ASYMPTOTICS OF GEOMETRIC WHITTAKER COEFFICIENTS

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Introduction

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Part I: The Key Calculation

1. Statement of the problem

1.1. **The geometric objects.**

1.1.1. Recall the stack $\overline{\mathrm{Bun}}_B$ defined as in [BG1]. Recall that $\overline{\mathrm{Bun}}_B$ admits a decomposition into locally closed substacks, parameterized by elements $\lambda \in \Lambda^{\mathrm{pos}}$,

$$\overline{\mathrm{Bun}}_B = \bigcup_{\lambda \in \Lambda^{\mathrm{pos}}} \overline{\mathrm{Bun}}_B^{=\lambda}.$$

For each $\lambda \in \Lambda^{\mathrm{pos}}$ we let

$$\jmath^\lambda : \overline{\mathrm{Bun}}_B^{=\lambda} \hookrightarrow \overline{\mathrm{Bun}}_B$$

denote the corresponding locally closed embedding.

We have a canonical identification

$$\overline{\mathrm{Bun}}_B^{=\lambda} \simeq X^\lambda \times \mathrm{Bun}_B\,.$$

We let $\overline{\mathrm{Bun}}_B^{\leq\lambda}$ denote the open substack of $\overline{\mathrm{Bun}}_B$ equal to

$$\bigcup_{0 \leq \mu \leq \lambda} \overline{\mathrm{Bun}}_B^{=\mu}.$$

1.1.2. We let

$$\overline{\mathsf{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G \text{ and } \overline{\mathsf{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$$

denote the canonical projections, and by

$$\mathsf{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G \text{ and } \mathsf{q} : \mathrm{Bun}_B \to \mathrm{Bun}_T$$

their restrictions to $\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$, respectively.

Note that we have a commutative diagram

$$\begin{CD}
X^\lambda \times \mathrm{Bun}_B @>{\sim}>> \overline{\mathrm{Bun}}_B^{=\lambda} \\
@V{\mathrm{id} \times \mathsf{q}}VV @VV{\overline{\mathsf{q}} \circ \jmath^\lambda}V \\
X^\lambda \times \mathrm{Bun}_T @>>> \mathrm{Bun}_T,
\end{CD}$$

where the bottom horizontal arrow is the map

$$D, \mathcal{P}_T \mapsto \mathcal{P}_T(-D),$$

where $D \in X^\lambda$ is a Λ^{pos}-colored divisor.

1.1.3. Let $\mathcal{P}_T$ be a T-bundle. We let $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ denote the fiber product

$$\overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{pt},$$

where the map $\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ is $\overline{\mathfrak{q}}$ and $\mathrm{pt} \to \mathrm{Bun}_T$ is given by $\mathcal{P}_T$.

The group T acts by automorphisms of $\mathcal{P}_T$; by functoriality, we obtain a T-action on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$. The quotient stack $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T$ identifies with

$$\overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{pt}/T;$$

in particular, the resulting map

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T \to \overline{\mathrm{Bun}}_B$$

is a closed embedding.

We let $\mathfrak{r}$ denote the tautological projection $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T$.

The stack $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ inherits a decomposition into locally closed substacks with

$$\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T} \simeq X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B,$$

where the map $X^\lambda \to \mathrm{Bun}_T$ is

$$D \mapsto \mathcal{P}_T(D).$$

By a slight abuse of notation, we shall use the same symbol $\jmath^\lambda$ to denote also the maps

$$\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T} \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \text{ and } \overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}/T \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T,$$

respectively.

Similarly, by a slight abuse of notation we will denote by $\mathrm{id} \times \mathfrak{q}$ also the maps

$$\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T} \simeq X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \overset{\mathrm{id} \times \mathfrak{q}}{\longrightarrow} X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_T = X^\lambda$$

and

$$\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}/T \simeq (X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B)/T \overset{\mathrm{id} \times \mathfrak{q}}{\longrightarrow} (X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_T)/T = X^\lambda \times \mathrm{pt}/T, \tag{1.1}$$

respectively.

1.2. **The basic character sheaf.**

1.2.1. In what follows we fix one and for all a line bundle $\omega_X^{\frac{1}{2}}$ equipped with an isomorphsim

$$(\omega_X^{\frac{1}{2}})^{\otimes 2} \simeq \omega_X.$$

We let $\rho(\omega_X)$ denote the T-bundle on X induced from the line bundle $\omega_X^{\frac{1}{2}}$ by means of the cocharacter $2\rho : \mathbb{G}_m \to T$.

Consider the stack

$$\mathrm{Bun}_{N,\rho(\omega_X)}\,.$$

As in [FGV], there exists a canonical map

$$\mathrm{ev} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathbb{A}^1.$$

1.2.2. The map ev is constructed as follows. Consider the map

$$B \to B/[N,N]Z_G \simeq \prod_i (\mathbb{G}_m \ltimes \mathbb{G}_a),$$

where i runs through the set of vertices of the Dynkin diagram of G. From here we obtain a map of stacks

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to \prod_i \left(\mathrm{Bun}_{\mathbb{G}_m \ltimes \mathbb{G}_a} \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \mathrm{pt}\right),$$

where $\mathrm{pt} \to \mathrm{Bun}_{\mathbb{G}_m}$ is given by the line bundle ω_X.

Now, the stack $\mathrm{Bun}_{\mathbb{G}_m \ltimes \mathbb{G}_a} \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \mathrm{pt}$ classfies short exact sequences

$$0 \to \omega_X \to \mathcal{E} \to \mathcal{O}_X \to 0,$$

and hence is equipped with a canonocal map to $\mathbb{A}^1$.

Finally, we let ev be the composition

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to \prod_i \left(\mathrm{Bun}_{\mathbb{G}_m \ltimes \mathbb{G}_a} \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \mathrm{pt}\right) \to \prod_i \mathbb{A}^1 \xrightarrow{\mathrm{sum}} \mathbb{A}^1. \tag{1.2}$$

1.2.3. Set

$$\psi := \mathrm{ev}^*(\text{A-Sch}),$$

where A-Sch $\in$ D-mod$(\mathbb{A}^1)$ is the algebraic-Schreier sheaf (i.e., the exponential D-module, shifted cohomologically by $[-1]$).

Set

$$\overline{\psi} := (\jmath^0)_!(\psi) \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}).$$

Remark 1.2.4. As was observed in [FGV], the extension $\psi \rightsquigarrow \overline{\psi}$ is clean, i.e., the canonical map

$$(\jmath^0)_!(\psi) \to (\jmath^0)_*(\psi) \tag{1.3}$$

is an isomorphism.

1.2.5. Consider the object

$$\mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T).$$

For each $\lambda \in \Lambda^{\mathrm{pos}}$ consider

$$(\jmath^\lambda)^! \circ \mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(\overline{\mathrm{Bun}}^{=\lambda}_{N,\rho(\omega_X)}/T).$$

Consider now the object

$$\mathcal{M}^\lambda := (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! \circ \mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(X^\lambda \times \mathrm{pt}/T), \tag{1.4}$$

where $\mathrm{id} \times \mathsf{q} : \overline{\mathrm{Bun}}^{=\lambda}_{N,\rho(\omega_X)}/T \to X^\lambda \times \mathrm{pt}/T$ is as in (1.1).

1.2.6. The goal of Part I of this paper is concerned with the object $\mathcal{M}^\lambda$ of (1.4). On the one hand, in Theorem 1.3.6, we will calculate $\mathcal{M}^\lambda$ more or less explicitly.

On the other hand, and which is more important, in Theorem 1.4.6, we will relate $\mathcal{M}^\lambda$ to the calculation of a certain *Constant Term* functor.

The proofs of Theorems 1.3.6 and 1.4.6 will share a common core, given by Theorem 2.2.3. The latter theorem (like most of the assertions of this kind) is a corollary of Braden's theorem about hyperbolic restrictions; the deduction will be the subject of Part II of the paper.

In Part III of the paper we will explain how Theorem 1.4.6 fits into the geometric Langlands program.

1.2.7. *Example.* Let us describe explicitly the object $\mathcal{M}^0$, i.e., $\mathcal{M}^\lambda$ for $\lambda = 0$. This is easy to do "by hand". Namely, we claim that $\mathcal{M}^0 \in \text{D-mod}(\text{pt}/T)$ identifies with $H_c(T)$, where the latter is the direct image with compact supports of $k \in \text{Vect} = \text{D-mod}(\text{pt})$ under the map

$$\text{triv} : \text{pt} \to \text{pt}/T.$$

Indeed, note that in the composition (1.2), the first map is a smooth unipoten gerbe (see [DrGa1] for what this means), so the functors of *-pullback and *-pushforward define mutually inverse equivalences of categories.

Hence, $\mathcal{M}^0$ identifies with the direct image under $\mathbb{A}^r/T \to \text{pt}/T$ of the direct image with compact supports under $\mathbb{A}^r \to \mathbb{A}^r/T$ of

$$\text{A-Sch}^{\boxtimes r} \in \text{D-mod}(\mathbb{A}^r),$$

where r denotes the semi-simple rank of G.

Hence, $\mathcal{M}^0$ identifies with the direct image under $\mathbb{A}^r/T \to \text{pt}/T$ of the direct image with compact supports under $\mathbb{A}^r \to \mathbb{A}^r/T$ of

$$\text{A-Sch}^{\boxtimes r} \in \text{D-mod}(\mathbb{A}^r).$$

By the contraction principle (see [DrGa2]), the operation of direct image under $\mathbb{A}^r/T \to \text{pt}/T$ can be replaced by that of *-pullback under $\text{pt}/T \to \mathbb{A}^r/T$.

The stated answer for $\mathcal{M}^0$ follows now by base change from the Cartesian square

$$\begin{tikzcd} \text{pt} \arrow[r] \arrow[d, "\text{triv}"'] & \mathbb{A}^r \arrow[d] \\ \text{pt}/T \arrow[r] & \mathbb{A}^r/T. \end{tikzcd}$$

1.3. **The sheaves Ω^λ.**

1.3.1. In this subsection we recall the objects

$$\Omega^\lambda \in \text{D-mod}(X^\lambda)^\heartsuit, \quad \lambda \in \Lambda^{\text{pos}},$$

introduced in [BG2].

The object Ω^λ is characterised by the requirement that its !-fiber at

$$D = \sum_i \lambda_i \cdot x_i \in X^\lambda, \quad x_i \neq x_j,$$

identifies with

$$\bigotimes_i (\text{C}^\cdot(\check{\mathfrak{n}}))^{-\lambda_i},$$

where $\text{C}^\cdot(\check{\mathfrak{n}})$ is the cohomological Chevalley complex of the Lie algebra $\check{\mathfrak{n}}$, and superscript $-\lambda_i$ means the $(-\lambda_i)$-graded component with respect to the adjoint action of $\check{T}$. The sheaf structure on Ω^λ is given by the structure on $\text{C}^\cdot(\check{\mathfrak{n}})$ of commutative DG algebra.

1.3.2. *Examples.* For $G = SL_2$ and $\lambda = n \in \mathbb{Z}^{\geq 0}$ (so that X^λ is the n-th symmetric power $X^{(n)}$), the sheaf Ω^λ is the (clean) extension of the sign local system on $\overset{\circ}{X}{}^{(n)}$.

The same happens for any G for $\lambda = n \cdot \alpha_i$, where α_i is a simple coroot.

Consider now $G = SL_3$ with its two simple roots α_1 and α_2. For $\lambda = \alpha_1 + \alpha_2$ we have $X^\lambda = X \times X$, and we have

$$\Omega^{\alpha_1+\alpha_2} = j_*(k_X \boxtimes k_X)[2],$$

where $j : X \times X - \Delta(X) \hookrightarrow X \times X$.

1.3.3. Although this will not be necessary for the sequel, let us add a few more pieces of information regarding the sheaves Ω^λ.

First, the assigment

$$\lambda \mapsto \Omega^\lambda$$

has a factorization property:

$$\Omega^{\lambda_1+\lambda_2}|_{(X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}}} \simeq \Omega^{\lambda_1} \boxtimes \Omega^{\lambda_2}|_{(X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}}}, \tag{1.5}$$

where

$$(X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}} \subset X^{\lambda_1} \times X^{\lambda_2}$$

is the open subset corresponding to pairs

$$(D_1, D_2), \quad \mathrm{supp}(D_1) \cap \mathrm{supp}(D_2) = \emptyset.$$

Note that the addition map

$$X^{\lambda_1} \times X^{\lambda_2} \to X^{\lambda_1+\lambda_2}$$

is étale when retsricted to $(X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}}$, so in the left-hand side of (1.5) we can take either the ! or the *-retsriction.

1.3.4. The isomorphism (1.5) allows to describe the sheaves Ω^λ inductively. Indeed, suppose we know Ω^λ for $\lambda' < \lambda$. Then (1.5) implies that we know the restriction of Ω^λ to $X^\lambda - \Delta(X) \overset{j}{\hookrightarrow} X^\lambda$.

According to [BG2, Proposition 3.2], we have:

$$\Omega^\lambda = \begin{cases} k_X[1] \text{ if } \lambda \text{ is a simple coroot,} \\ j_{!*}(\Omega^\lambda|_{X^\lambda - \Delta(X)}) \text{ if } \exists\, w \in W,\ \lambda = \rho - w(\rho) \text{ with } \ell(w) = 2, \\ j_{!*}(\Omega^\lambda|_{X^\lambda - \Delta(X)}) \simeq H^0(j_*(\Omega^\lambda|_{X^\lambda - \Delta(X)})),\ \text{if } \exists\, w \in W,\ \lambda = \rho - w(\rho) \text{ with } \ell(w) \geq 3, \\ H^0(j_*(\Omega^\lambda|_{X^\lambda - \Delta(X)})) \simeq j_*(\Omega^\lambda|_{X^\lambda - \Delta(X)}) \text{ if } \nexists\, w \in W,\ \lambda = \rho - w(\rho), \end{cases}$$

where H^0 referes to taking cohomology with respect to the D-module (i.e., perverse) t-structure.

1.3.5. The first main result of Part I of this paper reads:

Theorem 1.3.6. *There exists a canonical isomorphism in* D-mod$(X^\lambda \times \mathrm{pt}/T)$

$$\mathcal{M}^\lambda \simeq \Omega^\lambda[-2|\lambda|] \boxtimes H_c(T),$$

where $|\lambda|$ is the length of λ.

1.4. **Relation to the Constant Term functor(s).**

1.4.1. Recall that the map

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T \to \overline{\mathrm{Bun}}_B$$

is a closed embedding.

By a slight abuse of notation we shall identify objects of D-mod$(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T)$ with the corresponding objects of D-mod$(\overline{\mathrm{Bun}}_B)$.

Note that we have a commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{Bun}}^{=\lambda}_{N,\rho(\omega_X)}/T \arrow[r] \arrow[d, "\sim"'] & \overline{\mathrm{Bun}}^{=\lambda}_B \arrow[r, "\mathrm{id}\times\mathsf{q}"] & X^\lambda \times \mathrm{Bun}_T \arrow[d, "p_{X^\lambda}\times\mathrm{id}"] \\
(X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B)/T \arrow[r] & X^\lambda \times \mathrm{pt}/T \arrow[r, "\mathrm{AJ}"] & \mathrm{Bun}_T,
\end{tikzcd}$$

where AJ is a version of the Abel-Jacobi map,

$$D \mapsto \rho(\omega_X)(D),$$

and where $p_{X^\lambda} : X^\lambda \to \mathrm{pt}$.

1.4.2. Let $\mathcal{W} \in$ D-mod(Bun_G) denote the direct image of $\psi \in$ D-mod$(\mathrm{Bun}_{N,\rho(\omega_X)})$ with compact supports under the forgetful map

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G\,.$$

Tautologically,

$$\mathcal{W} \simeq \overline{\mathsf{p}}_! \circ \mathfrak{r}_!(\overline{\psi}).$$

Sometimes, the object $\mathcal{W} \in$ D-mod(Bun_G) goes under the same *the first Whittaker coefficient*.

1.4.3. Recall the *Constant Term* functor

$$\mathrm{CT}_* := \mathsf{q}_* \circ \mathsf{p}^! : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_T),$$

see [DrGa4].

Recall the object $\mathcal{M}^\lambda \in$ D-mod$(X^\lambda \times \mathrm{pt}/T)$, see Sect. 1.4. We claim that there exists a canonically defined map

$$\mathrm{AJ}_!(\mathcal{M}^\lambda) \to \mathrm{CT}_*(\mathcal{W}). \tag{1.6}$$

Indeed, let us write

$$\mathrm{AJ}_!(\mathcal{M}^\lambda) = (p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! \circ \mathfrak{r}_!(\overline{\psi}) \text{ and } \mathcal{W} = \overline{\mathsf{p}}_! \circ \mathfrak{r}_!(\overline{\psi}),$$

and we claim that there is a natural transformation

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! \to \mathrm{CT}_* \circ \overline{\mathsf{p}}_!. \tag{1.7}$$

1.4.4. Namely, consider the unit of the adjunction

$$\mathrm{Id} \to \overline{\mathsf{p}}^! \circ \overline{\mathsf{p}}_!,$$

and apply to both sides the functor

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! : \text{D-mod}(\overline{\mathrm{Bun}}_B) \to \text{D-mod}(\mathrm{Bun}_T).$$

We obtain a map

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! \to (p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (\jmath^\lambda)^! \circ \overline{\mathsf{p}}^! \circ \overline{\mathsf{p}}_! \simeq \tag{1.8}$$
$$\simeq (p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ \left(\overline{\mathsf{p}} \circ \jmath^\lambda\right)^! \circ \overline{\mathsf{p}}_!.$$

So, it is enough to construct a natural transformation

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ \left(\overline{\mathsf{p}} \circ \jmath^\lambda\right)^! \to \mathrm{CT}_* . \tag{1.9}$$

However,

$$\overline{\mathsf{p}} \circ \jmath^\lambda = \mathsf{p} \circ (p_{X^\lambda} \times \mathrm{id}) : X^\lambda \times \mathrm{Bun}_B \to \mathrm{Bun}_G,$$

so the expression in (1.9) is isomorphic to

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \circ (p_{X^\lambda} \times \mathrm{id})^! \circ \mathsf{p}^!. \tag{1.10}$$

Further, from the Cartesian diagram

$$\begin{array}{ccc} X^\lambda \times \mathrm{Bun}_B & \xrightarrow{p_{X^\lambda} \times \mathrm{id}} & \mathrm{Bun}_B \\ {\scriptstyle \mathrm{id} \times \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{q}} \\ X^\lambda \times \mathrm{Bun}_T & \xrightarrow{p_{X^\lambda} \times \mathrm{id}} & \mathrm{Bun}_T \end{array}$$

we obtain a natural transformation (in fact, an isomorphism, since the horizontal maps are proper)

$$(p_{X^\lambda} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathsf{q})_* \to \mathsf{q}_* \circ (p_{X^\lambda} \times \mathrm{id})_!.$$

Hence, the expression in (1.10) maps (in fact, isomorphically) to

$$\mathsf{q}_* \circ (p_{X^\lambda} \times \mathrm{id})_! \circ (p_{X^\lambda} \times \mathrm{id})^! \circ \mathsf{p}^!. \tag{1.11}$$

Finally, applying the co-unit of the adjunction $(p_{X^\lambda} \times \mathrm{id})_! \circ (p_{X^\lambda} \times \mathrm{id})^! \to \mathrm{Id}$, we obtain that the expression in (1.11) maps to

$$\mathsf{q}_* \circ \mathsf{p}^! = \mathrm{CT}_*,$$

as desired.

1.4.5. The main result of Part I of this paper is:

Theorem 1.4.6. *The map* (1.6) *is an isomorphism.*

2. Calculation via the Zastava spaces

2.1. **Zastava spaces: recollections.**

2.1.1. Let $Z^\lambda_{\mathcal{P}_T}$ denote the Zastava space corresponding to $\mathcal{P}_T \in \mathrm{Bun}_T$.

I.e., $Z^\lambda_{\mathcal{P}_T}$ is the open substack

$$\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-,\lambda+\deg(\mathcal{P}_T)}\right)^{\mathrm{gen.trans}} \subset \overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-,\lambda+\deg(\mathcal{P}_T)},$$

corresponding to B- and B^--reductions that are transversal at the generic point of the curve.

In the above formula $\mathrm{Bun}_{B^-,\lambda+\deg(\mathcal{P}_T)}$ is the connected component of Bun_{B^-} corresponding to B-bundles, for which the induced T-bundle has degree $\lambda + \deg(\mathcal{P}_T)$.

Remark 2.1.2. A basic feature of $Z^\lambda_{\mathcal{P}_T}$ is that it is actually a (quasi-projective) scheme.

2.1.3. We let

$$'\mathsf{p}^- : Z^\lambda_{\mathcal{P}_T} \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$$

denote the tautological projection.

For $0 \leq \mu \leq \lambda$, we let $'\jmath^\mu$ denote the locally closed embedding

$$Z^{\lambda,=\mu}_{\mathcal{P}_T} := Z^\lambda_{\mathcal{P}_T} \underset{\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}}{\times} \overline{\mathrm{Bun}}^{=\mu}_{N,\mathcal{P}_T} \hookrightarrow Z^\lambda_{\mathcal{P}_T}.$$

In particular, for $\mu = 0$ we let

$$\overset{\circ}{Z}{}^\lambda_{\mathcal{P}_T} \subset Z^\lambda_{\mathcal{P}_T}$$

denote the corresponding open subscheme.

2.1.4. According to [BFGM], there exists a canonical projection

$$\pi^\lambda : Z^\lambda_{\mathcal{P}_T} \to X^\lambda$$

that makes the diagram

$$\begin{tikzcd} Z^\lambda_{\mathcal{P}_T} \arrow[r] \arrow[d, "\pi^\lambda"'] & \mathrm{Bun}_{B^-,\lambda+\deg(\mathcal{P}_T)} \arrow[d] \\ X^\lambda \arrow[r] & \mathrm{Bun}_T \end{tikzcd}$$

commute, where the bottom horizontal arrow is the map $D \mapsto \mathcal{P}_T(D)$.

It is known that the map π^λ induces an isomorphism between $Z^{\lambda,=\lambda}_{\mathcal{P}_T}$ and X^λ. Hence, the map $'\jmath^\lambda$ provides a section of the projection π^λ. We shall also use the notation

$$\mathfrak{s}^\lambda := {}'\jmath^\lambda.$$

The composed map

$$X^\lambda \overset{\mathfrak{s}^\lambda}{\simeq} Z^{\lambda,=\lambda}_{\mathcal{P}_T} \xrightarrow{'\mathsf{p}^-} \overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T} = X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B$$

equals the map ι^λ.

2.2. **An adaptation of Braden's theorem.**

2.2.1. For any $\mathcal{P}_T$ consider the diagram

$$\begin{tikzcd}
X^\lambda \times \mathrm{pt}/T \arrow[r, "\mathfrak{s}^\lambda"] \arrow[d, "\iota^\lambda"'] & Z^\lambda_{\mathcal{P}_T}/T \arrow[r, "\pi^\lambda"] \arrow[d, "{'\mathsf{p}^-}"] & X^\lambda \times \mathrm{pt}/T \\
(X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B)/T \arrow[r, "\jmath^\lambda"] \arrow[d, "\mathrm{id}\times\mathsf{q}"'] & \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T & \\
X^\lambda \times \mathrm{pt}/T, & &
\end{tikzcd} \tag{2.1}$$

in which the inner square is Cartesian and the composite maps are both equal to $\mathrm{id}_{X^\lambda\times\mathrm{pt}/T}$.

By base change, we have an isomorphism of functors

$$(\mathrm{id}\times\mathsf{q})_*\circ(\jmath^\lambda)^!\circ({'\mathsf{p}^-})_*\circ(\pi^\lambda)^! \simeq \mathrm{Id}_{\text{D-mod}(X^\lambda\times\mathrm{pt}/T)}, \tag{2.2}$$

from which we obtain a natural transformation of functors

$$(\mathrm{id}\times\mathsf{q})_*\circ(\jmath^\lambda)^! \to (\pi^\lambda)_!\circ({'\mathsf{p}^-})^*, \quad \text{D-mod}(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T)\to\text{D-mod}(X^\lambda\times\mathrm{pt}/T). \tag{2.3}$$

2.2.2. We will apply Braden's theorem (see [DrGa3]) to prove:

Theorem 2.2.3. *The natural transformation* (2.3) *is an isomorphism.*

Theorem 2.2.3 will be proved in Sect. 5.1.

2.3. **Proof of Theorem 1.3.6.** In this subsection we will show how Theorem 2.2.3 implies Theorem 1.3.6.

2.3.1. Take $\mathcal{P}_T=\rho(\omega_X)$, and consider the corresponding Zastava space $Z^\lambda_{\rho(\omega_X)}$. Recall the map

$${'\mathsf{p}^-}: Z^\lambda_{\rho(\omega_X)} \to \overline{\mathrm{Bun}}_{N,\rho(\omega_X)}.$$

The key ingredient that connects $\mathcal{M}^\lambda$ with Ω^λ is provided by the followiing theorem (see [Ras]):

Theorem 2.3.2. *There exists a canonical isomorphism*

$$\pi^\lambda_!\circ({'\mathsf{p}^-})^*(\overline{\psi}) \simeq \Omega^\lambda[-2|\lambda|].$$

2.3.3. Thus, in order to prove Theorem 1.3.6, we need to construct a canonical isomorphism

$$(\mathrm{id}\times\mathsf{q})_*\circ(\jmath^\lambda)^!\circ\mathfrak{r}_!(\overline{\psi}) \simeq \pi^\lambda_!\circ({'\mathsf{p}^-})^*(\overline{\psi})\boxtimes H_c(T) \tag{2.4}$$

of objects of D-mod$(X^\lambda\times\mathrm{pt}/T)$.

Let us apply the natural isomorphism (2.3) to

$$\mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T).$$

We obtain that the left-hand side identifies with

$$(\pi^\lambda)_!\circ({'\mathsf{p}^-})^*\circ\mathfrak{r}_!(\overline{\psi}).$$

Applying base change along the lower square in the diagram

$$\begin{array}{ccc}
X^\lambda & \xrightarrow{\mathrm{id}\times\mathrm{triv}} & X^\lambda \times \mathrm{pt}/T \\
{\scriptstyle \pi^\lambda}\uparrow & & \uparrow{\scriptstyle \pi^\lambda} \\
Z^\lambda_{\mathcal{P}_T} & \xrightarrow{'\mathfrak{r}} & Z^\lambda_{\mathcal{P}_T}/T \\
{\scriptstyle '\mathsf{p}^-}\downarrow & & \downarrow{\scriptstyle '\mathsf{p}^-} \\
\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} & \xrightarrow{\mathfrak{r}} & \overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/T,
\end{array}$$

we rewrite

$$(\pi^\lambda)_! \circ ('\mathsf{p}^-)^* \circ \mathfrak{r}_!(\overline{\psi}) \simeq (\pi^\lambda)_! \circ ('\mathfrak{r})_! \circ ('\mathsf{p}^-)^*(\overline{\psi}) \simeq (\mathrm{id}\times\mathrm{triv})_! \circ (\pi^\lambda)_! \circ ('\mathsf{p}^-)^*(\overline{\psi}) \simeq$$
$$\simeq (\pi^\lambda)_! \circ ('\mathsf{p}^-)^*(\overline{\psi}) \boxtimes H_c(T),$$

as required.

2.4. **A reduction step towards the proof of Theorem 2.2.3.** As was mentioned earlier, Theorem 2.2.3 will be deduced from Braden's theorem. We would like to apply Braden's theorem to the algebraic *stack* $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$. The problem is that we cannot quite do this, because Braden's theorem is only known for algebraic *spaces*, and not general algebraic stacks.

In this subsection we will show that the assertion of Theorem 2.2.3 can be formally deduced from the case when the pair $(\mathcal{P}_T, \lambda)$ is such that $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}$ is an algebraic space, in which case Braden's theorem can be applied.

That said, we should say that, given Theorem 5.1.3 established in Part II of the paper, we can reprove an analog of Braden's theorem specifically for the stack $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$, using the method of [DrGa3]. I.e., the reduction to the case of algebraic spaces is not really necessary.

2.4.1. Let us be given a diagram of algebraic stacks

(2.5)
$$\begin{array}{ccccc}
\mathcal{Y}^0 & \xrightarrow{\iota^-} & \mathcal{Y}^- & \xrightarrow{\mathsf{q}^-} & \mathcal{Y}^0 \\
{\scriptstyle \iota^+}\downarrow & & \downarrow{\scriptstyle \mathsf{p}^-} & & \\
\mathcal{Y}^+ & \xrightarrow{\mathsf{p}^+} & \mathcal{Y} & & \\
{\scriptstyle \mathsf{q}^+}\downarrow & & & & \\
\mathcal{Y}^0, & & & &
\end{array}$$

where

$$\mathsf{q}^+ \circ \iota^+ = \mathrm{id}_{\mathcal{Y}^-} = \mathsf{q}^- \circ \iota^-.$$

and where the map

$$\mathcal{Y}^0 \to \mathcal{Y}^+ \underset{\mathcal{Y}}{\times} \mathcal{Y}^-$$

is an open embedding.

Then as in (2.2), we obtain a natural transformation

$$\mathsf{q}^+_* \circ (\mathsf{p}^+)^! \circ (\mathsf{p}^-)_* \circ (\mathsf{q}^-)^! \to \mathrm{Id}_{\text{D-mod}(\mathcal{Y}^0)},$$

from which we obtain a natural transformation

(2.6)
$$(\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \to (\mathsf{q}^-)_! \circ (\mathsf{p}^-)^*.$$

2.4.2. We wish to know when for a given

$$\mathcal{F} \in \text{D-mod}(\mathcal{Y})$$

the resulting map $(\mathsf{q}^+)_* \circ (\mathsf{p}^+)^!(\mathcal{F}) \to (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^!(\mathcal{F})$ is an isomorphism.

Suppose that we are given a similar diagram

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}^0 \arrow[r, "\widetilde{\iota}^-"] \arrow[d, "\widetilde{\iota}^+"'] & \widetilde{\mathcal{Y}}^- \arrow[r, "\widetilde{\mathsf{q}}^-"] \arrow[d, "\widetilde{\mathsf{p}}^-"] & \widetilde{\mathcal{Y}}^0 \\
\widetilde{\mathcal{Y}}^+ \arrow[r, "\widetilde{\mathsf{p}}^+"] \arrow[d, "\widetilde{\mathsf{q}}^+"'] & \widetilde{\mathcal{Y}} & \\
\mathcal{Y}^0. & &
\end{tikzcd} \tag{2.7}$$

2.4.3. Suppose that we are given a map from the diagram (2.7) to the diagram (2.5), such that:

- The maps $\phi^- : \widetilde{\mathcal{Y}}^- \to \mathcal{Y}^-$ and $\phi^0 : \widetilde{\mathcal{Y}}^0 \to \mathcal{Y}^0$ are isomorphisms.;
- The maps $\phi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ and $\phi^+ : \widetilde{\mathcal{Y}}^+ \to \mathcal{Y}^+$ are smooth;
- The diagram

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}^+ \arrow[r, "\widetilde{\mathsf{p}}^+"] \arrow[d, "\phi^+"'] & \widetilde{\mathcal{Y}} \arrow[d, "\phi"] \\
\mathcal{Y}^+ \arrow[r, "\mathsf{p}^+"] & \mathcal{Y}
\end{tikzcd}$$

  is Cartesian.

The following is an easy particular case of [DrGa4, Theorem 4.3.4]

Lemma 2.4.4. *Let $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ be such that for $\widetilde{\mathcal{F}} := \phi^*(\mathcal{F})$ the map*

$$(\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^!(\widetilde{\mathcal{F}}) \to (\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^!(\widetilde{\mathcal{F}})$$

is an isomorphism. Assume also that the maps

$$(\mathsf{q}^+)_* \circ (\mathsf{p}^+)^!(\mathcal{F}) \to (\mathsf{q}^+)_* \circ (\iota^+)_* \circ (\iota^+)^* \circ (\mathsf{p}^+)^!(\mathcal{F}) \simeq (\iota^+)^* \circ (\mathsf{p}^+)^!(\mathcal{F}) \tag{2.8}$$

and

$$(\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^!(\widetilde{\mathcal{F}}) \to (\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\iota}^+)_* \circ (\widetilde{\iota}^+)^* \circ (\widetilde{\mathsf{p}}^+)^!(\widetilde{\mathcal{F}}) \simeq (\widetilde{\iota}^+)^* \circ (\widetilde{\mathsf{p}}^+)^!(\widetilde{\mathcal{F}}) \tag{2.9}$$

are isomorphisms. Then the map

$$(\mathsf{q}^+)_* \circ (\mathsf{p}^+)^!(\mathcal{F}) \to (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^!(\mathcal{F})$$

is an isomorphism.

Proof. Diagram chase shows that we have the following commutative diagram of natural transformations

$$\begin{tikzcd}
(\phi^0)^* \circ (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \arrow[r, "(\phi^0)^*(2.6)"] \arrow[d] & (\phi^0)^* \circ (\mathsf{q}^-)_! \circ (\mathsf{p}^-)^* \arrow[d] \\
(\widetilde{\mathsf{q}}^+)_* \circ (\phi^+)^* \circ (\mathsf{p}^+)^! \arrow[d] & (\widetilde{\mathsf{q}}^-)_! \circ (\phi^-)^* \circ (\mathsf{p}^-)^* \arrow[d, "\sim"] \\
(\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^! \circ \phi^* \arrow[r, "(2.6)(\phi^*)"] & (\widetilde{\mathsf{q}}^-)_! \circ (\widetilde{\mathsf{p}}^-)^* \circ \phi^*.
\end{tikzcd}$$

We need to show that the top horizontal arrow in this diagram is an isomorphism when evaluated on $\mathcal{F}$. We shall do so by showing that all other maps are isomorphisms.

