WHAT ACTS ON GEOMETRIC EISENSTEIN SERIES

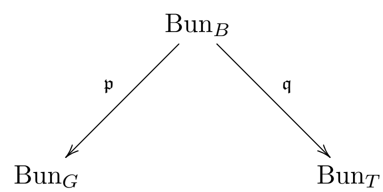
D. GAITSGORY

1. Introduction

1.1. **The Functor.** Geometric Eisenstein series is the functor $\mathrm{Eis}_! : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$ defined by

$$\mathcal{F} \mapsto \mathfrak{p}_! \circ \mathfrak{q}^*(\mathcal{F}),$$

(up to a cohomological shift), where



The question that we are concerned with in this paper is the description of $\mathrm{Hom}_{D(\mathrm{Bun}_G)}(\mathrm{Eis}_!(\mathcal{F}_1), \mathrm{Eis}_!(\mathcal{F}_2))$ in terms of $\mathcal{F}_1, \mathcal{F}_2$.

In some sense, the answer is tautological. Let CT_* denote the right adjoint to $\mathrm{Eis}_!$. The composition $\Phi := \mathrm{CT}_* \circ \mathrm{Eis}_!$ is a monad acting on $D(\mathrm{Bun}_T)$, and we have:

$$\mathrm{Hom}_{D(\mathrm{Bun}_G)}(\mathrm{Eis}_!(\mathcal{F}_1), \mathrm{Eis}_!(\mathcal{F}_2)) \simeq \mathrm{Hom}_{D(\mathrm{Bun}_T)}(\mathcal{F}_1, \Phi(\mathcal{F}_2)).$$

So, what we are after is to have a more detailed understanding of the monad Φ.

1.2. **The Monad.**

1.2.1. Parallel to what happens in the classical theory of automorphic functions, the functor Φ admits a canonical filtration by functors numbered by the Weyl group W (viewed as a poset with respect to the Bruhat order); we denote the subquotient functor corresponding to an element $w \in W$ by Φ_w. The term $\Phi_{main} := \Phi_1$ happens to be the most interesting; in fact Φ_{main} is itself a monad, and the canonical map $\Phi_{main} \to \Phi$ is a homomorphism. (By contrast, the term Φ_{w_0} is the simplest: it's given by the action of w_0 on Bun_T.)

In the classical theory, the analogue of the term Φ_{main} is a certain intertwining operator (acting on automorphic functions on the abelian group T), and it decomposes as a product of local intertwining operators.

We can now define our goal more precisely as follows: we'd like to describe

$$\Phi_{main} : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T)$$

in terms that are local with respect to the curve X. The latter phrase can be given a precise meaning as follows:

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Recall that for a reductive group M and $x \in X$ one can attach the Satake category at x, denoted $\mathrm{Sat}_{M,x}$ that acts on $D(\mathrm{Bun}_M)$ by Hecke functors. The assignment

$$x \mapsto \mathrm{Sat}_{M,x}$$

forms what is called a factorization (a.k.a. chiral) category over X, equipped with a compatible monoidal structure. We denote this category simply by Sat_M. Now, if $\mathcal{E} \in \mathrm{Sat}_M$ is a factorization (a.k.a. chiral) algebra equipped with a compatible associative algebra structure in this category, the chiral homology $H_{ch}(X, \mathcal{E})$ acts as a monad on $D(\mathrm{Bun}_M)$.

So, our first goal can be stated as follows:

Goal 1a: Describe explicitly the factorization algebra $\mathcal{E} \in \mathrm{Sat}_T$, such that $\Phi_{main} \simeq H_{ch}(X, \mathcal{E})$.

1.2.2. By the definition of the functor $\mathrm{Eis}_!$, its construction uses the stack Bun_B. However, Bun_B is a little "stupid" as a stack: it splits into connected components numbered by elements of Λ–the coroot lattice of T.

However, Bun_B admits a relative compactification

$$\begin{array}{ccccc} \mathrm{Bun}_B & \xrightarrow{\jmath} & \overline{\mathrm{Bun}}_B & & \\ & \overline{\mathfrak{p}} \swarrow & & \searrow \overline{\mathfrak{q}} & \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_T \end{array}$$

with $\mathfrak{p} = \overline{\mathfrak{p}} \circ \jmath$ and $\mathfrak{q} = \overline{\mathfrak{q}} \circ \jmath$.

The stack $\overline{\mathrm{Bun}}_B$ is stratified by locally closed substacks of the form

$$\iota^\lambda : X^\lambda \times \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B,$$

where X^λ is the space of configuration on Λ^{pos}-colored divisors of total degree λ. (Here $\Lambda^{pos} \subset \Lambda$ is the semi-group spanned by positive simple roots.)

Our second goal can be stated as follows:

Goal 1b: Describe the factorization algebra $\mathcal{E}$ in terms adjunction of the strata in $\overline{\mathrm{Bun}}_B$.

1.3. The "space" of rational reductions to B.

1.3.1. The situation with $\overline{\mathrm{Bun}}_B$ can be pushed even further. In 2004 Drinfeld proposed that there should exist a stack Bun_B^{rat} "of G-bundles equipped with a rational reduction to B". This stack is supposed to glue together the connected components Bun_B^μ of Bun_B for μ projecting to the same element of $\pi_1(G)$. In other words, the should exist a projection

$$\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_B^{rat}$$

that collapses each stratum $X^\lambda \times \mathrm{Bun}_B^\mu$ to just Bun_B^μ.

However, $\overline{\mathrm{Bun}}_B$ does not exist as an algebraic stack (the strata that need to collapse have wrong self-intersections). Nonetheless we will achieve:

Goal 2a: Construct a category $D(\mathrm{Bun}_B^{rat})$, equipped with a "direct image" functor $\mathrm{Av}: D(\overline{\mathrm{Bun}}_B) \to D(\mathrm{Bun}_B^{rat})$ and a functor $\mathfrak{p}_!^{rat} : D(\mathrm{Bun}_B^{rat}) \to D(\mathrm{Bun}_G)$ so that $\overline{\mathfrak{p}}_! \simeq \mathfrak{p}_!^{rat} \circ \mathrm{Av}$.

1.3.2. As we shall see, the monad Φ_{main} will have a natural interpretation in terms of the category $D(\mathrm{Bun}_B^{rat})$. Namely, we'll achieve

Goal 2b: Interpret the monad Φ_{main} in terms of Homs in the category $D(\mathrm{Bun}_B^{rat})$.

1.4. **Compactified Eisenstein series.** The existence of the compactification $\overline{\mathrm{Bun}}_B$ has lead the authors of [BG1] and [FFKM] to consider another functor $D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$, namely that of compactified Eisenstein series:

$$\mathrm{Eis}_{!*}(\mathcal{F}) := \bar{\mathfrak{p}}_!(\bar{\mathfrak{q}}^*(\mathcal{F}) \otimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B}),$$

where $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ is the intersection cohomology sheaf on $\overline{\mathrm{Bun}}_B$.

Our next goal can be stated as follows:

Goal 3: From the action of the monad Φ_{main} on $\mathrm{Eis}_!$, find what monad acts on the functor $\mathrm{Eis}_{!*}$.

1.5. **The Langlands dual picture.**

1.5.1. Let us recall now that the functor $\mathrm{Eis}_!$ is supposed to fit into the Geometric Langlands picture

$$\begin{array}{ccc} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathrm{L}_G} & D(\mathrm{Bun}_G) \\ {\scriptstyle \mathrm{Eis}_{spec}}\uparrow & & \uparrow{\scriptstyle \mathrm{Eis}_! \circ -\rho(\omega_X)} \\ \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) & \xrightarrow{\mathrm{L}_T} & D(\mathrm{Bun}_T). \end{array} \tag{1.1}$$

Here $\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})$ is a certain modification of the category $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, described in [Sum].

The horizontal arrow L_G is the conjectural Langlands transform. The horizontal arrow L_T is the Langlands transform for T, which since T is abelian, is the Fourier-Mukai transform. The functor Eis_{spec} is the spectral Eisenstein series functor defined using the diagram

$$\begin{array}{ccccc} & & \mathrm{LocSys}_{\check{B}} & & \\ & {\scriptstyle \mathfrak{p}_{spec}}\swarrow & & \searrow{\scriptstyle \mathfrak{q}_{spec}} & \\ \mathrm{LocSys}_{\check{G}} & & & & \mathrm{LocSys}_{\check{T}} \end{array}$$

by

$$\mathrm{Eis}_{spec} := \mathfrak{p}_{spec*} \circ \mathfrak{q}^*_{spec}.$$

Finally, $-\rho(\omega_X)$ is the functor of shift by $-\rho(\omega_X) \in \mathrm{Bun}_T$ acting on $D(\mathrm{Bun}_T)$.

1.5.2. Our main goal can be stated as follows:

Goal 4a: Give an interpretation of the monad Φ_{main} and of the chiral algebra $\mathcal{E}$ on the Langlands dual side as a monad acting on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$.

1.5.3. Note that the map of stacks $\mathrm{pt}/\check{B} \to \mathrm{pt}/\check{T}$ admits a canonical section, and hence so does the map of stacks

$$\mathbf{e} : \mathrm{LocSys}_{\check{T}} \to \mathrm{LocSys}_{\check{B}}\,.$$

We'll show that if the diagram (1.2) takes place, then so does the diagram

$$\begin{array}{ccc} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathrm{L}_G} & D(\mathrm{Bun}_G) \\ {\scriptstyle \mathfrak{p}_{spec*} \circ \mathbf{e}_*}\uparrow & & \uparrow{\scriptstyle \mathrm{Eis}_{!*} \circ -\rho(\omega_X)} \\ \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) & \xrightarrow{\mathrm{L}_T} & D(\mathrm{Bun}_T). \end{array} \tag{1.2}$$

Our final goal is:

Goal 4b: Give an interpretation of the monad from *Goal 3* on the Langlands dual side as a monad acting on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$.

We observe that our *Goal 4b* is a categorical upgrade of the following remarkable fact discovered in [FFKM]: namely, that the object $\mathrm{Eis}_{!*}(\mathbb{C}_{\mathrm{Bun}_T})$ carries an action of the Langlands dual Lie algebra $\check{\mathfrak{g}}$.

1.6. **Structure of the paper.** This paper is divided into three parts, according to the level of sophistication at which we employ the machinery of factorization algebras and categories.

1.6.1. Part I is largely preparatory and is meant to set the scene for the more involved but similar in spirit manipulation in the later sections.

In particular, instead of the factorization category Sat_T, which is needed for the description of the monad Φ_{main}, we use its more simply-minded version that deals with factorization algebras of Λ^{pos}-vector space.

Although this restricted framework will not allow us to get to Φ_{main} or our goals that have to do with the Langlands dual picture, we will be able to observe some interesting phenomena, e.g. why the functor $\mathrm{Eis}_!$ factors on the Langlands dual side through a functor $\mathfrak{q}^*_{spec}$.

We should also remark that Part I is to a large extent a restatement of the results from [BG2] in the language of factorization algebras.

1.6.2. In Part II we'll achieve some of the goals stated above.

First, we'll show what the monad Φ_{main} has to do with the stacks $\overline{\mathrm{Bun}}_B$. Secondly, we'll define the category $D(\mathrm{Bun}_B^{rat})$ with the expected properties.

1.6.3. Finally, in Part III we'll state a certain local conjecture that will relate the monad Φ_{main} to the factorization category Sat_T, and assuming this conjecture, we'll be able to make a contact with the Langlands dual picture.

