$ and $0\le n'\le n$, the map

$$\mathbf{c}_{l',m',n'} \to \mathbf{c}_{l'-1,m',n'}$$

belongs to adm, for $0\le l'-1<l'\le l$.

5.5.3. By the assumption on Φ, the image of $\mathrm{Maps}^{adm}(([l]^{\mathrm{op}}\times[m])\times[n]^{\mathrm{op}},\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$ under the map (5.14) belongs to the subspace

$$\mathrm{Maps}^0([l]^{\mathrm{op}}\circledast[m]\circledast[n],\mathbb{S}) \subset \mathrm{Maps}([l]^{\mathrm{op}}\circledast[m]\circledast[n],\mathbb{S}),$$

consisting of functors such that for every fixed $0\le m'\le m$ and $0\le n'\le n$, the 1-morphism

$$s_{l',m',n'} \to s_{l'-1,m',n'}$$

admits a right adjoint, for $0\le l'-1<l'\le l$.

By [Chapter A.3, Corollary 3.1.7], we have a canonically defined map

$$\mathrm{Maps}^0([l]^{\mathrm{op}}\circledast[m]\circledast[n],\mathbb{S}) \to \mathrm{Maps}([m]\circledast[n]\circledast[l],\mathbb{S}).$$

Thus, we obtain a map in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$

$$(5.15)\qquad \mathrm{Maps}^{adm}(([l]^{\mathrm{op}}\times[m])\times[n]^{\mathrm{op}},\mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm} \to \mathrm{Maps}([m]\circledast[n]\circledast[l],\mathbb{S}),$$

functorial in $[l],[m],[n]\in\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$.

We claim:

Proposition 5.5.4. *The image of the map* (5.15) *belongs to* $\mathrm{Maps}([m]\circledast([n]\times[l]),\mathbb{S})$.

Proof. The statement of the proposition is equivalent to the following one: let

$$\begin{tikzcd}
\mathbf{c}_0 \arrow[r, "\epsilon"] \arrow[d, "\gamma_0"'] & \mathbf{c}_1 \arrow[d, "\gamma_1"] \\
\mathbf{c}'_0 \arrow[r, "\epsilon'"] & \mathbf{c}'_1,
\end{tikzcd}$$

be a commutative diagram with the vertical maps belong to *adm* and the horizontal ones to *co-adm* (in which case, its defect of Cartesianness automatically belongs to $adm \cap co\text{-}adm$, see Lemma 5.6.2 below). Consider the 2-morphism

$$\Phi^*_{co\text{-}adm}(\epsilon) \circ \Phi^!_{adm}(\gamma_1) \to \Phi^!_{adm}(\gamma_0) \circ \Phi^*_{co\text{-}adm}(\epsilon') \tag{5.16}$$

arising by adjunction from the map

$$\Phi(\gamma_0) \circ \Phi^*_{co\text{-}adm}(\epsilon) \to \Phi^*_{co\text{-}adm}(\epsilon') \circ \Phi(\gamma_1),$$

the latter being part of the data supplied by $\Phi^{adm \cap co\text{-}adm}_{vert;co\text{-}adm}$. Then the claim is that the 2-morphism (5.16) is an isomorphism.

The above statement can be split into two cases. One is when the above diagram is Cartesian, in which case, the assertion coincides with the assumption on $\Phi^{\text{isom}}_{vert;co\text{-}adm}$ from Sect. 5.2.2.

The second case is when we are dealing with the diagram of the form

$$\begin{tikzcd} \mathbf{c} \arrow[r, "\text{id}"] \arrow[d, "\text{id}"'] & \mathbf{c} \arrow[d, "\delta"] \\ \mathbf{c} \arrow[r, "\delta"] & \mathbf{c}', \end{tikzcd} \tag{5.17}$$

where $\delta \in adm \cap co\text{-}adm$. But such a diagram is automatically Cartesian by the condition of Sect. 5.1.2.

□

Remark 5.5.5. Note that from diagram (5.17) it follows that for $\delta \in adm \cap co\text{-}adm$ we have a canonical isomorphism

$$\Phi^!_{adm}(\delta) \simeq \Phi^*_{co\text{-}adm}(\delta), \tag{5.18}$$

characterized uniquely by the property that the isomorphism

$$\text{id} \simeq \Phi^*_{co\text{-}adm}(\delta) \circ \Phi(\delta)$$

defines the unit of an adjunction.