The assumption on the map of diagrams implies that the lower left vertical arrow and the upper right vertical arrows are isomorphisms. The bottom horizontal arrow is an isomorphism by assumption. Hence, it remains to show that the upper left vertical arrow is an isomorphism.

However, this follows from the commutative diagram

$$\begin{tikzcd}
(\phi^0)^* \circ (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \arrow[r] \arrow[d] & (\phi^0)^* \circ (\iota^+)^* \circ (\mathsf{p}^+)^! \arrow[d, "\sim"] \\
(\tilde{\mathsf{q}}^+)_* \circ (\phi^+)^* \circ (\mathsf{p}^+)^! \arrow[r] & (\iota^+)^* \circ (\phi^+)^* \circ (\mathsf{p}^+)^!
\end{tikzcd}$$

and the assumption that the maps (2.8) and (2.9) are isomorphisms.

□

2.4.5. Let us first take the diagram (2.5) to be (2.1). To prove Theorem 2.2.3 we have to show that the corresponding natural transformation (2.6) is an isomorphism.

For a point $x \in X$, let

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{\mathrm{good}_x} \subset \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$$

be the open substack, where we do not allow the generalized B-reduction to degenerate at $x \in X$.

Denote also

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{\leq\lambda,\mathrm{good}_x} \text{ and } \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{=\lambda,\mathrm{good}_x}$$

the corresponding open substacks of $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{\leq\lambda}$ and $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{=\lambda}$, respectively. We have

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{=\lambda,\mathrm{good}_x} \simeq (X-x)^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B\,.$$

It is enough to show that the natural transformation (2.6) is an isomorphism for the diagram

$$\begin{tikzcd}
(X-x)^\lambda \times \mathrm{pt}/T \arrow[r] \arrow[d] & Z^\lambda_{\mathcal{P}_T}/T \underset{X^\lambda}{\times} (X-x)^\lambda \arrow[r] \arrow[d] & (X-x)^\lambda \\
\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{=\lambda,\mathrm{good}_x}/T \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}^{\leq\lambda,\mathrm{good}_x}/T & \\
(X-x)^\lambda & &
\end{tikzcd} \tag{2.10}$$

for any x. We shall do so by applying Lemma 2.4.4.

Note that the natural transformation (2.8) is an isomorphism, by the *contraction principle*, see [DrGa3].

2.4.6. Let μ be a dominant coweight, and set $\mathcal{P}'_T := \mathcal{P}_T(-\mu\cdot x)$. We claim that there exists a canonically defined commutative (in fact Cartesian) diagram

$$\begin{tikzcd}
\overline{\mathrm{Bun}}^{=\lambda,\mathrm{good}_x}_{N,\mathcal{P}'_T} \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}'_T} \arrow[d] \\
\overline{\mathrm{Bun}}^{=\lambda,\mathrm{good}_x}_{N,\mathcal{P}_T} \arrow[r] & \overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}_T},
\end{tikzcd}$$

in which the vertical arrows are smooth.

Indeed, let $\mathrm{Bun}_{B,\mathcal{P}_T}(\mathcal{D}_x)$ (resp., $\mathrm{Bun}_{B,\mathcal{P}'_T}(\mathcal{D}_x)$) denote the stack of B-bundles on the formal disc around x in X, equipped with an identification of the induced T-bundle with $\mathcal{P}_T|_{\mathcal{D}_x}$ (resp., $\mathcal{P}'_T|_{\mathcal{D}_x}$).

We have canonically defined maps

$$\overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}_T} \to \mathrm{Bun}_{B,\mathcal{P}_T}(\mathcal{D}_x) \text{ and } \overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}'_T} \to \mathrm{Bun}_{B,\mathcal{P}'_T}(\mathcal{D}_x).$$

We also have a smooth map

$$\mathrm{Bun}_{B,\mathcal{P}'_T}(\mathcal{D}_x) \to \mathrm{Bun}_{B,\mathcal{P}_T}(\mathcal{D}_x),$$

and a Cartesian square

$$\begin{tikzcd}
\overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}'_T} \arrow[r] \arrow[d] & \mathrm{Bun}_{B,\mathcal{P}'_T}(\mathcal{D}_x) \arrow[d] \\
\overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}_T} \arrow[r] & \mathrm{Bun}_{B,\mathcal{P}_T}(\mathcal{D}_x).
\end{tikzcd}$$

2.4.7. Note also that there is a canonical isomorphism

$$Z^\lambda_{\mathcal{P}_T} \underset{X^\lambda}{\times} (X-x)^\lambda \simeq Z^\lambda_{\mathcal{P}'_T} \underset{X^\lambda}{\times} (X-x)^\lambda$$

(the Zastava space is "local" in X).

By construction, the diagram

$$\begin{tikzcd}
\overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}'_T} \arrow[d] & Z^\lambda_{\mathcal{P}'_T} \underset{X^\lambda}{\times} (X-x)^\lambda \arrow[l, "'\mathsf{p}^-"'] \arrow[d] \\
\overline{\mathrm{Bun}}^{\mathrm{good}_x}_{N,\mathcal{P}_T} & Z^\lambda_{\mathcal{P}_T} \underset{X^\lambda}{\times} (X-x)^\lambda \arrow[l, "'\mathsf{p}^-"']
\end{tikzcd}$$

commutes as well.

Hence, we obtain a map from the diagram

$$\begin{tikzcd}
(X-x)^\lambda \times \mathrm{pt}/T \arrow[r] \arrow[d] & Z^\lambda_{\mathcal{P}'_T}/T \underset{X^\lambda}{\times} (X-x)^\lambda \arrow[r] \arrow[d] & (X-x)^\lambda \\
\overline{\mathrm{Bun}}^{=\lambda,\mathrm{good}_x}_{N,\mathcal{P}'_T}/T \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}^{\leq\lambda,\mathrm{good}_x}_{N,\mathcal{P}'_T}/T & \\
(X-x)^\lambda & &
\end{tikzcd} \tag{2.11}$$

to that in (2.11).

2.4.8. Applying Lemma 2.4.4, we obtain that the assertion of Theorem 2.2.3 for a given pair $\mathcal{P}_T$ and λ follows from the validity of Theorem 2.2.3 for $\mathcal{P}'_T := \mathcal{P}_T(-\mu \cdot x)$ with $\mu \in \Lambda^+$, and the same λ for all $x \in X$.

Choose μ so that $\deg(\mathcal{P}_T) + \lambda - \mu$ is anti-dominant and regular. Note that in this case points of $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}'_T}$ admit no non-trivial automorphisms, so the open substack $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}'_T}$ is an algebraic space.

Hence, we obtain that assertion of Theorem 2.2.3 for a given λ follows from the case when $\mathcal{P}_T$ is such that $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}$ is an algebraic space.

3. Proof of Theorem 1.4.6

3.1. **Compatibility between diagrams.**

3.1.1. Let us be given be a pair of diagrams (2.5) and (2.7) and a map between them. Then the construction in Sect. 1.4.4 gives rise to a natural transformation

$$(\phi^0)_! \circ (\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^! \to (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \circ \phi_!. \tag{3.1}$$

Let us make the following assumptions:

- The diagram

$$\begin{tikzcd} \widetilde{\mathcal{Y}}^0 \arrow[r, "\widetilde{\iota}^+"] \arrow[d, "\phi^0"'] & \widetilde{\mathcal{Y}}^+ \arrow[d, "\phi^+"] \\ \mathcal{Y}^0 \arrow[r, "\iota^+"] & \mathcal{Y}^+ \end{tikzcd} \tag{3.2}$$

  is Cartesian;
- The map $\widetilde{\mathcal{Y}}^- \to \widetilde{\mathcal{Y}} \underset{\mathcal{Y}}{\times} \mathcal{Y}^-$ is an open embedding.

In particular, we obtain a natural transformation

$$(\phi^-)_! \circ (\widetilde{\mathsf{p}}^-)^* \to (\mathsf{p}^-)^* \circ \phi_!. \tag{3.3}$$

Furthermore, diagram chase shows that the following diagram of natural transformations is commutative:

$$\begin{tikzcd} (\phi^0)_! \circ (\widetilde{\mathsf{q}}^+)_* \circ (\widetilde{\mathsf{p}}^+)^! \arrow[r, "(3.1)"] \arrow[d, "(\phi^0)_!(2.6)"'] & (\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \circ \phi_! \arrow[dd, "(2.6)(\phi_!)"] \\ (\phi^0)_! \circ (\widetilde{\mathsf{q}}^-)_! \circ (\widetilde{\mathsf{p}}^-)^* \arrow[d, "\sim"'] & \\ (\mathsf{q}^-)_! \circ (\phi^-)_! \circ (\widetilde{\mathsf{p}}^-)^* \arrow[r, "(\mathsf{q}^-)_!(3.3)"] & (\mathsf{q}^-)_! \circ (\mathsf{p}^-)^* \circ \phi_!. \end{tikzcd} \tag{3.4}$$

3.1.2. We apply the commutative diagram (3.4) in the following situation. We take (2.7) to be the diagram

$$\begin{tikzcd}
X^\lambda \times \mathrm{pt}/T \arrow[r, "\mathfrak{s}^\lambda"] \arrow[d, "\iota^\lambda"'] & Z^\lambda_{\mathcal{P}_T}/T \arrow[r, "\pi^\lambda"] \arrow[d, "'\mathsf{p}^-"] & X^\lambda \times \mathrm{pt}/T \\
(X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B)/T \arrow[r, "j^\lambda"] \arrow[d, "\mathrm{id}\times\mathsf{q}"'] & \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T & \\
X^\lambda \times \mathrm{pt}/T, & &
\end{tikzcd}$$

of (2.1).

We take the diagram (2.5) to be the diagram

$$\begin{tikzcd}
\mathrm{Bun}_T \arrow[r] \arrow[d] & \mathrm{Bun}_{B^-} \arrow[r, "\mathsf{q}^-"] \arrow[d, "\mathsf{p}^-"] & \mathrm{Bun}_T \\
\mathrm{Bun}_B \arrow[r, "\mathsf{p}"] \arrow[d, "\mathsf{q}"'] & \mathrm{Bun}_G & \\
\mathrm{Bun}_T\,. & &
\end{tikzcd}$$

We evaluate the functors in (3.4) on the object

$$\mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(\overline{\mathrm{Bun}}_B).$$

The statement of Theorem 1.4.6 says that the top horizontal arrow in the resulting diagram (3.4) is an isomorphism. We shall do so by showing that all other arrows are isomorphisms.

3.1.3. The fact that the upper left vertical arrow is an isomorphism is the content of Theorem 2.2.3. The fact that the right vertical arrow is an isomorphism follows from the fact that the natural transformation

$$\mathsf{q}_* \circ \mathsf{p}^! \to (\mathsf{q}^-)_! \circ (\mathsf{p}^-)^*$$

is an isomorphism, see [DrGa4].

3.1.4. Thus, it remains to show that the natural transformation

$$(\mathsf{q}^-)_! \circ (\phi^-)_! \circ (\widetilde{\mathsf{p}}^-)^* \to (\mathsf{q}^-)_! \circ (\phi^-)_! \circ (\widetilde{\mathsf{p}}^-)^*,$$

induced by (3.3), yields an isomorphism when evaluated on the object

$$\mathfrak{r}_!(\overline{\psi}) \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}/T).$$

Concretely, we need to show the following. Consider the open embedding

$$\left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)^{\text{gen.trans.}} \overset{j}{\hookrightarrow} \left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right).$$

We need to show that the map

$$j_! \circ j^* \circ ('\mathsf{p}^-)^*(\psi) \to ('\mathsf{p}^-)^*(\psi)$$

induces an isomorphism after taking the direct image with compact supports along the projection

$$\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-} \to \mathrm{Bun}_{B^-} \xrightarrow{\mathsf{q}^-} \mathrm{Bun}_T\,.$$

3.1.5. The stack $\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}$ admits a decomposition into locally closed pieces indexed by the Weyl group:

$$\left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right) = \bigcup_w \left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)^w,$$

where the open piece $\left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)^{\text{gen.trans.}}$ corresponds to $w = 1$.

We will show that for each $w \neq 1$, the direct image with compact supports under the map

$$Z^w := \left(\mathrm{Bun}_{N,\rho(\omega_X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)^w \to \mathrm{Bun}_{B^-} \xrightarrow{\mathsf{q}^-} \mathrm{Bun}_T \tag{3.5}$$

of the *-restriction of ψ, vanishes.

3.2. Calculation for $w \neq 1$.

3.2.1. The stack Z^w maps to

$$X^{\Lambda^{\mathrm{pos}}} := \bigsqcup_{\mu \in \Lambda^{\mathrm{pos}}} X^\mu,$$

see [BG2, Sect. 10.9]. Let us denote this map by π^w.

The map (3.5) is the composition of the map π^w, followed by the Abel-Jacobi map

$$X^{\Lambda^{\mathrm{pos}}} \to \mathrm{Bun}_T, \quad D \mapsto \rho(\omega_X)(w(D)).$$

We will show that the object

$$(\pi^w)_!(\psi|_{Z^w}) \in \text{D-mod}(X^{\Lambda^{\mathrm{pos}}})$$

vanishes.

For that it suffices to show that $(\pi^w)_!(\psi|_{Z^w})$ vanishes, when restricted to $(X - x)^{\Lambda^{\mathrm{pos}}}$ for any $x \in X$.

3.2.2. Let B^w denote the subgroup $B \cap \mathrm{Ad}_w(B^-)$. It is equipped with a canonical projection to T. Let N^w denote the kernel of this projection, i.e., $N^w = N \cap \mathrm{Ad}_w(B^-)$.

For a test-scheme S and a map $z : S \to Z^w$ let D denote the resulting map $S \to X^{\Lambda^{\mathrm{pos}}}$. The point z corresponds to a G-bundle $\mathcal{P}_G$ on $S \times X$, equipped with a reduction α to B (for which the induced T-bundle is $\omega(\rho)$), and a reduction β to B^-, which are in position w at the generic point of X.

By the construction of the map π^w (see [BG2, Sect. 10.9]), the data of $(\mathcal{P}_G, \alpha, \beta)$, when restricted to the open subset

$$U := S \times X - \mathrm{Graph}_D,$$

is induced from a uniquely defined B^w-bundle $\mathcal{P}_{B^w}$, for which the induced T-bundle is identified with $\rho(\omega_X)$.

Let us denote

$$Z^{w,\mathrm{good}_x} := Z^w \underset{X^{\Lambda^{\mathrm{pos}}}}{\times} (X - x)^{\Lambda^{\mathrm{pos}}},$$

and let us choose a trivialization of the T-bundle $\rho(\omega_X)$ over the formal disc $\mathcal{D}_x$ around x in X.

Thus, we obtain a map of stacks

$$Z^{w,\mathrm{good}_x} \to \mathrm{Bun}_{N^w}(\mathcal{D}_x) \simeq \mathrm{pt}/N^w(\mathcal{D}_x),$$

where $\mathcal{D}_x$ denotes the formal disc around x, and where $N^w(\mathcal{D}_x)$ denotes the group-scheme classifying maps $\mathcal{D}_x \to N^w$.

3.2.3. Consider the fiber product

$$Z^{w,\mathrm{level}_x} := Z^{w,\mathrm{good}_x} \underset{\mathrm{pt}/N^w(\mathcal{D}_x)}{\times} \mathrm{pt}\,.$$

This is the moduli stack of $(\mathcal{P}_G, \alpha, \beta, \epsilon)$, where $(\mathcal{P}_G, \alpha, \beta)$ are as above, and ϵ is the datum of trivialization of $\mathcal{P}_{N^w}$ over $\mathcal{D}_x$. One can show that Z^{w,level_x} is in fact a scheme (of infinite type).

The stack (scheme) Z^{w,level_x} is acted on by the group ind-scheme $N^w(\mathcal{D}_x^\times)$, where $N^w(\mathcal{D}_x^\times)$ classifies maps from the formal punctured disc $\mathcal{D}_x^\times$ to N^w. This action preserves the map

$$Z^{w,\mathrm{level}_x} \to (X-x)^{\Lambda^{\mathrm{pos}}},$$

For any group-subscheme $\mathbf{N}' \subset \mathbf{N} := N^w(\mathcal{D}_x^\times)$, we can find a group-subscheme $\mathbf{N}'' \subset \mathbf{N}_0 := N^w(\mathcal{D}_x)$ of finite codimension, which is normal in $\mathbf{N}'$. Thus, we obtain an action of the (finite-dimensional) unipotent group $\mathbf{N}'/\mathbf{N}''$ on the stack (of finite type)

$$Z^{w,\mathrm{level}_x}/\mathbf{N}''.$$

This action preserves the projection

$$Z^{w,\mathrm{level}_x}/\mathbf{N}'' \to Z^{w,\mathrm{level}_x}/\mathbf{N}_0 = Z^{w,\mathrm{level}_x} \to (X-x)^{\Lambda^{\mathrm{pos}}}. \tag{3.6}$$

3.2.4. Consider the canonical projection

$$N \to \mathbb{G}_a^r.$$

Consider the map

$$N(\mathcal{D}_x^\times) \to \mathbb{G}_a^r(\mathcal{D}_x^\times) \xrightarrow{\mathrm{res}} \mathbb{G}_a^r \xrightarrow{\mathrm{sum}} \mathbb{G}_a,$$

where the map

$$\mathrm{res} : \mathbb{G}_a^r(\mathcal{D}_x^\times) \to \mathbb{G}_a^r$$

is defined using our chosen trivialization $\rho(\omega_X)|_{\mathcal{D}_x}$.

Since $w \neq 1$, the composition

$$\mathbf{N} = N^w(\mathcal{D}_x^\times) \hookrightarrow N(\mathcal{D}_x^\times) \to \mathbb{G}_a$$

is non-zero.

Let $\mathbf{N}' \subset N^w(\mathcal{D}_x^\times)$ be such that the composition

$$\mathbf{N}' \to \mathbb{G}_a(\mathcal{D}_x^\times) \to \mathbb{G}_a$$

is non-zero. Choose a subgroup $\mathbf{N}'' \subset \mathbf{N}_0$ as above. Thus, we obtain a non-trivial homomorphism

$$\mathbf{N}'/\mathbf{N}'' \to \mathbb{G}_a. \tag{3.7}$$

3.2.5. Since $\mathbf{N}_0/\mathbf{N}''$ is unipotent, it is enough to show that the direct image with compact supports under the map (3.6) of

$$\psi|_{Z^{w,\mathrm{level}_x}/\mathbf{N}''} \in \text{D-mod}(Z^{w,\mathrm{level}_x}/\mathbf{N}'') \tag{3.8}$$

vanishes.

However, this follows from the fact that the object (3.8) is equivariant against the pull-back of

$$\text{A-Sch} \in \text{D-mod}(\mathbb{G}_a)$$

under a *non-trivial* homomorphism $\mathbf{N}'/\mathbf{N}'' \to \mathbb{G}_a$, namely, the homomorphism (3.7).

Part II: Geometry of $\mathbb{G}_m$-actions on moduli problems

4. $\mathbb{G}_m$-actions on classifying stacks

4.1. **Attractors and repellers.**

4.1.1. Let $\mathcal{Y}$ be an arbitrary prestack (i.e., a contravariant functor from the category of schemes to that of groupoids), equipped with an action of $\mathbb{G}_m$.

We let $\mathrm{Fixed}(\mathcal{Y})$, $\mathrm{Attr}(\mathcal{Y})$ and $\mathrm{Repel}(\mathcal{Y})$ denote the prestacks given by

$$\mathrm{Hom}(S, \mathrm{Fixed}(\mathcal{Y})) = \mathrm{Hom}^{\mathbb{G}_m}(S, \mathcal{Y}),$$

and

$$\mathrm{Hom}(S, \mathrm{Attr}(\mathcal{Y})) = \mathrm{Hom}^{\mathbb{G}_m}(S \times \mathbb{A}^1, \mathcal{Y}), \quad \mathrm{Hom}(S, \mathrm{Repel}(\mathcal{Y})) = \mathrm{Hom}^{\mathbb{G}_m}(S \times \mathbb{A}^1, \mathcal{Y}),$$

respectively, where in the case of attractors the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ is the standard one, and in the case of repellers its inverse.

4.1.2. We have the natural forgetful map $\mathrm{Fixed}(\mathcal{Y}) \to \mathcal{Y}$, as well as the maps

$$\mathrm{Attr}(\mathcal{Y}) \to \mathcal{Y} \text{ and } \mathrm{Repel}(\mathcal{Y}) \to \mathcal{Y},$$

given by evaluation at $1 \in \mathbb{A}^1$, and the maps

$$\mathrm{Fixed}(\mathcal{Y}) \to \mathrm{Attr}(\mathcal{Y}) \text{ and } \mathrm{Fixed}(\mathcal{Y}) \to \mathrm{Repel}(\mathcal{Y}),$$

given by the projection $\mathbb{A}^1 \to \mathrm{pt}$.

The latter maps admit respective left inverses

$$\mathrm{Attr}(\mathcal{Y}) \to \mathrm{Fixed}(\mathcal{Y}) \text{ and } \mathrm{Repel}(\mathcal{Y}) \to \mathrm{Fixed}(\mathcal{Y}),$$

given by evaluation at $0 \in \mathbb{A}^1$.

4.1.3. It is shown in [Dr] that if $\mathcal{Y}$ is a (resp., separated) scheme of finite type, then so are $\mathrm{Fixed}(\mathcal{Y})$, $\mathrm{Attr}(\mathcal{Y})$ and $\mathrm{Repel}(\mathcal{Y})$.

If $\mathcal{Y}$ is a scheme, on which the $\mathbb{G}_m$-action is locally linear (see [DrGa3] for what this means), then the above representability is much easier to establish.

When $\mathcal{Y}$ is an affine scheme, the maps

$$\mathrm{Fixed}(\mathcal{Y}) \to \mathrm{Attr}(\mathcal{Y}) \to \mathcal{Y}$$

are all closed embeddings.

4.1.4. We have the following general observation:

Proposition 4.1.5. *Let N be a unipotent group, acted on by $\mathbb{G}_m$ with strictly positive eigevalues on the Lie algebra. Consider the resulting $\mathbb{G}_m$-action on the stack* pt/N. *Then:*
(a) $\operatorname{Fixed}(\operatorname{pt}/N) \simeq \operatorname{pt}$.
(b) *The forgetful map* $\operatorname{Attr}(\operatorname{pt}/N) \to \operatorname{pt}/N$ *is an isomorphism.*
(c) *The maps* $\operatorname{Fixed}(\operatorname{pt}/N) \to \operatorname{Repel}(\operatorname{pt}/N) \to \operatorname{Fixed}(\operatorname{pt}/N)$ *are isomorphisms.*

This proposition implies that we have the following commutative diagrams

$$\begin{array}{ccc} \operatorname{pt} & \longrightarrow & \operatorname{Fixed}(\operatorname{pt}/N) \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{pt}/N & \longrightarrow & \operatorname{Attr}(\operatorname{pt}/N) \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{pt} & \longrightarrow & \operatorname{Fixed}(\operatorname{pt}/N) \end{array}$$

and

$$\begin{array}{ccc} \operatorname{pt} & \longrightarrow & \operatorname{Fixed}(\operatorname{pt}/N) \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{pt} & \longrightarrow & \operatorname{Repel}(\operatorname{pt}/N) \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{pt} & \longrightarrow & \operatorname{Fixed}(\operatorname{pt}/N) \end{array}$$

with horizontal arrows being isomorphisms.

Proof. Filtering N, we can assume that $N = \mathbb{G}_a$ with $\mathbb{G}_m$ acting by the n-th power power of the standard character, where $n > 0$.

Let S be an arbitrary test-scheme, and let $\mathcal{E}$ be an $\mathbb{G}_a$-bundle on S (resp., $S \times \mathbb{A}^1$) is equipped with a structure of $\mathbb{G}_m$-equivariance with respect to the above $\mathbb{G}_m$-action on $\operatorname{pt}/\mathbb{G}_a$ (resp., and the corresponding $\mathbb{G}_m$-action on $\mathbb{A}^1$).

With no restriction of generality we can assume that S is affine. Then $\mathcal{E}$ is non-canonically trivial. Let V denote the $\mathbb{G}_m$-module of automorphisms of $\mathcal{E}$. In case (a) we have

$$V = \Gamma(S, \mathcal{O}_S),$$

with the standard action of $\mathbb{G}_m$. In cases (b) and (c), we have

$$V = \Gamma(S, \mathcal{O}_S) \otimes k[t],$$

where $\mathbb{G}_m$ acts by the n-th power of standard character on $\Gamma(S, \mathcal{O}_S)$ and where t has degree -1 (resp., 1).

Note that $\mathcal{E}$ admits a $\mathbb{G}_m$-equivariant trivialization: indeed an osbtruction would be an element of $H^1(\mathbb{G}_m, V)$, whereas this group is 0.

Now, points (a) and (c) of the lemma follow from the fact that in these cases $H^0(\mathbb{G}_m, V) = 0$. Point (b) follows from the fact that in this case evaluation at $1 \in \mathbb{A}^1$ defines an isomorphism

$$H^0(\mathbb{G}_m, V) \to \Gamma(S, \mathcal{O}_S).$$

$\square$

4.2. **A digression: some maps related to parabolic subgroups.** This subsection will contain some tautological manipulations and notation, which will be used in Sect. 4.3.

4.2.1. In what follows we will use the following observation:

For a parabolic P with Levi quotient M there exists a canonically defined map[1]

$$\mathrm{pt}/M \to \mathrm{pt}/P, \tag{4.1}$$

corresponding to *split* P-bundles, right inverse to the tautological projection $\mathrm{pt}/P \to \mathrm{pt}/M$.

Indeed, since the different splittings of the projection $P \to M$ are *uniquely* conjugate by means of the of the *unipotent radical* of P, they give rise to canonically isomorphic splittings of the map $\mathrm{pt}/P \to \mathrm{pt}/M$.

4.2.2. Let $\gamma : \mathbb{G}_m \to G$ be a homomorphism. Let M_γ be the centralizer of γ, which is the same as $\mathrm{Fixed}(G)$, where $\mathbb{G}_m$ acts on G by conjugation via γ. Let P_γ be the subgroup $\mathrm{Attr}(G)$.

It is easy to see that P_γ is a parabolic subgroup, and the maps

$$M_\gamma \to P_\gamma \to M_\gamma$$

realize M_γ as both the Levi subgroup and the Levi quotient group of P_γ.

4.2.3. Note that the homomorphism $\gamma : \mathbb{G}_m \to Z(M_\gamma)$ defines an action of $\mathbb{G}_m$ on the identity automorphism of pt/M_γ. Hence, we obtain a map

$$\mathrm{pt}/M_\gamma \to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma), \tag{4.2}$$

where for two prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$ we let $\mathrm{Maps}(\mathcal{Y}_1, \mathcal{Y}_2)$ denote the prestack

$$\mathrm{Hom}(S, \mathrm{Maps}(\mathcal{Y}_1, \mathcal{Y}_2)) = \mathrm{Hom}(S \times \mathcal{Y}_1, \mathcal{Y}_2).$$

Consider the fiber product

$$\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \underset{\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma)}{\times} \mathrm{pt}/M_\gamma,$$

where $\mathrm{pt}/M_\gamma \to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma)$ is the map (4.2).

Lemma 4.2.4. *The map*

$$\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \underset{\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma)}{\times} \mathrm{pt}/M_\gamma \to$$
$$\to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma) \underset{\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma)}{\times} \mathrm{pt}/M_\gamma = \mathrm{pt}/M_\gamma$$

is an isomorphism.

Proof. Follows from Proposition 4.1.5(a). ∎

[1] We learned this from J. Lurie.

4.2.5. Consider now the fiber product

$$\mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \underset{\mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/M_\gamma)}{\times} \mathrm{pt}/M_\gamma,$$

where $\mathrm{pt}/M_\gamma \to \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/M_\gamma)$ is the map

$$\mathrm{pt}/M_\gamma \overset{(4.2)}{\longrightarrow} \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma) \to \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/M_\gamma).$$

Evaluation at $1 \in \mathbb{A}^1$ defines a map

$$\mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \underset{\mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/M_\gamma)}{\times} \mathrm{pt}/M_\gamma \to \mathrm{pt}/P_\gamma. \tag{4.3}$$

Lemma 4.2.6. *The map* (4.3) *is an isomorphism.*

Proof. Follows from Proposition 4.1.5(b). □

4.2.7. From Lemma 4.2.4 we obtain a map

$$\mathrm{pt}/M_\gamma \to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/P_\gamma). \tag{4.4}$$

From Lemma 4.2.6 we obtain a map

$$\mathrm{pt}/P_\gamma \to \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/P_\gamma). \tag{4.5}$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{pt}/M_\gamma & \xrightarrow{(4.4)} & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \\ \downarrow & & \downarrow \\ \mathrm{pt}/P_\gamma & \xrightarrow{(4.5)} & \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/P_\gamma) \\ \downarrow & & \downarrow \\ \mathrm{pt}/M_\gamma & \xrightarrow{(4.4)} & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/P_\gamma), \end{array} \tag{4.6}$$

where the upper left vertical map is given by (4.1).

4.3. **Attractors on the classifying stack.**

4.3.1. Consider the stack pt/G as endowed with the trivial action of $\mathbb{G}_m$. In this subsection we will describe the prestacks

$$\mathrm{Fixed}(\mathrm{pt}/G) \text{ and } \mathrm{Attr}(\mathrm{pt}/G).$$

Note that we have

$$\mathrm{Fixed}(\mathrm{pt}/G) = \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G) \text{ and } \mathrm{Attr}(\mathrm{pt}/G) = \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G).$$

4.3.2. Composing the maps in the diagram (4.6) with the map

$$\mathrm{pt}/P_\gamma \to \mathrm{pt}/G,$$

from (4.2) we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{pt}/M_\gamma & \longrightarrow & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G) \\ \downarrow & & \downarrow \\ \mathrm{pt}/P_\gamma & \longrightarrow & \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G) \\ \downarrow & & \downarrow \\ \mathrm{pt}/M_\gamma & \longrightarrow & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G). \end{array} \tag{4.7}$$

We have:

Theorem 4.3.3 (V. Drinfeld)**.** *The diagram* (4.7) *gives rise to isomorphisms*

$$\begin{array}{ccc} \bigsqcup\limits_\gamma \mathrm{pt}/M_\gamma & \xrightarrow{\sim} & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G) \\ \downarrow & & \downarrow \\ \bigsqcup\limits_\gamma \mathrm{pt}/P_\gamma & \xrightarrow{\sim} & \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G) \\ \downarrow & & \downarrow \\ \bigsqcup\limits_\gamma \mathrm{pt}/M_\gamma & \xrightarrow{\sim} & \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G), \end{array}$$

where the disjoint unions are taken with respect to the set of conjugacy classes of homomorphisms $\gamma : \mathbb{G}_m \to G$.

The rest of this subsection is devoted to the proof of this theorem. [2]

4.3.4. The asserion that

$$\bigsqcup_\gamma \mathrm{pt}/M_\gamma \to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G) \tag{4.8}$$

is an isomorphism is equivalent to the following two maps being isomorphisms:

$$\mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G)/\mathrm{Ad}_G \to \mathrm{Maps}(\mathrm{pt}/\mathbb{G}_m, \mathrm{pt}/G). \tag{4.9}$$

and

$$\bigsqcup_\gamma \mathrm{pt}/M_\gamma \to \mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G)/\mathrm{Ad}_G\,. \tag{4.10}$$

The fact that the map (4.9) is an isomorphsm is tautological: any G-bundle on a test-scheme S is étale-locally trivial, and the obstruction to this trivialization being $\mathbb{G}_m$-equivariant is given by a map of group-schemes $S \times \mathbb{G}_m \to S \times G$ over S.

The fact that (4.10) is an isomorphism is equivalent to the next assertion: for a homomorphism $\gamma : \mathbb{G}_m \to G$, the resulting map from G/M_γ to $\mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G)$ is an open embedding. To prove this we argue as follows.

[2]The proof is a minor modification of the one communicated to us by V. Drinfeld.

Note that $\mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G)$ is an ind-scheme of ind-finite type. Write it as a union of G-invariant closed subschemes of finite type, denote them Y_i. Note that the map

$$G/M_\gamma \to \mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G)$$

induces an isomorphism of tangent spaces at each point. Hence, for every indiex i, for which this map factors as

$$G/M_\gamma \to Y_i \to \mathrm{Maps}_{\mathrm{groups}}(\mathbb{G}_m, G),$$

the resulting map $G/M_\gamma \to Y_i$ is an open embedding, as desired.

4.3.5. The proof that

$$\bigsqcup_\gamma \mathrm{pt}/P_\gamma \to \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G)$$

is an isomorphism relies on the following statement:

Proposition 4.3.6. *For an affine test-scheme S, any $\mathbb{G}_m$-equivariant G-bundle on $S \times \mathbb{A}^1$ is (non-canonically) isomorphic to one pulled back under the map $S \times \mathbb{A}^1 \to S$.*

Assuming this proposition, we finish the proof as follows:

Knowing that (4.8) is an isomorphism, we need to show that for a test-scheme S and the trivial G-bundle on $S \times \mathbb{A}^1$, endowed with a structure of $\mathbb{G}_m$-equivariance given by $\gamma : \mathbb{G}_m \to G$, the group of its $\mathbb{G}_m$-equivariant automorphisms is isomorphic to $\mathrm{Hom}(S, P_\gamma)$. However, the group in question is

$$\mathrm{Hom}^{\mathbb{G}_m}(S \times \mathbb{A}^1, G),$$

and the assertion follows from the fact that $P_\gamma := \mathrm{Attr}(G)$.