We note that the results "proved" in Part III heavily result on the yet unpublished theory of factorization categories.

Part I

2. Factorization algebras in a simplified context

This section introduces a language of factorization algebras graded by a semi-lattice Λ^{pos}. [1] This is nothing but a particular case of the set-up of factorization algebras from [CHA], with the difference that the presence of Λ allows to replace the Ran space by genuine schemes.

2.1. **The graded Ran space.**

2.1.1. For $\lambda \in \Lambda^{pos}$, let X^λ denote the corresponding partially symmetrized power of X. For $\lambda = \lambda_1 + \lambda_2$ let

$$\mathrm{add}_{\lambda_1,\lambda_2} : X^{\lambda_1} \times X^{\lambda_2} \to X^\lambda$$

denote the canonical map.

For $\mathcal{F}_i \in D(X^{\lambda_i})$, with $\lambda = \lambda_1 + \lambda_2$ we'll denote by $\mathcal{F}_1 \star \mathcal{F}_2$ the object of $D(X^\lambda)$ equal to

$$(\mathrm{add}_{\lambda_1,\lambda_2})_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

2.1.2. We regard the disjoint union $\underset{\lambda}{\cup} X^\lambda$ is a Λ^{pos}-graded version of the Ran space $\mathrm{Ran}(X)$ and denote it by $\mathrm{Ran}(X, \Lambda^{pos})$.

The category

$$D(\mathrm{Ran}(X, \Lambda^{pos})) := \{\lambda \mapsto \mathcal{F}^\lambda \in D(X^\lambda)\}$$

has a natural monoidal structure with respect to $\star$: for two families $\{\mathcal{F}_1^\lambda\}$ and $\{\mathcal{F}_2^\lambda\}$ the value of their tensor product on X^λ is

$$\underset{\lambda=\lambda_1+\lambda_2}{\oplus} \mathcal{F}_1^{\lambda_1} \star \mathcal{F}_2^{\lambda_2}.$$

This monoidal structure is naturally symmetric.

2.1.3. For $\lambda \in \Lambda^{pos}$ consider the natural closed embedding $\Delta^\lambda : X \to X^\lambda$.

The functor $\Delta_* := \{\Delta_*^\lambda\}$ makes the category $D(X)^{\Lambda^{pos}}$ of Λ^{pos}-graded objects of $D(X)$ a full subcategory of $D(\mathrm{Ran}(X, \Lambda^{pos}))$. This embedding has a right adjoint given by $\Delta^! := \{\Delta^{\lambda !}\}$.

The category $D(X)^{\Lambda^{pos}}$ has a natural symmetric monoidal structure given by $\overset{!}{\otimes}$. The functor $\Delta^!$ is strictly monoidal.

By adjunction, for $\mathcal{F}_1, ..., \mathcal{F}_n, \tilde{\mathcal{F}} \in D(X)^{\Lambda^{pos}}$ we have a natural map

$$\mathrm{Hom}_{D(X)^{\Lambda^{pos}}}(\tilde{\mathcal{F}}, \mathcal{F}_1 \overset{!}{\otimes} ... \overset{!}{\otimes} \mathcal{F}_n) \to \mathrm{Hom}_{D(\mathrm{Ran}(X,\Lambda^{pos}))}(\Delta_*(\tilde{\mathcal{F}}), \Delta_*(\mathcal{F}_1) \star ... \star \Delta_*(\mathcal{F}_n)), \tag{2.1}$$

and these maps are compatible with iterated tensor products. However, we have the following straightforward assertion:

Lemma 2.1.4. *The maps* (2.1) *are isomorphisms.*

2.1.5. One can speak about associative, commutative and Lie algebras and co-algebras in either $D(X)^{\Lambda^{pos}}$ or $D(\mathrm{Ran}(X, \Lambda^{pos}))$. The functor $\Delta^!$ maps such objects in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ to objects of a similar nature in $D(X)^{\Lambda^{pos}}$.

In addition, Lemma 2.1.4 implies that for an object $M \in D(X)^{\Lambda^{pos}}$, a structure of *co-algebra* of any kind on it is equivalent to one on $\Delta_*(M)$.

[1] Unless specified otherwise, we consider Λ^{pos} without the 0 element.

2.1.6. The usual Koszul duality defines equivalences

$$\mathrm{KD}_{\mathrm{A}\to\mathrm{cA}} : \mathrm{Assoc.\,Alg} \to \mathrm{co\text{-}Assoc.\,co\text{-}Alg},$$
$$\mathrm{KD}_{\mathrm{cC}\to\mathrm{L}} : \mathrm{co\text{-}Com.\,co\text{-}Alg} \to \mathrm{Lie\ alg},$$

and

$$\mathrm{KD}_{\mathrm{C}\to\mathrm{cL}} : \mathrm{Com.\,Alg} \to \mathrm{Lie\ co\text{-}alg}$$

in either context.

NB: We'll use the subscripts $\star$ or $\overset{!}{\otimes}$ to indicate which category we're in when an ambiguity is likely to occur.

Explicitly, $\mathrm{KD}_{\mathrm{A}\to\mathrm{cA}}$ attaches to an associative augmented algebra A the co-associative co-algebra $\mathrm{Bar}(A)$ that computes Tor's of the augmentation module with itself over A. The inverse functor sends a co-associative co-algebra $A^\vee$ to $\mathrm{coBar}(A^\vee)$, where the latter computes Exts's of the augmentation module with itself over $A^\vee$.

The inverse to $\mathrm{KD}_{\mathrm{cC}\to\mathrm{L}}$ sends a Lie algebra L to its homological Chevalley complex $\mathrm{C}_\cdot(L)$. The inverse to $\mathrm{KD}_{\mathrm{C}\to\mathrm{cL}}$ sends a Lie co-algebra $L^\vee$ to its homological Chevalley complex $\mathrm{C}^\cdot(L)$.

2.2. **Factorization algebras.**

2.2.1. By definition, a Λ^{pos}-factorization algebra is the same as a chiral algebra on X, graded by Λ^{pos}. The corresponding functor is given by

$$A \mapsto \mathfrak{C}(A),$$

where $\mathfrak{C}$ stands for the chiral Chevalley-Cousin complex, following the conventions from [CHA].

2.2.2. Explicitly, we can think of a Λ^{pos}-graded factorization algebra $\mathcal{A}$ as follows. To each $\lambda \in \Lambda^{pos}$ we assign $\mathcal{A}^\lambda \in D(X^\lambda)$ and whenever $\lambda = \lambda_1 + \lambda_2$, we have an isomorphism

$$\mathcal{A}^\lambda|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}} \simeq \mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2}|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}}, \tag{2.2}$$

satisfying the natural compatibilities.

NB: The restriction of the map $\mathrm{add}_{\lambda_1,\lambda_2}$ to the open subscheme $(X^{\lambda_1}\times X^{\lambda_2})_{disj} \subset X^{\lambda_1}\times X^{\lambda_2}$ is étale, so the notation $\mathcal{F} \mapsto \mathcal{F}|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}}$ is unambiguous.

2.2.3. Whenever we have two factorization algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ their $\star$-product $\mathcal{A}_1 \star \mathcal{A}_2$ has a natural factorization algebra structure.

2.3. **Commutative factorization algebras.**

2.3.1. We shall say that $\mathcal{A}$ is *commutative* if the above isomorphisms have been extended to maps

$$\mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2} \to \mathrm{add}^!_{\lambda_1,\lambda_2}(\mathcal{A}^\lambda),$$

which also satisfy the natural compatibilities.

By adjunction, for a commutative $\mathcal{A}$, we obtain the maps

$$\mathcal{A}^{\lambda_1} \star \mathcal{A}^{\lambda_2} \to \mathcal{A}^{\lambda_1+\lambda_2},$$

which make $\mathcal{A}$ into a commutative algebra in $D(\mathrm{Ran}(X,\Lambda^{pos}))$ with respect to $\star$.

Note that commutative factorization algebras form a full subcategory among commutative algebras in $D(\mathrm{Ran}(X,\Lambda^{pos}))$. Indeed, for a commutative algebra $\mathcal{A} := \{\mathcal{A}^\lambda\} \in D(\mathrm{Ran}(X,\Lambda^{pos}))$ we have the maps

$$\mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2}|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}} \to \mathcal{A}^\lambda|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}}$$

and $\mathcal{A}$ is a commutative factorization algebra if and only if these maps are isomorphisms.

2.3.2. As in [CHA], one can prove the following:

Proposition 2.3.3. *The assignment*

$$\mathcal{A} \mapsto \Delta^!(\mathcal{A})$$

establishes an equivalence between the category of commutative factorization algebras and that of commutative algebras in $D(X)^{\Lambda^{pos}}$.

NB: Note that the chiral algebra corresponding to such $\mathcal{A}$ is $\Delta^!(\mathcal{A})[-1]$.

For a commutative algebra A in $D(X)^{\Lambda^{pos}}$, we'll denote by A_{Ran} the corresponding commutative factorization algebra.

2.3.4. The following proposition is due to [JNKF].

Proposition 2.3.5. *The following diagram of functors is commutative:*

$$\begin{array}{ccc}
\text{Lie co-alg}(D(X)^{\Lambda^{pos}}, \overset{!}{\otimes}) & \xrightarrow{\mathrm{KD}_{\mathrm{cL}\to\mathrm{C}}(D(X)^{\Lambda^{pos}}, \overset{!}{\otimes})} & \mathrm{Com.\,Alg}(D(X)^{\Lambda^{pos}}, \overset{!}{\otimes}) \\
\Delta_* \downarrow & & \downarrow A\mapsto A_{\mathrm{Ran}} \\
\text{Lie co-alg}(D(\mathrm{Ran}(X, \Lambda^{pos}), \star) & \xrightarrow{\mathrm{KD}_{\mathrm{cL}\to\mathrm{C}}(D(\mathrm{Ran}(X,\Lambda^{pos}), \star)} & \mathrm{Com.\,Alg}(D(\mathrm{Ran}(X, \Lambda^{pos}), \star).
\end{array}$$

The above proposition reads as follows: for a Lie co-algebra $L^\vee \in D(X)^{\Lambda^{pos}}$ with respect to the $\overset{!}{\otimes}$ tensor structure, consider the following two procedures:

- Consider the commutative algebra $\mathrm{C}^\cdot(L^\vee) \in D(X)^{\Lambda^{pos}}$ with respect to the $\overset{!}{\otimes}$ tensor structure, and apply to it the equivalence of Proposition 2.3.3, i.e., $\mathrm{C}^\cdot(L^\vee)_{\mathrm{Ran}}$.
- Consider the Lie co-algebra $\Delta_*(L^\vee) \in D(\mathrm{Ran}(X, \Lambda^{pos}))$, and consider its cohomological Chevalley complex $\mathrm{C}^\cdot(\Delta_*(L^\vee))$ with respect to the $\star$ tensor structure.

We'll denote the resulting commutative factorization algebra by $\Omega(L)$, i.e.,

$$\mathrm{C}^\cdot(L^\vee)_{\mathrm{Ran}} \simeq \Omega(L) \simeq \mathrm{C}^\cdot(\Delta_*(L^\vee)).$$

It follows from the construction that if $L^\vee$ is such that $L^\vee[-1]$ is a D-module (belongs to the heart of the t-structure), then so does $\Omega(L)$.