5.6. **Step B: the construction.** We will now turn the idea described in Sect. 5.4.5 into a formal construction. I.e., we will define the functor

$$\Phi_{\text{Factor}_{\bullet,\bullet}} : \text{Factor}_{\bullet,\bullet}(\mathbf{C}) \to \text{Sq}_{\bullet,\bullet}(\mathbb{S}).$$

This will combine the construction from Sect. 5.5 and the manipulation that was employed in the proof of Theorem 2.1.3, namely, we will use *clusters*.

5.6.1. Let $\mathbf{defGrid}^{vert}(\mathbf{C})^{adm \cap co\text{-}adm}_{co\text{-}adm, adm}$ denote the following bi-simplicial $(\infty,1)$-*category*. For each m, n, the $(\infty,1)$-category $\mathbf{defGrid}^{vert}_{m,n}(\mathbf{C})^{adm \cap co\text{-}adm}_{co\text{-}adm, adm}$ is a 1-full subcategory in

$$\mathbf{Maps}([m]^{\text{op}} \times [n]^{\text{op}}, \mathbf{C}).$$

Its objects are commutative diagrams $\underline{\mathbf{c}}$ that satisfy:

(1) For every i, the map $\mathbf{c}_{i,j} \to \mathbf{c}_{i,j-1}$ belongs to *adm*.
(2) For every j, the map $\mathbf{c}_{i,j} \to \mathbf{c}_{i-1,j}$ belongs to *co-adm*.

As 1-morphisms we allow those maps between diagrams $\underline{\mathbf{c}} \to \underline{\mathbf{c}}'$ that satisfy:

(1) For every i and j, the map $\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$ belongs to *vert*.

(2) For a fixed j, the defect of Cartesianness of the square

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i-1,j} \arrow[d] \\ \mathbf{c}'_{i,j} \arrow[r] & \mathbf{c}'_{i-1,j} \end{tikzcd}$$

belongs to $adm \cap co\text{-}adm$.

We note:

Lemma 5.6.2. *For a commutative square*

$$\begin{tikzcd} \mathbf{c}_0 \arrow[r] \arrow[d] & \mathbf{c}_1 \arrow[d] \\ \mathbf{c}'_0 \arrow[r] & \mathbf{c}'_1, \end{tikzcd}$$

in which the vertical maps belong to adm and the horizontal to vert or horiz (resp., the horizontal ones to co-adm and vertical ones to vert), its defect of Cartesianness belongs to adm (resp., co-adm).

Proof. Follows from the '2 out of 3' property of the classes $co\text{-}adm$ and adm. □

Hence, we obtain that there is canonical isomorphism

$$\mathrm{Seq}_l(\mathbf{defGrid}^{vert}_{m,n}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm}) \simeq \mathrm{Maps}^{adm}(([n]^{\mathrm{op}} \times [l]) \times [m]^{\mathrm{op}}, \mathbf{C})^{adm\cap co\text{-}adm}_{vert;co\text{-}adm},$$

see Sect. 5.5.2 for the notation $\mathrm{Maps}^{adm}(-,-)^{adm\cap co\text{-}adm}_{vert;co\text{-}adm}$.

Therefore, from Proposition 5.5.4, we obtain a map

$$\mathrm{Seq}_l(\mathbf{defGrid}^{vert}_{m,n}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm}) \to \mathrm{Maps}([l] \circledast ([m] \times [n]), \mathbb{S}), \tag{5.19}$$

functorial in $[l],[m],[n] \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$.

5.6.3. Let Q be a cluster, see Sect. 2.6.2 for what this means. Let $\mathbf{Q}$ be the category underlying Q.

We define the $(\infty,1)$-category

$$\mathbf{defQ}^{vert}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm}$$

analogously to $\mathbf{defGrid}^{vert}_{m,n}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm}$, so that we recover the latter when

$$Q = (\{0,...,m\} \times \{0,...,n\})^{\mathrm{op}}.$$

As in Sect. 2.7, the map (5.19), gives maps

$$\mathrm{Seq}_l(\mathbf{defQ}^{vert}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm}) \to \mathrm{Maps}([l] \circledast \mathbf{Q}^{\mathrm{op}}, \mathbb{S}), \tag{5.20}$$

functorial in $[l] \in \mathbf{\Delta}^{\mathrm{op}}$ and the cluster Q.