4.3.7. *Proof of Proposition 4.3.6.* Let $S = \mathrm{Spec}(A)$ and $\mathbb{A}^1 = \mathrm{Spec}(k[t])$. First, we note that the Beauville-Laszlo theorem implies that we can replace $S \times \mathbb{A}^1$ by $\mathrm{Spec}(A[[t]])$. Further, since we are dealing with G-bundles, we can replace $\mathrm{Spec}(A[[t]])$ by the formal scheme

$$\mathrm{Spf}(A[[t]]) = \underset{i}{\mathrm{colim}}\, \mathrm{Spec}(A[t]/t^i).$$

Now, the required assertion follows by induction on i by standard deformation theory.

□

4.4. **Fixed points on the affine Grassmannian.** The contents of this subsection are not needed for the rest of the paper.

4.4.1. Recall that the affine Grassmannian Gr_G is the indscheme whose S-points are pairs consisting of a G-bundle on $S \times \mathbb{A}^1$ and its trivialization on $S \times (\mathbb{A}^1 - 0)$. Note that the $\mathbb{G}_m$-action on $\mathbb{A}^1$ defines a $\mathbb{G}_m$ on Gr_G. This is the so-called action by "loop rotation".

In this subsection we will use Theorem 4.3.3 to describe the loci $\mathrm{Fixed}(\mathrm{Gr}_G)$ and $\mathrm{Attr}(\mathrm{Gr}_G)$.

4.4.2. Note that we have a canonical identification

$$\mathrm{Fixed}(\mathrm{Gr}_G) \simeq \mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G) \underset{\mathrm{pt}/G}{\times} \mathrm{pt},$$

where the map $\mathrm{Maps}(\mathbb{A}^1/\mathbb{G}_m, \mathrm{pt}/G) \to \mathrm{pt}/G$ is given by restriction to $(\mathbb{A}^1 - 0)/\mathbb{G}_m \simeq \mathrm{pt}$.

Thus, from Theorem 4.3.3, we recover the following assertion from [BD]:

Corollary 4.4.3. *There exists a canonical isomorphism*

$$\bigsqcup_{\gamma} G/P_\gamma \to \mathrm{Fixed}(\mathrm{Gr}_G),$$

where the disjoint unions are taken with respect to the set of conjugacy classes of homomorphisms $\gamma : \mathbb{G}_m \to G$.

4.4.4. Unwinding the definitions, we obtain that for a given $\gamma : \mathbb{G}_m \to G$, the corresponding map

$$G/P_\gamma \to \mathrm{Fixed}(\mathrm{Gr}_G)$$

is defined as follows.

It sends the point $1 \in G/P_\gamma$ to the point $t^\gamma \in \mathrm{Gr}_G$, where t^γ is induced by means of γ from the point of

$$t \in \mathrm{Gr}_{\mathbb{G}_m}(k),$$

corresponding to the trivial line bundle on $\mathbb{A}^1$ with the trivialization over $\mathbb{A}^1 - 0$ given by the coordinate function t.

The entire map $G/P_\gamma \to \mathrm{Fixed}(\mathrm{Gr}_G)$ is extended by G-equivariance, where G acts on Gr_G via the embedding $G \hookrightarrow G[\![t]\!]$.

4.4.5. Let $\mathrm{Gr}_G^\gamma \subset \mathrm{Gr}_G$ be the $G[\![t]\!]$ oribit of the point t^γ. We will now rederive the following result of [BD]:

Corollary 4.4.6. *For every γ we have a canonical identification*

$$\mathrm{Gr}_G^\gamma = \mathrm{Attr}(\mathrm{Gr}_G) \underset{\mathrm{Fixed}(\mathrm{Gr}_G)}{\times} G/P_\gamma$$

as sub-prestacks of Gr_G.

Proof. Note that the forgetful map $\mathrm{Attr}(G[\![t]\!]) \to G[\![t]\!]$ is an isomorphism. Hence, we obtain that the $G[\![t]\!]$-action on Gr_G lifts to a $G[\![t]\!]$-action on the functor $\mathrm{Attr}(\mathrm{Gr}_G)$. In particular, we obtain that each

$$\mathrm{Attr}(\mathrm{Gr}_G) \underset{\mathrm{Fixed}(\mathrm{Gr}_G)}{\times} G/P_\gamma$$

is $G[\![t]\!]$-invariant.

Since $t^\gamma \in \mathrm{Attr}(\mathrm{Gr}_G) \underset{\mathrm{Fixed}(\mathrm{Gr}_G)}{\times} G/P_\gamma$, we obtain an inclusion

$$\mathrm{Gr}_G^\gamma \hookrightarrow \mathrm{Attr}(\mathrm{Gr}_G) \underset{\mathrm{Fixed}(\mathrm{Gr}_G)}{\times} G/P_\gamma. \tag{4.11}$$

Since every k-point of Gr_G belongs to some (in fact, unique) Gr_G^γ, we obtain that (4.11) induces a bijection on k-points.

By [DrGa3], the functor $\mathrm{Attr}(\mathrm{Gr}_G) \underset{\mathrm{Fixed}(\mathrm{Gr}_G)}{\times} G/P_\gamma$ is representable by an ind-scheme. Hence, in order to show that (4.11) is an isomorphism, it suffices to show that the map (4.11) induces an isomorphism of tangent spaces at t^γ. The latter is straightforward.

□

5. Attractors and repellers on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$

5.1. $\mathbb{G}_m$-action on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$.

5.1.1. Fix a regular dominant cocharacter $\gamma : \mathbb{G}_m \to T$. This defines an action of $\mathbb{G}_m$ on the map

$$\mathcal{P}_T : \mathrm{pt} \to \mathrm{Bun}_T,$$

and hence on the stack

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T} = \overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{pt}\,.$$

Consider the corresponding presatcks

$$\mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right), \mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \text{ and } \mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right).$$

5.1.2. The goal of this section is to prove the following result:

Theorem 5.1.3. *The diagram*

$$\begin{tikzcd} X^\lambda \arrow[r, "\mathfrak{s}^\lambda"] \arrow[d, "\iota^\lambda"'] & Z^\lambda_{\mathcal{P}_T} \arrow[r, "\pi^\lambda"] \arrow[d, "'\mathsf{p}^-"] & X^\lambda \\ X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \arrow[r, "\jmath^\lambda"] \arrow[d, "\mathrm{id}\times\mathsf{q}"'] & \overline{\mathrm{Bun}}_{N,\mathcal{P}_T} & \\ X^\lambda, & & \end{tikzcd} \tag{5.1}$$

identifies canonically with the diagram

$$\begin{tikzcd} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \arrow[r] \arrow[d] & \mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \arrow[r] \arrow[d] & \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \\ \mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_{N,\mathcal{P}_T} & \\ \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right). & & \end{tikzcd}$$

Theorem 5.1.3 is the combination of Propositions 5.2.4, 5.3.2 and Theorem 5.4.5, proved in this bulk of this section.

Let us show how Theorem 5.1.3 implies Theorem 2.2.3.

Proof of Theorem 2.2.3. As was established in Sect. 2.4.8, it is enough to prove Theorem 2.2.3 under the assumption that $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}$ is an algebraic space.

In this case the fact that (2.3) is an isomorphism follows from Braden's theorem via Theorem 5.1.3. Namely, the version of Braden's theorem for algebraic spaces (see [DrGa3]) says that for an algebraic space $\mathcal{Y}$ equipped with an action of $\mathbb{G}_m$, in the diagram

$$
\begin{tikzcd}
\mathrm{Fixed}(\mathcal{Y}) \arrow[r, "\iota^-"] \arrow[d, "\iota^+"'] & \mathrm{Repel}(\mathcal{Y}) \arrow[r, "\mathsf{q}^-"] \arrow[d, "\mathsf{p}^-"] & \mathrm{Fixed}(\mathcal{Y}) \\
\mathrm{Attr}(\mathcal{Y}) \arrow[r, "\mathsf{p}^+"] \arrow[d, "\mathsf{q}"'] & \mathcal{Y} & \\
\mathrm{Fixed}(\mathcal{Y}) & &
\end{tikzcd}
\tag{5.2}
$$

the resulting natural transformation

$$(\mathsf{q}^+)_* \circ (\mathsf{p}^+)^! \to (\mathsf{q}^-)_! \circ (\mathsf{p}^-)^*$$

is an isomorphism *when evaluated on $\mathbb{G}_m$-monodromic objects.*

In particular, if the action of $\mathbb{G}_m$ on the terms of (5.2) is obtained from an action of some group H via a homomorphism $\mathbb{G}_m \to H$, then the corresponding natural transfomation of functors

$$\text{D-mod}(\mathcal{Y}/H) \rightrightarrows \text{D-mod}(\mathrm{Fixed}(\mathcal{Y})/H)$$

is an isomorphism.

□

5.2. **The fixed-point locus on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$.**

5.2.1. The $\mathbb{G}_m$-action on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ preserves the locally closed substacks $\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}$. Consider also the corresponding prestacks

$$\mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right), \mathrm{Attr}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \text{ and } \mathrm{Repel}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right).$$

Consider the map, denoted ι^λ,

$$X^\lambda \simeq X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_T \to X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B, \tag{5.3}$$

where the map $\mathrm{Bun}_T \to \mathrm{Bun}_B$ is induced by (4.1).

It is easy to see that the resulting map

$$X^\lambda \xrightarrow{\iota^\lambda} X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \simeq \overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}$$

naturally factors as

$$X^\lambda \to \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right). \tag{5.4}$$

We claim:

Lemma 5.2.2.

(a) *The map* (5.4) *is an isomorphism.*

(b) *The map* $\mathrm{Attr}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \to \overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}$ *is an isomorphism.*

Proof. Follows from points (a) and (b) of Proposition 4.1.5, respectively.

□

5.2.3. Consider now the map

$$\bigsqcup_{\lambda\in\Lambda^{\mathrm{pos}}} X^\lambda \simeq \bigsqcup_{\lambda\in\Lambda^{\mathrm{pos}}} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \to \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right), \tag{5.5}$$

where the first arrow is the isomorphism of Lemma 5.2.2(a).

We are going to prove:

Proposition 5.2.4. *The maps in* (5.5) *are isomorphisms.*

Remark 5.2.5. The decomposition into locally closed $\mathbb{G}_m$-invariant substacks

$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \simeq \bigcup_{\lambda\in\Lambda^{\mathrm{pos}}} \overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}$$

defines the corresponding decomposition of $\mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right)$ as a union of locally closed sub-prestacks

$$\mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \simeq \bigcup_{\lambda\in\Lambda^{\mathrm{pos}}} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right).$$

So, the assertion of Proposition 5.2.4 is equivalent to the fact that the above locally closed sub-prestacks are disjoint, i.e., that each is open.

Remark 5.2.6. Assume for a moment that the pair $(\mathcal{P}_T,\lambda)$ is such that $\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}$ is a separated algebraic space. In this case the assertion that the decomposition into locally closed subsets

$$\mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}\right) \simeq \bigcup_{0\leq\mu\leq\lambda} X^\mu$$

is in fact the union of connected components is immediate: indeed each X^μ is a proper scheme, and hence must be closed as a sub-algebraic space.

5.2.7. *Proof of Proposition 5.2.4.* We will explictly construct an inverse map to

$$\bigsqcup_{\lambda\in\Lambda^{\mathrm{pos}}} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \to \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right).$$

For a test-scheme S, let us be given a $\mathbb{G}_m$-equivariant map $S \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$, i.e., $\mathbb{G}_m$-equivariant a G-bundle $\mathcal{P}_G$ on $S\times X$, equipped with a generalized reduction to B, with the corresponding T-bundle being $\mathcal{P}_T|_{S\times X}$, on which the structure of $\mathbb{G}_m$-equivariance is defined by γ by means of (4.2).

Let $U\subset S\times X$ be the open subset, dense in every fiber over S, over which our reduction to B is genuine. Note that by Lemma 4.2.4, this reduction to B comes from a canonical reduction β to T. Furthermore, this reduction is $\mathbb{G}_m$-equivariant, where the structure of $\mathbb{G}_m$-equivariance on the corresponding T-bundle is defined by γ.

We need to show that $\mathcal{P}_G$ admits a reduction β' to a T-bundle $\mathcal{P}'_T$ on all of $S\times X$, such that β' coincides with β when restricted to U.

Note that by Theorem 4.3.3, there exists a homomorphism $\gamma':\mathbb{G}_m\to G$ (uniquely defined up to conjugacy) and a reduction β'' of $\mathcal{P}_G$ to an $M_{\gamma'}$-bundle, on which the structure of equivariance is given by γ' by means of (4.2).

Comparing the two pieces of data over U, the injectivity of the map in Theorem 4.3.3 implies that $\gamma'=\gamma$, $M_{\gamma'}=T$ and $\beta=\beta''$. We set $\beta':=\beta''$, thereby finishing the proof.

□

5.3. Attractors on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$.

5.3.1. For $\lambda \in \Lambda^{\mathrm{pos}}$ let

$$\mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right)$$

denote the preimage of

$$X^\lambda \subset \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right)$$

under the canonical projection

$$\mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \to \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right).$$

We have canonically defined maps

$$X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \to \mathrm{Attr}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \to \mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right), \tag{5.6}$$

where the first arrow is the isomorphism of Lemma 5.2.2(b).

We claim:

Proposition 5.3.2. *The maps in* (5.6) *are isomorphisms.*

Note that Propositions 5.2.4 and 5.3.2 imply that there exists a commutative diagram

$$\begin{tikzcd}
\underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} X^\lambda \arrow[r, "\sim"] \arrow[d] & \underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \arrow[r, "\sim"] \arrow[d] & \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \arrow[d] \\
\underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} X^\lambda \underset{\mathrm{Bun}_T}{\times} \mathrm{Bun}_B \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\times\mathsf{q}"'] & \underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} \mathrm{Attr}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \arrow[r, "\sim"] \arrow[d] & \mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right) \arrow[d] \\
\underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} X^\lambda \arrow[r, "\sim"] & \underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} \mathrm{Fixed}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \arrow[r, "\sim"] & \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right)
\end{tikzcd} \tag{5.7}$$

with the horizontal arrows being isomorphisms.

Remark 5.3.3. It is not difficult to show directly that the map (5.6) is an isomorphism at the level of k-points, and further that it induces an isomorphism at the level of the corresponding reduced prestacks. This would suffice for the applications in this paper, since we will be dealing with D-modules.

5.3.4. *Proof of Proposition 5.3.2.* We will construct an inverse to the map

$$\underset{\lambda\in\Lambda^{\mathrm{pos}}}{\bigsqcup} \mathrm{Attr}\left(\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}\right) \to \mathrm{Attr}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right).$$

For a test-scheme S, let us be given a $\mathbb{G}_m$-equivariant map $S \times \mathbb{A}^1 \to \overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$, i.e., $\mathbb{G}_m$-equivariant a G-bundle $\mathcal{P}_G$ on $S \times X \times \mathbb{A}^1$, equipped with a generalized reduction α to B, with the corresponding T-bundle being $\mathcal{P}_T|_{S\times X}$, on which the structure of $\mathbb{G}_m$-equivariance is defined by γ by means of (4.2).

Let $U \subset S \times X \times \mathbb{A}^1$ be the open subset, dense in every fiber over $S \times \mathbb{A}^1$ over which our reduction to B is genuine. Clearly, the open subset U is $\mathbb{G}_m$-invariant and $\alpha|_U$ is $\mathbb{G}_m$-equivariant.

We need to show that $\mathcal{P}_G$ admits a reduction α' to a B-bundle $\mathcal{P}'_B$ on all of $S \times X \times \mathbb{A}^1$, such that α' coincides with α when restricted to U.

Note that by Theorem 4.3.3, there exists a homomorphism $\gamma' : \mathbb{G}_m \to G$ (uniquely defined up to conjugacy) and a reduction α'' of $\mathcal{P}_G$ to an $P_{\gamma'}$-bundle $\mathcal{P}_{P_{\gamma'}}$, such that the structure of equivarience on the induced $M_{\gamma'}$-bundle is given by γ' by means of (4.2).

Comparing the two pieces of data over U, the injectivity of the map in Theorem 4.3.3 implies that $\gamma' = \gamma$, $M_{\gamma'} = T$ and $\alpha = \alpha''$. We set $\alpha' := \alpha''$, finishing the proof.

5.4. **Repellers on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$.**

5.4.1. Recall the Zastava space $Z^\lambda_{\mathcal{P}_T}$. The action of $\mathbb{G}_m$ on $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ is compatible with the projection
$$\overline{\mathrm{Bun}}_{N,\mathcal{P}_T} \to \mathrm{Bun}_G,$$
where $\mathbb{G}_m$ acts trivially on Bun_G. Hence, it induces an action on the Zastava space $Z^\lambda_{\mathcal{P}_T}$.

The following is [BFGM, ???]:

Lemma 5.4.2. *The forgetful map* $\mathrm{Repel}(Z^\lambda_{\mathcal{P}_T}) \to Z^\lambda_{\mathcal{P}_T}$ *is an isomorphism, and we have a commutative diagram*
$$\begin{array}{ccc}
\mathrm{Fixed}(Z^\lambda_{\mathcal{P}_T}) & \xrightarrow{\sim} & X^\lambda \\
\downarrow & & \downarrow \mathfrak{s}^\lambda \\
\mathrm{Repel}(Z^\lambda_{\mathcal{P}_T}) & \xrightarrow{\sim} & Z^\lambda_{\mathcal{P}_T} \\
\downarrow & & \downarrow \pi^\lambda \\
\mathrm{Fixed}(Z^\lambda_{\mathcal{P}_T}) & \xrightarrow{\sim} & X^\lambda .
\end{array}$$

5.4.3. Consider the prestack $\mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}\right)$. For a given $\lambda \in \Lambda^{\mathrm{pos}}$ let
$$\mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right)$$
denote the preimage of the connected component
$$X^\lambda \subset \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right)$$
under the canonical projection
$$\mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right) \to \mathrm{Fixed}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right).$$

5.4.4. From Lemma 5.4.2 we obtain that the map
$$'\mathfrak{p}^- : Z^\lambda_{\mathcal{P}_T} \to \overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}$$
gives rise to a map
$$Z^\lambda_{\mathcal{P}_T} \to \mathrm{Repel}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}, X^\lambda\right) \tag{5.8}$$
that makes the diagram
$$\begin{array}{ccc}
X^\lambda & \xrightarrow{\mathrm{id}} & X^\lambda \\
\downarrow \mathfrak{s}^\lambda & & \downarrow \\
Z^\lambda_{\mathcal{P}_T} & \xrightarrow{'\mathfrak{p}^-} & \mathrm{Repel}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}, X^\lambda\right) \\
\downarrow \pi^\lambda & & \downarrow \\
X^\lambda & \xrightarrow{\mathrm{id}} & X^\lambda
\end{array}$$
commute.

We claim:

Theorem 5.4.5. *The map* (5.8) *is an isomorphism.*

5.4.6. *Proof of Theorem 5.4.5.* We will construct a map

$$\mathrm{Repel}\left(\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}, X^\lambda\right) \to \mathrm{Repel}(Z^\lambda_{\mathcal{P}_T}) \simeq Z^\lambda_{\mathcal{P}_T},$$

inverse to (5.8).

Let us be given an S-point of $\mathrm{Repel}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}, X^\lambda\right)$. That is, we have a $\mathbb{G}_m$-equivariant G-bundle $\mathcal{P}_G$ on $S \times X \times \mathbb{A}^1$ equipped with a generalized reduction to B, such that the corresponding T-bundle is the pullback of $\mathcal{P}_T$ under $S \times X \times \mathbb{A}^1 \to X$.

The projection

$$\mathrm{Repel}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}, X^\lambda\right) \to X^\lambda$$

defines a map $D : S \to X^\lambda$. Consider the open subset

$$U := (S \times X - \mathrm{Graph}_D) \times \mathbb{A}^1 \subset S \times X \times \mathbb{A}^1,$$

and let us restrict our data to it.

First, we claim that the resulting generalized B-reduction is non-degenerate, when restricted to U. This follows from the fact that it is non-degenerate when restricted to

$$U \underset{\mathbb{A}^1}{\times} \{0\} \subset U,$$

and the $\mathbb{G}_m$-equivariance.

Now, Proposition 4.1.5(c) implies that our data over U admits a unique isomorphism with one induced from the T-bundle $\mathcal{P}_T|_U$. In particular, our G-bundle, when restricted to U admits a canonical $\mathbb{G}_m$-equivariant reduction α to B^-, such that the induced T-bundle is $\mathcal{P}_T|_U$.

Consider the T-bundle $\mathcal{P}'_T := \mathcal{P}_T(D)$ on $S \times X$. We claim that the above B^--reduction α extends to a B^--reduction over the entire $S \times X \times \mathbb{A}^1$ with the induced T-bundle being $\mathcal{P}'_T|_{S\times X\times\mathbb{A}^1}$. Such an extension is automatically unique.

The existence of such an extension, provides the required map

$$S \to \mathrm{Repel}(Z^\lambda_{\mathcal{P}_T}),$$

and it is clear from the construction that the resulting map

$$\mathrm{Repel}\left(\overline{\mathrm{Bun}}^{\leq\lambda}_{N,\mathcal{P}_T}, X^\lambda\right) \to \mathrm{Repel}(Z^\lambda_{\mathcal{P}_T}) \simeq Z^\lambda_{\mathcal{P}_T}$$

is the inverse of (5.8).

5.4.7. The data of a $\mathbb{G}_m$-equivariant bundle $\mathcal{P}_G$ on $S \times X \times \mathbb{A}^1$ amounts to a map

$$S \times X \times \mathbb{A}^1/\mathbb{G}_m \to \mathrm{pt}/G.$$

By Theorem 4.3.3, there exists a homomorphism $\gamma' : \mathbb{G}_m \to G$ (uniquely defined up to conjugacy) and a reduction α' of $\mathcal{P}_G$ to a $P_{\gamma'}$-bundle $\mathcal{P}_{P_{\gamma'}}$ such that on the induced $M_{\gamma'}$-bundle $\mathcal{P}_{M_{\gamma'}}$ the structure of $\mathbb{G}_m$-equivariance is given by γ' by means of (4.2). Furthermore, the pair $(\gamma', \mathcal{P}_{M_{\gamma'}})$ is determined by the restriction of $\mathcal{P}_G$ to

$$S \times \mathbb{A}^1 \times 0 \subset S \times \mathbb{A}^1 \times \mathbb{A}^1.$$

Hence, $\gamma' = \gamma$, $M = T$ and $\mathcal{P}_{M_{\gamma'}} = \mathcal{P}'_T$. Furthermore, $P_{\gamma'} = B^-$ (the effect of replacing the standard $\mathbb{G}_m$-action by its inverse swaps the attractor and the repeller for the $\mathbb{G}_m$-action on G, given by means of the conjugation by γ).

Comparing the data over U, the injectivity of the map in Theorem 4.3.3 implies that $\alpha' = \alpha$, as required.

□

5.5. **The enhanced flag variety.** This subsection is not necessary for the rest of the paper. We will explain an alternative (and in a sense, more direct) way to prove Propositions 5.2.4, 5.3.2 and Theorem 5.4.5, and thus Theorem 5.1.3.

In order to slightly simplify the exposition, we will assume that G is such that $[G, G]$ is simply connected.

5.5.1. Consider the enhanced flag variety G/N; it is known to be quasi-affine, and we let $\overline{G/N}$ denote its affine closure.

We regard $\overline{G/N}$ as equipped with a G-action on the left and a T-action on the right.

Recall that the assumption that $[G, G]$ be simply connected implies [3] that $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ is defined as the open substack of

$$\mathrm{Maps}(X, G\backslash(\overline{G/N})/T) \underset{\mathrm{Maps}(X,\mathrm{pt}/T)}{\times} \mathrm{pt},$$

where the map $\mathrm{pt} \to \mathrm{Maps}(X, \mathrm{pt}/T)$ is given by $\mathcal{P}_T$.

The corresponding open substack

$$\overset{\circ}{\mathrm{Maps}}(X, G\backslash(\overline{G/N})/T) \subset \mathrm{Maps}(X, G\backslash(\overline{G/N})/T)$$

is given by the followowing condition:

For a test-scheme S and a map $S \times X \to G\backslash(\overline{G/N})/T$, we require that for every geometric point $s \in S$, the corresponding map

$$X \to G\backslash\overline{G/N}/T,$$

is such that a non-empty open subset of X hits the open substack $G\backslash(G/N)/T \subset G\backslash(\overline{G/N})/T$.

5.5.2. Fix a Cartan subgroup $T \subset B \subset G$. We will identify the set of cocharacters γ up to conjugation with the set of dominant cocharacters of T.

Let $\overline{T}$ denote the affine variety $\mathrm{Spec}(\check{\Lambda}^+)$. We have an open embedding $T \hookrightarrow \overline{T}$, and an action of T on $\overline{T}$ that extends the natural T-action on itself. The quotient stack

$$T\backslash\overline{T}$$

identifies with $\prod_i \mathbb{A}^1/\mathbb{G}_m$, where the product is taken over the set of vertices of the Dynkin diagram of G.

Consider the stack $\mathrm{Maps}(X, T\backslash\overline{T})$; it splits into connected components

$$\mathrm{Maps}(X, T\backslash\overline{T}) \simeq \bigsqcup_{\lambda} \mathrm{Maps}(X, T\backslash\overline{T})^\lambda,$$

accprding to the degree of the composed map $X \to \mathrm{pt}/T$.

[3]For a general G, the definition of $\overline{\mathrm{Bun}}_{N,\mathcal{P}_T}$ is slightly different, see [Sch].

For each λ, let $\overset{\circ}{\mathrm{Maps}}(X, T\backslash\overline{T})^\lambda$ denote the open substack of $\mathrm{Maps}(X, T\backslash\overline{T})^\lambda$ singled out by the following condition:

For a test-scheme S and a map $S \times X \to T\backslash\overline{T}$, we require that for every geometric point $s \in S$, the corresponding map
$$X \to T\backslash\overline{T},$$
is such that a non-empty open subset of X hits the open substack $\mathrm{pt} = T/T \subset T\backslash\overline{T}$.

Note that $\overset{\circ}{\mathrm{Maps}}(X, T\backslash\overline{T})^\lambda$ is empty unless $\lambda \in \Lambda^{\mathrm{pos}}$, and in the latter case we have a canonical identification
$$X^\lambda \simeq \overset{\circ}{\mathrm{Maps}}(X, T\backslash\overline{T})^\lambda.$$

The closed embedding $T \hookrightarrow G/N$ extends to a closed embedding
$$\overline{T} \hookrightarrow \overline{G/N}.$$

In fact, $\overline{T}$ identifies with the set of T-fixed points in $\overline{G/N}$ with respect to the action given by the diagonal embedding
$$T \mapsto T \times T \hookrightarrow G \times T.$$

5.5.3. Let γ be a fixed dominant regular cocharacter $\mathbb{G}_m \to T$. Consider the resulting action of $\mathbb{G}_m$ on the stack $G\backslash\overline{G/N}$.

According to Theorem 4.3.3, we have
$$\mathrm{Fixed}(G\backslash(\overline{G/N})) \simeq \bigsqcup_{\gamma'} \mathrm{Fixed}(G\backslash(\overline{G/N})) \underset{\mathrm{Fixed}(\mathrm{pt}/G)}{\times} \mathrm{pt}/M_{\gamma'},$$
where each $\mathrm{Fixed}(G\backslash(\overline{G/N})) \underset{\mathrm{Fixed}(\mathrm{pt}/G)}{\times} \mathrm{pt}/M_{\gamma'}$ identifies with the stack
$$M_{\gamma'}\backslash \mathrm{Fixed}_{\gamma',\gamma}(\overline{G/N}),$$
where the $\mathrm{Fixed}_{\gamma',\gamma}(\overline{G/N})$ denotes the fixed-point locus of the $\mathbb{G}_m$-action on $\overline{G/N}$ given by
$$\mathbb{G}_m \xrightarrow{\gamma',\gamma} T \times T \hookrightarrow G \times T.$$

The following is easy to see from the definitions:

Lemma 5.5.4. *The only connected components of* $\mathrm{Fixed}(G\backslash(\overline{G/N}))$ *that have a non-empty intersection with* $\mathrm{Fixed}(G\backslash(G/N))$ *are those with* $\gamma' = \gamma$. *Furthermore, the inclusion*
$$\overline{T} \to \mathrm{Fixed}_{\gamma,\gamma}(\overline{G/N})$$
is an isomorphism.

5.5.5. We note that Lemma 5.5.4 readily implies Proposition 5.2.4. We shall now deduce Proposition 5.3.2 and Theorem 5.4.5.

Note that the stacks $\overline{\mathrm{Bun}}^{=\lambda}_{N,\mathcal{P}_T}$ and $Z^\lambda_{\mathcal{P}_T}$ identify with the stacks
$$\overset{\circ}{\mathrm{Maps}}{}^\lambda(X, B\backslash\overline{T}/T) \underset{\mathrm{Maps}(X,\mathrm{pt}/T)}{\times} \mathrm{pt} \text{ and } \overset{\circ}{\mathrm{Maps}}{}^\lambda(X, B^-\backslash(\overline{G/N})/T) \underset{\mathrm{Maps}(X,\mathrm{pt}/T)}{\times} \mathrm{pt},$$
respectively, where

- $\overset{\circ}{\mathrm{Maps}}(X, -) \subset \mathrm{Maps}(X, -)$ is the usual open condition that says that a non-empty open subset of X must hit $B\backslash T/T \subset B\backslash\overline{T}/T$ (resp., $B^-\backslash(G/N)/T \subset B^-\backslash(\overline{G/N})/T$).

- The superscript λ in $\overset{\circ}{\mathrm{Maps}}{}^{\lambda}$ indicates that we are taking the union of those connected components for which the resulting composed maps
$$X \to B\backslash\overline{T}/T \to \mathrm{pt}/B \to \mathrm{pt}/T \text{ and } X \to B^-\backslash(\overline{G/N})/T \to \mathrm{pt}/B^- \to \mathrm{pt}/T$$
are of degree λ.

Thus, in order to prove Proposition 5.3.2 and Theorem 5.4.5, it suffices to show that the maps
$$B\backslash\overline{T} \to \mathrm{Attr}(G\backslash(\overline{G/N})) \underset{\mathrm{Fixed}(G\backslash(\overline{G/N}))}{\times} T\backslash\overline{T}$$
and
$$B^-\backslash(\overline{G/N}) \to \mathrm{Repel}(G\backslash(\overline{G/N})) \underset{\mathrm{Fixed}(G\backslash(\overline{G/N}))}{\times} T\backslash\overline{T}$$
are isomorphisms.

5.5.6. Using Theorem 4.3.3 and Lemma 5.5.4 we identify the relevant connected components of $\mathrm{Attr}(G\backslash(\overline{G/N}))$ and $\mathrm{Repel}(G\backslash(\overline{G/N}))$ with
$$B\backslash\mathrm{Attr}(\overline{G/N}) \text{ and } B^-\backslash\mathrm{Repel}(\overline{G/N}),$$
where the $\mathbb{G}_m$-action on $\overline{G/N}$ is given by
$$\mathbb{G}_m \xrightarrow{\gamma,\gamma} T\times T \hookrightarrow G\times T.$$

Thus, it is sufficient to show that for the above $\mathbb{G}_m$-action on $\overline{G/N}$, the map
$$\overline{T} \to \mathrm{Attr}(\overline{G/N}) \text{ and } \mathrm{Repel}(\overline{G/N}) \to \overline{G/N}$$
are isomorphisms, both of which assertions are obvious.

Part III: Whittaker categories

6. Spaces of rational reductions

6.1. **The framework.** In this subsection G will be an arbitrary algebraic group.

6.1.1. Let Y a separated scheme equipped with an action of G. Let $\mathrm{Maps}(X, G\backslash Y)$ denote the prestack classifying maps $X \to G\backslash Y$. In particular, for $Y = \mathrm{pt}$ we recover Bun_G.

We let $\mathrm{Maps}^{\mathrm{gen}}(X, G\backslash Y) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G)$ denote the following prestack:

For a test-scheme S, the groupoid of S-points consists of pairs $(\mathcal{P}_G, f)$, where:

- $\mathcal{P}_G$ is a G-bundle on $S\times X$;
- f is a *relatively generically defined* section of the associated bundle $\mathcal{P}_G \overset{G}{\times} Y$.

By a relatively generically defined section we mean a section defined over an open subset $U \subset S\times X$, which is dense in the fiber over any k-point of S. Note that such U is schematically dense in $S\times X$.

We emphasize that the open subset U is *not* part of the data. I.e., we identify f_1 defined over U_1 and f_2 defined over U_2 if $f_1|_{U_1\cap U_2} = f_2|_{U_1\cap U_2}$.

Note that we have a tautological monomorphism
$$\mathrm{Maps}(X, G\backslash Y) \to \mathrm{Maps}^{\mathrm{gen}}(X, G\backslash Y) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G),$$
corresponding to maps that are defined on all of $S\times X$.

6.1.2. It is not difficult to see that if Y is affine, then the forgetful map

$$\mathrm{Maps}^{\mathrm{gen}}(X, G\backslash Y) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G) \to \mathrm{Bun}_G$$

is ind-schematic (i.e., the base change by an affine scheme yields an ind-scheme).

If Y admits a G-equivariant quasi-projective (resp., projective) embedding, then it is shown in [Bar] that, up to fppf sheafification, $\mathrm{Maps}^{\mathrm{gen}}(X, G\backslash Y) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G)$ can be presented as a quotient of an algebraic stack, equal to the disjoint union of algebraic stacks quasi-projective (resp., projective) over Bun_G, by a proper equivalence relation.