2.3.6. The following construction will be used in the sequel. Let $L^\vee$ be as above. Consider $\Delta_*(L^\vee)$ as a Lie co-algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ with respect to $\star$, and consider its universal co-enveloping co-algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}))$; denote the resulting object of $D(\mathrm{Ran}(X, \Lambda^{pos}))$ by $U^\vee(L)_{\mathrm{Ran}}$. [2]

By construction, $U^\vee(L)_{\mathrm{Ran}}$ is a commutative algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ with a compatible co-associative co-algebra structure, i.e., it's a commutative Hopf algebra.

In addition, $U^\vee(L)$ has the following properties:

Lemma 2.3.7.
(1) *As a commutative algebra in* $D(\mathrm{Ran}(X, \Lambda^{pos}))$, $U^\vee(L)_{\mathrm{Ran}}$ *is factorizable.*
(2) *The Hopf algebra* $\Delta^!(U^\vee(L)_{\mathrm{Ran}}) \in D(X)^{\Lambda^{pos}}$ *identifies with the universal co-enveloping co-algebra* $U^\vee(L)$ *of* $L^\vee$*, considered as a Lie co-algebra with respect to the* $\overset{!}{\otimes}$ *tensor structure.*

[2] We emphasize that $U^\vee(L)$ is not the Verdier dual of the chiral universal envelope of the Verdier dual L of $L^\vee$, assuming it was dualizable. As we shall see, $U^\vee(L)$ is that for the loop object of $L[-1]$.

Note also that the universal co-enveloping co-algebra $U^\vee(L)_{\mathrm{Ran}}$ of a Lie co-algebra $L^\vee$ (in any tensor category) has the following additional interpretations:

2.3.8. For a Lie co-algebra $L^\vee \in D(X)^{\Lambda^{pos}}$ consider its suspension $L^\vee[1]$. This is a co-algebra object in the category of Lie co-algebras in $D(X)^{\Lambda^{pos}}$. Then we have:

$$U^\vee(L)_{\mathrm{Ran}} \simeq \Omega(L^\vee[1]),$$

as commutative factorization algebras in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ equipped with a co-algebra structure.

2.3.9. Another (in a sense, tautologically equivalent) interpretation is as follows: for a Lie co-algebra $L^\vee$, consider $\mathrm{C}^\cdot(L^\vee)$ as an associative algebra. Consider the co-associative co-algebra $\mathrm{Bar}\,(\mathrm{C}^\cdot(L^\vee))$. The commutative algebra structure on $\mathrm{C}^\cdot(L^\vee)$ makes $\mathrm{Bar}\,(\mathrm{C}^\cdot(L^\vee))$ into a commutative Hopf algebra. We have a canonical isomorphism

$$U^\vee(L) \simeq \mathrm{Bar}\,(\mathrm{C}^\cdot(L^\vee))\,,$$

as commutative Hopf algebras. So, we have:

$$U^\vee(L)_{\mathrm{Ran}} \simeq \mathrm{Bar}(\Omega(L)),$$

as commutative factorization algebras in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ equipped with a co-algebra structure.

2.4. **Co-commutative factorization algebras.**

2.4.1. We say that a Λ^{pos}-factorization algebra is co-commutative if the factorization isomorphisms (2.2) come by adjunction from maps

$$\mathcal{A}^{\lambda_1+\lambda_2} \to \mathcal{A}^{\lambda_1} \star \mathcal{A}^{\lambda_2},$$

which make $\mathcal{A}$ into a co-commutative co-algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}))$.

As in the case of commutative factorization algebras, the category of co-commutative factorization algebras is a full subcategory in the category of co-commutative co-algebras in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ with respect to the $\star$ tensor structure.

NB: We refrain from formulating the structure of co-commutative factorization algebra as a map on $X^{\lambda_1} \times X^{\lambda_2}$ because the latter would involve the functor $\mathrm{add}^*_{\lambda_1,\lambda_2}$, which is not defined for all D-modules.

2.4.2. The following is parallel to Proposition 2.3.5 combined with Proposition 2.3.3:

Proposition 2.4.3.
(1) *For a Lie-* algebra L in $D(X)^{\Lambda^{pos}}$, the co-commutative co-algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}))$ with respect to the $\star$ tensor structure, given by $\mathrm{C}.(\Delta_*(L))$ is factorizable.*
(2) *The above assignment $L \mapsto \mathrm{C}.(\Delta_*(L))$ is an equivalence between the category of Lie-* algebras in $D(X)^{\Lambda^{pos}}$ and co-commutative factorization algebras.*

For a Lie-* algebra L in $D(X)^{\Lambda^{pos}}$ let us denote the resulting co-commutative factorization algebra by $\Upsilon(L)$.

It follows from the construction that it L is such that $L[1]$ is a D-module (i.e., belongs to the heart of the t-structure), then so does $\Upsilon(L)$.

Let $\mathfrak{U}(L)$ denote the chiral universal enveloping algebra of L; this is a Λ^{pos}-graded chiral algebra on X. By construction, we have:

$$\mathfrak{C}(\mathfrak{U}(L)) \simeq \Upsilon(L).$$

2.4.4. The following is parallel to Sect. 2.3.6. For a Lie-* algebra L in $D(X)^{\Lambda^{pos}}$, consider $\Delta_*(L)$ as a Lie algebra in $D(\mathrm{Ran}(X,\Lambda^{pos}))$ with respect to the $\star$ tensor structure, and let $U(L)_{\mathrm{Ran}}$ denote its universal enveloping algebra. This is a co-commutative factorization algebra in $D(\mathrm{Ran}(X,\Lambda^{pos}))$ with a compatible associative algebra structure with respect to $\star$.

We can also think of $U(L)_{\mathrm{Ran}}$ as $\mathrm{coBar}(\Upsilon(L))$, with its natural structure of co-commutative Hopf algebra. In addition, we have a canonical isomorphism

$$\Upsilon(L[-1]) \simeq U(L)_{\mathrm{Ran}},$$

as co-commutative Hopf algebras.

2.5. **Verdier duality.**

2.5.1. Parallel to the above discussion, we can consider the semi-group Λ^{neg}. For a compact object $\mathcal{F} \in D(X^\lambda)$, we'll think of its Verdier dual $\mathbb{D}(\mathcal{F})$ as an object of $D(X^{-\lambda})$.

We shall say that an object $\mathcal{F} = \{\mathcal{F}^\lambda\}$ of $D(\mathrm{Ran}(X,\Lambda^{pos}))$ is locally compact if each of its components $\mathcal{F}^\lambda$ is compact as an object of $D(X^\lambda)$.

Thus, we obtain that Verdier duality defines a contravariant equivalence between the sub-categories $D(\mathrm{Ran}(X,\Lambda^{pos}))$ and $D(\mathrm{Ran}(X,\Lambda^{neg}))$, consisting of locally compact objects.

2.5.2. The $\star$ tensor product sends (locally) compact objects to (locally) compact ones, and satisfies:

$$\mathbb{D}(\mathcal{F}_1 \star \mathcal{F}_2) \simeq \mathbb{D}(\mathcal{F}_1) \star \mathbb{D}(\mathcal{F}_2).$$

In particular, Verdier duality defines anti-equivalences between the categories of locally compact associative/commutative/Lie algebras in $D(\mathrm{Ran}(X,\Lambda^{pos}))$ with respect to the $\star$ tensor structure and the corresponding co-algebras in $D(\mathrm{Ran}(X,\Lambda^{neg}))$.

Moreover, Koszul duaility in any of the contexts: associative/commutative/Lie sends locally compact objects to locally compact ones, and we have:

$$\mathbb{D} \circ \mathrm{KD} \simeq \mathrm{KD} \circ \mathbb{D}.$$

2.5.3. Let $\mathcal{A}$ be a factorization algebra which is locally compact. It is clear that $\mathbb{D}(\mathcal{A})$ has a natural structure of factorization algebra.

The following is evident from the definitions:

Lemma 2.5.4. *Let $\mathcal{A}$ be a factorization algebra which is locally compact. Then the structure on it of commutative/co-commutative factorization algebra is equivalent to the structure of co-commutative/commutative factorization algebra on $\mathbb{D}(\mathcal{A})$.*

2.5.5. Let L be a Lie-* algebra in $D(X)^{\Lambda^{pos}}$, which is compact in every degree. Then $L^\vee := \mathbb{D}(L) \in D(X)^{\Lambda^{neg}}$ is also compact in every degree, and has a natural structure of Lie co-algebra with respect to the $\overset{!}{\otimes}$ tensor structure.

From Sect. 2.5.2, we obtain that the objects $\Upsilon(L)$, $\Omega(L)$, $U(L)_{\mathrm{Ran}}$ and $U^\vee(L)_{\mathrm{Ran}}$ are all locally compact. Moreover,

Lemma 2.5.6. *We have canonical isomorphisms $\mathbb{D}(\Upsilon(L)) \simeq \Omega(L)$ as commutative algebras, and $\mathbb{D}(U(L)) \simeq U^\vee(L)$ as commutative Hopf algebras in the $\star$ tensor structure.*

3. Eisenstein series, take I (a summary of [BG2])

3.1. **Action of** $\mathrm{Ran}(X,\Lambda^{pos}))$ **on** Bun_T**.**

3.1.1. We assume now that X is compact and that Λ^{pos} maps to the lattice Λ of coweights of some torus T. Consider the category $D(\mathrm{Bun}_T)$.

We denote the action of $D(\mathrm{Ran}(X,\Lambda^{pos}))$ as a monoidal category on $D(\mathrm{Bun}_T)$ by

$$(\mathcal{F} := \{\mathcal{F}^\lambda\}, \mathcal{S}) \mapsto \mathcal{F}\star\mathcal{S} := \underset{\lambda}{\oplus}(\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T})_*\left(\pi^{\lambda !}(\mathcal{F}^\lambda)\overset{!}{\otimes}\mathrm{mult}^!_\lambda(\mathcal{S})\right),$$

where mult_λ is the natural map

$$X^\lambda \times \mathrm{Bun}_T \overset{\mathrm{AJ}^\lambda\times\mathrm{id}}{\longrightarrow} \mathrm{Bun}_T\times\mathrm{Bun}_T \overset{\mathrm{mult}}{\longrightarrow} \mathrm{Bun}_T,$$

where

$$\mathrm{AJ}^\lambda : X^\lambda \to \mathrm{Bun}_T$$

is the Abel-Jacobi map.

3.1.2. Similarly, we define an action of $D(\mathrm{Ran}(X,\Lambda^{neg}))$. Since the maps mult_λ are smooth and proper, we have:

Lemma 3.1.3. *For a compact object* $\mathcal{F}\in D(\mathrm{Ran}(X,\Lambda^{pos}))$*, the functors*

$$\mathcal{S}\mapsto\mathcal{F}\star\mathcal{S} \text{ and } \mathcal{S}'\mapsto\mathbb{D}(\mathcal{F})\star\mathcal{S}'$$

are both left and right adjoint of one another.

3.2. **Action on Drinfeld's compactifications.**

3.2.1. For $\lambda\in\Lambda^{pos}$ denote by

$$\bar{\iota}^\lambda : X^\lambda\times\overline{\mathrm{Bun}}_B\to\overline{\mathrm{Bun}}_B$$

the corresponding map obtained by "adding zeros". This is a finite map. We let ι^λ denote its restriction to the open substack $X^\lambda\times\mathrm{Bun}_B$.