In other words, we obtain canonically defined functors

$$\mathbf{defQ}^{vert}(\mathbf{C})^{adm\cap co\text{-}adm}_{co\text{-}adm,adm} \to \mathrm{Funct}(\mathbf{Q}^{\mathrm{op}}, \mathbb{S})_{\text{right-lax}} \tag{5.21}$$

that depend functorially on Q.

5.6.4. Taking $Q = ((\{0,...,n\} \times \{0,...,n\})^{\geq \text{dgnl}})^{\text{op}}$, from (5.21) we obtain a map

$$\mathbf{defGrid}^{vert}_{n,n}(\mathbf{C})^{adm \cap co\text{-}adm}_{co\text{-}adm, adm} \to \text{Funct}(([n]\times[n])^{\geq \text{dgnl}}, \mathbb{S})_{\text{right-lax}},$$

and composing with the diagonal embedding $[n] \to ([n]\times[n])^{\geq \text{dgnl}}$, we obtain a map

$$\mathbf{defGrid}^{vert}_{n,n}(\mathbf{C})^{adm \cap co\text{-}adm}_{co\text{-}adm, adm} \to \text{Funct}([n], \mathbb{S})_{\text{right-lax}}. \tag{5.22}$$

Note that we have a tautologically defined functor

$$\text{Seq}_l(\text{Factor}_{m,n}(\mathbf{C})) \to \text{Maps}([l]\times[m], \mathbf{defGrid}^{vert}_{n,n}(\mathbf{C})^{adm \cap co\text{-}adm}_{co\text{-}adm, adm}).$$

Composing with (5.22), we obtain a functor

$$\text{Seq}_l(\text{Factor}_{m,n}(\mathbf{C})) \to \text{Maps}([l]\times[m], \text{Funct}([n], \mathbb{S})_{\text{right-lax}}) =$$
$$= \text{Maps}(([l]\times[m]) \circledast [n], \mathbb{S}) \to \text{Maps}([l]\circledast[m]\circledast[n], \mathbb{S}) = \text{Seq}_l(\text{Funct}([m]\circledast[n], \mathbb{S})_{\text{right-lax}}),$$

i.e., a functor

$$\text{Factor}_{m,n}(\mathbf{C}) \to \text{Funct}([m]\circledast[n], \mathbb{S})_{\text{right-lax}}. \tag{5.23}$$

We claim:

Lemma 5.6.5. *The functor* (5.23) *sends every arrow in* $\text{Factor}_{m,n}(\mathbf{C})$ *to an isomorphism in* $\text{Funct}([m]\circledast[n], \mathbb{S})_{\text{right-lax}}$.

Proof. Follows from the condition that for a map of objects in $\text{Factor}_{m,n}(\mathbf{C})$, i.e., a natural transformation between functors

$$[m] \times ([n]^{\text{op}} \times [n]^{\text{op}})^{\geq \text{dgnl}} \to \mathbf{C},$$

the induced natural transformation of the functors

$$[m]\times[n]^{\text{op}} \to [m] \times ([n]^{\text{op}} \times [n]^{\text{op}})^{\geq \text{dgnl}} \to \mathbf{C}$$

is an isomorphism. □

5.6.6. The above lemma implies that the functor (5.23) factors though

$$(\text{Funct}([m]\circledast[n], \mathbb{S})_{\text{right-lax}})^{\text{Spc}} \simeq \text{Maps}([m]\circledast[n], \mathbb{S}) =: \text{Sq}_{m,n}(\mathbb{S}).$$

The resulting functor

$$\text{Factor}_{m,n}(\mathbf{C}) \to \text{Sq}_{m,n}(\mathbb{S})$$

is the sought-for functor $\Phi_{\text{Factor}_{m,n}}$ of (5.8).