6.1.3. In most (but not all!) of our applications, we will be interested in the case when Y is of the form G/H, where H is a subgroup. In this case we shall denote

$$\mathrm{Bun}_G^{H\text{-gen}} := \mathrm{Maps}^{\mathrm{gen}}(X, G\backslash(G/H)) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G).$$

In this case we can think of $\mathrm{Bun}_G^{H\text{-gen}}$ as the stack classifying G-bundles, equipped with a generically defined reduction to H.

Note that the prestack $\mathrm{Maps}(X, G\backslash(G/H))$ identifies with Bun_H.

6.1.4. Consider the particular case of $Y = G$, i.e., the prestack $\mathrm{Bun}_G^{1\text{-gen}}$. This is one of the versions of the Beilinson-Drinfeld Grassmannian: it classifies G-bundles, equipped with a generic trivialization.

We will sometimes write

$$\mathrm{Gr}_{G,\mathrm{gen}} := \mathrm{Bun}_G^{1\text{-gen}}.$$

6.2. **The space of generic parabolic reductions.** In what follows we will let G be a reductive group.

6.2.1. Let us take $Y = G/P$. One of our principal actors will be the corresponding prestack $\mathrm{Bun}_G^{P\text{-gen}}$.

6.2.2. Note that in addition to $\mathrm{Bun}_G^{P\text{-gen}}$, one can consider the prestack $\widetilde{\mathrm{Bun}}_P$, defined as in [BG1]. We have a tautologically defined map

$$\widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G^{P\text{-gen}},$$

which is schematic, proper and surjective at the level of k-points.

6.3. **Two versions of the Whittaker space.**

6.3.1. Consider the subgroups $N \subset G$ and $N \cdot Z_G \subset G$, and the corresponding prestacks

$$\mathrm{Bun}_G^{N\text{-gen}} \text{ and } \mathrm{Bun}_G^{N\cdot Z_G\text{-gen}},$$

respectively. Note that we have a natural forgetful map $\to$ between the two, and also forgetful maps from both to $\mathrm{Bun}_G^{B\text{-gen}}$.

We can think of $\mathrm{Bun}_G^{N\text{-gen}}$ (resp., $\mathrm{Bun}_G^{N\cdot Z_G\text{-gen}}$) as the prestack, classifying the data of $(\mathcal{P}_G, \beta, \gamma)$, where:

- $\mathcal{P}_G$ is a G-bundle $S \times X$;
- β is a reduction of $\mathcal{P}_G$ to a B-bundle $\mathcal{P}_B$, defined over an open subset $U \subset S \times X$, dense in every fiber over S;
- γ is the datum of:

– (in the case of $\mathrm{Bun}_G^{N\text{-gen}}$) a trivialization of the induced T-bundle $T \overset{B}{\times} \mathcal{P}_B$;
– (in the case of $\mathrm{Bun}_G^{N\cdot Z_G\text{-gen}}$) a trivialization γ_i of the line bundle $\check{\alpha}_i(\mathcal{P}_B)$ for every simple root $\check{\alpha}_i$.

6.3.2. We let $\mathcal{Q}_G$ (resp., $\mathcal{Q}_{G,G}$) denote a variant of $\mathrm{Bun}_G^{N\text{-gen}}$ (resp., $\mathrm{Bun}_G^{N\cdot Z_G\text{-gen}}$), where we modify the meaning of γ as follows:

- (in the case of $\mathrm{Bun}_G^{N\text{-gen}}$) an isomorphism between the T-bundle $T \overset{B}{\times} \mathcal{P}_B$ and the pullback from X of the T-bundle $\rho(\omega_X)$.
- (in the case of $\mathrm{Bun}_G^{N\cdot Z_G\text{-gen}}$) an identification γ_i between the line bundle $\check{\alpha}_i(\mathcal{P}_B)$ and the pullback from X of the line bundle ω_X, for each simple root $\check{\alpha}_i$.

As before, we have the forgetful maps

$$\mathcal{Q}_G \to \mathcal{Q}_{G,G} \to \mathrm{Bun}_G^{B\text{-gen}}.$$

6.3.3. Consider the fiber product

$$\mathcal{Q}_G \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B.$$

We have the tautological Cartesian squares

$$\begin{CD}
\mathcal{Q}_G \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B @>>> \mathcal{Q}_G \\
@VVV @VVV \\
\mathrm{Bun}_B @>>> \mathrm{Bun}_G^{B\text{-gen}},
\end{CD}$$

and another Cartesian square

$$\begin{CD}
\mathcal{Q}_G \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B @>>> \mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}} \\
@VVV @VVV \\
\mathrm{Bun}_B @>>> \mathrm{Bun}_T,
\end{CD}$$

where $\mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}}$ is the prestack classifying T-bundles, equipped with a generic identification with $\rho(\omega_X)$.

Note, however, that since T is commutative, we can canonically identify

$$\mathrm{Gr}_{T,\mathrm{gen}} \simeq \mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}}$$

via tensoring by the T-bundle $\rho(\omega_X)$.

In particular, for every m-tuple $\underline{\lambda}$ of elements $\lambda_1, ..., \lambda_m$ of Λ, we have a canonically defined map

$$X^n \to \mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}}, \quad (x_1, ..., x_m) \mapsto \rho(\omega_X)(\Sigma\, \lambda_i \cdot x_i),$$

and hence a map

$$\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} X^m \to \mathcal{Q}_G \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B \to \mathcal{Q}_G.$$

Removing the diagonals, and symmetrizing along those λ_i's that are pair-wise equal, we obtain the maps

$$\overset{\circ}{X}{}^{\underline{\lambda}} \to \mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}},$$

and the decomposition of $\mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}}$ into the strata:

$$\mathrm{Gr}_{T,\rho(\omega_X),\mathrm{gen}} \simeq \bigcup_{\underline{\lambda}} \overset{\circ}{X}{}^{\underline{\lambda}}.$$

Hence, we also obtain a decomposition of $\mathcal{Q}_G$ into the strata:

$$\mathcal{Q}_G \simeq \bigcup_{\underline{\lambda}} \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \overset{\circ}{X}{}^{\underline{\lambda}}. \tag{6.1}$$

6.3.4. Similarly, consider

$$\mathcal{Q}_{G,G} \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B,$$

and the Cartesian squares

$$\begin{tikzcd}
\mathcal{Q}_{G,G} \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B \arrow[r] \arrow[d] & \mathcal{Q}_{G,G} \arrow[d] \\
\mathrm{Bun}_B \arrow[r] & \mathrm{Bun}_G^{B\text{-gen}}.
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{Q}_{G,G} \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \mathrm{Bun}_B \arrow[r] \arrow[d] & \mathrm{Gr}_{T/Z_G,\rho(\omega_X),\mathrm{gen}} \arrow[d] \\
\mathrm{Bun}_B \arrow[r] & \mathrm{Bun}_{T/Z_G}.
\end{tikzcd}$$

Thus, we obtain the decomposition of $\mathcal{Q}_{G,G}$ into the strata:

$$\mathcal{Q}_{G,G} \simeq \bigcup_{\underline{\lambda}} \mathrm{Bun}_B \underset{\mathrm{Bun}_{T/Z_G}}{\times} \overset{\circ}{X}{}^{\underline{\lambda}}, \tag{6.2}$$

where the λ's are now elements of the quotient lattice Λ/Λ_{Z_G}.

Note that the group T/Z_G, being the Cartan of the adjoint quotient of G, is canonically isomorphic to the proeduct $\prod_i \mathbb{G}_m$, where the i-the copy of $\mathbb{G}_m$ mapsto into T/Z_G is the i-th fundamental coweight ϖ_i.

6.4. **The degenerate Whittaker space.**

6.4.1. Let $P \subset G$ be again a parabolic subgroup with Levi quotient M. Note that Z_M is well-defined a subgroup of the abstract Cartan T.

Consider the subgroup $N \cdot Z_M \subset G$, and the corresponding prestack

$$\mathrm{Bun}_G^{N\cdot Z_M\text{-gen}}.$$

We have the natural forgetful map $\mathrm{Bun}_G^{N\cdot Z_M\text{-gen}} \to \mathrm{Bun}_G^{B\text{-gen}}$. This allows to think of $\mathrm{Bun}_G^{N\cdot Z_M\text{-gen}}$ as the prestack classifying the data of $(\mathcal{P}_G, \beta, \gamma)$, where:

- $\mathcal{P}_G$ is a G-bundle $S \times X$;
- β is a reduction of $\mathcal{P}_G$ to a B-bundle $\mathcal{P}_B$, defined over an open subset $U \subset S \times X$, dense in every fiber over S;
- γ is the datum of trivialization γ_i of the line bundle $\check{\alpha}_i(\mathcal{P}_B)$ for every simple root $\check{\alpha}_i$, which i is a vertex of the Dynkin diagram of M.

6.4.2. We shall denote by $\mathcal{Q}_{G,P}$ the variant of $\mathrm{Bun}_G^{N\cdot Z_M\text{-gen}}$, where we modify the data γ_i as follows:

Instead of being a trivialization of $\check{\alpha}_i(\mathcal{P}_B)$, it is now an isomorphism with the pullback from X of the line bundle ω_X.

Note that for $P = G$ we recover the prestack $\mathcal{Q}_{G,G}$ introduced in Sect. 6.3.2. At the other extreme, i.e., for $P = B$ we have $\mathcal{Q}_{G,B} \simeq \mathrm{Bun}_G^{B\text{-gen}}$.

6.4.3. Consider the composed map

$$\mathcal{Q}_{G,P} \to \mathrm{Bun}_G^{B\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}},$$

and denote

$$\mathcal{Q}_{P,M} := \mathcal{Q}_{G,P} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P,$$

so that we have a Cartesian square

$$\begin{tikzcd} \mathcal{Q}_{P,M} \arrow[r] \arrow[d] & \mathcal{Q}_{G,P} \arrow[d] \\ \mathrm{Bun}_P \arrow[r] & \mathrm{Bun}_G^{P\text{-gen}}. \end{tikzcd} \tag{6.3}$$

Note however, the we also have the following Cartesian square:

$$\begin{tikzcd} \mathcal{Q}_{P,M} \arrow[r, "'\mathfrak{q}_P"] \arrow[d] & \mathcal{Q}_{M,M} \arrow[d] \\ \mathrm{Bun}_P \arrow[r, "\mathfrak{q}_P"] & \mathrm{Bun}_M. \end{tikzcd} \tag{6.4}$$

6.5. **The extended Whittaker space.** We now come to the definition of our principal actor–the prestack in D-modules on which we will realize the extended Whittaker category.

In this subsection we will take G to be a reductive group with a connected center.

6.5.1. Let $\mathfrak{ch}_G$ denote the scheme $\mathrm{Spec}(k[\check{\Lambda}^{\mathrm{pos}}])$. Note that we have an open embedding

$$T/Z_G =: \overset{\circ}{\mathfrak{ch}}_G \hookrightarrow \mathfrak{ch}_G.$$

In fact, $\mathfrak{ch}_G$ can be written as the union of strata indexed by the parabolics of G,

$$\mathfrak{ch}_G \simeq \bigcup_P \overset{\circ}{\mathfrak{ch}}_M.$$

We have a natural action of T on $\mathfrak{ch}_G$, which extends the tautological action of T on $\overset{\circ}{\mathfrak{ch}}_G = T/Z_G$, and coincides with the tautological action of T on each $\overset{\circ}{\mathfrak{ch}}_M \simeq T/Z_M$.

6.5.2. Consider the G-scheme

$$G/N \overset{T}{\times} \mathfrak{ch}_G,$$

and the corresponding prestack

$$\operatorname{Maps}^{\rm gen}(X, G\backslash(G/N \overset{T}{\times} \mathfrak{ch}_G)) \underset{\operatorname{Maps}^{\rm gen}(X,\operatorname{pt}/G)}{\times} \operatorname{Maps}(X, \operatorname{pt}/G). \tag{6.5}$$

By definition, the above prestack classifies the data of $(\mathcal{P}_G, \gamma, \beta)$, where:

- $\mathcal{P}_G$ is a G-bundle $S \times X$;
- β is a reduction of $\mathcal{P}_G$ to a B-bundle $\mathcal{P}_B$, defined over an open subset $U \subset S \times X$, dense in every fiber over S;
- γ is the datum of *section* γ_i of the line bundle $-\check{\alpha}_i(\mathcal{P}_B)$ for every simple root $\check{\alpha}_i$, which *is a root of* M;

For every parabolic P, the locally closed embedding $\overset{\circ}{\mathfrak{ch}}_M \hookrightarrow \mathfrak{ch}_G$ gives rise to a locally closed embedding of prestacks

$$\operatorname{Bun}_G^{N\cdot Z_M\text{-gen}} \hookrightarrow \operatorname{Maps}^{\rm gen}(X, G\backslash(G/N \overset{T}{\times} \mathfrak{ch}_G)) \underset{\operatorname{Maps}^{\rm gen}(X,\operatorname{pt}/G)}{\times} \operatorname{Maps}(X, \operatorname{pt}/G).$$

In terms of the modular description as $(\mathcal{P}_G, \gamma, \beta)$, the above locally closed embedding corresponds to those γ, for which γ_i is zero for i not in the Dynkin diagram of M, and is invertible for i in the Dynkin diagram of M.

6.5.3. We let $\mathcal{Q}^{\rm ext}_{G,G}$ denote the variant of the prestack (6.5), where we modify the meaning of γ_i as follows: instead of being a section of the line bundle $-\check{\alpha}_i(\mathcal{P}_B)$, it is now a map from the line bundle $\check{\alpha}_i(\mathcal{P}_B)$ to the pullback of ω_X.

As before, we have a decomposition of $\mathcal{Q}^{\rm ext}_{G,G}$ into locally closed sub-prestacks $\mathcal{Q}_{G,P}$. In addition, we have a tautological projection

$$\mathcal{Q}^{\rm ext}_{G,G} \to \operatorname{Bun}_G^{B\text{-gen}}.$$

7. Induced and Whittaker categories

7.1. **The principal induced category.** In this subsection we will introduce the most "elementary" of the categories discussed in this category:

$$\mathrm{I}(G,B) := \text{D-mod}(\operatorname{Bun}_G^{B\text{-gen}})^{\mathbf{N}} \subset \text{D-mod}(\operatorname{Bun}_G^{B\text{-gen}}).$$

This subcategory is defined by imposing a certain equivariance (or, rather, invariance) condition. The first step in the definition is to consider a version of this condition on the "good locus".

7.1.1. *Definition on the good locus.* Let $\overline{x} = (x_1, ..., x_n)$ be a finite non-empty collection of points of X. We define the sufunctors

$$(\operatorname{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}} \subset \operatorname{Bun}_G^{B\text{-gen}}$$

to consist of the data $(\mathcal{P}_G, \beta)$, where we require that β be defined on a Zariski neighborhood of $\underline{x}$ (i.e., on a Zariski neighborhood of each $S \times x_i$ for a test-scheme S).

Recall the notation from Sect. 3.2. Denote

$$\mathcal{D}_{\underline{x}} := \underset{k=1,\dots,n}{\sqcup} \mathcal{D}_{x_k}.$$

Restriction under $\mathcal{D}_{\underline{x}} \hookrightarrow X$ defines a map

$$(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}} \to \mathrm{pt}/B(\mathcal{D}_{\underline{x}}).$$

Denote

$$(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\underline{x}}} := (\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}} \underset{\mathrm{pt}/B(\mathcal{D}_{\underline{x}})}{\times} \mathrm{pt}.$$

The natural action of the group $\mathbf{B}_0 := B(\mathcal{D}_{\underline{x}})$ on $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\underline{x}}}$ extends to an action of $\mathbf{B} := B(\mathcal{D}^\times_{\underline{x}})$. Denote also $\mathbf{N}_0 := N(\mathcal{D}_{\underline{x}})$ and $\mathbf{N} := N(\mathcal{D}^\times_{\underline{x}})$, and similarly for T.

We identify

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right) = \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{B}_0}$$

via the operation of *-pullback.

The full subcategory

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}} \subset \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)$$

is by definition

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N}\cdot\mathbf{T}_0}.$$

7.1.2. *Reformulation in terms of finite-dimensional groups.* Let us spell this out in more detail, while appealing only to prestacks locally of finite type:

For a given $\mathbf{N}' \subset \mathbf{N}$ (see notation in Sect. 3.2.3) we can choose $\mathbf{N}'' \subset \mathbf{N}_0$ and $\mathbf{T}'' \subset \mathbf{T}_0$ small enough so that the subgroup $\mathbf{N}'' \cdot \mathbf{T}''$ is normal in $\mathbf{N}' \cdot \mathbf{T}_0$. Then the quotient group

$$\mathbf{N}' \cdot \mathbf{T}_0/\mathbf{N}'' \cdot \mathbf{T}''$$

acts on the quotient

$$(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}} := (\mathbf{N}'' \cdot \mathbf{T}'')\backslash(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\underline{x}}}.$$

We identify

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right) \simeq \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}}\right)^{\mathbf{B}_0/\mathbf{N}''\cdot\mathbf{T}''}$$

via the operation of *-pullback (which, up to a shift, is the same as the !-pullback, as the morphism $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}} \to (\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}$ is smooth).

Set

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}'} := \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}}\right)^{\mathbf{N}'\cdot\mathbf{T}_0/\mathbf{N}''\cdot\mathbf{T}''} \subset$$
$$\subset \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}}\right)^{\mathbf{B}_0/\mathbf{N}''\cdot\mathbf{T}''} \simeq \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right).$$

It is easy to see that the above subcategory is independent of the choice of $\mathbf{N}''$ and $\mathbf{T}''$. Note that since $\mathbf{N}'/\mathbf{N}''$ is unipotent, the forgetful functor

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}'} \to \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)$$

is fully faithful.

Finally, we set

$$\text{D-mod}\left((\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}\right)^{\mathbf{N}} := \underset{\mathbf{N}'}{lim}\, \text{D-mod}\left((\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}\right)^{\mathbf{N}'} \simeq \bigcap_{\mathbf{N}'} \text{D-mod}\left((\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}\right)^{\mathbf{N}'} \subset \text{D-mod}\left((\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}\right),$$

where the index $\mathbf{N}'$ runs through the poset of group-subschemes of $\mathbf{N}$.

7.1.3. *Preservation by functors.* Let $\mathcal{Y}$ be a sub-prestack of $(\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}$, whose preimage in $(\text{Bun}_G^{B\text{-gen}})^{\text{level}_{\underline{x}}}$ is preserved by the action of $N(\mathcal{D}_{\underline{x}}^\times)$.

Then by the same token, it makes sense to define

$$\text{D-mod}(\mathcal{Y})^{\mathbf{N}} \subset \text{D-mod}(\mathcal{Y}).$$

Let $f : \mathcal{Y}' \to \mathcal{Y}$ be an inlcusion of sub-prestacks as above. Then the functor $f^!$ sends $\text{D-mod}(\mathcal{Y})^{\mathbf{N}}$ to $\text{D-mod}(\mathcal{Y}')^{\mathbf{N}}$.

If f is schematic, so that the functor $f_* : \text{D-mod}(\mathcal{Y}') \to \text{D-mod}(\mathcal{Y})$ is defined, then this functor also sends $\text{D-mod}(\mathcal{Y}')^{\mathbf{N}}$ to $\text{D-mod}(\mathcal{Y})^{\mathbf{N}}$.

Further, suppose $\mathcal{F}' \in \text{D-mod}(\mathcal{Y}')$ be an object such that the partially defined left adjoint $f_!$ to $f^!$ is defined on $\mathcal{F}'$. Then if $\mathcal{F}' \in \text{D-mod}(\mathcal{Y}')^{\mathbf{N}}$, then $f_!(\mathcal{F}') \in \text{D-mod}(\mathcal{Y})^{\mathbf{N}}$.

Similarly, assume that f is schematic, and let $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ be such that the partially defined left adjoint f^* to f_* is defined on $\mathcal{F}$. Then if $\mathcal{F} \in \text{D-mod}(\mathcal{Y})^{\mathbf{N}}$, then $f^*(\mathcal{F}) \in \text{D-mod}(\mathcal{Y}')^{\mathbf{N}}$.

If $\alpha \mapsto \mathcal{Y}_\alpha \subset \mathcal{Y}$ is a collection of invariant sub-prestacks as above, and assume that $\mathcal{Y}(k) = \underset{\alpha}{\cup} \mathcal{Y}_\alpha(k)$. The followimg easily results from the definitions:

Lemma 7.1.4. *Under the above circumstances, an object of* $\text{D-mod}(\mathcal{Y})$ *belongs to* $\text{D-mod}(\mathcal{Y})^{\mathbf{N}}$ *if and only if its !-restriction to each* $\mathcal{Y}_\alpha$ *belongs to* $\text{D-mod}(\mathcal{Y}_\alpha)^{\mathbf{N}}$.

7.1.5. *Strata-wise description.* Recall the inclusion

$$\text{Bun}_B \hookrightarrow \text{Bun}_G^{B\text{-gen}},$$

and note that it factrs through $(\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}$ for any $\underline{x}$.

We have the following general assertion:

Lemma 7.1.6. *Let* $\mathcal{Y}$ *be a* $\mathbf{N}$*-invariant substack of* Bun_B. *Then:*

(a) $\mathcal{Y}$ *is the preimage under the map* $\mathsf{q}_B : \text{Bun}_B \to \text{Bun}_T$ *of a uniquely defined sub-prestack* $\mathcal{Z} \subset \text{Bun}_T$*;*

(b) *The functors*

$$(\mathsf{q}_B)^* : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y})^{\mathbf{N}} : (\mathsf{q}_B)_*$$

are mutually inverse equivalences of categories;

(c) *Under the identifiaction of point (b), the functor*

$$(\mathsf{q}_B)_* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Z})$$

goes over to the right adjoint to the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N}} \hookrightarrow \text{D-mod}(\mathcal{Y})$.

Proof. We will us the notation of Sect. 7.1.2. The assertion follows from the fact that (over a given quasi-compact substack of Bun_T), we can choose the subgroup $\mathbf{N}'$ large enough so that the map

$$(\mathbf{N}' \cdot \mathbf{T}_0/\mathbf{N}'' \cdot \mathbf{T}'')\backslash(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{level}_{\mathbf{N}''\cdot\mathbf{T}''}} \to \mathrm{Bun}_T$$

is a unipotent gerbe.

□

Remark 7.1.7. Note that Lemmas 7.1.4 and 7.1.6 give a "hands-on" description of

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}} \subset \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right).$$

Namely, an object of $\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)$ belongs to $\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}}$ if and only if its !-restriction to Bun_B belongs to the essential image of the fullu faithful functor

$$\mathsf{q}^* : \text{D-mod}(\mathrm{Bun}_T) \to \text{D-mod}(\mathrm{Bun}_B).$$

7.1.8. *Independence of the choice of $\underline{x}$.* Let $\underline{x} \subset \underline{x}'$ be an inclusion of non-empty collection of points. We can regard $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}'}}$ as a sub-prestack of $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}$.

Let $\mathcal{Y}$ be a sub-prestack of $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}'}}$. Thus, there are a priori two different meanings of $\text{D-mod}(\mathcal{Y})^{\mathbf{N}}$, depending on whether we regard it with respect to $\underline{x}'$ or $\underline{x}$.

However, intersecting $\mathcal{Y}$ with Bun_B, and applying Lemmas 7.1.4 and 7.1.6, we obtain that the above two conditions on an object of $\text{D-mod}(\mathcal{Y})$ to belong to $\text{D-mod}(\mathcal{Y})^{\mathbf{N}}$ coincide.

7.1.9. We are finally ready to define the subcategory

$$\mathrm{I}(G,B) := \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})^{\mathbf{N}} \subset \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}).$$

Namely, we let it to be the full subcategory, consisting of those objects, whose restriction to $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}$ belongs to $\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N}}$ for any non-empty collection of points $\underline{x}$.

Remark 7.1.10. As written, the condition of belonging to the Whittaker category involves checking a huge family of conditions: one for each collection $\underline{x}$. However, Sect. 7.1.8 says that it suffices to check a much smaller set of conditions:

Note that the sub-prestacks $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{x}}$ for $x \in X$ form a Zariski cover of $\mathrm{Bun}_G^{B\text{-gen}}$. Now, Sect. 7.1.8 implies that to check whether a given object of $\text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})$ belongs to $\text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})^{\mathbf{N}}$, it is enough to check that its restriction to each $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{x}}$ belongs to the corresponding subcategory

$$\text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{x}}\right)^{\mathbf{N}} \subset \text{D-mod}\left((\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}_{x}}\right).$$

7.1.11. Let $\mathcal{Y}$ be a sub-prestack of $\mathrm{Bun}_G^{B\text{-gen}}$, such that its intersection with any $(\mathrm{Bun}_G^{B\text{-gen}})^{\mathrm{good}\,\underline{x}}$ has the invariance property described in Sect. 7.1.3. We shall call such sub-prestacks $\mathbf{N}$-*invariant.*

For a $\mathbf{N}$-invariant sub-prestack we define $\text{D-mod}(\mathcal{Y})^{\mathbf{N}} \subset \text{D-mod}(\mathcal{Y})$ by the same procedure as when $\mathcal{Y}$ is all of $\mathrm{Bun}_G^{B\text{-gen}}$. The assighment $\mathcal{Y} \mapsto \text{D-mod}(\mathcal{Y})^{\mathbf{N}}$ has properties analogous to those described in Sects. 7.1.3-7.1.5. In particular, Lemmas 7.1.4 and 7.1.6 and Remark 7.1.7 apply to the present context.

In particular, we have a well-defined functor

$$(\iota_T)^! : \mathrm{I}(G,B) \to \text{D-mod}(\mathrm{Bun}_T)$$

that makes the diagram

$$\begin{tikzcd}
\text{D-mod}(\text{Bun}_G^{B\text{-gen}}) \arrow[r, "(\iota_B)^!"] & \text{D-mod}(\text{Bun}_B) \\
\text{D-mod}(\text{Bun}_G^{B\text{-gen}})^{\mathbf{N}} \arrow[u] \arrow[r] & \text{D-mod}(\text{Bun}_B)^{\mathbf{N}} \arrow[u] \\
\mathrm{I}(G,B) \arrow[u, "="] \arrow[r, "(\iota_T)^!"] & \text{D-mod}(\text{Bun}_T) \arrow[u, "(\mathsf{q}_B)^*"']
\end{tikzcd}$$

commute, where ι_B denotes the inclusion $\text{Bun}_B \hookrightarrow \text{Bun}_G^{B\text{-gen}}$.

7.2. **Averaging.** In this subsection we will show that the tautological embedding

$$\mathrm{I}(G,B) := \text{D-mod}(\text{Bun}_G^{B\text{-gen}})^{\mathbf{N}} \subset \text{D-mod}(\text{Bun}_G^{B\text{-gen}}).$$

admits a *continuous* right adjoint, denoted $\text{Av}_*^{\mathbf{N}}$; morever, this right adjoint is "nicely behaved".

7.2.1. Let $\mathcal{Y}$ be an $\mathbf{N}$-invariant sub-prestack of $\text{Bun}_G^{B\text{-gen}}$. We will decsribe the functor $\text{Av}_*^{\mathbf{N}}$ right adjoint to the tautological embedding $\text{D-mod}(\mathcal{Y})^{\mathbf{N}} \hookrightarrow \text{D-mod}(\mathcal{Y})$.

We shall first to do when $\mathcal{Y}$ is a sub-prestack of $(\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}$ for some $\underline{x}$.

7.2.2. Recall the notation of Sect. 7.1.2. Set

$$\mathcal{Y}^{\text{level}_{\mathbf{N}''\cdot\mathbf{T}''}} := \mathcal{Y} \underset{(\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}}{\times} (\text{Bun}_G^{B\text{-gen}})^{\text{level}_{\mathbf{N}''\cdot\mathbf{T}''}};$$

this is a $\mathbf{B}_0/\mathbf{N}''\cdot\mathbf{T}''$-torsor over $\mathcal{Y}$.

Let

$$\text{D-mod}(\mathcal{Y}^{\text{level}_{\mathbf{N}''}})^{\mathbf{N}'\cdot\mathbf{T}_0/\mathbf{N}''\cdot\mathbf{T}''} \subset \text{D-mod}(\mathcal{Y}^{\text{level}_{\mathbf{N}''}})^{\mathbf{B}_0/\mathbf{N}''\cdot\mathbf{T}''} \simeq \mathcal{Y}$$

be the corresponding equivariant category. This category is independent of the choice of $\mathbf{N}''\cdot\mathbf{T}''$; we shall denote it by $\text{D-mod}(\mathcal{Y})^{\mathbf{N}'}$.

The embedding of this subactegory admits a right adjoint, given by *-averaging along the (unipotent, finite-dimensional) group $\mathbf{N}'/\mathbf{N}''$ on $\mathcal{Y}^{\text{level}_{\mathbf{N}''\cdot\mathbf{T}''}}$. We shall denote this right adjoint by $\text{Av}_*^{\mathbf{N}'}$.

7.2.3. By definition

$$\text{D-mod}(\mathcal{Y})^{\mathbf{N}} = \underset{\mathbf{N}'}{\cap} \text{D-mod}(\mathcal{Y})^{\mathbf{N}'} \subset \text{D-mod}(\mathcal{Y}).$$

A priori, the right adjoint to the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N}} \hookrightarrow \text{D-mod}(\mathcal{Y})$ is given by

$$\mathcal{F} \mapsto \underset{\mathbf{N}'}{\mathit{lim}}\, \text{Av}_*^{\mathbf{N}'}(\mathcal{F}). \tag{7.1}$$

Thus, our task is to show that the functor in (7.1) is continuous.

7.2.4. We claim that for a quasi-compact test-scheme S and an S-point of $\mathcal{Y}$, the family of functors that takes $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ to

$$\mathbf{N}' \mapsto \text{Av}_*^{\mathbf{N}'}(\mathcal{F})|_S$$

stabilizes.

Indeed, by since the restriction functor under $\text{Bun}_B \hookrightarrow \text{Bun}_G^{B\text{-gen}}$ is conservative, it is enough to assume that our S-point factors through Bun_B.

Now the stabilization follows from the same argument as in the proof of Lemma 7.1.6: over a quasi-compact substack of Bun_B, equivariance with respect to $\mathbf{N}$ is equivalent to equivariance with respect to some $\mathbf{N}'$, provided that $\mathbf{N}' \subset \mathbf{N}$ is large enough.

7.2.5. The above stabilization property implies that for an inclusion $f : \mathcal{Y}' \to \mathcal{Y}$, we have a commutative diagram

$$\begin{CD} \text{D-mod}(\mathcal{Y}) @>{f^!}>> \text{D-mod}(\mathcal{Y}') \\ @V{\text{Av}_*^{\mathbf{N}}}VV @VV{\text{Av}_*^{\mathbf{N}}}V \\ \text{D-mod}(\mathcal{Y})^{\mathbf{N}} @>{f^!}>> \text{D-mod}(\mathcal{Y}')^{\mathbf{N}}. \end{CD} \tag{7.2}$$

In addition, if f is schematic, the following diagram is also commutative:

$$\begin{CD} \text{D-mod}(\mathcal{Y}') @>{f_*}>> \text{D-mod}(\mathcal{Y}) \\ @V{\text{Av}_*^{\mathbf{N}}}VV @VV{\text{Av}_*^{\mathbf{N}}}V \\ \text{D-mod}(\mathcal{Y}')^{\mathbf{N}} @>{f_*}>> \text{D-mod}(\mathcal{Y})^{\mathbf{N}}. \end{CD}$$

7.2.6. The above properties imply that the functor $\text{Av}_*^{\mathbf{N}}$ is continuous also in the context of $\mathcal{Y}$ being an $\mathbf{N}$-invariant sub-prestack of all of $\text{Bun}_G^{B\text{-gen}}$ (rather than of $(\text{Bun}_G^{B\text{-gen}})^{\text{good}_{\underline{x}}}$).

Furthermore, the commutativity of the diagrams in Sect. 7.2.5 continues to hold in this context as well.

7.2.7. Composing the functor

$$(\mathsf{p}^{\text{enh}})^! : \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_G^{B\text{-gen}})$$

with the functor

$$\text{Av}_*^{\mathbf{N}} : \text{D-mod}(\text{Bun}_G^{B\text{-gen}}) \to \text{D-mod}(\text{Bun}_G^{B\text{-gen}})^{\mathbf{N}} =: \text{I}(G, B),$$

we obtain a functor that we denote

$$\text{CT}_B^{\text{enh}} : \text{D-mod}(\text{Bun}_G) \to \text{I}(G, B).$$

We call it the *enhanced principal Constant Term* functor. The commutation of the diagram (7.2) and Lemma 7.1.6(c) imply:

Corollary 7.2.8. *There exists a canonical isomorphism of functors* $\text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_T)$

$$(\iota_T)^! \circ \text{CT}_B^{\text{enh}} \simeq \text{CT}_B,$$

where we remind that $\text{CT}_B := (\mathsf{q}_B)_* \circ (\mathsf{p}_B)^!$, *and* $(\iota_T)^!$ *is as in Sect. 7.1.11.*

7.3. **General parabolic induced categories.** In this subsection we fix a parabolic subgroup P. We will generalize the discussion of the previous subsection, and define a certain full subcategory
$$\mathrm{I}(G,P) := \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N}(\mathbf{P})} \subset \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}}).$$

7.3.1. The definition is similar to that of $\mathrm{I}(G,B)$ with the following modifications:

- Level structure is taken with respect to the group P (instead of B);
- Extra equivariance is imposed with respect to the group $\mathbf{N}(\mathbf{P}) := N(P)(\mathcal{D}_{\underline{x}}^\times)$ (instead of $\mathbf{N} := N(\mathcal{D}_{\underline{x}}^\times)$;

7.3.2. Let $\mathcal{Y} \subset \mathrm{Bun}_G^{P\text{-gen}}$ be a sub-prestack, such that its preimage in any $(\mathrm{Bun}_G^{P\text{-gen}})^{\mathrm{good}_x}$ is invariant with respect to the action of the corresponding group-indscheme $\mathbf{N}(\mathbf{P})$. We shall call such sub-prestacks $\mathbf{N}(\mathbf{P})$-invariant.