The above procedure defines an action of $D(\mathrm{Ran}(X,\Lambda^{pos}))$ as a monoidal category on $D(\overline{\mathrm{Bun}}_B)$ by

$$(\mathcal{F} := \{\mathcal{F}^\lambda\},\mathcal{S})\mapsto\mathcal{F}\star\mathcal{S} := \underset{\lambda}{\oplus}\bar{\iota}^\lambda{}_*(\mathcal{F}^\lambda\boxtimes\mathcal{S}).$$

3.2.2. Let $\bar{\mathfrak{p}}$, $\bar{\mathfrak{q}}$ denote the projections:

$$\mathrm{Bun}_G \overset{\bar{\mathfrak{p}}}{\longleftarrow} \overline{\mathrm{Bun}}_B \overset{\bar{\mathfrak{q}}}{\longrightarrow} \mathrm{Bun}_T\,.$$

We define the functor of Eisenstein series

$$\mathrm{Eis} : D(\overline{\mathrm{Bun}}_B)\times D(\mathrm{Bun}_T)\to D(\mathrm{Bun}_G)$$

by

$$\mathcal{T},\mathcal{F}\mapsto\bar{\mathfrak{p}}_*\left(\mathcal{T}\overset{!}{\otimes}\bar{\mathfrak{q}}^!(\mathcal{F})\right).$$

The following is a diagram chase:

Proposition 3.2.3. *For* $\mathcal{S}\in D(\mathrm{Ran}(X,\Lambda^{pos}))$ *there exists a canonical isomorphism:*

$$\mathrm{Eis}(\mathcal{S}\star\mathcal{T},\mathcal{F})\simeq\mathrm{Eis}(\mathcal{T},\mathcal{S}\star\mathcal{F}).$$

3.3. **A factorization algebra attached to** $\check{\mathfrak{n}}^-$**.**

3.3.1. Consider the Lie algebra $\check{\mathfrak{n}}^-$; consider the constant Lie-* algebra $\check{\mathfrak{n}}^-_X := \check{\mathfrak{n}}^- \otimes \mathbb{C}_X$, which is graded by Λ^{neg}, and its Verdier dual $(\check{\mathfrak{n}}^-_X)^\vee \simeq (\check{\mathfrak{n}}^-)^\vee \otimes \mathbb{C}_X[2]$ which is graded by Λ^{pos}. Consider the corresponding commutative algebra $\Omega(\check{\mathfrak{n}}^-_X)$ in $D(\operatorname{Ran}(X, \Lambda^{pos}))$ with respect to $\star$.

By Sect. 2.3.4, $\Omega^\lambda(\check{\mathfrak{n}}^-_X)$ is a D-module (i.e., belongs to the heart of the t-structure) for every λ.

3.3.2. Let $\jmath$ denote the open embedding $\operatorname{Bun}_B \hookrightarrow \overline{\operatorname{Bun}}_B$, and consider the object

$$\jmath_!(\mathrm{IC}_{\operatorname{Bun}_B}) \in D(\overline{\operatorname{Bun}}_B).$$

NB: Since Bun_B is smooth, $\mathrm{IC}_{\operatorname{Bun}_B}$ is isomorphic to $\mathbb{C}_{\operatorname{Bun}_B}[\dim(\operatorname{Bun}_B)]$, where we apply the cohomological shift by the corresponding amount on each connected component.

The following has been established in [BG2]:

Theorem 3.3.3. *The object $\jmath_!(\mathrm{IC}_{\operatorname{Bun}_B}) \in D(\overline{\operatorname{Bun}}_B)$ is naturally a $\Omega(\check{\mathfrak{n}}^-_X)$-module, with respect to the above action of $D(\operatorname{Ran}(X, \Lambda^{pos}))$ on $D(\overline{\operatorname{Bun}}_B)$. Moreover, the map*

$$\Omega^\lambda(\check{\mathfrak{n}}^-_X) \boxtimes \mathrm{IC}_{\operatorname{Bun}_B} \to \iota^{\lambda!} \circ \jmath_!(\mathrm{IC}_{\operatorname{Bun}_B}),$$

arising by adjunction from

$$\iota^\lambda{}_! \left(\Omega^\lambda(\check{\mathfrak{n}}^-_X) \boxtimes \mathrm{IC}_{\operatorname{Bun}_B}\right) \simeq \bar{\iota}^\lambda{}_! \left(\Omega^\lambda(\check{\mathfrak{n}}^-_X) \boxtimes \jmath_!(\mathrm{IC}_{\operatorname{Bun}_B})\right) \to \jmath_!(\mathbb{C}_{\operatorname{Bun}_B}),$$

identifies

$$\Omega^\lambda(\check{\mathfrak{n}}^-_X) \boxtimes \mathrm{IC}_{\operatorname{Bun}_B} \simeq H^0\left(\iota^{\lambda!}(\jmath_!(\mathrm{IC}_{\operatorname{Bun}_B}))\right).$$

3.3.4. Let $\mathrm{Eis}_!$ denote the functor $D(\operatorname{Bun}_T) \to D(\operatorname{Bun}_G)$ defined as $\mathrm{Eis}(\jmath_!(\mathrm{IC}_{\operatorname{Bun}_B}), -)$, i.e.,

$$\mathrm{Eis}_!(\mathcal{F}) = \mathfrak{p}_! \left(\mathrm{IC}_{\operatorname{Bun}_B} \overset{!}{\otimes} \mathfrak{q}^!(\mathcal{F})\right).$$

As a corollary of Theorem 3.3.3 we obtain:

Corollary 3.3.5. *There exists a natural transformation*

$$\mathrm{Eis}_!(\Omega(\check{\mathfrak{n}}^-_X) \star \mathcal{F}) \to \mathrm{Eis}_!(\mathcal{F}),$$

compatible with the algebra structure on $\Omega(\check{\mathfrak{n}}^-_X)$.

3.4. **Intermediate Eisenstein series.**

3.4.1. Let $\Omega(\check{\mathfrak{n}}^-_X)$-mod$(D(\operatorname{Bun}_T))$ denote the category of $\Omega(\check{\mathfrak{n}}^-_X)$-modules in $D(\operatorname{Bun}_T)$. We have the natural forgetful functor

$$\Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\operatorname{Bun}_T)) \to D(\operatorname{Bun}_T),$$

which admits a left adjoint, denoted $\mathbf{ind}^{\Omega(\check{\mathfrak{n}}^-_X)}$,

$$\mathcal{F} \mapsto \Omega(\check{\mathfrak{n}}^-_X) \star \mathcal{F}.$$

Corollary 3.3.5 implies:

Corollary 3.4.2. *The functor $\mathrm{Eis}_!$ canonically extends to a functor*

$$\mathrm{Eis}_!^{int} : \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\operatorname{Bun}_T)) \to D(\operatorname{Bun}_G),$$

so that $\mathrm{Eis}_! \simeq \mathrm{Eis}_!^{int} \circ \mathbf{ind}^{\Omega(\check{\mathfrak{n}}^-_X)}$.

3.4.3. Let's give an interpretation of the category $\Omega(\check{\mathfrak{n}}^-_X)$-mod$(D(\mathrm{Bun}_T))$ in terms of geometric Langlands.

Recall that we have an equivalence (Fourier-Mukai transform):

$$\Phi_T : D(\mathrm{Bun}_T) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}).$$

The following results from deformation theory:

Proposition 3.4.4. *There exists a canonical equivalence*

$$\Phi_B : \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

making the diagrams

$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) & \xrightarrow{\Phi_B} & \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \\ \downarrow & & \downarrow{\scriptstyle \mathfrak{q}_{spec*}} \\ D(\mathrm{Bun}_T) & \xrightarrow{\Phi_T} & \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}), \end{array}$$

(where the left vertical arrow is the forgetful functor) and

$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) & \xrightarrow{\Phi_B} & \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \\ {\scriptstyle \mathbf{ind}^{\Omega(\check{\mathfrak{n}}^-_X)}}\uparrow & & \uparrow{\scriptstyle \mathfrak{q}^*_{spec}} \\ D(\mathrm{Bun}_T) & \xrightarrow{\Phi_T} & \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \end{array}$$

commute.

3.4.5. The above achieves the stated goal 2 mentioned in the introduction:

The functor $\mathrm{Eis}^{int}_! \circ \Phi_B$ can be interpreted as a functor

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \to D(\mathrm{Bun}_G),$$

which is the thought-for $\Psi_G \circ \mathfrak{p}_{spec*}$.

The composition

$$\mathrm{Eis}^{int}_! \circ \Phi_B \circ \mathfrak{q}^*_{spec} : \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to D(\mathrm{Bun}_G)$$

identifies with $\mathrm{Eis}_! \circ \Phi_T$ and is supposed to be isomorphic to the composition

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \xrightarrow{\mathrm{Eis}_{spec}} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\Phi_G} D(\mathrm{Bun}_G).$$

3.5. Compactified Eisenstein series.

3.5.1. Let's recall the second main result of [BG2]. Consider the Bar-construction

$$\mathrm{Bar}\left(\Omega(\check{\mathfrak{n}}^-_X), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right) \in D(\overline{\mathrm{Bun}}_B). \tag{3.1}$$

Theorem 3.5.2. *There exists a canonical isomorphism in $D(\overline{\mathrm{Bun}}_B)$:*

$$\mathrm{Bar}\left(\Omega(\check{\mathfrak{n}}^-_X), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right) \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_B}.$$

3.5.3. Let

$$\mathbf{triv}^{\Omega(\check{\mathfrak{n}}^-_X)} : D(\mathrm{Bun}_T) \to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))$$

be the functor that associates to $\mathcal{F} \in D(\mathrm{Bun}_T)$ the same object endowed with the trivial action of $\Omega(\check{\mathfrak{n}}^-_X)$. From Theorem 3.5.2 we obtain:

Corollary 3.5.4. *There exists a canonical isomorphism of functors $D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$:*

$$\mathrm{Eis}_{!*} \simeq \mathrm{Eis}^{int}_! \circ \mathbf{triv}^{\Omega(\check{\mathfrak{n}}^-_X)}.$$

3.5.5. By Sect. 2.3.6, the object (3.1) acquires a natural co-action of the co-associative co-algebra $U^\vee(\check{\mathfrak{n}}^-_X)$ (again, with respect to the above action of $D(\operatorname{Ran}(X,\Lambda^{pos}))$ on $D(\overline{\operatorname{Bun}}_B)$). As a corollary of Theorem 3.5.2, we obtain:

Corollary 3.5.6. *The object* $\operatorname{IC}_{\overline{\operatorname{Bun}}_B}\in D(\overline{\operatorname{Bun}}_B)$ *is naturally a* $U^\vee(\check{\mathfrak{n}}^-_X)$*-comodule, or, equivalently, a co-module for the Lie co-algebra* $(\check{\mathfrak{n}}^-_X)^\vee\in D(\operatorname{Ran}(X,\Lambda^{pos}))$.

3.5.7. We define the functor of compactified Eisenstein series $\operatorname{Eis}_{!*}:D(\operatorname{Bun}_T)\to D(\operatorname{Bun}_G)$ as $\operatorname{Eis}(\operatorname{IC}_{\overline{\operatorname{Bun}}_B},-)$, i.e.,

$$\operatorname{Eis}_!(\mathcal{F})=\overline{\mathfrak{p}}_*\left(\operatorname{IC}_{\overline{\operatorname{Bun}}_B}\overset{!}{\otimes}\overline{\mathfrak{q}}^!(\mathcal{F})\right).$$

From Corollary 3.5.6, we obtain:

Corollary 3.5.8. *There exists a natural transformation*

$$\operatorname{Eis}_{!*}(\mathcal{F})\to\operatorname{Eis}_{!*}(U^\vee(\check{\mathfrak{n}}^-_X)\star\mathcal{F})$$

compatible with the co-algebra structure on $U^\vee(\check{\mathfrak{n}}^-_X)$.