5.7. **Step C.**

5.7.1. Let us first explain the idea of the proof when $m = 0$ and $n = 1$. In this case, the category $\text{Factor}_{0,1}(\mathbf{C})$ has as objects diagrams

$$\begin{tikzcd} & \mathbf{c}_{0,1} \arrow[dr, "\gamma"] & \\ \mathbf{c}_{1,1} \arrow[ur, "\epsilon"] & & \mathbf{c}_{0,0}. \end{tikzcd}$$

and morphisms are given by diagrams

$$\begin{tikzcd}
& \mathbf{c}_{0,1} \arrow[dr, "\gamma"] \arrow[dd, "\beta_{0,1}"] & \\
\mathbf{c}_{1,1} \arrow[ur, "\epsilon"] \arrow[dd, "\beta_{0,0}"'] & & \mathbf{c}_{0,0} \arrow[dd, "\beta_{1,1}"] \\
& \mathbf{c}'_{0,1} \arrow[dr, "\gamma'"] & \\
\mathbf{c}'_{1,1} \arrow[ur, "\epsilon'"] & & \mathbf{c}'_{0,0},
\end{tikzcd}$$

where $\beta_{0,0}$ and $\beta_{1,1}$ are isomorphisms and $\beta_{0,1} \in adm$.

The category $\mathrm{defGrid}_{0,1}(\mathbf{C})^{adm}_{vert,horiz}$ is space of arrows $\mathbf{c}_0 \overset{\alpha}{\to} \mathbf{c}_1$. Hence, we see that the fiber of the functor

$$\mathrm{Factor}_{0,1}(\mathbf{C}) \to \mathrm{defGrid}_{0,1}(\mathbf{C})^{adm}_{vert,horiz}$$

over a given $(\mathbf{c}_0 \overset{\alpha}{\to} \mathbf{c}_1) \in \mathrm{defGrid}_{0,1}(\mathbf{C})^{adm}_{vert,horiz}$ is the category $\mathrm{Factor}(\alpha)$ from Sect. 5.1.4.

Hence, the contractibility of the fiber follows from the assumption in Sect. 5.1.4. The proof in the general case will be a rather straightforward combinatorial game.

5.7.2. The proof of Step C will proceed by induction on m. We shall perform the induction step, because the base of the induction (i.e., the case of $m = 0$) is similar, but simpler. So, we assume that the assertion is valid for $m - 1$, and we will now pass to the case of m.

Consider the map

$$\mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert,horiz} \to \mathrm{defGrid}_{m-1,n}(\mathbf{C})^{adm}_{vert,horiz},$$

given by restriction along

$$[m-1] \times [n] \to [m] \times [n],$$

where $[m-1] \to [m]$ is given by $i \mapsto i + 1$.

Note that the resulting map

$$\mathrm{Factor}_{m,n}(\mathbf{C}) \to \mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert,horiz} \underset{\mathrm{defGrid}_{m-1,n}(\mathbf{C})^{adm}_{vert,horiz}}{\times} \mathrm{Factor}_{m-1,n}(\mathbf{C})$$

is a co-Cartesian fibration. Hence, by induction, it suffices to show that it has contractible fibers.

Fix an object of

$$\mathbf{d} \in \mathrm{defGrid}_{m,n}(\mathbf{C})^{adm}_{vert,horiz} \underset{\mathrm{defGrid}_{m-1,n}(\mathbf{C})^{adm}_{vert,horiz}}{\times} \mathrm{Factor}_{m-1,n}(\mathbf{C}),$$

and we will analyze the fiber of $\mathrm{Factor}_{m,n}(\mathbf{C})$ over this object, denote this fiber by $\overline{\mathbf{C}}$.