By a similar token, for a $\mathbf{N}(\mathbf{P})$-invariant sub-prestack $\mathcal{Y}$ we define the full subcategory
$$\text{D-mod}(\mathcal{Y})^{\mathbf{N}(\mathbf{P})} \subset \text{D-mod}(\mathcal{Y}).$$
The assignment $\mathcal{Y} \mapsto \text{D-mod}(\mathcal{Y})^{\mathbf{N}(\mathbf{P})}$ has properties similar to those described in in Sects. 7.1.3-7.1.5.

Analogously to Lemma 7.1.6, we have:

Lemma 7.3.3. *Let $\mathcal{Y}$ be a $\mathbf{N}(\mathbf{P})$-invariant substack of* Bun_P*. Then:*
(a) *$\mathcal{Y}$ is the preimage under the map $\mathsf{q}_P : \mathrm{Bun}_P \to \mathrm{Bun}_M$ of a uniquely defined sub-prestack $\mathcal{Z} \subset \mathrm{Bun}_M$;*
(b) *The functors*
$$(\mathsf{q}_P)^* : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y})^{\mathbf{N}(\mathbf{P})} : (\mathsf{q}_P)_*$$
are mutually inverse equivalences of categories;
(c) *Under the identifiaction of point (b), the functor*
$$(\mathsf{q}_P)_* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Z})$$
goes over to the right adjoint to the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N}(\mathbf{P})} \hookrightarrow \text{D-mod}(\mathcal{Y})$.

In particular, we have a well-defined functor
$$(\iota_M)^! : \mathrm{I}(G,P) \to \text{D-mod}(\mathrm{Bun}_M)$$
that makes the diagram
$$\begin{tikzcd}
\text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}}) \arrow[r, "(\iota_P)^!"] & \text{D-mod}(\mathrm{Bun}_P) \\
\text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N}} \arrow[r] \arrow[u] & \text{D-mod}(\mathrm{Bun}_P)^{\mathbf{N}(\mathbf{P})} \arrow[u] \\
\mathrm{I}(G,P) \arrow[r, "(\iota_M)^!"] \arrow[u, "="] & \text{D-mod}(\mathrm{Bun}_M) \arrow[u, "(\mathsf{q}_P)^*"']
\end{tikzcd}$$
commute, where ι_P denotes the inclusion $\mathrm{Bun}_P \hookrightarrow \mathrm{Bun}_G^{P\text{-gen}}$.

7.3.4. For any $\mathcal{Y}$ as in Lemma 7.3.3 we shall denote by $\mathrm{Av}_*^{\mathbf{N}(\mathbf{P})}$ the right adjoint to the embedding
$$\text{D-mod}(\mathcal{Y})^{\mathbf{N}(\mathbf{P})} \hookrightarrow \text{D-mod}(\mathcal{Y}).$$

It has properties parallel to those of the functor $\mathrm{Av}_*^{\mathbf{N}}$ in Sect. 7.2.

7.3.5. We define the *enhanced parabolic Constant Term* functor

$$\mathrm{CT}_P^{\mathrm{enh}} : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N(P)}} = \mathrm{I}(G,P)$$

to be the composition

$$\mathrm{Av}_*^{\mathbf{N(P)}} \circ (\mathsf{p}_P^{\mathrm{enh}})^!.$$

As in Corollary 7.2.8 we have a canonical isomorphism of functors

$$(\iota_M)^! \circ \mathrm{CT}_P^{\mathrm{enh}} \simeq \mathrm{CT}_P : \quad \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M), \tag{7.3}$$

where $\mathrm{CT}_P := (\mathsf{q}_P)_* \circ (\mathsf{p}_P)^!$.

7.4. **Two versions of the (non-degenerate) Whittaker category.** In this subsection we will discuss yet another version of the category $\text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})^{\mathbf{N}}$, by imposing equivariance against a non-degenerate character.

We will define full subcategories

$$\mathrm{Whit}(G) := \text{D-mod}(\mathcal{Q}_G)^{\mathbf{N},\chi_G} \subset \text{D-mod}(\mathcal{Q}_G)$$

and

$$\mathrm{Whit}(G,G) := \text{D-mod}(\mathcal{Q}_{G,G})^{\mathbf{N},\chi_G} \subset \text{D-mod}(\mathcal{Q}_{G,G}).$$

First, we need the following additional ingredient.

7.4.1. As in Sect. 3.2, we choose a trivialization of the T-bundle $\rho(\omega_X)$. Recall (see Sect. 3.2.4) that such a trivialization defines a homomorphism

$$\mathbf{N} \to \mathbb{G}_a, \tag{7.4}$$

which is trivial on $\mathbf{N}_0$, and hence a homomorphism $\mathbf{N}'/\mathbf{N}'' \to \mathbf{G}_a$ for any

$$\mathbf{N}'' \subset \mathbf{N}_0 \subset \mathbf{N}' \subset \mathbf{N}$$

as in Sect. 3.2.3. Let $\chi_{G,\mathbf{N}'/\mathbf{N}''}$ denote the corresponding character sheaf on $\mathbf{N}'/\mathbf{N}''$, equal to the pull-back of the A-Sch under the homomorphism (7.4).

7.4.2. Now the definition of the full subcategories

$$\text{D-mod}(\mathcal{Q}_G)^{\mathbf{N},\chi_G} \subset \text{D-mod}(\mathcal{Q}_G) \text{ and } \text{D-mod}(\mathcal{Q}_{G,G})^{\mathbf{N},\chi_G} \subset \text{D-mod}(\mathcal{Q}_{G,G})$$

follows the procedure in Sect. 7.1, with the following modifications in Sect. 7.1.2:

- Level structure is considered with respect to the group N;
- Instead of plain equivariance with respect to the group $\mathbf{N}' \cdot \mathbf{T}_0/\mathbf{N}'' \cdot \mathbf{T}''$, we require equivariance against the character sheaf $\chi_{G,\mathbf{N}'/\mathbf{N}''}$ on the group $\mathbf{N}'/\mathbf{N}''$.

7.4.3. Let $\mathcal{Y}$ be a $\mathbf{N}$-invariant sub-prestack of $\mathcal{Q}_G$ (resp., $\mathcal{Q}_{G,G}$). By a similar token we introduce the full subcategory

$$\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_G} \subset \text{D-mod}(\mathcal{Y}).$$

The assignment $\mathcal{Y} \mapsto \text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_G}$ has properties similar to those described in Sect. 7.1.3, and in particular, the analog of Lemma 7.1.4 holds.

In addition, the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_G} \hookrightarrow \text{D-mod}(\mathcal{Y})$ admits a continuous right adjoint, denoted $\mathrm{Av}_*^{\mathbf{N},\chi_G}$, with properties parallel to those of the functor $\mathrm{Av}_*^{\mathbf{N}}$ in Sect. 7.2.

Note that for the forgetful map $f : \mathcal{Q}_G \to \mathcal{Q}_{G,G}$, the functor $f^!$ sends $\mathrm{Whit}(G,G)$ to $\mathrm{Whit}(G)$, and the diagram

$$\begin{array}{ccc} \text{D-mod}(\mathcal{Q}_G) & \xrightarrow{\mathrm{Av}_*^{\mathbf{N},\chi_G}} & \mathrm{Whit}(G) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^!} \\ \text{D-mod}(\mathcal{Q}_{G,G}) & \xrightarrow{\mathrm{Av}_*^{\mathbf{N},\chi_G}} & \mathrm{Whit}(G,G). \end{array} \tag{7.5}$$

7.4.4. *The functor(s) of non-degenerate Whittaker coefficients.* We define the functors of *non-degenerated Whittaker coefficients*:

$$\text{W-coeff}_G : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G) \text{ and } \text{W-coeff}_{G,G} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,G)$$

to be

$$\mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_G)^! \circ (\mathfrak{p}_B^{\mathrm{enh}})^! \text{ and } \mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_{G,G})^! \circ (\mathfrak{p}_B^{\mathrm{enh}})^!,$$

respectively.

We remind that $\mathfrak{r}_G$ and $\mathfrak{r}_{G,G}$ denote the maps

$$\mathcal{Q}_G \to \mathrm{Bun}_G^{B\text{-gen}} \text{ and } \mathcal{Q}_{G,G} \to \mathrm{Bun}_G^{B\text{-gen}},$$

respectively.

7.4.5. *Strata-wise description.* One substantial difference between the Whittaker categories (i.e., $\text{D-mod}(\mathcal{Q}_G)^{\mathbf{N},\chi_G}$ or $\text{D-mod}(\mathcal{Q}_{G,G})^{\mathbf{N},\chi_G}$) and $\text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})$ is their strata-wise description, cf. Sect. 7.1.5.

Recall that the prestack $\mathcal{Q}_G$ (resp., $\mathcal{Q}_{G,G}$) is the union of the strata given by (6.1) (resp., (6.2)), i.e., of the form

$$\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \overset{\circ}{X}{}^{\underline{\lambda}} \text{ and } \mathrm{Bun}_B \underset{\mathrm{Bun}_{T/Z_G}}{\times} \overset{\circ}{X}{}^{\underline{\lambda}}, \tag{7.6}$$

respectively. Let g denote the map from such a stratum to the corresponding $\overset{\circ}{X}{}^{\underline{\lambda}}$.

We have the following version of Lemma 7.1.6 (cf. [FGV]):

Lemma 7.4.6. *Let $\mathcal{Y}$ be a $\mathbf{N}$-invariant sub-prestack of one of the strata in* (7.6).

(a) *$\mathcal{Y}$ is the preimage under g of a uniquely defined sub-prestack $\mathcal{Z} \subset \overset{\circ}{X}{}^{\underline{\lambda}}$.*

(b) *The category $\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_G}$ is zero unless $\underline{\lambda}$ is dominant, i.e., if $\underline{\lambda} = \lambda_1, ..., \lambda_m$, then all $\lambda_j \in \Lambda^+$.*

(c) *If $\underline{\lambda}$ is dominant, we have a canonically defined map $\mathrm{ev} : \mathcal{Y} \to \mathbb{A}^1$, and the operation*

$$\mathcal{F} \in \text{D-mod}(\mathcal{Z}) \mapsto g^*(\mathcal{F}) \overset{*}{\otimes} \mathrm{ev}^*(\text{A-Sch}) \in \text{D-mod}(\mathcal{Y})$$

defines an equivalence

$$\text{D-mod}(\mathcal{Z}) \to \text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_G}.$$

7.5. **Degenerate Whittaker categories.** Let P be again a parabolic subgroup. Consider the prestack $\mathcal{Q}_{G,P}$. In this subsection we will define a full subcategory

$$\mathrm{Whit}(G,P) := \text{D-mod}(\mathcal{Q}_{G,P})^{\mathbf{N},\chi_P} \subset \text{D-mod}(\mathcal{Q}_{G,P}),$$

which combines some of the ingredients of $\mathrm{I}(G,P)$ and $\mathrm{Whit}(G,G)$.

7.5.1. The definition of $\mathrm{Whit}(G,P)$ is similar to that of $\mathrm{Whit}(G,G)$, with the following modifications:

- Level structure is taken with respect to the group $N \cdot Z_M$ (instead of $N \cdot Z_G$);
- Instead of the (non-degenerate) homomorphism $N \to \mathbb{G}_a$ used in Sect. 7.4.1, we use the homomorphism
$$N \to N_M \to \mathbb{G}_a,$$
where $N_M \simeq N/N(P)$ is the unipotent radical of the Borel subgroup of the Levi quotient M, and $N_M \to \mathbb{G}_a$ is the corresponding (non-degenerate) homomorphism for M.
- Equivariance is imposed with respect to subgroups of the form
$$\mathbf{N}' \cdot (\mathbf{Z_M})_0/\mathbf{N}'' \cdot (\mathbf{Z_M})''$$
for
$$\mathbf{N}'' \subset \mathbf{N}_0 \subset \mathbf{N}' \subset \mathbf{N} \text{ and } (\mathbf{Z_M})'' \subset (\mathbf{Z_M})_0$$
so that $\mathbf{N}'' \cdot (\mathbf{Z_M})''$ is normal in $\mathbf{N}' \cdot (\mathbf{Z_M})_0$.

Note that for $P = B$, we get $\mathrm{Whit}(G,B) = \mathrm{I}(G,B)$, and for $P = G$ we recover $\mathrm{Whit}(G,G)$.

7.5.2. Let $\mathcal{Y}$ be a $\mathbf{N}$-invariant sub-prestack of $\mathcal{Q}_{G,P}$. By a similar token we introduce the full subcategory
$$\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_P} \subset \text{D-mod}(\mathcal{Y}).$$

The assignment $\mathcal{Y} \mapsto \text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_P}$ has properties similar to those described in Sect. 7.1.3, and in particular, the analog of Lemma 7.1.4 holds.

In addition, the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_P} \hookrightarrow \text{D-mod}(\mathcal{Y})$ admits a continuous right adjoint, denoted $\mathrm{Av}_*^{\mathbf{N},\chi_P}$, with properties parallel to those of the functor $\mathrm{Av}_*^{\mathbf{N}}$ in Sect. 7.2.

7.5.3. Recall Cartesian diagram (6.4):
$$\begin{array}{ccc}
\mathcal{Q}_{P,M} & \xrightarrow{'\mathsf{q}_P} & \mathcal{Q}_{M,M} \\
{\scriptstyle \mathfrak{r}_{P,M}}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}_{M,M}} \\
\mathrm{Bun}_P & \xrightarrow{\mathsf{q}_P} & \mathrm{Bun}_M .
\end{array}$$

As in Lemma 7.3.3, we obtain the following:

Lemma 7.5.4. *Let $\mathcal{Y}$ be a sub-prestack of $\mathcal{Q}_{P,M}$, invariant with respect to $\mathbf{N}$. Then:*

(a) *$\mathcal{Y}$ is the preimage under the map $'\mathsf{q}_P$ of a uniquely defined sub-prestack $\mathcal{Z} \subset \mathrm{Bun}_M$ invariant with respect to $\mathbf{N_M}$,*

(b) *The functors*
$$('\mathsf{q}_P)^* : \text{D-mod}(\mathcal{Z})^{\mathbf{N_M},\chi_M} \rightleftarrows \text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi_P} : ('\mathsf{q}_P)_*$$
are mutually inverse equivalences of categories.

7.5.5. We define the functor of *P-degenerate Whittaker coefficients*:

$$\text{W-coeff}_{G,P} : \text{D-mod}(\text{Bun}_G) \to \text{Whit}(G, P)$$

to be

$$\text{Av}_*^{\mathbf{N},\chi_P} \circ \mathfrak{r}_{G,P}^! \circ (\mathsf{p}_P^{\text{enh}})^!,$$

where $\mathfrak{r}_{G,P}$ denotes the projection $\mathcal{Q}_{G,P} \to \text{Bun}_G^{P\text{-gen}}$.

In addition, we have the functor $\text{W-coeff}_{P,P} : \text{I}(G, P) \to \text{Whit}(G, P)$, defined as

$$\text{Av}_*^{\mathbf{N},\chi_P} \circ \mathfrak{r}_{G,P}^!.$$

It follows from the definitions that we have

$$\text{W-coeff}_{G,P} \simeq \text{W-coeff}_{P,P} \circ \text{CT}_P^{\text{enh}}. \tag{7.7}$$

7.5.6. Consider the Cartesian diagram (6.3)

$$\begin{tikzcd}
\mathcal{Q}_{P,M} \arrow[r, "'\iota_P^!"] \arrow[d, "\mathfrak{r}_{P,M}"'] & \mathcal{Q}_{G,P} \arrow[d, "\mathfrak{r}_{G,P}"] \\
\text{Bun}_P \arrow[r, "\iota_P^!"] & \text{Bun}_G^{P\text{-gen}}.
\end{tikzcd}$$

It follows from the definitions and Lemma 7.5.4 that we have the following isomorphism of functors

$$\text{I}(G, P) \to \text{Whit}(M, M),$$

namely,

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ \text{W-coeff}_{P,P} \simeq \text{W-coeff}_{M,M} \circ (\iota_M)^!, \tag{7.8}$$

where we recall that $(\iota_M)^!$ denotes the functor $\text{I}(G, P) \to \text{D-mod}(\text{Bun}_M)$ equal to $(\mathsf{q}_P)_* \circ (\iota_P)^!$.

In particular, combing (7.8) and (7.7), we obtain the following isomorphism of functors

$$\text{D-mod}(\text{Bun}_G) \to \text{Whit}(M, M),$$

namely:

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ \text{W-coeff}_{G,P} \simeq \text{W-coeff}_{M,M} \circ \text{CT}_P. \tag{7.9}$$

7.5.7. Let us summarize the above discussion. We have the usual Constant Term functor

$$\text{CT}_P : \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_M),$$

but this functor loses a lot of information. Its enhancement is the functor

$$\text{CT}_P^{\text{enh}} : \text{D-mod}(\text{Bun}_G) \to \text{I}(G, P),$$

and the functor CT_P is obtained from CT^{enh} by restriction along $\iota_P : \text{Bun}_P \hookrightarrow \text{Bun}_G^{P\text{-gen}}$, i.e., $\text{CT}_P \simeq (\iota_M)^! \circ \text{CT}_P^{\text{enh}}$.

We can further compose the functor CT_P with the functor of non-degenerate Whittaker coefficients for M, i.e., we obtain the functor

$$\text{W-coeff}_{M,M} \circ \text{CT}_P : \text{D-mod}(\text{Bun}_G) \to \text{Whit}(M, M).$$

This functor also loses information because CT_P does. Its enhancement is the functor

$$\text{W-coeff}_{G,P} : \text{D-mod}(\text{Bun}_G) \to \text{Whit}(G, P).$$

The relationship between $\text{W-coeff}_{M,M} \circ \text{CT}_P$ and $\text{W-coeff}_{G,P}$ is also given by retsriction to the strata along the $'\iota_P$, i.e., by the isomorphism (7.9).

7.6. **The extended Whittaker category.** Our final goal in this section is to define the full category

$$\mathrm{Whit}^{\mathrm{ext}}(G) \subset \text{D-mod}(\mathcal{Q}^{\mathrm{ext}}).$$

This category will contain the categories $\mathrm{Whit}(G,P)$ as strata, where P ranges through the poset of parabolics subgroups of G.

7.6.1. We shall write the definition by appealing to group-schemes and prestacks of infinite type as in Sect. 7.1.1, leaving the reformulation in terms of objects of finite type (in the spirit of Sect. 7.1.2) to the reader.

For a fixed non-empty collection of points $\underline{x}$, let $(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}}$ be the corresponding sub-prestack of $\mathcal{Q}^{\mathrm{ext}}$, where we require that the data of both β and γ (see Sect. 6.5.3) be defined near $\underline{x}$.

7.6.2. We have a canonically defined map $(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}} \to \mathrm{pt}/\mathbf{B}_0$. Consider the fibe product

$$(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}} := (\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}} \underset{\mathrm{pt}/\mathbf{B}_0}{\times} \mathrm{pt}\,.$$

The action of $\mathbf{B}_0$ on $(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}$ canonically extends to an actio of $\mathbf{B}$.

Note now that the data of γ in the definition of $\mathcal{Q}^{\mathrm{ext}}$ gives rise to a canonically defined map

$$(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}} \to \prod_i \omega_{\mathcal{D}_{\underline{x}}} =: \mathfrak{ch}_{G,\underline{x}}. \tag{7.10}$$

The map (7.10) is equivariant with respect to the group $\mathbf{T}_0$, where the latter acts on $\mathfrak{ch}_{G,\underline{x}}$ via the map

$$T \to T/Z_G \simeq \prod_i \mathbb{G}_m.$$

The scheme $\mathfrak{ch}_{G,\underline{x}}$ identifies with the scheme of group homomorphisms $\mathbf{N} \to \mathbb{G}_a$ that are trivial on $\mathbf{N}_0$. Thus, the group-indscheme $\mathbf{N} \times \mathfrak{ch}_{G,\underline{x}}$ over $\mathfrak{ch}_{G,\underline{x}}$ comes equipped with a homomorphism, denoted χ^{ext}, to the constant group-scheme $\mathbb{G}_a \times \mathfrak{ch}_{G,\underline{x}}$. Furthemore, χ^{ext} is invariant with respect to the action of $\mathbf{T}_0$, which acts by conjugation on $\mathbf{N}$, and in the way specified above on $\mathfrak{ch}_{G,\underline{x}}$.

Thus, viewing $(\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}$ as a prestack equipped with a $\mathbf{N}$-action over $\mathfrak{ch}_{G,\underline{x}}$ by means of the map (7.10), it makes sense to consider the equivariant category

$$\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N},\chi^{\mathrm{ext}}},$$

equipped with a fully faithful forgetful functor to $\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N}_0}$.

Furthemore, since we have a compatible system of actions of $\mathbf{T}_0$ on all of the above objects, we can consider the category

$$\left(\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N},\chi^{\mathrm{ext}}}\right)^{\mathbf{T}_0},$$

which is equipped with a fully faithful forgetful functor to

$$\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N}_0\cdot\mathbf{T}_0} \simeq \text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}}\right).$$

We set

$$\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N},\chi^{\mathrm{ext}}} := \left(\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}\right)^{\mathbf{N},\chi^{\mathrm{ext}}}\right)^{\mathbf{T}_0}.$$

7.6.3. The definition of the category

$$\mathrm{Whit}^{\mathrm{ext}}(G) := \text{D-mod}(\mathcal{Q}^{\mathrm{ext}})^{\mathbf{N},\chi^{\mathrm{ext}}} \subset \text{D-mod}(\mathcal{Q}^{\mathrm{ext}})$$

is obtained from that of

$$\text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}}\right)^{\mathbf{N},\chi^{\mathrm{ext}}} \subset \text{D-mod}\left((\mathcal{Q}^{\mathrm{ext}})^{\mathrm{good}_{\underline{x}}}\right)$$

by the procedure parallel to that in Sects. 7.1.9.

In particular, the definition of the subcategory

$$\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi^{\mathrm{ext}}} \subset \text{D-mod}(\mathcal{Y})$$

makes sense for any $\mathbf{N}$-invariant sub-prestack of $\mathcal{Y} \in \mathcal{Q}^{\mathrm{ext}}$, and properties similar to those described in Sect. 7.1.3, and the analog of Lemma 7.1.4 hold.

Further, the inclusion $\text{D-mod}(\mathcal{Y})^{\mathbf{N},\chi^{\mathrm{ext}}} \hookrightarrow \text{D-mod}(\mathcal{Y})$ admits a continuous right adjoint, denoted $\mathrm{Av}_*^{\mathbf{N},\chi^{\mathrm{ext}}}$ with properties analogous to those described in Sect. 7.2.

7.6.4. *Recovering the degenerate Whittaker categories.* Recall that for a fixed parabolic P we have a canonically defined locally closed embedding

$$\mathbf{i}_P : \mathcal{Q}_{G,P} \to \mathcal{Q}^{\mathrm{ext}}.$$

Thus, taking $\mathcal{Y} = \mathcal{Q}_{G,P}$, we obtain a full subactegory

$$\text{D-mod}(\mathcal{Q}_{G,P})^{\mathbf{N},\chi^{\mathrm{ext}}} \subset \text{D-mod}(\mathcal{Q}_{G,P}).$$

Proposition 7.6.5. *We have*

$$\text{D-mod}(\mathcal{Q}_{G,P})^{\mathbf{N},\chi^{\mathrm{ext}}} = \mathrm{Whit}(G,P)$$

as subcategories of $\text{D-mod}(\mathcal{Q}_{G,P})$.

Proof. By the construction of both categories, it is enough to identify the two subactegories of $\text{D-mod}(\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}})$, i.e.,

$$\text{D-mod}(\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}})^{\mathbf{N},\chi_P} \text{ and } \text{D-mod}(\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}})^{\mathbf{N},\chi^{\mathrm{ext}}}.$$

The two categories in question are defined as follows. For $\text{D-mod}(\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}})^{\mathbf{N},\chi_P}$ we consider the prestack

$$\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}} \underset{\mathrm{pt}/\mathbf{N}_0\cdot(\mathbf{Z_M})_0}{\times} \mathrm{pt},$$

and impose equivariance for the group $\mathbf{N}\cdot(\mathbf{Z_M})_0$ against the character

$$\chi_P : \mathbf{N}\cdot(\mathbf{Z_M})_0 \to \mathbb{G}_a.$$

For $\text{D-mod}(\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}})^{\mathbf{N},\chi^{\mathrm{ext}}}$ we consider the prestack

$$\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}} \underset{\mathrm{pt}/\mathbf{B}_0}{\times} \mathrm{pt},$$

equipped with a map to $\mathfrak{ch}_{G,\underline{x}}$, and impose equivariance with respect to $\mathbf{N}$ against the family of characters χ^{ext}, together with an equivariance with respect to $\mathbf{T}_0$.

Note, however, that the restriction of the map (7.10) to

$$\mathcal{Q}_{G,P}^{\mathrm{good}_{\underline{x}}} \underset{\mathrm{pt}/\mathbf{B}_0}{\times} \mathrm{pt} \subset (\mathcal{Q}^{\mathrm{ext}})^{\mathrm{level}_{\underline{x}}}$$

maps to the locally closed subscheme

$$\overset{\circ}{\mathfrak{ch}}_{M,\underline{x}} \subset \mathfrak{ch}_{G,\underline{x}}$$

(see the notation in Sect. 6.5.1).

Note also that the action of $\mathbf{T}_0$ on $\overset{\circ}{\mathfrak{ch}}_{M,\underline{x}}$ is transitive. A choice of a trivialization of $\rho(\omega_X)$ on $\mathcal{D}_{\underline{x}}$ fixes a point in $\overset{\circ}{\mathfrak{ch}}_{M,\underline{x}}$ whose stabilizer equals $(\mathbf{Z_M})_0$. The preimage of this point in

$$\mathcal{Q}^{\mathrm{good}_{\underline{x}}}_{G,P} \underset{\mathrm{pt}/\mathbf{B}_0}{\times} \mathrm{pt}$$

identifies canonically with

$$\mathcal{Q}^{\mathrm{good}_{\underline{x}}}_{G,P} \underset{\mathrm{pt}/\mathbf{N}_0\cdot(\mathbf{Z_M})_0}{\times} \mathrm{pt}\,.$$

These makes the identification of the two categories in question manifest.

□

7.6.6. *The functor of extended Whittaker coefficients.* Let $\mathfrak{r}^{\mathrm{ext}}$ denote the map

$$\mathcal{Q}^{\mathrm{ext}} \to \mathrm{Bun}_G^{B\text{-gen}}.$$

We define the functor of *extended Whittaker coefficients*

$$\text{W-coeff}^{\mathrm{ext}}_G : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G)$$

to be the composition

$$\mathrm{Av}_*^{\mathbf{N},\chi^{\mathrm{ext}}} \circ (\mathfrak{r}^{\mathrm{ext}})^! \circ (\mathfrak{p}_B^{\mathrm{enh}})^!.$$

By Sect. 7.6.4, for a fixed parabolic P we have

$$\mathbf{i}_P^! \circ \text{W-coeff}^{\mathrm{ext}}_G \simeq \text{W-coeff}_{G,P},$$

as functors $\text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,P)$.

8. Existence of certain left adjoints

8.1. **What we want to do.**

8.1.1. In the previous section we introduced several categories

$$\mathrm{I}(G,P) \subset \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}});\ \mathrm{Whit}(G) \subset \text{D-mod}(\mathcal{Q}_G);\ \mathrm{Whit}(G,P) \subset \text{D-mod}(\mathcal{Q}_{G,P})$$

and

$$\mathrm{Whit}^{\mathrm{ext}}(G) \subset \text{D-mod}(\mathcal{Q}^{\mathrm{ext}}).$$

These categories came along with the naturally defined functors

$$(\iota_M)^! : \mathrm{I}(G,P) \to \text{D-mod}(\mathrm{Bun}_M);$$
$$\mathrm{CT}_P^{\mathrm{enh}} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{I}(G,P);$$
$$\text{W-coeff}_G : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G);$$
$$\text{W-coeff}_{G,G} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,G);$$
$$f^! : \mathrm{Whit}(G,G) \to \mathrm{Whit}(G);$$
$$\text{W-coeff}_{P,P} : \mathrm{I}(G,P) \to \mathrm{Whit}(G,P);$$
$$\text{W-coeff}_{G,P} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,P);$$
$$('\mathfrak{q}_P)_* \circ ('\iota_P)^! : \mathrm{Whit}(G,P) \to \mathrm{Whit}(M,M);$$
$$\mathbf{i}_P^! : \mathrm{Whit}^{\mathrm{ext}}(G) \to \mathrm{Whit}(G,P);$$

$$\text{W-coeff}^{\text{ext}}_G : \text{D-mod}(\text{Bun}_G) \to \text{Whit}^{\text{ext}}(G).$$

Our goal in this section is to show that all of the above functors admit left adjoints. Such questions are non-obvious because the !-pushforward is not in general defined even for schematic morphisms, when one works with non-holonomic D-modules.

The existence of these left adjoints is important for geometric Langlands, because objects obtained by applying these functors are compact generators of many categories of interest. Moreover, the left adjoints have clear counterparts on the spectral side.

8.1.2. The left adjoints of the functors in Sect. 8.1.1 can be split into three groups, in each of which the reasons of its existence is different:

Group one is

$$(\iota_M)^!,\ \text{CT}^{\text{enh}}_P \text{ and } ('\mathsf{q}_P)_* \circ ('\iota_P)^!.$$

Here the idea behind the existence of the left adjoint in question is Drinfeld's compactification $\widetilde{\text{Bun}}_P$ and a certain ULA property, see Sect. 8.2.1 below.

Group two consists of the functor $f^!$. In this case, the reason for the existence of the left agoint is a certain property of the prestack $\text{Maps}^{\text{gen}}(X, \mathbb{G}_m)$.

Group three consists of W-coeff_G, $\text{W-coeff}_{G,G}$ and $\mathbf{i}^!_P$. In this case the reason for the existence of the left adjoint in question is the Hecke action.

8.1.3. The existence of the left adjoint to $\text{W-coeff}_{P,P}$, to be denoted $\text{Poinc}_{P,P}$, is a formal corollary of the existence of the other functors, specifically, $\text{W-coeff}_{G,G}$, $\iota^!_M$ and $('\mathsf{q}_P)_* \circ ('\iota_P)^!$.

Indeed, since the functor $('\mathsf{q}_P)_* \circ ('\iota_P)^!$ is conservative and admits a left adjoint, in order to prove the existence of the left adjoint to $\text{W-coeff}_{P,P}$, it suffices to prove the existence of a left adjoint of the composed functor

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ \text{W-coeff}_{P,P}.$$

Now, using (7.8), the result follows from the existence of the left adjoints to $\iota^!_M$ and $\text{W-coeff}_{M,M}$ (so we are using the existence of the left adjoint to $\text{W-coeff}_{G,G}$ for the group M).

Note that the existence of the left adjoints of $\text{W-coeff}_{P,P}$ and of CT^{enh}_P, respectively, implies the existence of the left adjoint of $\text{W-coeff}_{G,P}$, to be denoted $\text{Poinc}_{G,P}$, since

$$\text{W-coeff}_{G,P} = \text{W-coeff}_{P,P} \circ \text{CT}^{\text{enh}}_P.$$

8.1.4. Finally, we note that the existence of the left adjoint of $\mathbf{i}^!_P$ and $\text{W-coeff}_{G,P}$ implies the existence of a left adjoint of $\text{W-coeff}^{\text{ext}}_G$; the latter willbe deonoted by $\text{Poinc}^{\text{ext}}_G$.

8.2. **Using Drinfeld's compactification.**

8.2.1. Recall the map

$$\mathbf{k}_P : \widetilde{\text{Bun}}_P \to \text{Bun}^{P\text{-gen}}_G,$$

see Sect. 6.2.2. Let $\widetilde{\iota}_P$ denote the open embedding

$$\text{Bun}_P \hookrightarrow \widetilde{\text{Bun}}_P;$$

we have $\mathbf{k}_P \circ \widetilde{\iota}_P = \iota_P$.

Let $\widetilde{\mathsf{q}}_P$ denote the map

$$\widetilde{\text{Bun}}_P \to \text{Bun}_M.$$

We will use the following key property of ths object

$$(\widetilde{\iota}_P)_!(k_{\text{Bun}_P}) \in \text{D-mod}(\widetilde{\text{Bun}}_P).$$

Namely, it is established in [BG1] that $(\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P})$. is ULA with respect to the map $\widetilde{\mathsf{q}}$.

8.2.2. Let us show that the functor

$$(\iota_M)^! : \mathrm{I}(G,P) \to \text{D-mod}(\mathrm{Bun}_M)$$

admits a left adjoint, to be denoted $(\iota_M)_!$.