From Lemma 3.1.3, we obtain:

Corollary 3.5.9. *There exists a natural transformation*

$$\operatorname{Eis}_{!*}(U(\check{\mathfrak{n}}^-_X)\star\mathcal{F})\to\operatorname{Eis}_!(\mathcal{F})$$

compatible with the algebra structure on $U(\check{\mathfrak{n}}^-_X)$.

3.5.10. Let

$$(\check{\mathfrak{n}}^-_X)^\vee\text{-comod}(D(\operatorname{Bun}_T))=\check{\mathfrak{n}}^-_X\text{-mod}(D(\operatorname{Bun}_T))$$

denote the category of $(\check{\mathfrak{n}}^-_X)^\vee$-comodules or (which is equivalent by Lemma 3.1.3) of $\check{\mathfrak{n}}^-_X$-modules in $D(\operatorname{Bun}_T)$. Let

$$\mathbf{ind}^{\check{\mathfrak{n}}^-_X}:D(\operatorname{Bun}_T)\to\check{\mathfrak{n}}^-_X\text{-mod}(D(\operatorname{Bun}_T))$$

denote the left adjoint to the forgetful functor. From Corollary 3.5.9 we obtain:

Corollary 3.5.11. *The functor* $\operatorname{Eis}_{!*}$ *canonically extends to a functor*

$$\operatorname{Eis}^{int}_{!*}:\check{\mathfrak{n}}^-_X\text{-mod}(D(\operatorname{Bun}_T))\to D(\operatorname{Bun}_G),$$

so that $\operatorname{Eis}_{!*}\simeq\operatorname{Eis}^{int}_{!*}\circ\mathbf{ind}^{\check{\mathfrak{n}}^-_X}$.

4. Relationship between two kinds of Eisenstein series and ϵ-factors

4.1. The renormalized categories.

4.1.1. Let $\mathcal{A}$ be an associative algebra in $D(\operatorname{Ran}(X,\Lambda^{pos}))$, which is locally compact as an object of this category. Let $\mathcal{A}^\vee$ be the Verdier dual co-algebra in $D(\operatorname{Ran}(X,\Lambda^{neg}))$.

Note that the forgetful functor

$$\mathcal{A}\text{-mod}(D(\operatorname{Bun}_T))\to D(\operatorname{Bun}_T)$$

does not send compact objects to compact ones.

We define the renormalized version of category $\mathcal{A}\text{-mod}(D(\operatorname{Bun}_T))$, denoted

$$\mathcal{A}\text{-mod}(D(\operatorname{Bun}_T))^{ren}$$

to be the ind-completion of the full subcategory of $\mathcal{A}\text{-mod}(D(\operatorname{Bun}_T))$ consisting of objects whose image under the forgetful functor in $D(\operatorname{Bun}_T)$ is compact.

We have a tautological functor

$$\Xi^{\mathcal{A}} : \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T)).$$

4.1.2. By construction, the forgetful functor

$$\Xi^{\mathcal{A}} : \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to D(\mathrm{Bun}_T)$$

sends compact objects to compact ones. Hence, it admits a right adjoint that commutes with direct sums, which we will denote by $\mathbf{coind}^{\mathcal{A}}$. Explicitly,

$$\mathbf{coind}^{\mathcal{A}}(\mathcal{F}) \simeq \mathcal{A}^\vee \star \mathcal{F},$$

where we regard $\mathcal{A}^\vee := \mathbb{D}(\mathcal{A})$ as an $\mathcal{A}$-module.

In addition to the above functor $\Xi^{\mathcal{A}}$, we have a functor

$$\Psi^{\mathcal{A}} : \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T)) \to \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren}$$

defined so that

$$\Psi^{\mathcal{A}} \circ \mathbf{ind}^{\mathcal{A}} \simeq \mathbf{coind}^{\mathcal{A}}.$$

4.1.3. Let $\mathbf{triv}^{\mathcal{A}}$ denote the functor

$$D(\mathrm{Bun}_T) \to \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))$$

that attaches to an object of $D(\mathrm{Bun}_T)$ the same object endowed with the trivial action of $\mathcal{A}$.

This functor factors naturally as $\Xi^{\mathcal{A}} \circ \mathbf{triv}^{\mathcal{A},ren}$. The resulting functor

$$\mathbf{triv}^{\mathcal{A},ren} : D(\mathrm{Bun}_T) \to \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren}$$

has the property that it sends compact objects to compact ones. Hence, it admits a right adjoint that commutes with direct sums. We'll denote this right adjoint by $\mathbf{inv}^{\mathcal{A}}$.

4.2. Koszul dualities.

4.2.1. For $\mathcal{A} \in D(\mathrm{Ran}(X, \Lambda^{pos}))$ as above, let $\mathcal{B}^\vee \in D(\mathrm{Ran}(X, \Lambda^{pos}))$ be the Koszul dual co-algebra. By the local compactness assumption, $\mathcal{B}^\vee$ is also locally compact. Let $\mathcal{B} \in D(\mathrm{Ran}(X, \Lambda^{neg}))$ be its Verdier dual algebra.

We have the following Koszul duality type result:

Proposition 4.2.2.
(1) *The functor*

$$\mathbf{inv}^{\mathcal{A}} : \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to D(\mathrm{Bun}_T)$$

canonically factors through a functor

$$\mathbf{inv}^{\mathcal{A}}_{enh} : \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \mathcal{B}\text{-mod}(D(\mathrm{Bun}_T)),$$

followed by the forgetful functor.
(2) *The functor* $\mathbf{inv}^{\mathcal{A}}_{enh}$ *is an equivalence. Its inverse is the functor*

$$\mathbf{coinv}^{\mathcal{B}}_{enh} : \mathcal{B}\text{-mod}(D(\mathrm{Bun}_T)) \to \mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren},$$

whose composition with the forgetful functor is the functor

$$\mathbf{coinv}^{\mathcal{B}} : \mathcal{B}\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T),$$

left adjoint to $\mathbf{triv}^{\mathcal{B}}$*.*

(3) *The following diagram of functors is commutative:*

$$
\begin{array}{ccc}
\mathcal{A}\text{-mod}(D(\mathrm{Bun}_T)) & \longrightarrow & \mathcal{B}\text{-mod}(D(\mathrm{Bun}_T))^{ren} \\
{\scriptstyle \Xi^{\mathcal{A}}}\big\uparrow & & {\scriptstyle \Psi^{\mathcal{B}}}\big\uparrow \\
\mathcal{A}\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \longrightarrow & \mathcal{B}\text{-mod}(D(\mathrm{Bun}_T)).
\end{array}
\tag{4.1}
$$

4.2.3. Let's apply the above discussion to $\mathcal{A}$ being $\Omega(\check{\mathfrak{n}}_X^-)$. In this case $\mathcal{B} = U(\check{\mathfrak{n}}_X^-)$. We obtain:

Corollary 4.2.4.

(1) *The functor*

$$\mathbf{inv}^{\Omega(\check{\mathfrak{n}}_X^-)} : \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to D(\mathrm{Bun}_T)$$

canonically factors through a functor

$$\mathbf{inv}_{enh}^{\Omega(\check{\mathfrak{n}}_X^-)} : \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T)),$$

followed by the forgetful functor.

(2) *The functor* $\mathbf{inv}_{enh}^{\Omega(\check{\mathfrak{n}}_X^-)}$ *is an equivalence. Its inverse is the functor*

$$\mathbf{coinv}_{enh}^{\check{\mathfrak{n}}_X^-} : \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T)) \to \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T))^{ren},$$

whose composition with the forgetful functor is the functor

$$\mathbf{coinv}^{\check{\mathfrak{n}}_X^-} : \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T),$$

left adjoint to $\mathbf{triv}^{\check{\mathfrak{n}}_X^-}$.

(3) *The functor*

$$\mathbf{inv}^{\check{\mathfrak{n}}_X^-} : \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to D(\mathrm{Bun}_T)$$

canonically factors through a functor

$$\mathbf{inv}_{enh}^{\check{\mathfrak{n}}_X^-} : \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)),$$

followed by the forgetful functor.

(4) *The functor* $\mathbf{inv}_{enh}^{\check{\mathfrak{n}}_X^-}$ *is an equivalence. Its inverse is the functor*

$$\mathbf{coinv}_{enh}^{\Omega(\check{\mathfrak{n}}_X^-)} : \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) \to \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T))^{ren},$$

whose composition with the forgetful functor is the functor

$$\mathbf{coinv}^{\Omega(\check{\mathfrak{n}}_X^-)} : \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T),$$

left adjoint to $\mathbf{triv}^{\Omega(\check{\mathfrak{n}}_X^-)}$.

(5) *The above equivalences make the following diagrams commutative:*

$$
\begin{array}{ccc}
\Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) & \longrightarrow & \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T))^{ren} \\
{\scriptstyle \Xi^{\Omega(\check{\mathfrak{n}}_X^-)}}\big\uparrow & & {\scriptstyle \Psi^{\check{\mathfrak{n}}_X^-}}\big\uparrow \\
\Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \longrightarrow & \check{\mathfrak{n}}_X^-\text{-mod}(D(\mathrm{Bun}_T))
\end{array}
\tag{4.2}
$$

and

(4.3)
$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T))^{ren} \\ \Psi^{\Omega(\check{\mathfrak{n}}^-_X)}\downarrow & & \Xi^{\check{\mathfrak{n}}^-_X}\downarrow \\ \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)) \end{array}$$

4.3. **Implications for Eisenstein series.**

4.3.1. Consider the functor

$$\check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)) \to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))$$

given by (either of the) two circuits of the diagram (4.2).

From Corollary 3.5.4, we obtain:

Corollary 4.3.2. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)) & \longrightarrow & \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \\ \mathrm{Eis}^{int}_{!*}\downarrow & & \mathrm{Eis}^{int}_{!}\downarrow \\ D(\mathrm{Bun}_G) & \xrightarrow{\mathrm{Id}} & D(\mathrm{Bun}_G) \end{array}$$

4.3.3. On the other hand, we should point out that the diagram involving the functors $\mathrm{Eis}^{int}_!$ and $\mathrm{Eis}^{int}_{!*}$, and the diagram (4.3) will not commute. However, by Corollary 4.3.2, the calculation of the resulting functor

$$\Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \rightrightarrows \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)) \overset{\mathrm{Eis}^{int}_{!*}}{\longrightarrow} D(\mathrm{Bun}_G)$$

boils down to the calculation of the composite functor

$$\Xi^{\Omega(\check{\mathfrak{n}}^-_X)} \circ \Psi^{\Omega(\check{\mathfrak{n}}^-_X)} : \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)). \tag{4.4}$$

4.4. **ϵ-factors.**

4.4.1. In this subsection we'll study the functor (4.4), introduced above, and the corresponding functor

$$\Xi^{\check{\mathfrak{n}}^-_X} \circ \Psi^{\check{\mathfrak{n}}^-_X} : \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)) \to \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T)). \tag{4.5}$$

Note that by Corollary 4.2.4, the study of the above functors is equivalent to that of the composed functors

$$\Psi^{\Omega(\check{\mathfrak{n}}^-_X)} \circ \Xi^{\Omega(\check{\mathfrak{n}}^-_X)} : \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren}$$

and

$$\Psi^{\check{\mathfrak{n}}^-_X} \circ \Xi^{\check{\mathfrak{n}}^-_X} : \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \check{\mathfrak{n}}^-_X\text{-mod}(D(\mathrm{Bun}_T))^{ren}.$$

We'll calculate these compositions on certain subcategories in both cases.

