FILTERED COLIMITS OF ∞-CATEGORIES

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0.1. The ∞-category of (small) ∞-categories admits all limits and colimits. In general, colimits are difficult to describe in explicit terms. The purpose of this note is to give an explicit description of filtered colimits of ∞-categories.

Our conventions regarding ∞-categories follow those of [DG]. In particular, whenever we say category, unless otherwise specified, we will mean an ∞-category. Furthermore, for objects x, y in a category C, we will denote by $\mathrm{Maps}(x, y)$ the ∞-groupoid of maps from x to y, which we will also refer to as the mapping space from x to y.

0.2. Suppose that I is a filtered category (see [Lu, §5.3.1] for this notion in the ∞-categorical context), and we have a diagram of ∞-categories

$$I \to \infty\text{-}\mathrm{Cat}, \;\; i \mapsto \mathbf{C}_i.$$

Let $\mathbf{C}$ be the colimit $\underset{i\in I}{colim}\, \mathbf{C}_i$. Objects of $\mathbf{C}$ are given as images of objects of $\mathbf{C}_i$ under the tautological maps $\mathbf{C}_i \to \mathbf{C}$. Let $c_{i_0} \in \mathbf{C}_{i_0}$, $d_{i_1} \in \mathbf{C}_{i_1}$ be two objects and let c, d be their images in $\mathbf{C}$. For a map $i_0 \to i'_0$ (resp. $i_1 \to i'_1$) in I, we will denote by $c_{i'_0}$ (resp. $d_{i'_1}$) the image of c_{i_0} (resp. d_{i_1}) in $\mathbf{C}_{i'_0}$ (resp. $\mathbf{C}_{i'_1}$).

Let $I_{(i_0,i_1)/}$ be the category of diagrams $i_0 \to j \leftarrow i_1$ in I, i.e.

$$I_{(i_0,i_1)/} := I_{i_0/} \underset{I}{\times} I_{i_1/}.$$

Slightly abusing notation, we will denote an object $i_0 \to j \leftarrow i_1$ of $I_{(i_0,i_1)/}$ by j.

The goal of this note is to describe the mapping spaces in $\mathbf{C}$:

Lemma 0.2.1. *In the situation above, there is a natural isomorphism*

$$\mathrm{Maps}_{\mathbf{C}}(c, d) \simeq \underset{j\in I_{(i_0,i_1)/}}{colim} \mathrm{Maps}_{\mathbf{C}_j}(c_j, d_j).$$

0.3. We will give two proofs of this lemma. The first will be in the spirit of Lurie's book a model category theoretic proof somewhat in the spirit of [Lu], using the Joyal model structure on simplicial sets. The second proof will be intrinsic to the theory of ∞-categories without reference to model categories.

0.4. Initial observations. Since I is filtered, the category $I_{(i_0,i_1)/}$ is non-empty. Let $j \in I_{(i_0,i_1)/}$. The category $I_{j/}$ is filtered and the functor $I_{j/} \to I$ is cofinal. We can therefore replace I by $I_{j/}$ and c_{i_0} (resp. d_{i_1}) by c_j (resp. d_j). Thus, we can assume without loss of generality, that I has an initial object j and $i_0 \simeq i_1 \simeq j$. In this case, we wish to show

$$\mathrm{Maps}_{\mathbf{C}}(c, d) \simeq \underset{i\in I}{colim} \mathrm{Maps}_{\mathbf{C}_i}(c_i, d_i).$$

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0.5. First proof. By [Lu, Proposition 5.3.1.6], we may further assume that I is a poset. We can then represent the functor

$$I \to \infty\text{-}\mathrm{Cat}$$

by a functor

$$I \to sSet$$

into the (1-)category of simplicial sets with the Joyal model structure. The colimit ∞-category is given by the homotopy colimit of this functor. Replacing by a cofibrant resolution (in the projective model structure on diagrams) if necessary, we can assume that the homotopy colimit is given by the colimit in $sSet$.

Recall that in $sSet$ with the Joyal model structure, we have the fibrant replacement functor given by $X \mapsto N(|\mathfrak{C}[X]|)$ (in the notation of [Lu, §1.1.5]). This functor commutes with filtered colimits in $sSet$. In particular, we can assume that each simplicial set in the diagram is a quasi-category (and so is their colimit).

Now, given a quasi-category K and objects (i.e. 0-simplices) $x, y \in K^0$, the mapping space $\mathrm{Maps}_K(x, y)$ can be described as the following simplicial set (which is a Kan complex). Consider the maps

$$\mathfrak{d}_n : K^{n+1} \to K^0 \times K^n$$

from the $(n+1)$-simplices of K to the product of the 0-simplices and the n-simplices of K given by the maps $[0] \ni 0 \mapsto 0 \in [n+1]$ and $[n] \ni i \mapsto (i+1) \in [n+1]$. The n-simplicies of $\mathrm{Maps}_K(x, y)$ are then given by the fiber of $\mathfrak{d}_n$ over the point (x, y), where the y in the second component denotes the degenerate n-simplex. The desired result now follows from the fact that finite limits (e.g. fiber products) and filtered colimits commute in sets. □

0.6. Second proof. By [Re] and [JT], we have a fully faithful functor (of ∞-categories)

$$CS : \infty\text{-}\mathrm{Cat} \to (\infty\text{-}Grpd)^{\Delta^{op}}$$

given by $\mathbf{C} \mapsto (I \mapsto \mathrm{Maps}_{\infty\text{-}\mathrm{Cat}}(I, \mathbf{C}))$ for a category $\mathbf{C}$ and $I \in \Delta$. In other words, the space of n-simplices of $CS(\mathbf{C})$ is given by the mapping space from $[n]$ to $\mathbf{C}$. This functor admits a left adjoint. The essential image of this functor, defined by Rezk, is given by "complete Segal spaces."

The condition for a simplicial ∞-groupoid to be a complete Segal space is given by the requirement that certain finite diagrams are limit diagrams (see [Re]). In particular, since finite limits commute with filtered colimits in $\infty\text{-}Grpd$, we have that a filtered colimit of complete Segal spaces is a complete Segal space. Thus, CS preserves filtered colimits.

Given two objects $x, y \in \mathbf{C}$, the mapping space $\mathrm{Maps}(x, y)$ is isomorphic to the fiber of the map $CS(\mathbf{C})^1 \to CS(\mathbf{C})^0 \times CS(\mathbf{C})^0$ (given by source and target) over the point (x, y). Now, since $CS(\mathbf{C}) = colim\, CS(\mathbf{C}_i)$ and filtered colimits commute with finite limits, we have that

$$\mathrm{Maps}_{\mathbf{C}}(c, d) = colim\ \mathrm{Maps}_{\mathbf{C}_i}(c_i, d_i)$$

as desired.

□

References

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[Lu] J. Lurie, *Higher topos theory.*
[Re] C. Rezk, *A model for the homotopy theory of homotopy theory.*