Since the map $\mathbf{k}_P$ is schematic and proper, the functor

$$(\mathbf{k}_P)_! : \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}}) \to \text{D-mod}(\widetilde{\mathrm{Bun}}_P),$$

left adjoint to $(\mathbf{k}_P)^!$, is defined. Hence, to prove the existence of the left adjoint to

$$(\iota_M)^! = (\mathsf{q})_* \circ (\iota_P)^!,$$

it suffices to show that the left adjoint to

$$(\mathsf{q})_* \circ (\widetilde{\iota}_P)^! : \text{D-mod}(\widetilde{\mathrm{Bun}}_P) \to \text{D-mod}(\mathrm{Bun}_M)$$

exists.

Now, the ULA property of $(\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P})$ immediately implies that the functor

$$(\widetilde{\iota}_P)_! \circ (\mathsf{q}_P)^* : \text{D-mod}(\mathrm{Bun}_M) \to \text{D-mod}(\widetilde{\mathrm{Bun}}_P)$$

is defined. In fact

$$(\widetilde{\iota}_P)_! \circ (\mathsf{q}_P)^*(\mathcal{F}) \simeq (\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P}) \overset{!}{\otimes} (\mathsf{q}_P)^!(\mathcal{F})[2\dim(\mathrm{Bun}_M)].$$

8.2.3. We will now show that the functor

$$({}'\mathsf{q}_P)_* \circ ({}'\iota_P)^! : \mathrm{Whit}(G,P) \to \mathrm{Whit}(M,M)$$

admits a left adjoint, to be denoted $({}'\iota_P)_! \circ ({}'\mathsf{q}_P)^*$.

We will show more generally that the functor

$$({}'\mathsf{q}_P)_* \circ ({}'\iota_P)^! : \text{D-mod}(\mathcal{Q}_{G,P}) \to \text{D-mod}(\mathcal{Q}_{M,M})$$

admits a left adjoint.

Note that the morphism

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{Q}_{M,M} \to \mathcal{Q}_{P,M} \to \mathcal{Q}_{G,P}$$

extends to a morphism

$$\mathbf{k}_P^{\mathcal{Q}} : \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{Q}_{M,M} \to \mathcal{Q}_{G,P}.$$

Moreover, the morphism $\mathbf{k}_P^{\mathcal{Q}}$ is schematic and proper. Hence, it is enough to show that the functor

$$\text{D-mod}(\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{Q}_{M,M}) \overset{(\widetilde{\iota}_P \times \mathrm{id}_{\mathcal{Q}_{M,M}})^!}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{Q}_{M,M}) \overset{(\mathsf{q}_P \times \mathrm{id}_{\mathcal{Q}_{M,M}})_*}{\longrightarrow} \text{D-mod}(\mathcal{Q}_{M,M})$$

admits a left adjoint, i.e., that the functor

$$(\widetilde{\iota}_P \times \mathrm{id}_{\mathcal{Q}_{M,M}})_! \circ (\mathsf{q}_P \times \mathrm{id}_{\mathcal{Q}_{M,M}})^*$$

is defined.

The existence of the latter functor follows again from the fact that

$$(\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P}) \in \text{D-mod}(\widetilde{\mathrm{Bun}}_P)$$

is ULA with respect to the map $\widetilde{\mathsf{q}}$. Indeed, the left adjoint in question is isomorphic to

$$\mathcal{F} \mapsto (\mathrm{id}_{\widetilde{\mathrm{Bun}}_P} \times \mathfrak{r}_{M,M})^! \circ (\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P}) \overset{!}{\otimes} (\widetilde{\mathsf{q}}_P \times \mathrm{id}_{\mathcal{Q}_{M,M}})^!(\mathcal{F})[2\dim(\mathrm{Bun}_M)].$$

8.2.4. We will now show that the functor

$$\mathrm{CT}_P^{\mathrm{enh}} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{I}(G, P)$$

admits a left adjoint, to be denoted $\mathrm{Eis}_P^{\mathrm{enh}}$.

We shall give two (closely related) proofs.

The first proof uses the well-definedness of the functor $(\iota_M)_!$. Since the functor $(\iota_M)^!$ is conservative and admits a left adjoint, it suffices to show that the composed functor

$$(\iota_M)^! \circ \mathrm{CT}_P^{\mathrm{enh}} : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M)$$

admits a left adjoint.

Recall that $(\iota_M)^! \circ \mathrm{CT}_P^{\mathrm{enh}} \simeq \mathrm{CT}_P$ (see (7.3)), and the latter functor does admit a left adjoint:

$$\mathrm{Eis}_P := (\mathsf{p}_P)_! \circ (\mathsf{q}_P)^*,$$

see [DrGa4].

Note, however, that the existence of Eis_P also used the stack $\widetilde{\mathrm{Bun}}_P$ and the ULA property of $(\widetilde{\iota}_P)_!(k_{\mathrm{Bun}_P}) \in \text{D-mod}(\widetilde{\mathrm{Bun}}_P)$.

8.2.5. The second proof of the existence of $\mathrm{Eis}_P^{\mathrm{enh}}$ is in a sense more streamlined. We will show that the functor

$$(\mathsf{p}_P^{\mathrm{enh}})^! : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}})$$

already admits a left adjoint, to be denoted $(\mathsf{p}_P^{\mathrm{enh}})_!$.

Since the morphism $\mathsf{k}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G^{P\text{-gen}}$ is surjective on field-valued points, the functor $(\mathsf{k}_P)^!$ is conservative. Hence, it is enough to show that the composition

$$(\mathsf{k}_P)^! \circ (\mathsf{p}_P^{\mathrm{enh}})^! = (\widetilde{\mathsf{p}}_P)^!$$

admits a left adjoint, where

$$\widetilde{\mathsf{p}}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G .$$

The latter is, however, immediate since the morphism $\widetilde{\mathsf{p}}_P$ is schematic and proper.

8.3. **Rational maps into a torus.**

8.3.1. For a connected algebraic group H consider the group-prestack

$$\mathrm{Maps}^{\mathrm{gen}}(X, H)$$

and the prestack

$$\mathrm{Gr}_{H,\mathrm{gen}} := \mathrm{Bun}_H \underset{\mathrm{Maps}^{\mathrm{gen}}(X, \mathrm{pt}/H)}{\times} \mathrm{pt}.$$

We have have a natural projection

$$\mathrm{Gr}_{H,\mathrm{gen}} \to \mathrm{Bun}_H . \tag{8.1}$$

For any test-scheme S and an S-point of Bun_H, the set of its left to an S-point of $\mathrm{Gr}_{H,\mathrm{gen}}$ is a torsor over the group of S-points of $\mathrm{Maps}^{\mathrm{gen}}(X, H)$. Furthermore, according to [DS], this torsor is non-empty locally in the Zariski topology on S.

The following is the main result of [Ga] (with (b) being an easy corollary of (a)):

Theorem 8.3.2.
(a) *The functor* Vect $\to$ D-mod$(\text{Maps}^{\text{gen}}(X,H))$ *is fully faithful.*
(b) *The !-pullback functor along* (8.1)

$$\text{D-mod}(\text{Bun}_H) \to \text{D-mod}(\text{Gr}_{H,\text{gen}}).$$

is fully faithful.

8.3.3. We will need the following two additional properties of the prestack $\text{Gr}_{H,\text{gen}}$:

Lemma 8.3.4.
(a) $\text{Gr}_{H,\text{gen}}$ *can be written as a colimit of prestacks, each representable by a scheme of finite type, with transition maps being proper*
(b) *Assume that H is reductive. Then* $\text{Gr}_{H,\text{gen}}$ *can be written as a colimit of prestacks, each representable by a projective scheme.*

8.3.5. Assume now that H is isomorphic to the product of several copies of $\mathbb{G}_m$. From Lemma 8.3.4 we obtain:

Corollary 8.3.6. *The prestack*

$$\text{Gr}_{H,\text{gen}} \underset{\text{Bun}_H}{\times} \text{pt}/H \simeq \text{Maps}^{\text{gen}}(X,H)/H$$

can be written as a colimit of prestacks, each representable by a projective scheme.

8.3.7. Recall the map

$$f : \mathcal{Q}_G \to \mathcal{Q}_{G,G}$$

we claim:

Proposition 8.3.8. *The functor* $f^! : \text{D-mod}(\mathcal{Q}_G) \to \text{D-mod}(\mathcal{Q}_{G,G})$ *is fully faithful and admits a left adjoint (to be denoted $f_!$).*

Proof. By assumption, the group Z_G is connected, and so is isomorphic to the product of several copies of $\mathbb{G}_m$. For a test-scheme S and an S-point of $\mathcal{Q}_{G,G}$, the groupoid of its lifts to an S-point of $\mathcal{Q}_G$ is a torsor over the group of S-points of $\text{Maps}^{\text{gen}}(X,Z_G)$. Moreover, this groupoid is non-empty in the Zariski topology on S.

Thus, the map f Zariski-locally splits as a product of the base times $\text{Maps}^{\text{gen}}(X,Z_G)$. This implies that $f^!$ is fully faithful in view of Theorem 8.3.2(a).

To prove the existence of $f_!$, we note that the map f factors as

$$\mathcal{Q}_G \to \mathcal{Q}_G \times \text{pt}/Z_G \overset{f'}{\to} \mathcal{Q}_{G,G},$$

so it suffices to show that the functor

$$f'_! : \text{D-mod}(\mathcal{Q}_G \times \text{pt}/Z_G) \to \text{D-mod}(\mathcal{Q}_{G,G})$$

is defined.

Now, the map f' Zariski-locally splits as a product of the base times the prestack $\text{Maps}^{\text{gen}}(X,Z_G)/Z_G$, and the assertion follows from Corollary 8.3.6.

□

Combining Proposition 8.3.8 with diagram (7.5), we obtain:

Corollary 8.3.9. *The functor $f_!$ sends* $\text{Whit}(G)$ *to* $\text{Whit}(G,G)$ *and provides a left adjoint to* $f^! : \text{Whit}(G,G) \to \text{Whit}(G)$.

Remark 8.3.10. It is not difficult to show that the functor $f_!$ intertwines the functors $\mathrm{Av}_*^{\mathbf{N},\chi_G}$, acting on $\text{D-mod}(\mathcal{Q}_G)$ and $\text{D-mod}(\mathcal{Q}_{G,G})$, respectively.

8.4. **Hecke action.**

8.4.1. Fix an integer n, and consider the version of the Hecke stack with n points

$$\begin{tikzcd} & \mathcal{H}^n \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ X^n \times \mathrm{Bun}_G & & \mathrm{Bun}_G\,. \end{tikzcd}$$

The (ind)-algebraic stack $\mathcal{H}_{\mathrm{Bun}_G}$ classifies the data of:

- an n-tuple of points $\overline{y}$ of X;
- a pair of G-bundles $\mathcal{P}_G$ and $\mathcal{P}'_G$;
- an isomorphism α between $\mathcal{P}_G$ and $\mathcal{P}'_G$ defined away from $\underline{x}$.

8.4.2. Recall also that to an n-tuple $\overline{\lambda} = \lambda^1, ..., \lambda^n$ of elements of Λ^+ we associate an object

$$\mathcal{S}^{\overline{\lambda}} \in \text{D-mod}(\mathrm{Bun}_G)^{\heartsuit}[\dim(\mathrm{Bun}_G)].$$

The Verdier dual of $\mathcal{S}^{\overline{\lambda}}$ identifies with $\mathcal{S}^{-w_0(\overline{\lambda})}[-2\dim(\mathrm{Bun}_G)]$, where w_0 is the longest element of the Weyl group of G.

The key property of $\mathcal{S}_{\overline{\lambda}}$ is that it is ULA with respect to the projections $\overleftarrow{h}$ and $\overrightarrow{h}$.

8.4.3. We let $\mathrm{H}^{\overline{\lambda},\to}$ be the functor $\text{D-mod}(X^n \times \mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_G)$ defined by

$$\mathcal{F} \mapsto \overrightarrow{h}_*(\overleftarrow{h}^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{S}^{\overline{\lambda}})[2n].$$

We let $\mathrm{H}^{\overline{\lambda},\leftarrow}$ be the functor $\text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(X^n \times \mathrm{Bun}_G)$ defined by

$$\mathcal{F} \mapsto \overleftarrow{h}_*(\overrightarrow{h}^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{S}^{\overline{\lambda}}).$$

The ULA property of $\mathcal{S}^{\overline{\lambda}}$ and the fact that both maps $\overleftarrow{h}$ and $\overrightarrow{h}$ are schematic and proper imply:

Lemma 8.4.4. *The functor* $\mathrm{H}^{\overline{\lambda},\to}$ *is the left adjoint of the functor and* $\mathrm{H}^{-w_0(\overline{\lambda}),\leftarrow}$.

8.4.5. Let Y be a scheme acted on by G. Consider the corresponding prestack

$$\mathcal{Y} := \mathrm{Maps}^{\mathrm{gen}}(X, G\backslash Y) \underset{\mathrm{Maps}^{\mathrm{gen}}(X,\mathrm{pt}/G)}{\times} \mathrm{Maps}(X, \mathrm{pt}/G),$$

equipped with its natural projection to $\mathrm{Maps}(X, \mathrm{pt}/G) = \mathrm{Bun}_G$, see Sect. 6.1.1.

It follows from the definitions that we can create a diagram

$$\begin{tikzcd} & \mathcal{H}^n_{\mathcal{Y}} \arrow[dl, "\overleftarrow{h}_Y"'] \arrow[dd] \arrow[dr, "\overrightarrow{h}_Y"] & \\ X^n \times \mathcal{Y} \arrow[dd] & & \mathcal{Y} \arrow[dd] \\ & \mathcal{H}^n \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ X^n \times \mathrm{Bun}_G & & \mathrm{Bun}_G, \end{tikzcd}$$

in which both the left and the right diamonds are Cartesian.

We define the Hecke functors

$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Y}} : \text{D-mod}(X^n \times \mathcal{Y}) \to \text{D-mod}(\mathcal{Y}) \text{ and } \mathrm{H}^{\overline{\lambda},\leftarrow}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(X^n \times \mathcal{Y})$$

to be

$$\mathcal{F} \mapsto (\overrightarrow{h}_Y)_*((\overleftarrow{h}_Y)^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{S}^{\overline{\lambda}}_Y)[2n] \text{ and } \mathcal{F} \mapsto \overleftarrow{h}_*(\overrightarrow{h}^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{S}^{\overline{\lambda}}_Y),$$

respectively, where $\mathcal{S}^{\overline{\lambda}}_{\mathcal{Y}}$ denotes the pullback of $\mathcal{S}^{\overline{\lambda}}$ to $\mathcal{H}^n_{\mathcal{Y}}$.

Our key tool will be the following assertion (whose proof is the same as that of Lemma 8.4.4):

Lemma 8.4.6. *The functor* $\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Y}}$ *is the left adjoint of* $\mathrm{H}^{-w_0(\overline{\lambda}),\leftarrow}_{\mathcal{Y}}$.

8.4.7. Taking $\mathcal{Y}$ to be one of the prestacks

$$\mathrm{Bun}_G^{P\text{-gen}}; \mathcal{Q}_G, \mathcal{Q}_{G,P} \text{ or } \mathcal{Q}^{\mathrm{ext}},$$

we obtain the corresponding pairs of adjoint functors

$$\mathrm{H}^{\overline{\lambda},\to}_{\mathrm{Bun}_G^{P\text{-gen}}} : \text{D-mod}(X^n \times \mathrm{Bun}_G^{P\text{-gen}}) \rightleftarrows \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}}) : \mathrm{H}^{\overline{\lambda},\leftarrow}_{\mathrm{Bun}_G^{P\text{-gen}}};$$

$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Q}_G} : \text{D-mod}(X^n \times \mathcal{Q}_G) \rightleftarrows \text{D-mod}(\mathcal{Q}_G) : \mathrm{H}^{\overline{\lambda},\leftarrow}_{\mathcal{Q}_G};$$

$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Q}_{G,P}} : \text{D-mod}(X^n \times \mathcal{Q}_{G,P}) \rightleftarrows \text{D-mod}(\mathcal{Q}_{G,P}) : \mathrm{H}^{\overline{\lambda},\leftarrow}_{\mathcal{Q}_{G,P}};$$

and

$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Q}^{\mathrm{ext}}} : \text{D-mod}(X^n \times \mathcal{Q}^{\mathrm{ext}}) \rightleftarrows \text{D-mod}(\mathcal{Q}^{\mathrm{ext}}) : \mathrm{H}^{\overline{\lambda},\leftarrow}_{\mathcal{Q}^{\mathrm{ext}}}.$$

Furthermore, the definitions of Sect. 7 can be adapted to define the full subcategories

$$\text{D-mod}(X^n \times \mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N}} \subset \text{D-mod}(X^n \times \mathrm{Bun}_G^{P\text{-gen}});$$

$$\text{D-mod}(X^n \times \mathcal{Q}_G)^{\mathbf{N},\chi_G} \subset \text{D-mod}(X^n \times \mathcal{Q}_G);$$

$$\text{D-mod}(X^n \times \mathcal{Q}_{G,P})^{\mathbf{N},\chi_P} \subset \text{D-mod}(X^n \times \mathcal{Q}_{G,P})$$

and

$$\text{D-mod}(X^n \times \mathcal{Q}^{\mathrm{ext}})^{\mathbf{N}^{\mathrm{ext}}} \subset \text{D-mod}(X^n \times \mathcal{Q}^{\mathrm{ext}}).$$

It follows from the definitions that the above Hecke functors preserve the corresponding subcategories, i.e., define (automatically, mutually adjoint) functors

$$\text{D-mod}(X^n \times \mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N}} \rightleftarrows \text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbf{N}} = \mathrm{I}(G,P);$$

$$\text{D-mod}(X^n \times \mathcal{Q}_G)^{\mathbf{N},\chi_G} \rightleftarrows \text{D-mod}(\mathcal{Q}_G)^{\mathbf{N},\chi_G} = \mathrm{Whit}(G);$$

$$\text{D-mod}(X^n \times \mathcal{Q}_{G,P})^{\mathbf{N},\chi_P} \rightleftarrows \text{D-mod}(\mathcal{Q}_{G,P})^{\mathbf{N},\chi_P} = \mathrm{Whit}(G,P);$$

$$\text{D-mod}(X^n \times \mathcal{Q}^{\mathrm{ext}})^{\mathbf{N}^{\mathrm{ext}}} \rightleftarrows \text{D-mod}(\mathcal{Q}^{\mathrm{ext}})^{\mathbf{N}^{\mathrm{ext}}} = \mathrm{Whit}^{\mathrm{ext}}(G).$$

Furthermore, the Hecke functors commute in the natural sense with the averaging functors

$$\mathrm{Av}^{\mathbf{N}}_*, \ \mathrm{Av}^{\mathbf{N},\chi_G}_*, \ \mathrm{Av}^{\mathbf{N},\chi_P}_* \text{ and } \mathrm{Av}^{\mathbf{N},\chi^{\mathrm{coeff}}}_*.$$

From here, we obtain that the Hecke functors commute in the natural sense with the functors listed in Sect. 8.1.1.

8.4.8. Hence, if
$$\mathsf{G} : \text{D-mod}(\mathcal{Y}_1)^{\mathbf{N},?} \to \text{D-mod}(\mathcal{Y}_2)^{\mathbf{N},??}$$
is one of the functors from Sect. 8.1.1, and G^L is its (potentially partially defined) left adjoint, which is defined on an object $\mathcal{F}_2 \in \text{D-mod}(\mathcal{Y}_2)^{\mathbf{N},??}$ is an object, then for any
$$n, \quad \overline{\lambda} \text{ and } \mathcal{F} \in \text{D-mod}(X^n),$$
the functor G^L is defined also on
$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Y}_2}(\mathcal{F} \boxtimes \mathcal{F}_2) \in \text{D-mod}(\mathcal{Y}_2)^{\mathbf{N},??},$$
and we have
$$\mathsf{G}^L \circ \mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Y}_2}(\mathcal{F} \boxtimes \mathcal{F}_2) \simeq \mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Y}_2}(\mathcal{F} \boxtimes \mathsf{G}^L(\mathcal{F}_2)).$$

8.5. **The functor of Poincaré series.** In this subsection we will show that the functors
$$\text{W-coeff}_G : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G) \text{ and } \text{W-coeff}_{G,G} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,G)$$
both admit left adjoints; to be denoted
$$\mathrm{Poinc}_G \text{ and } \mathrm{Poinc}_{G,G},$$
respectively.

8.5.1. Recall that
$$\text{W-coeff}_G = \mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_G)^! \circ (\mathsf{p}_B^{\mathrm{enh}})^! \text{ and } \text{W-coeff}_{G,G} = \mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_{G,G})^! \circ (\mathsf{p}_B^{\mathrm{enh}})^!$$

Recall (see Sect. 8.2.5) that we have already shown that the functor $(\mathsf{p}_B^{\mathrm{enh}})^!$ admits a left adjoint. We will now show that the functors
$$\mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_G)^! : \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) \to \mathrm{Whit}(G)$$
and
$$\mathrm{Av}_*^{\mathbf{N},\chi_G} \circ (\mathfrak{r}_{G,G})^! : \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) \to \mathrm{Whit}(G,G)$$
admit left adjoints.

I.e., we have to show that the *partially defined* functors
$$(\mathfrak{r}_G)_! : \text{D-mod}(\mathcal{Q}_G) \to \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}) \text{ and } (\mathfrak{r}_{G,G})_! : \text{D-mod}(\mathcal{Q}_{G,G}) \to \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}})$$
are actually defined on the full subactegories
$$\mathrm{Whit}(G) \subset \text{D-mod}(\mathcal{Q}_G) \text{ and } \mathrm{Whit}(G,G) \subset \text{D-mod}(\mathcal{Q}_{G,G}),$$
respectively.

Note that since we know that the functor $f_!$, left adjoint to $f^!$ is well-defined, and $f^!$ is conservative, the assertions for $(\mathfrak{r}_G)_!$ and $(\mathfrak{r}_{G,G})_!$ are equivalent. We will prove one involving $(\mathfrak{r}_G)_!$.

8.5.2. The plan of the proof is the following: we will show that there exists a particular object
$$\psi^{\mathcal{Q}} \in \mathrm{Whit}(G),$$
on which the functor $(\mathfrak{r}_G)_!$ is defined for reasons of holonomicity, and that the category $\mathrm{Whit}(G)$ is generated by the images of objects of the form
$$\mathcal{F} \boxtimes \psi^{\mathcal{Q}}, \quad \mathcal{F} \in \text{D-mod}(X^n)$$
under the Hecke functors
$$\mathrm{H}^{\overline{\lambda},\to}_{\mathcal{Q}_G} : \text{D-mod}(X^n \times \mathcal{Q}_G)^{\mathbf{N},\chi_G} \to \mathrm{Whit}(G).$$

This would imply that the functor Poinc_G is defined on all of $\mathrm{Whit}(G)$ is view of Sect. 8.4.8.

8.5.3. Recall the (algebraic) stack $\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}$, see Sect. 1.1.3 for the notation. Note that we have a canonically defined schematic and proper map

$$\mathbf{m}_G : \overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \mathcal{Q}_G.$$

Recall the object

$$\overline{\psi} \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}).$$

Set

$$\psi^{\mathcal{Q}} := (\mathbf{m}_G)_!(\overline{\psi}).$$

It follows from the definitions that

$$\psi^{\mathcal{Q}} \in \text{D-mod}(\mathcal{Q}_G)^{\mathbf{N},\chi_G}.$$

8.5.4. We claim the functor $(\mathfrak{r}_G)_!$ is defined on $\psi^{\mathcal{Q}}$. Indeed, its suffices to show that the functor

$$(\mathfrak{r}_G \circ \mathbf{m}_G)_! : \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)}) \to \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}),$$

left adjoint to the pullback functor $(\mathfrak{r}_G \circ \mathbf{m}_G)^!$, is defined on $\overline{\psi}$.

We note that the morphism $\mathfrak{r}_G \circ \mathbf{m}_G$ factors as

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \overline{\mathrm{Bun}}_B \overset{\mathbf{k}_B}{\to} \text{D-mod}(\mathrm{Bun}_G^{B\text{-gen}}),$$

where $\mathbf{k}_B$ is schematic and proper. Hence, it remains to show that the !-pushforward along

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \simeq \overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{pt} \to \overline{\mathrm{Bun}}_B,$$

is defined on $\overline{\psi}$.

However, the above morphism is a schematic morphism between algebraic stacks, while $\overline{\psi} \in \text{D-mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})$ is holonomic. This implies the well-definedness of the !-pushforward.

8.5.5. Thus, for the existence of Poinc_G, it remains to show that the category $\mathrm{Whit}(G)$ is generated by the essential images of objects

$$\mathcal{F} \boxtimes \psi^{\mathcal{Q}}, \quad \mathcal{F} \in \text{D-mod}(X^n)$$

under the Hecke functors

$$\mathrm{H}_{\mathcal{Q}_G}^{\overline{\lambda},\to} : \text{D-mod}(X^n \times \mathcal{Q}_G)^{\mathbf{N},\chi_G} \to \mathrm{Whit}(G).$$

This results from Lemma 7.4.6 the following assertion:

Lemma 8.5.6.
(a) *The image in $\mathcal{Q}_G$ under the map $\overrightarrow{h}_{\mathcal{Q}_G}$ of the support of*

$$(X^n \times \mathrm{Im}(\mathbf{m}_G)) \underset{\mathrm{Bun}_G}{\times} \mathrm{supp}(\mathcal{S}^{\overline{\lambda}}) \subset \mathcal{H}^n_{\mathcal{Q}_G}$$

intersects a dominant stratum

$$\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \overset{\circ}{X}{}^{\underline{\mu}} \subset \mathcal{Q}_G$$

(see (6.1)*) only if $m \leq n$ and $\Sigma\, \mu_i \leq \Sigma\, \lambda^j$.*
(b) *For $m = n$ and $\underline{\mu} = \overline{\lambda}$, the functor*

$$\mathcal{F} \in \text{D-mod}(X^n) \mapsto \mathrm{H}_{\mathcal{Q}_G}^{\overline{\lambda},\to}(\mathcal{F} \boxtimes \psi^{\mathcal{Q}})|_{\mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \overset{\circ}{X}{}^{\underline{\mu}}}$$

identifies, in terms of the equivalence of Lemma 7.4.6(c) with the functor

$$\text{D-mod}(X^n) \to \text{D-mod}(\overset{\circ}{X}{}^n) \to \text{D-mod}(\overset{\circ}{X}{}^{\underline{\mu}}),$$

given by restriction to the complement of the diagonal divisor, followed by that of direct image along the partial symmetriczation map $\overset{\circ}{X}{}^n \to \overset{\circ}{X}{}^{\underline{\mu}}$.

8.6. **Functors associated to the extended Whittaker category: the case of $P = G$.** The goal of this subsection and the next is to prove the existence of the left adjoint of the functor

$$(\mathbf{i}_P)^! : \text{Whit}^{\text{ext}}(G) \to \text{Whit}(G, P).$$

Namely, we will show that the partially defined left adjoint $(\mathbf{i}_P)_!$ to

$$(\mathbf{i}_P)^! : \text{D-mod}(\mathcal{Q}^{\text{ext}}) \to \text{D-mod}(\mathcal{Q}_{G,P})$$

is defined on the full subcategory

$$\text{Whit}(G, P) \subset \text{D-mod}(\mathcal{Q}_{G,P}).$$

8.6.1. We shall first consider the case when $P = G$, and then reduce the general case to this one. The proof of the fact that the functor $(\mathbf{i}_G)^!$ is defined on

$$\text{Whit}(G, G) \subset \text{D-mod}(\mathcal{Q}_{G,G})$$

is based on the Hecke action. Namely, as in Sect. 8.5, it suffices that the functor $(\mathbf{i}_G)^!$ is defined on the object

$$f_!(\psi^{\mathcal{Q}}) \in \text{D-mod}(\mathcal{Q}_{G,G}).$$

I.e., we need to show that the functor $(\mathbf{i}_G \circ f \circ \mathbf{m}_G)_!$ is defined on $\overline{\psi} \in \text{D-mod}(\overline{\text{Bun}}_{N,\rho(\omega_X)})$.

8.6.2. Note that the morphism

$$\mathbf{i}_G \circ f \circ \mathbf{m}_G : \overline{\text{Bun}}_{N,\rho(\omega_X)} \to \mathcal{Q}^{\text{ext}}$$

can be canonically factored as

$$\overline{\text{Bun}}_{N,\rho(\omega_X)} \to (\overline{\text{Bun}}_{N,\rho(\omega_X)} \times \prod_i \mathbb{A}^1)/T \xrightarrow{\mathbf{k}^{\text{ext}}} \mathcal{Q}^{\text{ext}},$$

where the map $\mathbf{k}^{\text{ext}}$ is schematic and proper. In the above formula, T acts on $\prod_i \mathbb{A}^1$ via

$$T \to T/Z_G = \prod_i \mathbb{G}_m.$$

8.6.3. Hence, in order to show that $(\mathbf{i}_G \circ f \circ \mathbf{m}_G)_!$ is defined on $\overline{\psi} \in \text{D-mod}(\overline{\text{Bun}}_{N,\rho(\omega_X)})$, it suffices to show that the direct image with compact supports along

$$\overline{\text{Bun}}_{N,\rho(\omega_X)} \to (\overline{\text{Bun}}_{N,\rho(\omega_X)} \times \prod_i \mathbb{A}^1)/T$$

is defined on $\overline{\psi}$.

However, the map in question is between algebraic stacks, and the object $\overline{\psi}$ is holonomic. This implies the well-definedness of the !-pushforward.

8.7. **Functors associated to the extended Whittaker category: the general case.** We will now prove that the functor

$$(\mathbf{i}_P)_! : \text{D-mod}(\mathcal{Q}_{G,P}) \to \text{D-mod}(\mathcal{Q}^{\text{ext}})$$

is defined on the full subcategory

$$\text{Whit}(G,P) \subset \text{D-mod}(\mathcal{Q}_{G,P})$$

for any parabolic P. We shall take as an input the fact that this is the case for the functor $\mathbf{i}_M$, i.e., when G is replaced by M.

8.7.1. First, since the functor

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! : \text{Whit}(G,P) \to \text{Whit}(M,M)$$

is conservative and admits a left adjoint, it suffices to show that the partially defined composed functor

$$(\mathbf{i}_P \circ {}'\iota_P)_! \circ ('\mathsf{q}_P)^* : \text{D-mod}(\mathcal{Q}_{M,M}) \to \text{D-mod}(\mathcal{Q}^{\text{ext}})$$

is defined on

$$\text{Whit}(M,M) \subset \text{D-mod}(\mathcal{Q}_{M,M}).$$

We will show more generally, that for any $\mathcal{F} \in \text{D-mod}(\mathcal{Q}_{M,M})$, for which the functor $(\mathbf{i}_M)_!$ is defined on $\mathcal{F}$, the functor $(\mathbf{i}_P \circ {}'\iota_P)_! \circ ('\mathsf{q}_P)^*$ is also defined on $\mathcal{F}$.

8.7.2. Let $\mathcal{Q}_G^{\text{ext},\leq P} \subset \mathcal{Q}_G^{\text{ext}}$ be the closed subfunctor corresponding to the condition that the γ_i's vanish for i *not* in the Dynkin diagram of M. The map $\mathbf{i}_P$ factors as

$$\mathcal{Q}_{G,P} \to \mathcal{Q}_G^{\text{ext},\leq P} \to \mathcal{Q}_G^{\text{ext}},$$

where the first arrow is an open embedding.

Let $\mathbf{j}_P$ denote the composed map

$$\mathcal{Q}_{P,M} \xrightarrow{'\iota_P} \mathcal{Q}_{G,P} \to \mathcal{Q}_G^{\text{ext},\leq P}.$$

It suffices to show that the (partially defined) functor $(\mathbf{j}_P)_! \circ ('\mathsf{q}_P)^*$ is defined on $\mathcal{F}$ as above.

8.7.3. Note that we have a Cartesian diagram

$$\begin{array}{ccccc}
\mathcal{Q}_{P,M} & \xrightarrow{'\mathbf{i}_M} & \text{Bun}_P \underset{\text{Bun}_G^{P\text{-gen}}}{\times} \mathcal{Q}_G^{\text{ext},\leq P} & \xrightarrow{'\mathbf{j}_P} & \mathcal{Q}_G^{\text{ext},\leq P} \\
{\scriptstyle '\mathsf{q}_P}\downarrow & & \downarrow{\scriptstyle '\mathsf{q}_P^{\mathcal{Q}}} & & \\
\mathcal{Q}_{M,M} & \xrightarrow{\mathbf{i}_M} & \mathcal{Q}_M^{\text{ext}}, & &
\end{array}$$

where the composed horizontal map equals $\mathbf{j}_P$.

Now, as in Sect. 8.2, one shows that the functor

$$('\mathbf{j}_P)_! \circ ('\mathsf{q}_P^{\mathcal{Q}})^* : \text{D-mod}(\mathcal{Q}_M^{\text{ext}}) \to \text{D-mod}(\mathcal{Q}_G^{\text{ext},\leq P}),$$

left adjoint to $('\mathsf{q}_P^{\mathcal{Q}})_* \circ ('\mathbf{j}_P)^!$, is defined.

Finally, we note that the partially defined functors

$$('\mathbf{i}_M)_! \circ ('\mathsf{q}_P)^* \text{ and } ('\mathsf{q}_P^{\mathcal{Q}})^* \circ (\mathbf{i}_M)_!$$

are canonically isomorphic, because their respective right adjoints

$$('\mathsf{q}_P)_* \circ ('\mathbf{i}_M)^! \text{ and } (\mathbf{i}_M)^! \circ ('\mathsf{q}_P^{\mathcal{Q}})_*$$

are isomorphic by base change.