4.4.2. For every coroot α consider the functor $D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T)$ given by

$$D(\mathrm{Bun}_T) \overset{\mathrm{mult}^!}{\longrightarrow} D(\mathrm{Bun}_T \times \mathrm{Bun}_T) \overset{(\mathrm{id} \times i_\alpha)^!}{\longrightarrow} \\ \to D(\mathrm{Bun}_T \times \mathrm{Pic}(X)) \to D(\mathrm{Bun}_T \times \mathrm{Pic}'(X)) \overset{\mathrm{pr}_*}{\longrightarrow} D(\mathrm{Bun}_T),$$

where

$$i_\alpha : \mathrm{Pic}(X) \to \mathrm{Bun}_T$$

is the map induced by α; $\mathrm{Pic}'(X) \to \mathrm{Pic}(X)$ is the coarse moduli space (i.e., the Picard scheme).

Let $D_{reg}(\mathrm{Bun}_T) \subset D(\mathrm{Bun}_T)$ denote the full subcategory spanned by objects annihilated by the above functors for all coroots α. I.e., this is the right orthogonal of the category generated by the images of the pull-push functors corresponding to the diagram

$$D(\mathrm{Bun}'_{T_\alpha}) \leftarrow D(\mathrm{Bun}'_T) \to D(\mathrm{Bun}_T),$$

where T_α is the quotient torus of T by the corresponding copy of $\mathbb{G}_m$, and for a torus T', $\mathrm{Bun}_{T'}$ denotes the corresponding coarse moduli space.

Thus, the embedding $D_{reg}(\mathrm{Bun}_T) \hookrightarrow D(\mathrm{Bun}_T)$ admits a right adjoint, so we can think of $D_{reg}(\mathrm{Bun}_T)$ as a localization of $D_{reg}(\mathrm{Bun}_T)$.

4.4.3. Let

$$\Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T)),\ \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren}, \\ \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T)) \text{ and } \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren}$$

be the preimages of in the corresponding categories of $D_{reg}(\mathrm{Bun}_T) \subset D(\mathrm{Bun}_T)$ under the forgetful functors.

We'll prove:

Proposition 4.4.4.
(1) *The vertical functors in the diagram*

$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T)) & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren} \\ {\scriptstyle \Psi^{\Omega(\check{\mathfrak{n}}^-_X)}}\downarrow & & {\scriptstyle \Xi^{\check{\mathfrak{n}}^-_X}}\downarrow \\ \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren} & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T)) \end{array}$$

are localizations, and the vertical functors in the diagram

$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T)) & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren} \\ {\scriptstyle \Xi^{\Omega(\check{\mathfrak{n}}^-_X)}}\uparrow & & {\scriptstyle \Psi^{\check{\mathfrak{n}}^-_X}}\uparrow \\ \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren} & \longrightarrow & \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T)) \end{array}$$

are fully faithful.
(2) *The composition* (4.5) *restricted to* $\check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))$, *is isomorphic to the shift functor*

$$\mathcal{F} \mapsto (-2\rho(\omega_X)) \star \mathcal{F}[(2g-2)\dim(\mathfrak{n}^-)],$$

where $2\rho(\omega_X)$ *is the point of* Bun_T *induced from* $\omega_X \in \mathrm{Pic}(X)$, *using the cocharacter* 2ρ.

Te rest of this section is devoted to the proof of this proposition.

4.4.5. Let us apply the Fourier-Mukai equivalence

$$\Psi_T : D(\mathrm{Bun}_T) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}).$$

Under this equivalence, $\check{\mathfrak{n}}^-_X$-mod$(D(\mathrm{Bun}_T))$ corresponds to quasi-coherent sheaves, endowed with an action of the sheaf of DG Lie-algebras $\check{\mathfrak{n}}^-_{univ}$:

$$E \mapsto \Gamma(X, \check{\mathfrak{n}}^-_E).$$

The category $\check{\mathfrak{n}}^-_X$-mod$(D(\mathrm{Bun}_T))^{ren}$ is the ind-completion of the full subcategory of $\check{\mathfrak{n}}^-_X$-mod$(D(\mathrm{Bun}_T))$, conisiting of objects that are compact as objects of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$.

Note also that under Fourier-Mukai, $D_{reg}(\mathrm{Bun}_T)$ corresponds to localization on the open substack

$$\mathsf{QCoh}_{reg}(\mathrm{LocSys}_{\check{T}}) \subset \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$$

consisting of $\check{T}$-local systems E, such that the 1-dimensional local system $\alpha(E)$ is non-trivial for every coroot α of G (=root α of $\check{G}$).

Over this substack, the sheaf $\check{\mathfrak{n}}^-_{univ}$ is concentrated in cohomological degree 1. This implies that

$$\Xi^{\check{\mathfrak{n}}^-_X} : \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren} \simeq \check{\mathfrak{n}}^-_{univ}\text{-mod}(\mathsf{QCoh}_{reg}(\mathrm{LocSys}_{\check{T}}))^{ren} \to \\ \to \check{\mathfrak{n}}^-_{univ}\text{-mod}(\mathsf{QCoh}_{reg}(\mathrm{LocSys}_{\check{T}}))^{ren} \simeq \check{\mathfrak{n}}^-_X\text{-mod}(D_{reg}(\mathrm{Bun}_T))$$

is a localization.

The composition $\Xi^{\check{\mathfrak{n}}^-_X} \circ \Psi^{\check{\mathfrak{n}}^-_X}$ is given by the $\check{\mathfrak{n}}^-_{univ}$-module equal to $U(\check{\mathfrak{n}}^-_{univ})^\vee$, where $\mathcal{F} \mapsto \mathcal{F}^\vee$ denotes the natural duality on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$. Since $\check{\mathfrak{n}}^-_{univ}[1]$ is a locally free coherent sheaf, we have a canonical isomorphism of $\check{\mathfrak{n}}^-_X$-modules:

$$U(\check{\mathfrak{n}}^-_{univ})^\vee \simeq U(\check{\mathfrak{n}}^-_{univ}) \otimes \det(\check{\mathfrak{n}}^-_{univ}[1]^\vee)[\mathrm{rk}(\check{\mathfrak{n}}^-_{univ})].$$

By Riemann-Roch, $\mathrm{rk}(\check{\mathfrak{n}}^-_{univ}) = (2g-2)\dim(\mathfrak{n}^-)$. Finally, we need to identify the line bundle $\det(\check{\mathfrak{n}}^-_{univ}[1])$ with the line bundle

$$E \mapsto \langle E, -2\rho(\omega_X)\rangle \simeq \prod_\alpha \langle -\alpha(E), \omega_X\rangle,$$

where $\langle -, - \rangle$ is the Weil pairing.

The required identification follows from the next general observation:

Lemma 4.4.6. *Let E' be a non-trivial 1-dimensional local system on X. Then we have a canonical isomorphism*

$$\det(H^1(X, E')) \simeq \langle E', \omega_X\rangle.$$

5. Verdier dualily and the functional equation

5.1. The dual categories.

5.1.1. Let us return to the general setting of Sect. 4.1. For an associative algebra $\mathcal{A}^+ \in D(\mathrm{Ran}(X, \Lambda^{pos}))$, let's denote by $\mathcal{A}^- \in D(\mathrm{Ran}(X, \Lambda^{neg}))$ the algebra obtained from the tautological map

$$\lambda \mapsto -\lambda : \Lambda^{neg} \to \Lambda^{pos}. \tag{5.1}$$

5.1.2. Note that Verdier duality on $D(\mathrm{Bun}_T)$ gives rise to a canonical identification

$$\left(\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))\right)^\vee \simeq \mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T)),$$

in such a way that the following diagram is commutative

$$\begin{array}{ccc} (D(\mathrm{Bun}_T))^c & \xrightarrow{\mathbb{D}} & (D(\mathrm{Bun}_T))^c \\ {\scriptstyle \mathbf{ind}^{\mathcal{A}^+}}\big\downarrow & & {\scriptstyle \mathbf{ind}^{\mathcal{A}^-}}\big\downarrow \\ (\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T)))^c & \xrightarrow{\mathbb{D}} & (\mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T)))^c, \end{array}$$

where we use the notation $\mathbb{D}$ to denote the canonical anti-equivalence

$$\mathbf{C}^c \to (\mathbf{C}^\vee)^c$$

for a compactly generated category $\mathbf{C}$ and its subcategory $\mathbf{C}^c$ of compact objects.

In other words, the dual of the forgetful functor $\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T)$ is the induction functor $\mathbf{ind}^{\mathcal{A}^-}$, and vice versa.

5.1.3. In addition, we have a canonical identification

$$\left(\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))^{ren}\right)^\vee \simeq \mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T))^{ren},$$

in such a way that the diagram

$$\begin{array}{ccc} (D(\mathrm{Bun}_T))^c & \xrightarrow{\mathbb{D}} & (D(\mathrm{Bun}_T))^c \\ \big\uparrow & & \big\uparrow \\ (\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))^{ren})^c & \xrightarrow{\mathbb{D}} & (\mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T))^{ren})^c, \end{array}$$

where the vertical arrows are the forgetful functors. In other words, the functor dual to the forgetful functor $\mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to D(\mathrm{Bun}_T)$ is the co-induction functor $\mathbf{coind}^{\mathcal{A}^-}$, and vice versa.

The next assertion follows from the definitions:

Lemma 5.1.4. *The dual of the functor*

$$\Xi^{\mathcal{A}^+} : \mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))^{ren} \to \mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T))$$

is

$$\Psi^{\mathcal{A}^-} : \mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T)) \to \mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T))^{ren},$$

and vice versa.

5.1.5. Let $\mathcal{B}^+$ be the Koszul dual algebra to $\mathcal{A}^+$. Then $\mathcal{B}^-$ is the Koszul dual of $\mathcal{A}^-$.

Let us observe that the equivalences of Proposition 4.2.2 are compatible with those above, in the sense that the duals of the functors

$$\mathbf{coinv}^{\mathcal{A}^+}_{enh} : \mathcal{A}^+\text{-mod}(D(\mathrm{Bun}_T)) \leftrightarrows \mathcal{B}^+\text{-mod}(D(\mathrm{Bun}_T))^{ren} : \mathbf{inv}^{\mathcal{B}^+}_{enh}$$

are

$$\mathbf{inv}^{\mathcal{B}^-}_{enh} : \mathcal{B}^-\text{-mod}(D(\mathrm{Bun}_T))^{ren} \leftrightarrows \mathcal{A}^-\text{-mod}(D(\mathrm{Bun}_T)) : \mathbf{coinv}^{\mathcal{B}^+}_{enh}.$$

5.2. Verdier duality on $\overline{\mathrm{Bun}}_B$.

5.2.1. Theorems 3.3.3 and 3.5.2 have the following Verdier dual cousins. Consider the Λ^{pos}-graded Lie-* algebra $\check{\mathfrak{n}}^+_X$, and the corresponding co-commutative co-algebra $\Upsilon(\check{\mathfrak{n}}^+_X)$ in $D(\mathrm{Ran}(X,\Lambda^{pos}))$ with respect to $\star$.