5.7.3. Denote by $\underline{\mathbf{c}}$ the object of $\mathrm{Maps}([n]^{\mathrm{op}}, \mathbf{C})$ obtained from $\mathbf{d}$ by restriction along

$$\{1\} \times [n]^{\mathrm{op}} \to [m] \times [n]^{\mathrm{op}}.$$

Denote by $\overline{\mathbf{c}}'$ the object of $\mathrm{Maps}(([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}, \mathbf{C})$, obtained from $\mathbf{d}$ by restriction along

$$\{0\} \times ([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to [m-1] \times ([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}.$$

Denote by $\underline{\mathbf{c}}'$ the object of $\mathrm{Maps}([n]^{\mathrm{op}}, \mathbf{C})$ obtained from $\overline{\mathbf{c}}'$ be further restriction along

$$[n]^{\mathrm{op}} \to ([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}.$$

Note that the data of $\mathbf{d}$ gives a map

$$\underline{\mathbf{c}} \to \underline{\mathbf{c}}'. \tag{5.24}$$

The category $\overline{\mathbf{C}}$ is a 1-full subcategory in the category of functors, denoted $\overline{\mathbf{c}}$,

$$([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to \mathbf{C},$$

equipped with an identification of their restriction along $[n]^{\mathrm{op}} \to ([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$ with $\underline{\mathbf{c}}$, and with a natural transformation to $\overline{\mathbf{c}}'$, compatible with the identification $\overline{\mathbf{c}}'|_{[n]^{\mathrm{op}}} \simeq \underline{\mathbf{c}}'$ via the natural transformation (5.24).

The category $\overline{\mathbf{C}}$ is obtained by imposing the following conditions. At the level of objects we require:

(1) All maps $\mathbf{c}_{i,j} \to \mathbf{c}_{i-1,j}$ belong to *co-adm*;
(2) All maps $\mathbf{c}_{i,j-1} \to \mathbf{c}_{i,j-1}$ belong to *adm*;
(3) All maps $\mathbf{c}_{i,j} \to \mathbf{c}'_{i,j}$ belong to *vert*;
(4) The defect of Cartesianness of the squares

$$\begin{tikzcd} \mathbf{c}_{i,j} \arrow[r] \arrow[d] & \mathbf{c}_{i-1,j} \arrow[d] \\ \mathbf{c}'_{i,j} \arrow[r] & \mathbf{c}'_{i-1,j} \end{tikzcd}$$

belongs to $adm \cap co\text{-}adm$.

At the level of morphisms we allow maps $\overline{\mathbf{c}} \to \overline{\widetilde{\mathbf{c}}}$ such that for all i, j, the map

$$\mathbf{c}_{i,j} \to \widetilde{\mathbf{c}}_{i,j}$$

belongs to *adm*.

5.7.4. For $0 \leq n' \leq n$ let $\overline{\mathbf{C}}^{\leq n'}$ denote the following variant of $\overline{\mathbf{C}}$: instead of considering functors from $([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$ to $\mathbf{C}$, we consider functors defined on the full subcategory

$$([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}, \leq \mathrm{dgnl}+n'} \subset ([n]^{\mathrm{op}} \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}},$$

spanned by (i,j) with $j \leq i + n'$.

For example, for $n' = n$, we have $\overline{\mathbf{C}}^{\leq n} = \overline{\mathbf{C}}$, and for $n' = 0$, we have $\overline{\mathbf{C}}^{\leq n} = \{*\}$.

Restriction defines functors $\overline{\mathbf{C}}^{\leq n'} \to \overline{\mathbf{C}}^{\leq n'-1}$. We will prove by induction that the categories $\overline{\mathbf{C}}^{\leq n'}$ are contractible. We will use the following lemma:

Lemma 5.7.5. *Let $F : \mathbf{D}_1 \to \mathbf{D}_2$ be a functor between $(\infty,1)$-categories. Assume that:*

(a) *For every $\mathbf{d}_2 \in \mathbf{D}_2$, the category $\mathbf{D}_1 \underset{\mathbf{D}_2}{\times} \{\mathbf{d}_2\}$ is contractible;*

(b) *For every $\mathbf{d}_1 \in \mathbf{D}_1$ and a morphism $\beta : \mathbf{d}'_2 \to F(\mathbf{d}_1)$, the category of*

$$(\mathbf{d}'_1 \in \mathbf{D}_1, \alpha : \mathbf{d}'_1 \to \mathbf{d}_1, F(\alpha) \simeq \beta)$$

is contractible.

Then F induces an isomorphism between homotopy types.