Part IV: Gluing functors for the extended Whittaker category

9. Gluing functors: the statements

9.1. **The first functor.** Let $P_1 \subset P_2$ be a pair of parabolics. Our interest in this section is the gluing functor

$$(\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(G, P_1). \tag{9.1}$$

(Unfortunately), we will not be able to "describe" the above functor explicitly. This is primarily due to the fact that we do not really know if what terms we could give such a description.

Instead, however, we will able to describe rather explicitly two other functors, obtained from (9.1) by composition.

One of these functors is

$$\mathrm{Poinc}_{P_1,P_1} \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{I}(G, P_1).$$

The other is

$$({}'\mathsf{q}_{P_1})_* \circ ({}'\iota_{P_1})^! \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(M_1, M_1).$$

9.1.1. The description of both these functors uses the following ingredient. Consider the open substack

$$\mathcal{Q}_G^{\mathrm{ext}, \geq P_1} \subset \mathcal{Q}_G^{\mathrm{ext}},$$

given by the condition that the elements γ_i *do not vanish* for i being in the Dynkin diagram of M_1.

Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{Q}_{G,P_2} \arrow[r, "\mathbf{i}_{P_2}"] & \mathcal{Q}_G^{\mathrm{ext}, \geq P_1} \arrow[d, "\mathbf{t}_{P_1}"] & \mathcal{Q}_{G,P_1} \arrow[l, "\mathbf{i}_{P_1}"'] \\ & \mathcal{Q}_{G,P_1}, & \end{tikzcd}$$

where $\mathbf{t}_{P_1} \circ \mathbf{i}_{P_1}$ is the identity map.

Hence, we obtain a natural transformation

$$\text{D-mod}(\mathcal{Q}_G^{\mathrm{ext}, \geq P_1}) \to \text{D-mod}(\mathcal{Q}_{G,P_1}), \quad (\mathbf{i}_{P_1})^! \to (\mathbf{t}_{P_1})_!, \tag{9.2}$$

where the right-hand is only partially defined. Composing, we obtain a natural transformation

$$(\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! \to (\mathbf{t}_{P_1} \circ \mathbf{i}_{P_2})_! \tag{9.3}$$

If we apply the functors in (9.3) to an object $\mathcal{F} \in \mathrm{Whit}(G, P_2)$, we have

$$(\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_!(\mathcal{F}) \in \text{D-mod}(\mathcal{Q}_{G,P_1})^{\mathbf{N}, \chi_{P_1}} = \mathrm{Whit}(G, P_1).$$

Hence, by adjunction, we obtain a map

$$(\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_!(\mathcal{F}) \to \mathrm{Av}_*^{\mathbf{N}(\mathbf{P}_1)} \circ (\mathbf{t}_{P_1} \circ \mathbf{i}_{P_2})_!(\mathcal{F}) \tag{9.4}$$

In Sects. 10.2 and 10.3 we will prove:

Proposition 9.1.2.

(a) *The partially defined functor* $(\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_! : \text{D-mod}(\mathcal{Q}_{G,P_2}) \to \text{D-mod}(\mathcal{Q}_{G,P_1})$ *is defined on the full subcategory* $\mathrm{Whit}(G, P_2) \subset \text{D-mod}(\mathcal{Q}_{G,P_2})$.

(b) *The natural transformation of functors* $\mathrm{Whit}(G, P_2) \to \mathrm{Whit}(G, P_1)$ *given by* (9.4) *is an isomorphism.*

9.1.3. We will now describe the functor

$$\mathrm{Whit}(G, P_2) \overset{(\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_!}{\longrightarrow} \mathrm{Whit}(G, P_1) \overset{\mathrm{Poinc}_{P_1,P_1}}{\longrightarrow} \mathrm{I}(G, P_1). \tag{9.5}$$

Consider the composition

$$\mathrm{Whit}(G, P_2) \overset{\mathrm{Poinc}_{P_2,P_2}}{\longrightarrow} \mathrm{I}(G, P_2) \overset{\mathrm{CT}^{\mathrm{enh}}_{P_1,P_2}}{\longrightarrow} \mathrm{I}(G, P_1).$$

We claim that there is a canonically defined natural transformation between the above functors, namely,

$$\mathrm{Poinc}_{P_1,P_1} \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! \to \mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2}. \tag{9.6}$$

Indeed, we first apply the natural transformation (9.4) and obtain a map

$$\mathrm{Poinc}_{P_1,P_1} \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! \to \mathrm{Poinc}_{P_1,P_1} \circ \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_!. \tag{9.7}$$

Recall now that $\mathrm{Poinc}_{P_1,P_1} = (\mathfrak{r}_{G,P_1})_!$, where we remind that $\mathfrak{r}_{G,P_1}$ denotes the map $\mathcal{Q}_{G,P_1} \to \mathrm{Bun}_G^{P_1\text{-gen}}$.

We recall that

$$\mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} = \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^!,$$

so

$$\mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2} = \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^! \circ (\mathfrak{r}_{G,P_2})_!.$$

Let us note, however, that the map $\mathfrak{r}_{G,P_2} : \mathcal{Q}_{G,P_2} \to \mathrm{Bun}_G^{P_2\text{-gen}}$ factors as

$$\mathcal{Q}_{G,P_2} \overset{\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2}}{\longrightarrow} \mathcal{Q}_{G,P_1} \overset{\mathfrak{r}_{G,P_1}}{\longrightarrow} \mathrm{Bun}_G^{P_1\text{-gen}} \overset{\mathsf{p}^{\mathrm{enh}}_{P_1,P_2}}{\longrightarrow} \mathrm{Bun}_G^{P_2\text{-gen}}.$$

Hence, we rewrite

$$\mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2} \simeq \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^! \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})_! \circ (\mathfrak{r}_{G,P_1})_! \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_!. \tag{9.8}$$

Thus, in order to construct the natural transformation (9.6), we have to construct a natural transformation

$$(\mathfrak{r}_{G,P_1})_! \circ \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_! \to \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^! \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})_! \circ (\mathfrak{r}_{G,P_1})_! \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_!. \tag{9.9}$$

The latter is obtained as the composition

$$\begin{aligned}(\mathfrak{r}_{G,P_1})_! \circ \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_! \to \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathfrak{r}_{G,P_1})_! \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_! \to \\ \to \mathrm{Av}_*^{\mathbf{N(P_1)}} \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^! \circ (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})_! \circ (\mathfrak{r}_{G,P_1})_! \circ (\mathfrak{t}_{P_1} \circ \mathbf{i}_{P_2})_!,\end{aligned}$$

where the first arrow comes by adjunction from the fact that $(\mathfrak{r}_{G,P_1})_!$ maps

$$\text{D-mod}(\mathcal{Q}_{G,P_1})^{\mathbf{N(P_1)}} \to \text{D-mod}(\mathrm{Bun}_G^{P_1\text{-gen}})^{\mathbf{N(P_1)}} = \mathrm{I}(G, P_1),$$

and the second arrow comes from the $((\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})_!, (\mathsf{p}^{\mathrm{enh}}_{P_1,P_2})^!)$ adjunction.

We will prove:

Theorem 9.1.4. *The natural transformation* (9.6) *is an isomorphism.*

It is the description of the functor $\mathrm{Poinc}_{P_1,P_1} \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_!$ in terms as the functor $\mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2}$ that we mean to be an explicit description of the former functor.

9.2. **The second functor.** The goal of this subsection we will state a theorem that describes explicitly the functor

$$('\mathsf{q}_{P_1})_* \circ ('\imath_{P_1})^! \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(M_1, M_1). \tag{9.10}$$

9.2.1. First, we will reduce the study of this functor to the case when $P_2 = G$. This will use the following ingredient:

Note that we have a Cartesian diagram

$$\begin{tikzcd}
\mathcal{Q}_{G,P_2} \arrow[r, "\mathbf{i}_{P_2}"] & \mathcal{Q}_G^{\mathrm{ext},\leq P_2} \\
\mathcal{Q}_{P_2,M_2} \arrow[u, "'\imath_{P_2}"] \arrow[r, "'\mathbf{i}_{M_2}"] \arrow[d, "'\mathsf{q}_{P_2}"'] & \mathrm{Bun}_{P_2} \underset{\mathrm{Bun}_G^{P_2\text{-gen}}}{\times} \mathcal{Q}_G^{\mathrm{ext},\leq P_2} \arrow[u, "'\mathbf{j}_{P_2}"'] \arrow[d, "'\mathsf{q}^{\mathcal{Q}}_{P_2}"] \\
\mathcal{Q}_{M_2,M_2} \arrow[r, "\mathbf{i}_{M_2}"] & \mathcal{Q}^{\mathrm{ext}}_{M_2},
\end{tikzcd}$$

from which we obtain a natural transformation

$$('\mathsf{q}^{\mathcal{Q}}_{P_2})_* \circ ('\mathbf{i}_{M_2})_! \circ ('\imath_{P_2})^! \to ('\mathsf{q}^{\mathcal{Q}}_{P_2})_* \circ ('\mathbf{j}_{P_2})^! \circ (\mathbf{i}_{P_2})_!, \quad \mathrm{Whit}(G, P_2) \to \mathrm{Whit}^{\mathrm{ext}}(M_2). \tag{9.11}$$

In Sect. 10.1 we will prove:

Proposition 9.2.2. *The natural transformation* (9.11) *is an isomorphism.*

9.2.3. Note that the composition

$$\mathcal{Q}_{P_1,M_1} \xrightarrow{\imath_{P_1}} \mathcal{Q}_{G,P_1} \xrightarrow{\mathbf{i}_{P_1}} \mathcal{Q}_G^{\mathrm{ext}}$$

canonically factors as

$$\mathcal{Q}_{P_1,M_1} \to \mathrm{Bun}_{P_2} \underset{\mathrm{Bun}_G^{P_2\text{-gen}}}{\times} \mathcal{Q}_{G,P_1} \overset{\mathrm{id}\times \mathbf{i}_{P_1}}{\longrightarrow} \mathrm{Bun}_{P_2} \underset{\mathrm{Bun}_G^{P_2\text{-gen}}}{\times} \mathcal{Q}_G^{\mathrm{ext},\leq P_2} \overset{'\mathbf{j}_{P_2}}{\longrightarrow} \mathcal{Q}_G^{\mathrm{ext},\leq P_2} \hookrightarrow \mathcal{Q}_G^{\mathrm{ext}},$$

and where we have a Cartesian diagram

$$\begin{tikzcd}
\mathcal{Q}_{P_1,M_1} \arrow[r, "''\imath_{P_1(M_2)}"] \arrow[d] & \mathrm{Bun}_{P_2} \underset{\mathrm{Bun}_G^{P_2\text{-gen}}}{\times} \mathcal{Q}_{G,P_1} \arrow[r, "\mathrm{id}\times \mathbf{i}_{P_1}"] \arrow[d] & \mathrm{Bun}_{P_2} \underset{\mathrm{Bun}_G^{P_2\text{-gen}}}{\times} \mathcal{Q}_G^{\mathrm{ext},\leq P_2} \arrow[d, "'\mathsf{q}^{\mathcal{Q}}_{P_2}"] \\
\mathcal{Q}_{P_1(M_2),M_1} \arrow[r, "'\imath_{P_1(M_2)}"] \arrow[d, "'\mathsf{q}_{P_1(M_2)}"'] & \mathcal{Q}_{M_2,P_1(M_2)} \arrow[r, "\mathbf{i}_{P_1(M_2)}"] & \mathcal{Q}^{\mathrm{ext}}_{M_2} \\
\mathcal{Q}_{M_1,M_1}. & &
\end{tikzcd}$$

Hence, we can rewrite the functor $('\mathsf{q}_{P_1})_* \circ ('\imath_{P_1})^! \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_!$ of (9.10) as

$$('\mathsf{q}_{P_1})_* \circ (''\imath_{P_1(M_2)})^! \circ (\mathrm{id} \times \mathbf{i}_{P_1})^! \circ ('\mathbf{j}_{P_2})^! \circ (\mathbf{i}_{P_2})_! \simeq$$
$$\simeq ('\mathsf{q}_{P_1(M_2)})_* \circ ('\imath_{P_1(M_2)})^! \circ (\mathbf{i}_{P_1(M_2)})^! \circ ('\mathsf{q}^{\mathcal{Q}}_{P_2})_* \circ ('\mathbf{j}_{P_2})^! \circ (\mathbf{i}_{P_2})_!.$$

Using Proposition 9.2.2, we further rewrite it as

$$('\mathsf{q}_{P_1(M_2)})_* \circ ('\imath_{P_1(M_2)})^! \circ (\mathbf{i}_{P_1(M_2)})^! \circ ('\mathsf{q}^{\mathcal{Q}}_{P_2})_* \circ ('\mathbf{i}_{M_2})_! \circ ('\imath_{P_2})^!,$$

and finally, using the fact that for Whittaker sheaves

$$('\mathsf{q}^{\mathcal{Q}}_{P_2})_* \circ ('\mathbf{i}_{M_2})_! \simeq (\mathbf{i}_{M_2})_! \circ ('\mathsf{q}_{P_2})_*,$$

we obtain

$$('\mathsf{q}_{P_1})_* \circ ('\imath_{P_1})^! \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! \simeq ('\mathsf{q}_{P_1(M_2)})_* \circ ('\imath_{P_1(M_2)})^! \circ (\mathbf{i}_{P_1(M_2)})^! \circ (\mathbf{i}_{M_2})_! \circ ('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^!. \tag{9.12}$$

I.e., we obtain that the functor of (9.10) identifies with the composition of

$$('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^!, \quad \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(M_2, M_2),$$

and the functor

$$('\mathsf{q}_{P_1(M_2)})_* \circ ('\imath_{P_1(M_2)})^! \circ (\mathbf{i}_{P_1(M_2)})^! \circ (\mathbf{i}_{M_2})_! : \mathrm{Whit}(M_2, M_2) \to \mathrm{Whit}(M_1, M_1),$$

which is nothing but the functor (9.10) for the reductive group M_2 and its parabolic subgroup $P_1(M_2)$.

9.2.4. Thus, our current task is to describe the functor

$$('\mathsf{q}_P)_* \circ ('\imath_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_! : \mathrm{Whit}(G, G) \to \mathrm{Whit}(M, M). \tag{9.13}$$

Consider the fiber product

$$\mathcal{Q}_{G,P} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-},$$

where P^- is the parabolic opposite to P. Let

$$\mathrm{Zast}^{\mathrm{gen}}_P \subset \mathcal{Q}_{G,P} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}$$

denote the open substack, given by the condition that the generic reduction to B (which is part if the data for a point of $\mathcal{Q}_{G,P}$) and the reduction to P^- are transversal at the generic point of the curve.

Let u_P denote the projection $\mathrm{Zast}^{\mathrm{gen}}_P \to \mathcal{Q}_{G,P}$. Note now that the tautological map

$$\mathrm{Zast}^{\mathrm{gen}}_P \to \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$$

naturally lifts to a map

$$\mathrm{Zast}^{\mathrm{gen}}_P \to \mathcal{Q}_{M,M}.$$

We denote this map by v_P.

9.2.5. Consider the fiber product

$$\mathrm{Zast}^{\mathrm{gen}}_P \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{G,G},$$

where $\mathcal{Q}_{G,G} \to \mathcal{Q}_{G,P}$ is the map $\mathsf{t}_P \circ \mathbf{i}_G$. Denote the maps

$$\mathcal{Q}_{M,M} \leftarrow \mathrm{Zast}^{\mathrm{gen}}_P \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{G,G} \to \mathcal{Q}_{G,G}$$

by $'\mathsf{u}_P$ and $'\mathsf{v}_P$, respectively.

Consider the functor

$$('\mathsf{u}_P)_* \circ ('\mathsf{v}_P)^! : \text{D-mod}(\mathcal{Q}_{M,M}) \to \text{D-mod}(\mathcal{Q}_{G,G}),$$

and its partially defined left adjoint $('\mathsf{v}_P)_! \circ ('\mathsf{u}_P)^*$.

We claim that there is a canonically defined natural transformation of partially defined functors

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_! \to ('\mathsf{v}_P)_! \circ ('\mathsf{u}_P)^*, \quad \text{D-mod}(\mathcal{Q}_{G,G}) \to \text{D-mod}(\mathcal{Q}_{M,M}). \tag{9.14}$$

9.2.6. To define the map (9.14), we first apply the natural transformation (9.4) (which, according to Proposition 9.1.2, is in fact an isomorphism when evaluated on objects of $\text{Whit}(G,G)$):

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_! \to ('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ \text{Av}_*^{\mathbf{N(P)}} \circ (\mathsf{t}_P \circ \mathbf{i}_G)_!. \tag{9.15}$$

Note that

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ \text{Av}_*^{\mathbf{N(P)}} \circ (\mathsf{t}_P \circ \mathbf{i}_G)_! \simeq ('\mathsf{q}_P)_* \circ \text{Av}_*^{\mathbf{N(P)}} \circ ('\iota_P)^! \circ (\mathsf{t}_P \circ \mathbf{i}_G)_! \simeq ('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathsf{t}_P \circ \mathbf{i}_G)_!.$$

We now claim that there is a canonically defined natural transformation

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \to (\mathsf{v}_P)_! \circ (\mathsf{u}_P)^*, \quad \text{D-mod}(\mathcal{Q}_{G,P}) \to \text{D-mod}(\mathcal{Q}_{M,M}). \tag{9.16}$$

Once the natural transformation (9.16) is defined, the sought-for natural transformation (9.14) is set to be

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_! \to ('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathsf{t}_P \circ \mathbf{i}_G)_! \to$$
$$\to (\mathsf{u}_P)_! \circ (\mathsf{v}_P)^* \circ (\mathsf{t}_P \circ \mathbf{i}_G)_! \simeq ('\mathsf{v}_P)_! \circ ('\mathsf{u}_P)^*.$$

9.2.7. The natural transfrmation (9.16) is defined as follows. To specify it, by adjunction, we can equivalently specify a natural transformation

$$('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathsf{u}_P)_* \circ (\mathsf{v}_P)^! \to \text{Id}\,. \tag{9.17}$$

The natural transformation $('\mathsf{q}_P)_* \circ ('\iota_P)^! \circ (\mathsf{u}_P)_* \circ (\mathsf{v}_P)^!$ is given by pull-push along the Cartesian diagram

$$\begin{tikzcd}
\text{Zast}_P^{\text{gen}} \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{P,M} \arrow[r] \arrow[d] & \mathcal{Q}_{P,M} \arrow[r, "'\mathsf{q}_P"] \arrow[d, "'\iota_P"] & \mathcal{Q}_{M,M} \\
\text{Zast}_P^{\text{gen}} \arrow[r, "\mathsf{u}_P"] \arrow[d, "\mathsf{v}_P"'] & \mathcal{Q}_{G,P} & \\
\mathcal{Q}_{M,M}. & &
\end{tikzcd} \tag{9.18}$$

Note however, that the open embedding $\text{Bun}_M \hookrightarrow \text{Bun}_P \underset{\text{Bun}_M}{\times} \text{Bun}_{P^-}$ give rise to an open embedding

$$\mathcal{Q}_{M,M} \hookrightarrow \text{Zast}_P^{\text{gen}} \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{P,M}.$$

This gives rise to the natural transformation (9.17). This natural transformation is the key step in the construction of the natural transformation (9.14).

In Sect. 10.4 we will prove:

Proposition 9.2.8. *Both sides of* (9.16) *are well-defined, and the natural transformation* (9.16) *is an isomorphism.*

9.2.9. Thus, from Proposition 9.2.8 we obtain:

Theorem 9.2.10. *The natural transformation*

$$({}'\mathsf{q}_P)_* \circ ({}'\iota_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_! \to ({}'\mathsf{v}_P)_! \circ ({}'\mathsf{u}_P)^*$$

is an isomorphism when evaluated on objects of $\mathrm{Whit}(G,G) \subset \text{D-mod}(\mathrm{Bun}_G)$.

We regard the right-hand side in Theorem 9.2.10 as giving a more explicit description of the functor $({}'\mathsf{q}_P)_* \circ ({}'\iota_P)^! \circ (\mathbf{i}_P)^! \circ (\mathbf{i}_G)_!$, thus accomplishing the goal of this subsection.

9.3. Relation between Theorems 9.1.4 and 9.2.10.

9.3.1. Let us summarize the previous discussion. For a pair of parabolics $P_1 \subset P_2$ we consider the functor

$$\mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(G, P_1),$$

and the compositions

$$\mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{I}(G, P_1)$$

and

$$({}'\mathsf{q}_{P_1})_* \circ ({}'\iota_{P_1})^! \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(M_1, M_1).$$

In this subsection we will study the further compositions

$$(\imath_{M_1})^! \circ \mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \text{D-mod}(\mathrm{Bun}_{M_1}) \tag{9.19}$$

and

$$\mathrm{Poinc}_{M_1,M_1} \circ ({}'\mathsf{q}_{P_1})_* \circ ({}'\iota_{P_1})^! \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! : \mathrm{Whit}(G, P_2) \to \text{D-mod}(\mathrm{Bun}_{M_1}). \tag{9.20}$$

9.3.2. Recall that in Sect. 9.1 we have constructed a natural transformation

$$\mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \to \mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2},$$

and stated Theorem 9.1.4 that this natural transformation is an isomorphism.

Hence, composing with the functor $(\imath_{M_1})^! : \mathrm{I}(G, P_1) \to \text{D-mod}(\mathrm{Bun}_{M_1})$, we obtain a natural transformation (in fact, an isomorphism, according to Theorem 9.1.4):

$$\begin{aligned}(\imath_{M_1})^! \circ \mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \to (\imath_{M_1})^! \circ \mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2} \simeq \\ \simeq \mathrm{CT}_{P_1(M_2)} \circ (\imath_{M_2})^! \circ \mathrm{Poinc}_{P_2,P_2},\end{aligned} \tag{9.21}$$

where $\mathrm{CT}_{P_1(M_2)}$ is the Constant term functor $\text{D-mod}(\mathrm{Bun}_{M_2}) \to \text{D-mod}(\mathrm{Bun}_{M_1})$.

9.3.3. Recall also that in Sect. 9.2 we have constructed a natural transformation

$$({}'\mathsf{q}_{P_1})_* \circ ({}'\iota_{P_1})^! \circ (\mathbf{i}_{P_1})^! \circ (\mathbf{i}_{P_2})_! \to ({}'\mathsf{v}_{P_1(M_2)})_! \circ ({}'\mathsf{u}_{P_1(M_2)})^* \circ ({}'\mathsf{q}_{P_2})_* \circ ({}'\imath_{P_2})^!,$$

and Theorem 9.2.10 asserted that this natural transformation is an isomorphism.

Hence, composing with the functor $\mathrm{Poinc}_{M_1,M_1} : \mathrm{Whit}(M_1, M_1) \to \text{D-mod}(\mathrm{Bun}_{M_1})$, we obtain a natural transformation (in fact, an isomorphism according to Theorem 9.2.10):

$$\begin{aligned}&\mathrm{Poinc}_{M_1,M_1} \circ ({}'\mathsf{q}_{P_1})_* \circ ({}'\iota_{P_1})^! \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \to \\ &\simeq \mathrm{Poinc}_{M_1,M_1} \circ ({}'\mathsf{v}_{P_1(M_2)})_! \circ ({}'\mathsf{u}_{P_1(M_2)})^* \circ ({}'\mathsf{q}_{P_2})_* \circ ({}'\imath_{P_2})^! \simeq \\ &\simeq (\mathsf{p}^{\mathrm{enh}}_{B(M_1)} \circ \mathfrak{r}_{M_1,M_1} \circ {}'\mathsf{v}_{P_1(M_2)})_! \circ ({}'\mathsf{u}_{P_1(M_2)})^* \circ ({}'\mathsf{q}_{P_2})_* \circ ({}'\imath_{P_2})^!,\end{aligned} \tag{9.22}$$

where we note that the map $\mathsf{p}^{\mathrm{enh}}_{B(M_1)} \circ \mathfrak{r}_{M_1,M_1} \circ {}'\mathsf{v}_{P_1(M_2)}$ is just the projection

$$\mathrm{Zast}^{\mathrm{gen}}_{P_1(M_2)} \underset{\mathcal{Q}_{M_2,P_1(M_2)}}{\times} \mathcal{Q}_{M_2,M_2} \to \mathrm{Zast}^{\mathrm{gen}}_{P_1(M_2)} \to \mathcal{Q}_{M_2,M_2} \to \mathrm{Bun}_{M_2}.$$

9.3.4. We will now complete the natural transformations (9.21) and (9.22) to a commutative diagram of functors and natural transformations:

(9.23)
$$\begin{tikzcd}
(\imath_{M_1})^! \circ \mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \arrow[dr] & \\
& \mathrm{CT}_{P_1(M_2)} \circ (\imath_{M_2})^! \circ \mathrm{Poinc}_{P_2,P_2} \\
\mathrm{Poinc}_{M_1,M_1} \circ ('\mathsf{q}_{P_1})_* \circ ('\imath_{P_1})^! \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \arrow[uu] \arrow[dr] & \\
& (\mathsf{p}^{\mathrm{enh}}_{B(M_1)} \circ \mathfrak{r}_{M_1,M_1} \circ {}'\mathsf{v}_{P_1(M_2)})_! \circ ('\mathsf{u}_{P_1(M_2)})^* \circ ('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^! \arrow[uu]
\end{tikzcd}$$

In order to construct the vertical arrows in this diagram we need the following observation.

9.3.5. Note that for a parabolic P we have a natural transformation

$$\mathrm{Poinc}_{M,M} \circ ('\mathsf{q}_P)_* \circ ('\imath_P)^! \to (\imath_M)^! \circ \mathrm{Poinc}_{P,P}, \quad \mathrm{Whit}(G,P) \to \text{D-mod}(\mathrm{Bun}_M). \tag{9.24}$$

Namely, it comes from the Cartesian diagram

$$\begin{tikzcd}
\mathcal{Q}_{M,M} \arrow[d, "\mathfrak{r}_{M,M}"'] & \mathcal{Q}_{P,M} \arrow[l, "'\mathsf{q}_P"'] \arrow[r, "'\imath_P"] \arrow[d, "\mathfrak{r}_{P,M}"'] & \mathcal{Q}_{G,P} \arrow[d, "\mathfrak{r}_{G,P}"] \\
\mathrm{Bun}_M & \mathrm{Bun}_P \arrow[l, "\mathsf{q}_P"'] \arrow[r, "\imath_P"] & \mathrm{Bun}_G^{P-\mathrm{gen}}.
\end{tikzcd}$$

In Sect. 10.1 we will prove:

Proposition 9.3.6. *The natural transformation* (9.24) *is an isomorphism.*

9.3.7. We let the left vertical arrow in the diagram (9.23) to be given by the map (9.24) for $P = P_1$, precomposed with $\mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_!$.

The right vertical arrow in the diagram (9.23) is the composition of

$$\mathrm{CT}_{P_1(M_2)} \circ \mathrm{Poinc}_{M_2,M_2} \circ ('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^! \to (\imath_{M_2})^! \to \mathrm{CT}_{P_1(M_2)} \circ (\imath_{M_2})^! \circ \mathrm{Poinc}_{P_2,P_2},$$

given by the map (9.24) for $P = P_2$ composed with $\mathrm{CT}_{P_1(M_2)}$, and a map

$$(\mathsf{p}^{\mathrm{enh}}_{B(M_1)} \circ \mathfrak{r}_{M_1,M_1} \circ {}'\mathsf{v}_{P_1(M_2)})_! \circ ('\mathsf{u}_{P_1(M_2)})^* \circ ('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^! \to$$
$$\to \mathrm{CT}_{P_1(M_2)} \circ \mathrm{Poinc}_{M_2,M_2} \circ ('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^!,$$

obtained by composing $('\mathsf{q}_{P_2})_* \circ ('\imath_{P_2})^!$ with a natural transformation

$$(\mathsf{p}^{\mathrm{enh}}_{B(M_1)} \circ \mathfrak{r}_{M_1,M_1} \circ {}'\mathsf{v}_{P_1(M_2)})_! \circ ('\mathsf{u}_{P_1(M_2)})^* \to \mathrm{CT}_{P_1(M_2)} \circ \mathrm{Poinc}_{M_2,M_2}, \tag{9.25}$$

described below.

9.3.8. Note that in constructing the natural transformation (9.25), we are working with the reductive group M_2 and its parabolic $P_1(M_2)$. So, it is enough to construct this natural transformation for G and a parabolic P. I.e., we want to construct a natural transformation

$$(\mathsf{p}^{\mathrm{enh}}_B \circ \mathfrak{r}_{M,M} \circ {}'\mathsf{v}_P)_! \circ ('\mathsf{u}_P)^* \to \mathrm{CT}_P \circ \mathrm{Poinc}_{G,G}\,. \tag{9.26}$$

The crucial ingredient is provided by the natural transformation (in fact, an isomorphism) of [DrGa4]:

$$\mathrm{CT}_P \simeq (\mathsf{q}^-_P)_! \circ (\mathsf{p}^-_P)^*.$$

Thus, the right-hand side in (9.26) is given by *-pull and !-push along the diagram

$$\begin{tikzcd}
\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathcal{Q}_{G,G} \arrow[r] \arrow[d] & \mathcal{Q}_{G,G} \arrow[d] \\
\mathrm{Bun}_{P^-} \arrow[r, "\mathsf{p}_{P^-}"] \arrow[d, "\mathsf{q}_{P^-}"'] & \mathrm{Bun}_G \\
\mathrm{Bun}_M\,. &
\end{tikzcd}$$

Now, the left-hand side in (9.26) is given by *-pull and !-push along the diagram

$$\begin{tikzcd}
\mathrm{Zast}^{\mathrm{gen}}_P \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{G,G} \arrow[r] \arrow[d] & \mathcal{Q}_{G,G} \\
\mathrm{Zast}^{\mathrm{gen}}_P \arrow[d] & \\
\mathrm{Bun}_{P^-} \arrow[d, "\mathsf{q}_{P^-}"'] & \\
\mathrm{Bun}_M\,. &
\end{tikzcd}$$

The require natural transformation comes now from the fact that we have an open embedding

$$\mathrm{Zast}^{\mathrm{gen}}_P \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{G,G} \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathcal{Q}_{G,G},$$

which comes by base change from the open embedding

$$\mathrm{Zast}^{\mathrm{gen}}_P \hookrightarrow \mathcal{Q}_{G,P} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-}\,.$$

The next assertion is proved in a manner parallel to that of the isomorphism in Sect. 3.1.4:

Lemma 9.3.9. *The natural transformation* (9.26) *is an isomorphism.*

9.3.10. Thus, we have constructed all the arrows in the diagram (9.23). As in (3.4) we have:

Lemma 9.3.11. *The diagram* (9.23) *is commutative.*

9.3.12. Let us for a moment assume Proposition 9.3.6. We obtain:

Corollary 9.3.13. *Theorem 9.1.4 implies Theorem 9.2.10.*

Proof. It is enough to show that the natural transformation

$$\mathrm{Poinc}_{P_1,P_1} \circ \mathbf{i}^!_{P_1} \circ (\mathbf{i}_{P_2})_! \to \mathrm{CT}^{\mathrm{enh}}_{P_1,P_2} \circ \mathrm{Poinc}_{P_2,P_2}$$

becomes an isomorphism after applying the functor $(\imath_{M_1})^!$, since the latter is conservative. I.e., it is enough to show that the natural transformation (9.21) is an isomorphism.

This natural transformation is the top slanted arrow in the diagram (9.23). Now, Theorem 9.2.10 implies that the bottom slanted arrow in (9.23) is an isomorphism.

Now, Proposition 9.3.6 implies that the left vertical arrow in (9.23) is an isomorphism. Finally, Proposition 9.3.6 and Lemma 9.3.9 show that the right vertical arrow in (9.23) is also an isomorphism.

□

10. Gluing functors: proofs

In this section we will prove the following assertions stated in Sect. 9: Propositions 9.1.2, 9.2.2, 9.3.6 and 9.2.8.

Proposition 9.1.2 will be a rather general assertion about the interplay between Whittaker and Kirillov patterns.

We will group these assertions according to the ideas involved in their proofs.

Namely, Propositions 9.2.2 and 9.3.6 will be proved using a ULA consideration akin to one used in Sect. 8.2.

Proposition 9.2.8 will be an application of Braden's theorem, similar to the proof of Theorem 2.2.3 from Part I of the paper.

10.1. **Proof of Propositions 9.2.2 and 9.3.6: a ULA game.**

10.1.1. Consider the algebraic stack

$$\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P,$$

and let pr_1 and pr_2 denote its projections to $\widetilde{\mathrm{Bun}}_P$ and Bun_P, respectively.

Consider the object

$$(\mathrm{pr}_1)^! \circ (\widetilde{\imath}_P)_!(k_{\mathrm{Bun}_P}) \in \text{D-mod}(\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P).$$

The key tool in this subsection is the following assertion, which is proved in the same way as [BG1, Theorem ???]:

Proposition 10.1.2. *The object* $(\mathrm{pr}_1)^! \circ (\widetilde{\imath}_P)_!(k_{\mathrm{Bun}_P})$ *is ULA with respect to the map*

$$\widetilde{\mathsf{q}}_P \circ \mathrm{pr}_1 : \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_M .$$

We will now prove Proposition 9.3.6, and leave Proposition 9.2.2 as its proof is similar.

10.1.3. Since the essential image of the functor

$$({}'\imath_P)_! \circ ({}'\mathsf{q}_P)^* : \mathrm{Whit}(M,M) \to \mathrm{Whit}(G,P)$$

generates the target, it is enough to show that the natural transformation

$$\mathrm{Poinc}_{M,M} \circ ({}'\mathsf{q}_P)_* \circ ({}'\imath_P)^! \circ ({}'\imath_P)_! \circ ({}'\mathsf{q}_P)^* \to (\imath_M)^! \circ \mathrm{Poinc}_{P,P} \circ ({}'\imath_P)_! \circ ({}'\mathsf{q}_P)^*$$

is an isomorphism.