Theorem 5.2.2. *The object* $\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}) \in D(\overline{\mathrm{Bun}}_B)$ *is naturally a co-module with respect to* $\Upsilon(\check{\mathfrak{n}}^+_X)$. *Th corresponding map*

$$\iota^{\lambda *} \circ \jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}) \to \Upsilon^\lambda(\check{\mathfrak{n}}^+_X) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B}$$

identifies the latter with H^0 *of the former.*

5.2.3. Consider now the object

$$\mathrm{coBar}\left(\Upsilon(\check{\mathfrak{n}}^+_X), \jmath_*(\mathrm{IC}_{\mathrm{Bun}_B})\right) \in D(\overline{\mathrm{Bun}}_B). \tag{5.2}$$

NB: The formation of the co-Bar complex involves a limit, so such is the case in forming the object (5.2). However, the corresponding inverse system is easily seen to stabilize when restricted to every open substack of $\overline{\mathrm{Bun}}_B$ of finite type.

Theorem 5.2.4. *There exists a canonical isomorphism in* $D(\overline{\mathrm{Bun}}_B)$*:*

$$\mathrm{coBar}\left(\Upsilon(\check{\mathfrak{n}}^+_X), \jmath_*(\mathrm{IC}_{\mathrm{Bun}_B})\right) \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_B}.$$

Corollary 5.2.5. *The object* $\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \in D(\overline{\mathrm{Bun}}_B)$ *is naturally a* $U(\check{\mathfrak{n}}^+_X)$*-module, or, equivalently, a module for the Lie algebra* $\check{\mathfrak{n}}^+_X \in D(\mathrm{Ran}(X,\Lambda^{neg}))$.

5.2.6. We observe the following phenomenon: the object $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ has a structure of module with respect to $\check{\mathfrak{n}}^+_X$ and co-module with respect to $(\check{\mathfrak{n}}^-_X)^\vee$.

It is natural to ask:

Question 5.2.7. *Can one formulate in what sense* $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ *carries an action of the entire* $\check{\mathfrak{g}}$*?*

This is closely related to our goal 3 stated in the introduction. In fact, we'll consider on object (an algebra in $D(\mathrm{Ran}(X,\Lambda^{pos}))$), which is richer than $\Omega(\check{\mathfrak{n}}^-_X)$, which acts on $\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})$, and such as this action encodes the required structure.

5.3. Verdier dual picture for Eisenstein series.

5.3.1. In addition to $D(\mathrm{Bun}_G)$, we can consider its dual category, $D(\mathrm{Bun}_G)^\vee$. We will distinguish them notationally, by denoting the former by $D(\mathrm{Bun}_G)_!$ and the latter by $D(\mathrm{Bun}_G)_*$.

The functor $\mathrm{Eis}(\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}), -)$ is naturally a functor

$$\mathrm{Eis}_* : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)_*.$$

As in Corollary 3.4.2, from Theorem 5.2.2, we obtain:

Corollary 5.3.2. *The functor* Eis_* *canonically extends to a functor*

$$\mathrm{Eis}^{int}_* : \Omega(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G),$$

so that $\mathrm{Eis}_* \simeq \mathrm{Eis}^{int}_* \circ \mathbf{ind}^{\Omega(\check{\mathfrak{n}}^+_X)}$.

5.3.3. The following has been established in [Eis]:

Proposition 5.3.4. *The functor* $\mathrm{Eis}_! : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)_!$ *sends compact objects to compacts. For a compact* $\mathcal{F} \in D(\mathrm{Bun}_T)$ *we have:*

$$\mathbb{D} \circ \mathrm{Eis}_!(\mathcal{F}) \simeq \mathrm{Eis}_*(\mathbb{D}(\mathcal{F})).$$

Corollary 5.3.5. *The functors*

$$\mathrm{Eis}_!^{int} : \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_! \text{ and } \mathrm{Eis}_*^{int} : \Omega(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_*$$

both send compact objects to compacts, and for a compact $\mathcal{F} \in \Omega(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T))$ *we have:*

$$\mathbb{D} \circ \mathrm{Eis}_!^{int}(\mathcal{F}) \simeq \mathrm{Eis}_*^{int}(\mathbb{D}(\mathcal{F})).$$

5.3.6. In addition to Eis_*^{int}, one would wish to use Theorem 5.2.4 to define a functor

$$\mathrm{Eis}_{*!}^{int} : U(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_*,$$

so that

$$\mathrm{Eis}_{*!}^{int} \circ \mathbf{ind}^{\check{\mathfrak{n}}_X^+} \simeq \mathrm{Eis}_{*!},$$

where $\mathrm{Eis}_{*!}$ is *a* functor

$$D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)_*,$$

defined using the kernel $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. According to Theorem 5.2.4, this functor should also make the following diagram commutative:

$$\begin{array}{ccc} \Omega(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)) & \xrightarrow{\mathrm{Eis}_*^{int}} & D(\mathrm{Bun}_G)_* \\ {\scriptstyle \Psi^{\Omega(\check{\mathfrak{n}}_X^+)}}\downarrow & & \uparrow{\scriptstyle \mathrm{Eis}_{*!}^{int}} \\ \Omega(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \xrightarrow{\sim} & U(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)). \end{array} \tag{5.3}$$

Unfortunately, we do not know how to define $\mathrm{Eis}_{*!}$ (making sure that it commutes with direct sums). So, we do not know how to define $\mathrm{Eis}_{*!}^{int}$ either. However, we can define the functor $\mathrm{Eis}_{*!}^{int}$ (and, hence, $\mathrm{Eis}_{*!}$) on the subcategories

$$U(\check{\mathfrak{n}}_X^+)\text{-mod}(D_{reg}(\mathrm{Bun}_T)) \subset U(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)) \text{ and } D_{reg}(\mathrm{Bun}_T) \subset D(\mathrm{Bun}_T),$$

respectively.

Namely, we have the following assertion:

Lemma 5.3.7. *The functor* $\mathrm{Eis}_*^{int} : \Omega(\check{\mathfrak{n}}_X^+)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_*$*, when restricted to* $U(\check{\mathfrak{n}}_X^+)\text{-mod}(D_{reg}(\mathrm{Bun}_T))$ *factors through the localization*

$$U(\check{\mathfrak{n}}_X^+)\text{-mod}(D_{reg}(\mathrm{Bun}_T)) \to U(\check{\mathfrak{n}}_X^+)\text{-mod}(D_{reg}(\mathrm{Bun}_T))^{ren}.$$

5.4. Functional equation.

5.4.1. Recall that we have the functor

$$F : D(\mathrm{Bun}_G)_* \to D(\mathrm{Bun}_G)_!$$

with the following property:

$$F \circ \mathrm{Eis}_* \simeq \mathrm{Eis}_! \circ w_0, \tag{5.4}$$

where $w_0 : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T)$ denotes the functor corresponding to the action of w_0 on T.

5.4.2. Note that the action of w_0 on $D(\mathrm{Bun}_T)$ extends to an equivalence

$$w_0 : \Omega(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) \to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)).$$

Thus, from (5.4), we obtain the following form of the functional equation:

Corollary 5.4.3. *There exists a canonical isomorphism*

$$F \circ \mathrm{Eis}^{int}_* \simeq \mathrm{Eis}^{int}_! \circ w_0.$$

5.4.4. We shall now establish another form of the functional equation, this time for the functor $\mathrm{Eis}_{!*}$. Namely, we claim:

Proposition 5.4.5. *For $\mathcal{F} \in D_{reg}(\mathrm{Bun}_T)$, we have a canonical isomorphism*

$$F \circ \mathrm{Eis}_{*!}(\mathcal{F}') \simeq \mathrm{Eis}_{!*}(w_0(\mathcal{F})),$$

where $\mathcal{F}'$ is the shift of $\mathcal{F}$ equal to $(-2\rho(\omega_X)) \star \mathcal{F}[(2g-2)\dim(\mathfrak{n}^-)]$.

Proof. By Corollary 5.4.3, the RHS is isomorphic to

$$F \circ \mathrm{Eis}^{int}_* \left(\mathbf{triv}^{\Omega(\check{\mathfrak{n}}^+_X)}(\mathcal{F})\right).$$

Hence, the assertion follows from Proposition 4.4.4. □

6. Eisenstein series, take II

6.1. New algebras.

6.1.1. We'll change the framework of Sect. 2 slightly. Instead of $D(\mathrm{Ran}(X, \Lambda^{pos}))$, we'll consider the category $D(\mathrm{Ran}(X, \Lambda^{pos}) \times \mathrm{Bun}_T)$. The only difference is that the operation

$$\mathcal{F}^\lambda \in D(X^\lambda),\ \mathcal{F}^\mu \in D(X^\mu) \mapsto \mathcal{F}^\lambda \boxtimes \mathcal{F}^\mu \in D(X^\lambda \times X^\mu)$$

gets replaced by a twisted one, namely,

$$\mathcal{F}^\lambda \in D(X^\lambda \times \mathrm{Bun}_T),\ \mathcal{F}^\mu \in D(X^\mu \times \mathrm{Bun}_T) \mapsto$$
$$\mathcal{F}^\lambda \widetilde{\boxtimes} \mathcal{F}^\mu := (\mathrm{id}_{X^\lambda} \times \mathrm{mult}_\mu)^!(\mathcal{F}^\lambda) \overset{!}{\otimes} (\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T})^!(\mathcal{F}^\mu) \in D(X^\lambda \times X^\mu \times \mathrm{Bun}_T).$$

This makes $D(\mathrm{Ran}(X, \Lambda^{pos}) \times \mathrm{Bun}_T)$ into a monoidal category by means of

$$\mathcal{F}^\lambda, \mathcal{F}^\mu \mapsto \mathcal{F}^\lambda \star \mathcal{F}^\mu := (\mathrm{add}_{\lambda,\mu} \times \mathrm{id}_{\mathrm{Bun}_T})_*(\mathcal{F}^\lambda \widetilde{\boxtimes} \mathcal{F}^\mu).$$

As such it acts on $D(\mathrm{Bun}_T)$ via

$$\mathcal{F}^\lambda, \mathcal{S} \mapsto \mathcal{F}^\lambda \star \mathcal{S} := \mathrm{mult}_{\lambda\,*}(\mathcal{F}^\lambda \widetilde{\boxtimes} \mathcal{S}),$$

where

$$\mathcal{F}^\lambda \widetilde{\boxtimes} \mathcal{S} := \mathcal{F}^\lambda \overset{!}{\otimes} (\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T})^!(\mathcal{S}),$$

and similarly for $D(\overline{\mathrm{Bun}}_B)$.

The discussion of Koszul dualities (in the associative/co-associative setting) goes through without change.

We can also talk about factorization algebras in $D(\mathrm{Ran}(X, \Lambda^{pos}) \times \mathrm{Bun}_T)$. By this we mean an object $\mathcal{A} := \{\mathcal{A}^\lambda \in D(X^\lambda \times \mathrm{Bun}_T)\}$ endowed with an isomorphism

$$\mathcal{A}^\lambda \widetilde{\boxtimes} \mathcal{A}^\mu|_{(X^\lambda \times X^\mu)_{disj} \times \mathrm{Bun}_T} \simeq \mathrm{add}^!_{\lambda,\mu}(\mathcal{A}^{\lambda+\mu})|_{(X^\lambda \times X^\mu)_{disj} \times \mathrm{Bun}_T}, \tag{6.1}$$

satisfying a natural associativity condition.