5.7.6. We will show that the functors

$$\overline{\mathbf{C}}^{\leq n'} \to \overline{\mathbf{C}}^{\leq n'-1}$$

satisfy the conditions of Lemma 5.7.5. We will check condition (a), condition (b) being similar.

5.7.7. Fix an object $\overline{\mathbf{c}}^{\leq n'-1} \in \overline{\mathbf{C}}^{\leq n'-1}$. The fiber of $\overline{\mathbf{C}}^{\leq n'}$ is the product of the following categories, denoted $\mathbf{C}_i$, over the index $0 \leq i \leq n-n'-1$:

For each i, the category $\mathbf{C}_i$ is that of factorizations of the morphism

$$\mathbf{c}_{i+1,i+n'} \to \mathbf{c}_{i,i+n'-1}$$

(which is part of the data of $\overline{\mathbf{c}}^{\leq n'-1}$) as

$$\mathbf{c}_{i+1,i+n'} \xrightarrow{\epsilon} \mathbf{c}_{i,i+n'} \xrightarrow{\gamma} \mathbf{c}_{i,i+n'-1},$$

equipped with a datum of commutative diagram

$$\begin{tikzcd}
\mathbf{c}_{i+1,i+n'} \arrow[r] \arrow[d] & \mathbf{c}_{i,i+n'} \arrow[r] \arrow[d, "\beta"'] & \mathbf{c}_{i,i+n'-1} \arrow[d] \\
\mathbf{c}'_{i+1,i+n'} \arrow[r] & \mathbf{c}'_{i,i+n'} \arrow[r] & \mathbf{c}'_{i,i+n'-1}
\end{tikzcd}$$

such that

(1) $\epsilon \in co\text{-}adm$;
(2) $\gamma \in adm$;
(3) $\beta \in vert$;
(4) The datum of commutation of the outer square

$$\begin{tikzcd}
\mathbf{c}_{i+1,i+n'} \arrow[r] \arrow[d] & \mathbf{c}_{i,i+n'-1} \arrow[d] \\
\mathbf{c}'_{i+1,i+n'} \arrow[r] & \mathbf{c}'_{i,i+n'-1}
\end{tikzcd}$$

is that coming from $\overline{\mathbf{c}}^{\leq n-1}$;
(5) The defect of Cartesianness of the left square

$$\begin{tikzcd}
\mathbf{c}_{i+1,i+n'} \arrow[r] \arrow[d] & \mathbf{c}_{i,i+n'} \arrow[d, "\beta"'] \\
\mathbf{c}'_{i+1,i+n'} \arrow[r] & \mathbf{c}'_{i,i+n'}
\end{tikzcd}$$

belongs to $adm \cap co\text{-}adm$.

We claim that each of the category $\mathbf{C}_i$ is contractible.

5.7.8. Denote

$$\widetilde{\mathbf{c}}_{i,i+n'} := \mathbf{c}'_{i,i+n'} \underset{\mathbf{c}'_{i,i+n'-1}}{\times} \mathbf{c}_{i,i+n'-1}.$$

Then $\mathbf{C}_i$ is the category $\mathrm{Factor}(\alpha)$ of factorizations of the map

$$\alpha : \mathbf{c}_{i+1,i+n'} \to \widetilde{\mathbf{c}}_{i,i+n'}$$

as

$$\mathbf{c}_{i+1,i+n'} \to \mathbf{c}_{i,i+n'} \to \widetilde{\mathbf{c}}_{i,i+n'}$$

with the map $\mathbf{c}_{i+1,i+n'} \to \mathbf{c}_{i,i+n'}$ being in $co\text{-}adm$ and $\mathbf{c}_{i,i+n'} \to \widetilde{\mathbf{c}}_{i,i+n'}$ being in adm.

We note, however, that the morphisms $\mathbf{c}_{i+1,i+n'} \to \mathbf{c}_{i,i+n'-1}$ and $\mathbf{c}'_{i,i+n'} \to \mathbf{c}'_{i,i+n'-1}$ both belong to $horiz$. Hence, so does the morphism α, by the '2 out of 3' property.

Hence, the category $\mathrm{Factor}(\alpha)$ is contractible by the assumption in Sect. 5.1.4.