This is equivalent to showing that the natural transformation

$$(\mathfrak{r}_{P,M})_! \circ ({}'\imath_P)^! \circ ({}'\imath_P)_! \circ ({}'\mathsf{q}_P)_* \to (\imath_P)^! \circ (\imath_P)_! \circ (\mathsf{q}_P)^* \circ (\mathfrak{r}_{M,M})_! \tag{10.1}$$

that comes from the Cartesian diagram

$$\begin{tikzcd}
\mathcal{Q}_{M,M} \arrow[d, "\mathfrak{r}_{M,M}"'] & \mathcal{Q}_{P,M} \arrow[l, "{}'\mathsf{q}_P"'] \arrow[r, "{}'\imath_P"] \arrow[d, "\mathfrak{r}_{P,M}"'] & \mathcal{Q}_{G,P} \arrow[d, "\mathfrak{r}_{G,P}"] & \mathcal{Q}_{P,M} \arrow[l, "{}'\imath_P"'] \arrow[d, "\mathfrak{r}_{M,M}"] \\
\mathrm{Bun}_M & \mathrm{Bun}_P \arrow[l, "\mathsf{q}_P"'] \arrow[r, "\imath_P"] & \mathrm{Bun}_G^{P\text{-gen}} & \mathrm{Bun}_P \arrow[l, "\imath_P"']
\end{tikzcd}$$

is an isomorphism. We will show that (10.1) is an isomorphism for any object $\mathcal{F} \in \text{D-mod}(\mathcal{Q}_{M,M})$ on which the functor $(\mathfrak{r}_{M,M})_!$ is defined.

10.1.4. Recall the map $\mathbf{k}_P : \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_G^{P\text{-gen}}$. We have:

$$(\imath_P)^! \circ (\imath_P)_! \simeq (\imath_P)^! \circ (\mathbf{k}_P)_! \circ (\widetilde{\imath}_P)_!,$$

and further, using the properness of the maps $\mathbf{k}_P$ and

$$\mathrm{pr}_2 : \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_P$$

as

$$(\mathrm{pr}_2)_! \circ (\mathrm{pr}_1)^! \circ (\widetilde{\imath}_P)_!.$$

Hence, we can rewrite the functor $(\imath_P)^! \circ (\imath_P)_! \circ (\mathsf{q}_P)^*$ as the composition

$$(\mathrm{pr}_2)_! \circ (\mathrm{pr}_1)^! \circ (\widetilde{\imath}_P)_! \circ \mathsf{q}^*,$$

taken along the diagram

$$\begin{array}{ccccc}
\mathrm{Bun}_M & \xleftarrow{\mathsf{q}_P} & \mathrm{Bun}_P & \xrightarrow{\widetilde{\imath}_P} & \widetilde{\mathrm{Bun}}_P \\
 & & & & \uparrow \mathrm{pr}_1 \\
 & & & & \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \\
 & & & & \downarrow \mathrm{pr}_2 \\
 & & & & \mathrm{Bun}_P\,.
\end{array} \tag{10.2}$$

10.1.5. Recall now the map

$$\mathbf{k}_P^{\mathcal{Q}} : \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P \to \mathcal{Q}_{G,P}.$$

Consider now the fiber product

$$\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P),$$

formed using the map

$$\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \to \widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M\,.$$

The key observation is that the projection

$$\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) \to \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_P$$

canonically lifts to a map

$$\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) \to \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \simeq \mathcal{Q}_{P,M}$$

that makes the diagram

$$\begin{array}{ccc}
\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) & \xrightarrow{'\mathbf{k}_P^{\mathcal{Q}}} & \mathcal{Q}_{P,M} \\
\mathrm{id} \times \mathrm{pr}_1 \downarrow & & \downarrow {'\imath_P} \\
\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P & \xrightarrow{\mathbf{k}_P^{\mathcal{Q}}} & \mathcal{Q}_{G,P}
\end{array}$$

Cartesian.

Hence, we rewrite

$$({}'\imath_P)^! \circ ({}'\imath_P)_! \simeq ({}'\mathbf{k}_P^{\mathcal{Q}})_! \circ (\mathrm{id} \times \mathrm{pr}_1)^! \circ (\mathrm{id} \times \widetilde{\imath}_P)_!,$$

where $\mathrm{id} \times \widetilde{\imath}_P$ is the map

$$\mathcal{Q}_{P,M} \simeq \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \to \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P.$$

Hence, we can rewrite $({}'\imath_P)^! \circ ({}'\imath_P)_! \circ ({}'\mathsf{q}_P)_*$ as the composition

$$({}'\mathbf{k}_P^{\mathcal{Q}})_! \circ (\mathrm{id} \times \mathrm{pr}_1)^! \circ (\mathrm{id} \times \widetilde{\imath}_P)_! \circ (\mathrm{id} \times \mathsf{q}_P)^*$$

taken along the diagram

$$\begin{tikzcd}
\mathcal{Q}_{M,M} & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \arrow[l, "\mathrm{id} \times \mathsf{q}_P"'] \arrow[r, "\mathrm{id} \times \widetilde{\imath}_P"] & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P \\
& & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) \arrow[u, "\mathrm{id} \times \mathrm{pr}_1"'] \arrow[d, "{}'\mathbf{k}_P^{\mathcal{Q}}"] \\
& & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P .
\end{tikzcd} \tag{10.3}$$

10.1.6. Comparing the diagrams (10.2) and (10.3), and using the commutative diagram

$$\begin{tikzcd}
\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) \arrow[r, "{}'\mathbf{k}_P^{\mathcal{Q}}"] \arrow[d, "\mathfrak{r}_{M,M} \times \mathrm{id}"'] & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \arrow[d, "\mathfrak{r}_{M,M} \times \mathrm{id}"] \\
\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P \arrow[r, "\mathrm{pr}_2"] & \mathrm{Bun}_P,
\end{tikzcd}$$

we obtain that in ordet to show that (10.1) is an isomorphism, it is enought to show that the natural transfrmation

$$(\mathfrak{r}_{M,M} \times \mathrm{id})_! \circ (\mathrm{id} \times \mathrm{pr}_1)^! \to (\mathrm{pr}_1)^! \circ (\mathfrak{r}_{M,M} \times \mathrm{id})_!, \tag{10.4}$$

coming from the Cartesian diagram

$$\begin{tikzcd}
\mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P \arrow[d, "\mathfrak{r}_{M,M} \times \mathrm{id}"'] & \mathcal{Q}_{M,M} \underset{\mathrm{Bun}_M}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P) \arrow[l, "\mathrm{id} \times \mathrm{pr}_1"'] \arrow[d, "\mathfrak{r}_{M,M} \times \mathrm{id}"] \\
\widetilde{\mathrm{Bun}}_P & \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P, \arrow[l, "\mathrm{pr}_1"']
\end{tikzcd}$$

is an isomorphism, when evaluated on objects of the form

$$(\mathrm{id} \times \widetilde{\imath}_P)_! \circ (\mathrm{id} \times \mathsf{q}_P)^*(\mathcal{F}), \quad \mathcal{F} \in \text{D-mod}(\mathcal{Q}_{M,M}).$$

10.1.7. Note that the ULA property of $\widetilde{\imath}_!(k_{\mathrm{Bun}_P}) \in \text{D-mod}(\widetilde{\mathrm{Bun}}_P)$ with respect to the morphism $\widetilde{\mathsf{q}}$ implies that

$$(\mathrm{id} \times \widetilde{\imath}_P)_! \circ (\mathrm{id} \times \mathsf{q}_P)^*(\mathcal{F}) \simeq (\mathrm{id} \times \mathsf{q}_P)^!(\mathcal{F}) \overset{!}{\otimes} (\mathfrak{r}_{M,M} \times \mathrm{id})^!(\widetilde{\imath}_!(k_{\mathrm{Bun}_P}))[2\dim(\mathrm{Bun}_M)].$$

10.1.8. Consider the following general situation. Let $\mathcal{Y}$ be a smooth algebraic stack, and let

$$f : \mathcal{W}_2 \to \mathcal{W}_1$$

be a map of algebraic stacks over $\mathcal{Y}$. Let $\mathcal{G} \in \text{D-mod}(\mathcal{W}_1)$ have the property that it is ULA over $\mathcal{Y}$, and also $f^!(\mathcal{G}) \in \text{D-mod}(\mathcal{W}_2)$ is ULA over $\mathcal{Y}$.

Let $g : \mathcal{Z}' \to \mathcal{Z}''$ be an arbitrary map of prestacks over $\mathcal{Y}$. Let $\mathcal{F} \in \text{D-mod}(\mathcal{Z}')$ be such that $g_!(\mathcal{F})$ is defined.

Consider the Cartesian diagram

$$\begin{tikzcd}
\mathcal{Z}' \underset{\mathcal{Y}}{\times} \mathcal{W}_1 \arrow[d, "g\times \mathrm{id}"'] & \mathcal{Z}' \underset{\mathcal{Y}}{\times} \mathcal{W}_2 \arrow[l, "\mathrm{id}\times f"'] \arrow[d, "g\times \mathrm{id}"] \\
\mathcal{Z}'' \underset{\mathcal{Y}}{\times} \mathcal{W}_1 & \mathcal{Z}'' \underset{\mathcal{Y}}{\times} \mathcal{W}_2, \arrow[l, "\mathrm{id}\times f"']
\end{tikzcd}$$

and the resulting map

$$(g \times \mathrm{id})_! \circ (\mathrm{id} \times f)^!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}) \to (\mathrm{id} \times f)^! \circ (g \times \mathrm{id})_!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}). \tag{10.5}$$

We have:

Lemma 10.1.9. *Under the above circumstances, the map* (10.5) *is an isomorphism.*

Lemma 10.1.9 implies that (10.4) using Proposition 10.1.2.

Proof of Lemma 10.1.9. We have a commutative diagram

$$\begin{tikzcd}
(g \times \mathrm{id})_! \circ (\mathrm{id} \times f)^!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}) \arrow[r] \arrow[d, "\sim"'] & (\mathrm{id} \times f)^! \circ (g \times \mathrm{id})_!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}) \arrow[d] \\
(g \times \mathrm{id})_!(\mathcal{F} \overset{!}{\otimes} f^!(\mathcal{G})) \arrow[d] & (\mathrm{id} \times f)^!(g_!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}) \arrow[d, "\sim"'] \\
g_!(\mathcal{F}) \overset{!}{\otimes} f^!(\mathcal{G}) \arrow[r, "\mathrm{id}"] & g_!(\mathcal{F}) \overset{!}{\otimes} f^!(\mathcal{G}).
\end{tikzcd}$$

Hence, it is enough to show that $g : \mathcal{Z}' \to \mathcal{Z}'$ and $\mathcal{F} \in \text{D-mod}(\mathcal{Z}')$ as in the lemma, for an algebraic stack $\mathcal{W}$ over $\mathcal{Y}$ and $\mathcal{G} \in \text{D-mod}(\mathcal{W})$, which is ULA over $\mathcal{Y}$, the map

$$(g \times \mathrm{id})_!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}) \to g_!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}$$

is an isomorphism. This in turn follows from the commutative diagram

$$\begin{tikzcd}
(g \times \mathrm{id})_!(\mathcal{F} \overset{!}{\otimes} \mathcal{G}) \arrow[r] \arrow[d, "\sim"'] & g_!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G} \arrow[d, "\sim"'] \\
(g \times \mathrm{id})_!(\mathcal{F} \overset{*}{\otimes} \mathcal{G})[-2\dim(\mathcal{Y})] \arrow[r, "\sim"] & g_!(\mathcal{F}) \overset{*}{\otimes} \mathcal{G}[-2\dim(\mathcal{Y})].
\end{tikzcd}$$

□

10.2. **Proof of Proposition 9.1.2: a reduction step.**

10.2.1. First, as in Sects. 9.2.1 and 9.2.3, one reduces the assertion to the case when $P_2 = G$. Henceforth, we shall denote P_1 simply by P.

Next, using the Hecke action as in Sect. 8.5, we obtain that it is enough to establish both assertions of the proposition when evaluated on a single object of $\mathrm{Whit}(G,G)$, described below.

10.2.2. Denote $\overline{\mathcal{Q}}_{G,G} := \overline{\mathcal{Q}}_{G,G}$. Recall the map

$$\mathbf{m}_G : \overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \mathcal{Q}_G,$$

and let denote by the same symbol $\mathbf{m}_G$ the map

$$\overline{\mathcal{Q}}_{G,G} \to \mathcal{Q}_{G,G}.$$

The map $\mathbf{m}_G$ is schematic and proper. The required object of $\mathcal{Q}_{G,G}$ is

$$(\mathbf{m}_G)_!(\overline{\psi}),$$

where by a slight abuse of notation we denote by $\overline{\psi}$ the object of $\mathrm{D\text{-}mod}(\overline{\mathcal{Q}}_{G,G})$ which pulls back to the same-named object of $\mathrm{D\text{-}mod}(\overline{\mathrm{Bun}}_{N,\rho(\omega_X)})$ under the map

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \overline{\mathcal{Q}}_{G,G}.$$

10.2.3. Consider now the algebraic stack $\overline{\mathcal{Q}}_{G,P} := \overline{\mathrm{Bun}}_{N,\rho(\omega_X)}/Z_M$ and the morphism

$$\mathbf{m}_P : \overline{\mathcal{Q}}_{G,P} \to \mathcal{Q}_{G,P}.$$

The morphism $\mathbf{m}_P$ is also schematic and proper.

Note now that we have a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{Q}}_{G,G} \arrow[r, "\mathbf{m}_G"] \arrow[d, "f_P"'] & \mathcal{Q}_{G,G} \arrow[d, "\mathsf{t}_P \circ \mathbf{i}_G"] \\ \overline{\mathcal{Q}}_{G,P} \arrow[r, "\mathbf{m}_P"] & \mathcal{Q}_{G,P}, \end{tikzcd}$$

where the map f_P is schematic and proper.

This implies the statement of Proposition 9.1.2(a): the object $\overline{\psi} \in \mathrm{D\text{-}mod}(\overline{\mathcal{Q}}_{G,G})$ is holonomic, while

$$(\mathsf{t}_P \circ \mathbf{i}_G)_! \circ (\mathbf{m}_G)_!(\overline{\psi}) \simeq (\mathbf{m}_P)_! \circ (f_P)_!(\overline{\psi}).$$

10.2.4. To prove Proposition 9.1.2(a), we note that the morphism $\mathbf{m}_G$ can be canonically extended to a morphism

$$\mathbf{m}_G^{\mathrm{ext}} : \overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1)^{I_G}/(\mathbb{G}_m)^{I_G} \to \mathcal{Q}_G^{\mathrm{ext}},$$

which is also schematic and proper.

Furthermore, we have a Cartesian diagram

$$\begin{tikzcd}
\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G} \arrow[r, "\mathbf{m}_P"] \arrow[d] & \mathcal{Q}_{G,P} \arrow[d, "\mathbf{i}_P"] \\
\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1)^{I_G - I_M} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G} \arrow[r, "\mathbf{m}_G^{\mathrm{ext}}"] \arrow[d] & \mathcal{Q}_G^{\mathrm{ext},\geq P} \arrow[d] \\
\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1)^{I_G}\right)/(\mathbb{G}_m)^{I_G} \arrow[r, "\mathbf{m}_G^{\mathrm{ext}}"] & \mathcal{Q}_G^{\mathrm{ext}}, \\
\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_G}\right)/(\mathbb{G}_m)^{I_G} \arrow[r, "\mathbf{m}_G"] \arrow[u] & \mathcal{Q}_G \arrow[u, "\mathbf{i}_G"']
\end{tikzcd}$$

where the top left vertical map corresponds to $\{0\} \to (\mathbb{A}^1)^{I_G - I_M}$, and where we identify

$$\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G} \simeq \left(\overline{\mathcal{Q}}_{G,G}\right)/(\mathbb{G}_m)^{I_G - I_M} \simeq \overline{\mathcal{Q}}_{G,P}$$

and

$$\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_G}\right)/(\mathbb{G}_m)^{I_G} \simeq \overline{\mathcal{Q}}_{G,G}.$$

10.2.5. Thus, we have a diagram

$$\begin{tikzcd}
 & \left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G} \arrow[d, "i"] \\
\left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_G}\right)/(\mathbb{G}_m)^{I_G} \arrow[r, "j"] & \left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1)^{I_G - I_M} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G} \arrow[d, "t"] \\
 & \left(\overline{\mathcal{Q}}_{G,G} \times (\mathbb{A}^1 - \{0\})^{I_M}\right)/(\mathbb{G}_m)^{I_G},
\end{tikzcd}$$

which is equipped with a proper map to the diagram

$$\begin{tikzcd}
 & \mathcal{Q}_{G,P} \arrow[d, "\mathbf{i}_P"] \\
\mathcal{Q}_{G,G} \arrow[r, "\mathbf{i}_G"] & \mathcal{Q}_G^{\mathrm{ext},\geq P} \arrow[d, "\mathbf{t}_P"] \\
 & \mathcal{Q}_{G,P}.
\end{tikzcd}$$

As in (9.4), we have a canonically defined natural transformation

$$i^! \circ j_! \to \mathrm{Av}_*^{\mathbf{N},\chi_P} \circ (t \circ j)_!, \quad \text{D-mod}(\overline{\mathcal{Q}}_{G,G})^{\mathbf{N},\chi_G} \to \text{D-mod}(\overline{\mathcal{Q}}_{G,P})^{\mathbf{N},\chi_P}. \tag{10.6}$$

To prove Proposition 9.1.2(a), it suffices to show that the map (10.6) is an isomorphism when evaluated on $\overline{\psi} \in \text{D-mod}(\overline{\mathcal{Q}}_{G,G})^{\mathbf{N},\chi_G}$.

Remark 10.2.6. In fact, it is easy to see that the category $\text{D-mod}(\overline{\mathcal{Q}}_{G,G})^{\mathbf{N},\chi_G}$ is canonically equivalent to Vect with $\overline{\psi}$ being the generator.

10.3. **Proof of Proposition 9.1.2: Kirillov vs. Whittaker.** In this subsection we will prove that the map (10.6) is an isomorphism.

10.3.1. Unwinding the definitions, we arrive to the following set-up. Let $\mathcal{Y}$ be a prestack, equipped with an action of the group

$$(\mathbb{G}_m \ltimes \mathbb{G}_a)^I,$$

where I is some finite set. (In our situation we will take $I = I_G - I_M$).

Consider $(\mathbb{A}^1)^I$ as the variety of group-homomorphisms $(\mathbb{G}_a)^I \to \mathbb{A}_a$; in particular, we have a distinguished point $\chi := \mathbf{1}^I \in (\mathbb{A}^1)^I$.

Consider the diagram

$$\begin{array}{ccccc} \mathcal{Y} \simeq (\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)/(\mathbb{G}_m)^I & \xrightarrow{j} & (\mathcal{Y} \times (\mathbb{A}^1)^I)/(\mathbb{G}_m)^I & \xleftarrow{i} & \mathcal{Y}/(\mathbb{G}_m)^I \\ & & \Big\downarrow t & & \\ & & \mathcal{Y}/(\mathbb{G}_m)^I. & & \end{array}$$

It makes sense to consider the full subcategories

$$\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}} \subset \text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1)^I)$$

and

$$\left(\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \subset \text{D-mod}((\mathcal{Y} \times (\mathbb{A}^1)^I)/(\mathbb{G}_m)^I),$$

and the corresponding categories on $\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I$ and $\mathcal{Y}$.

Note that

$$\left(\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \simeq \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I, \chi}, \tag{10.7}$$

$$\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I, \chi^{\text{ext}}} \simeq \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I},$$

$$\left(\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \simeq \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m \ltimes \mathbb{G}_a)^I}.$$

10.3.2. As in (9.4), we have the partially defined functors and a natural transformation

$$i^! \to \text{Av}_*^{(\mathbb{G}_a)^I} \circ t_!, \quad \text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}} \to \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I}. \tag{10.8}$$

Proposition 10.3.3. *The natural transformation* (10.8) *evaluated to an isomorphism.*

This proposition will imply the isomorphism (10.6).

Remark 10.3.4. Note that the natural transformation (10.8) upgrades to a natural transformation of functors

$$\left(\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \to \left(\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \simeq \\ \simeq \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m \ltimes \mathbb{G}_a)^I}. \tag{10.9}$$

As such it can be interpreted as the composition

$$\left(\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I} \simeq \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I, \chi} \hookrightarrow \text{D-mod}(\mathcal{Y}) \xrightarrow{\text{Av}_!^{(\mathbb{G}_m)^I}} \\ \to \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m)^I} \xrightarrow{\text{Av}_*^{(\mathbb{G}_a)^I}} \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m \ltimes \mathbb{G}_a)^I}.$$

10.3.5. The proof of Proposition 10.3.3 relies on the following observation. Let act denote the action map

$$\mathcal{Y}\times(\mathbb{G}_a)^I\to\mathcal{Y}.$$

Note now that the functor

$$\text{D-mod}(\mathcal{Y})\overset{\text{act}^!}{\longrightarrow}\text{D-mod}(\mathcal{Y}\times(\mathbb{G}_a)^I)\overset{\text{FD}}{\to}\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I) \tag{10.10}$$

defines an equivalence

$$\text{D-mod}(\mathcal{Y})\to\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I,\chi^{\text{ext}}},$$

where FD is the functor of Fourier-Deligne transform along $(\mathbb{G}_a)^I$.

Proof of Proposition 10.3.3. We precompose both functors with the equialence (10.10). The composition

$$\text{D-mod}(\mathcal{Y})\overset{\text{act}^!}{\longrightarrow}\text{D-mod}(\mathcal{Y}\times\mathbb{G}_a)^I)\overset{\text{FD}}{\to}\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)\overset{t_!}{\longrightarrow}\text{D-mod}(\mathcal{Y})$$

is the identity functor. Hence, its further composition with $\text{Av}_*^{(\mathbb{G}_a)^I}$ is the functor $\text{Av}_*^{(\mathbb{G}_a)^I}$.

Now, the composition

$$\text{D-mod}(\mathcal{Y})\overset{\text{act}^!}{\longrightarrow}\text{D-mod}(\mathcal{Y}\times(\mathbb{G}_a)^I)\overset{\text{FD}}{\to}\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)\overset{i^!}{\longrightarrow}\text{D-mod}(\mathcal{Y})$$

is manifestly the functor $\text{Av}_*^{(\mathbb{G}_a)^I}$.

Thus, the natural transformation (10.8) is an endomorphism of the functor $\text{Av}_*^{(\mathbb{G}_a)^I}$. Furthermore, this endomorphism acts as the identity on $\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I}\subset\text{D-mod}(\mathcal{Y})$. Hence, it is canonically isomorphic to the identity map.

□

Remark 10.3.6. Denote $\text{Whit}(\mathcal{Y}):=\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_a)^I,\chi}$. Let us denote by $\text{Kir}_*(\mathcal{Y})$ (resp., $\text{Kir}_!(\mathcal{Y})$) the full subcategory of $\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m)^I}$ consisting of objects $\mathcal{F}$ such that $\text{Av}_*^{(\mathbb{G}_a)^I}(\mathcal{F})=0$ (resp., $\text{Av}_!^{(\mathbb{G}_a)^I}(\mathcal{F})=0$).

Note that the equivalence (10.10) gives rise to an equivalence

$$\text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m)^I}\simeq\left(\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I,\chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I}.$$

The above equivalence gives rise to an identification of $\text{Kir}_*(\mathcal{Y})$ with the full subcategory of $\left(\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I,\chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I}$ consisting of objects $\mathcal{F}'$ such that $i^!(\mathcal{F}')=0$. The functor j_* identifies the above category with $\left(\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1-\{0\})^I)^{(\mathbb{G}_a)^I,\chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I}$.

Thus, we obtain that (10.7) defines an equivalence

$$\text{Whit}(\mathcal{Y})\simeq\text{Kir}_*(\mathcal{Y}).$$

Explicitly, this equivalence is given by

$$\mathcal{F}\in\text{Whit}(\mathcal{Y})\mapsto\text{Av}_*^{(\mathbb{G}_m)^I}(\mathcal{F})\in\text{Kir}_*(\mathcal{Y}),\quad\widehat{\mathcal{F}}\in\text{Kir}_*(\mathcal{Y})\mapsto\text{Av}_*^{(\mathbb{G}_a)^I,\chi}(\widehat{\mathcal{F}})[2|I|]\simeq\text{Av}_!^{(\mathbb{G}_a)^I,\chi}(\widehat{\mathcal{F}}).$$

Simillary, $\text{Kir}_!(\mathcal{Y})$ identifies with the full subcategory of $\left(\text{D-mod}(\mathcal{Y}\times(\mathbb{A}^1)^I)^{(\mathbb{G}_a)^I,\chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I}$ consisting of objects $\mathcal{F}'$ such that $i^*(\mathcal{F}')=0$. The functor $j_!$ identifies the above category with the full subcategory of $\left(\text{D-mod}(\mathcal{Y} \times (\mathbb{A}^1 - \{0\})^I)^{(\mathbb{G}_a)^I, \chi^{\text{ext}}}\right)^{(\mathbb{G}_m)^I}$, consisting of objects $\mathcal{F}''$, for which $j_!(\mathcal{F}'')$ is defined.

Thus, we have a fully faithful embedding $\text{Kir}_!(\mathcal{Y}) \hookrightarrow \text{Whit}(\mathcal{Y})$, given by

$$\widehat{\mathcal{F}} \in \text{Kir}_!(\mathcal{Y}) \mapsto \text{Av}_*^{(\mathbb{G}_a)^I, \chi}(\widehat{\mathcal{F}}) \simeq \text{Av}_!^{(\mathbb{G}_a)^I, \chi}(\widehat{\mathcal{F}})[-2|I|].$$

Its essential image consists of those objects $\mathcal{F} \in \text{Whit}(\mathcal{Y})$, for which $\text{Av}_!^{(\mathbb{G}_m)^I}$ is defined, and

$$\mathcal{F} \mapsto \text{Av}_!^{(\mathbb{G}_m)^I}(\mathcal{F})$$

provides the inverse functor.

In terms of these identifications, the functor (10.9) can be interpreted as the composition

$$\text{Kir}_!(\mathcal{Y}) \hookrightarrow \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m)^I} \overset{\text{Av}_*^{(\mathbb{G}_m)^I}}{\longrightarrow} \text{D-mod}(\mathcal{Y})^{(\mathbb{G}_m \ltimes \mathbb{G}_a)^I}.$$

I.e., we are applying to an object of $\text{Kir}_!(\mathcal{Y})$ "the other" averaging functor with respect to $(\mathbb{G}_a)^I$, i.e., one does does not vanish.

10.4. **Proof of Proposition 9.2.8.** The idea of the proof is to replace the prestacks appearing in the diagram (9.18) by algebraic stacks, and then apply a version of Braden's theorem à la Theorem 2.2.3.

10.4.1. Let Div^+ be the scheme of effective divisors on X (i.e., the disjoint union of symmetric powers of X). Let $\text{Div}^+_{\omega_X}$ be the following twisted version of Div^+: it is the scheme that classifies pairs $(\mathcal{L}, \mathcal{L} \to \omega_X)$, where $\mathcal{L}$ is a line bundle on X and $\mathcal{L} \to \omega_X$ is a non-zero map.

Consider the forgetful map

$$\text{Div}^+ \times \text{Div}^+_{\omega_X} \to \text{Bun}_{\mathbb{G}_m}, \quad (D, \mathcal{L} \to \omega_X) \mapsto \mathcal{L}(-D) \to \omega_X.$$

Consider the following algebraic stacks:

$$\mathcal{Q}'_{G,P} := \overline{\text{Bun}}_B \underset{\text{Bun}_{T/Z_M}}{\times} \left(\text{Div}^+ \times \text{Div}^+_{\omega_X}\right)^{I_M},$$

where we identify $T/Z_M \simeq (\mathbb{G}_m)^{I_M}$ using the simple roots of M.

$$\mathcal{Q}'_{P,M} := \left(\text{Bun}_P \underset{\text{Bun}_M}{\times} \overline{\text{Bun}}_{B(M)}\right) \underset{\text{Bun}_{T/Z_M}}{\times} \left(\text{Div}^+ \times \text{Div}^+_{\omega_X}\right)^{I_M}.$$

$$\mathcal{Q}'_{M,M} := \overline{\text{Bun}}_{B(M)} \underset{\text{Bun}_{T/Z_M}}{\times} \left(\text{Div}^+ \times \text{Div}^+_{\omega_X}\right)^{I_M}.$$

We let Zast'_P be the corresponding open substack of $\mathcal{Q}'_{G,P} \underset{\text{Bun}_G}{\times} \text{Bun}_{P^-}$.

10.4.2. We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Zast}'_P \underset{\mathcal{Q}'_{G,P}}{\times} \mathcal{Q}'_{P,M} \arrow[r] \arrow[d] & \mathcal{Q}'_{P,M} \arrow[r, "{'\mathsf{q}'_P}"] \arrow[d, "{'\iota'_P}"] & \mathcal{Q}'_{M,M} \\
\mathrm{Zast}'_P \arrow[r, "\mathsf{u}'_P"] \arrow[d, "\mathsf{v}'_P"'] & \mathcal{Q}'_{G,P} & \\
\mathcal{Q}'_{M,M}. & &
\end{tikzcd}
\tag{10.11}
$$

The diagram (10.11) maps to the diagram

$$
\begin{tikzcd}
\mathrm{Zast}^{\mathrm{gen}}_P \underset{\mathcal{Q}_{G,P}}{\times} \mathcal{Q}_{P,M} \arrow[r] \arrow[d] & \mathcal{Q}_{P,M} \arrow[r, "{'\mathsf{q}_P}"] \arrow[d, "{'\iota_P}"] & \mathcal{Q}_{M,M} \\
\mathrm{Zast}^{\mathrm{gen}}_P \arrow[r, "\mathsf{u}_P"] \arrow[d, "\mathsf{v}_P"'] & \mathcal{Q}_{G,P} & \\
\mathcal{Q}_{M,M}. & &
\end{tikzcd}
$$

of (9.18) by means of schematic, proper and surjective maps.

Furthermore, the squares

$$
\begin{tikzcd}
\mathcal{Q}'_{P,M} \arrow[r, "{'\iota'_P}"] \arrow[d] & \mathcal{Q}'_{P,G} \arrow[d] \\
\mathcal{Q}_{P,M} \arrow[r, "{'\iota_P}"] & \mathcal{Q}_{P,G}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathrm{Zast}'_P \arrow[r, "\mathsf{u}'_P"] \arrow[d] & \mathcal{Q}'_{G,P} \arrow[d] \\
\mathrm{Zast}^{\mathrm{gen}}_P \arrow[r, "\mathsf{u}_P"] & \mathcal{Q}_{G,P}
\end{tikzcd}
$$

are Cartesian.

Hence, in order to prove Proposition 9.2.8, it suffices to prove the following:

Proposition 10.4.3. *The naturalt ransformation*

$$
({'\mathsf{q}'_P})_* \circ ({'\iota'_P})^! \to (\mathsf{v}'_P)_! \circ (\mathsf{u}'_P)^*, \quad \text{D-mod}(\mathcal{Q}'_{G,P}) \to \text{D-mod}(\mathcal{Q}'_{M,M})
\tag{10.12}
$$

that arises by adjunction from the natural transformation

$$
({'\mathsf{q}'_P})_* \circ ({'\iota'_P})^! \circ (\mathsf{u}'_P)_* \circ (\mathsf{v}'_P)^! \to \mathrm{Id}
$$

is an isomorphism.

10.4.4. The proof of Proposition 10.4.3 is completely parallel to the proof of Theorem 2.2.3. Namely, we choose a dominant regular co-character $\gamma : \mathbb{G}_m \to Z_M$, which defines an action of $\mathbb{G}_m$ on the stack $\overline{\mathrm{Bun}}_B$ *over* Bun_T, an in particular over $\mathcal{Q}'_{G,P}$.

Now, as in Sect. 5 one shows that the diagram (10.11) identifies with

$$\begin{CD}
\mathrm{Repel}(\mathcal{Q}'_{G,P}) \underset{\mathcal{Q}'_{G,P}}{\times} \mathrm{Attr}(\mathcal{Q}'_{G,P}) @>>> \mathrm{Attr}(\mathcal{Q}'_{G,P}) @>>> \mathrm{Fixed}(\mathcal{Q}'_{G,P}) \\
@VVV @VVV \\
\mathrm{Repel}(\mathcal{Q}'_{G,P}) @>>> \mathcal{Q}'_{G,P} \\
@VVV \\
\mathrm{Fixed}(\mathcal{Q}'_{G,P}).
\end{CD}$$

This allows to reduce the assertion of Proposition 10.4.3 to the situation of the usual Braden theorem.

References

[Bar]

[BD] A. Beilinson and V. Drinfeld, *Hitchin*.

[BG1] A. Braverman, D Gaitsgory, *Geometric Eisenstein Series*.

[BG2] A. Braverman, D Gaitsgory, *Deformations of Eisenstein Series*.

[BFGM] *IC of Drinfeld*.

[Dr] *On* $\mathbb{G}_m$*-actions*.

[DrGa1] *On some finiteness questions*.

[DrGa2] *Compact generation*.

[DrGa3] *Braden*.

[DrGa4] *Constant term functor(s)*.

[DS]

[FGV] *Whittaker patterns*.

[Ga] *Contractibility*

[Ras] *PhD Thesis*

[Sch] S. Schieder.