We'll say that a factorization algebra structure on $\mathcal{A}$ is compatible with an associative algebra (resp., co-associative co-algebra) structure if the morphisms

$$\mathcal{A}^\lambda \tilde{\boxtimes} \mathcal{A}^\mu \to \mathrm{add}^!_{\lambda,\mu}(\mathcal{A}^{\lambda+\mu}) \text{ and } \mathrm{add}^*_{\lambda,\mu}(\mathcal{A}^{\lambda+\mu}) \to \mathcal{A}^\lambda \tilde{\boxtimes} \mathcal{A}^\mu$$

corresponding to the algebra (resp., co-algebra) structure restrict to the map (6.1) on the open part $(X^\lambda \times X^\mu)_{disj} \times \mathrm{Bun}_T$.

6.1.2. We introduce an associative algebra $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$ in $D(\mathrm{Ran}(X, \Lambda^{pos}) \times \mathrm{Bun}_T)$ with a compatible factorization structure as follows:

Proposition-Construction 6.1.3. *There exists a canonically defined factorization algebra $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$ equipped with a structure of associative algebra in $D(\mathrm{Ran}(X, \Lambda^{pos}) \times \mathrm{Bun}_T)$ such that*

$$\iota^{\lambda !}(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})) \simeq \tilde{\Omega}(\check{\mathfrak{n}}^-_X)^\lambda \tilde{\boxtimes} \mathrm{IC}_{\mathrm{Bun}_B} := (\mathrm{id}_{X^\lambda} \times \mathfrak{q})^!(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)^\lambda) \overset{!}{\otimes} (\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_B})^!(\mathrm{IC}_{\mathrm{Bun}_B}),$$

and the resulting map

$$\tilde{\Omega}(\check{\mathfrak{n}}^-_X) \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B}) \to \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})$$

is an algebra action.

6.1.4. Let $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))$ denote the category of $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$-modules in $D(\mathrm{Bun}_T)$.

We obtain that the functor $\mathrm{Eis}_!$ can be canonically extended to a functor

$$\mathrm{Eis}^{ult}_! : \tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_!,$$

such that

$$\mathrm{Eis}^{ult}_! \circ \mathbf{ind}^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)} \simeq \mathrm{Eis}_! .$$

6.1.5. Let $\tilde{\Upsilon}(\check{\mathfrak{n}}^-_X)$ denote the Verdier dual of $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$; this is a factorization algebra in $D(\mathrm{Ran}(X, \Lambda^{neg}))$ with a compatible structure of co-associative co-algebra. Let $\tilde{\Upsilon}(\check{\mathfrak{n}}^+_X)$ (resp., $\tilde{\Omega}(\check{\mathfrak{n}}^+_X)$) be the corresponding objects obtained via (5.1).

Applying Verdier duality to Proposition 6.1.3, we obtain:

Corollary 6.1.6. *The object $\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}) \in D(\overline{\mathrm{Bun}}_B)$ is naturally a $\tilde{\Upsilon}(\check{\mathfrak{n}}^+_X)$-comodule.*

6.1.7. Consider the corresponding category

$$\tilde{\Upsilon}(\check{\mathfrak{n}}^+_X)\text{-comod}(D(\mathrm{Bun}_T)) \simeq \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)).$$

As in Sect. 5.1, we have a canonical equivalence

$$\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))\right)^\vee \simeq \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)).$$

As in Sect. 5.3, we can canonically extend the functor $\mathrm{Eis}_* : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)_*$ to a functor

$$\mathrm{Eis}^{ult}_* : \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)_*,$$

such that

$$\mathrm{Eis}^{ult}_* \circ \mathbf{ind}^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)} \simeq \mathrm{Eis}_* .$$

The functors $\mathrm{Eis}^{ult}_!$ and Eis^{ult}_* are Verdier conjugate of each other in the same sense as in Corollary 5.3.5.

Finally, the action of $w_0 \in W$ defines an equivalence

$$\tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) \to \tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)),$$

and as Corollary 5.4.3 we have the functional equation:

$$F \circ \mathrm{Eis}^{ult}_* \simeq \mathrm{Eis}^{ult}_! \circ w_0. \tag{6.2}$$

6.2. **Koszul duality for $\tilde{\Omega}$.**

6.2.1. A crucial observation concerning the quadruple of (co)-algebras

$$\tilde{\Omega}(\check{\mathfrak{n}}^+_X),\ \tilde{\Omega}(\check{\mathfrak{n}}^-_X),\ \tilde{\Upsilon}(\check{\mathfrak{n}}^+_X),\ \tilde{\Upsilon}(\check{\mathfrak{n}}^-_X)$$

is that it possesses an extra symmetry with respect to Koszul duality:

Proposition 6.2.2. *We have a canonical isomorphism of co-associative co-algebras:*

$$\mathrm{Bar}(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)) \simeq \tilde{\Upsilon}(\check{\mathfrak{n}}^+_X), \tag{6.3}$$

and of resulting co-modules in $D(\overline{\mathrm{Bun}}_B)$

$$\mathrm{Bar}\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right) \simeq \jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}). \tag{6.4}$$

Proof. Consider the object

$$\mathrm{Bar}\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right) \in D(\overline{\mathrm{Bun}}_B).$$

We claim that it is isomorphic to $\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B})$. Indeed, it suffices to show that when we apply $\iota^{\lambda !}$ to it for $\lambda \neq 0$, we obtain 0. However,

$$\iota^{\lambda !}\left(\mathrm{Bar}\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right)\right) \simeq \mathrm{Bar}\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X)), \tilde{\Omega}(\check{\mathfrak{n}}^-_X)^{\lambda}\right) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B} = 0.$$

Now, applying $\iota^{\lambda *}$ to both sides of (6.4), we obtain:

$$\mathrm{Bar}(\tilde{\Omega}(\check{\mathfrak{n}}^-_X))^{\lambda})\tilde{\boxtimes}\,\mathrm{IC}_{\mathrm{Bun}_B} \simeq \iota^{\lambda *}\left(\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B})\right),$$

Applying Verdier duality to Proposition 6.1.3, we obtain that

$$\iota^{\lambda *}\left(\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B})\right) \simeq \tilde{\Upsilon}(\check{\mathfrak{n}}^+_X) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B},$$

whence the identification (6.3).

□

6.2.3. Let

$$\Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)} : \tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \leftrightarrows \tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) : \Psi^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)}$$

and

$$\Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)} : \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \leftrightarrows \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) : \Psi^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}$$

be the corresponding categories and functors as in Sect. 4.1.

We have the corresponding commutative diagram of functors with the rows being mutually inverse equivalences:

$$\begin{array}{ccc}
\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) & \xrightarrow{\mathbf{coinv}^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)}_{enh}} & \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \\
\Psi^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)}\Big\downarrow & & \Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}\Big\downarrow \\
\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \xrightarrow{\mathbf{inv}^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)}_{enh}} & \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T))
\end{array}$$

and

$$\begin{array}{ccc}
\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) & \xleftarrow{\mathbf{inv}^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}_{enh}} & \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} \\
\Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}\Big\uparrow & & \Psi^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}\Big\uparrow \\
\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T))^{ren} & \xleftarrow{\mathbf{coinv}^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}_{enh}} & \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)).
\end{array}$$

6.2.4. Recall now that in addition to the functor F, there exists the tautological functor

$$\mathcal{T} : D(\mathrm{Bun}_G)_* \to D(\mathrm{Bun}_G)_!.$$

We have:

Proposition 6.2.5. *The following diagram of functors is commutative:*

$$\begin{array}{ccc} \tilde{\Omega}(\check{\mathfrak{n}}^+_X)\text{-mod}(D(\mathrm{Bun}_T)) & \xrightarrow{\Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)} \circ \mathbf{coinv}^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}_{enh}} & \tilde{\Omega}(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \\ {\scriptstyle \mathrm{Eis}^{ult}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{ult}_!} \\ D(\mathrm{Bun}_G)_* & \xrightarrow{\mathcal{T}} & D(\mathrm{Bun}_G)_!. \end{array}$$

Proof. The statement of the proposition is equivalent to the following one:

$$\jmath_*(\mathrm{IC}_{\mathrm{Bun}_B}) \simeq \mathrm{Bar}\left(\tilde{\Omega}(\check{\mathfrak{n}}^-_X), \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B})\right)$$

in a way compatible with the co-action of $\tilde{\Upsilon}(\check{\mathfrak{n}}^+_X)$. However, this follows from Proposition 6.2.2. □

6.3. **Summary of the functional equation.** We can now summarize what we have obtained so far regarding the functional equation (our goal 4):

For non-compactified Eisenstein series we have the isomorphisms:

$$F \circ \mathrm{Eis}_* \simeq \mathrm{Eis}_! \circ w_0, \;\; F \circ \mathrm{Eis}^{int}_* \simeq \mathrm{Eis}^{int}_! \circ w_0, \;\text{ and }\; \mathrm{Eis}^{ult}_* \simeq \mathrm{Eis}^{uld}_! \circ w_0,$$

$$\mathbb{D}_{\mathrm{Bun}_G} \circ \mathrm{Eis}_* \simeq \mathrm{Eis}_! \circ \mathbb{D}_{\mathrm{Bun}_T}, \;\; \mathbb{D}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^{int}_* \simeq \mathrm{Eis}^{int}_! \circ \mathbb{D}_{\mathrm{Bun}_T}, \;\text{ and }\; \mathbb{D}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^{ult}_* \simeq \mathrm{Eis}^{ult}_! \circ \mathbb{D}_{\mathrm{Bun}_T},$$

and

$$\mathcal{T} \circ \mathrm{Eis}^{ult}_* \simeq \mathrm{Eis}^{ult}_! \circ \left(\Xi^{\tilde{\Omega}(\check{\mathfrak{n}}^-_X)} \circ \mathbf{coinv}^{\tilde{\Omega}(\check{\mathfrak{n}}^+_X)}_{enh}\right).$$

For compactified Eisenstein series for $\mathcal{F} \in D_{reg}(\mathrm{Bun}_T)$ we have:

$$F \circ \mathrm{Eis}_{*!}(\mathcal{F}) \simeq \mathrm{Eis}_{!*} \circ w_0 \circ \mathrm{shift}(\mathcal{F}),$$

where shift is the shift functor on Bun_T by $-2\rho(\omega_X)$.

Combined from the "real" functional equation of [BG1]:

$$\mathrm{Eis}_{!*}(\mathcal{F}) \simeq \mathrm{Eis}_{!*} \circ w_0 \circ \mathrm{shift}(\mathcal{F})$$

for $\mathcal{F} \in D_{reg}(\mathrm{Bun}_T)$, we obtain yet one more isomorphism:

$$F \circ \mathrm{Eis}_{*!}(\mathcal{F}) \simeq \mathrm{Eis}_{!*}(\mathcal{F}).$$

7. The functor of constant term

7.1. **Two versions of the constant term functor.**

7.1.1. Recall that the functor

$$\mathrm{Eis}_! : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G),$$

has a right adjoint, known as the constant term functor $\mathrm{CT}_* : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_T)$, given by

$$\mathcal{F}' \in D(\mathrm{Bun}_G) \mapsto \mathfrak{q}_* \circ \mathfrak{p}^!(\mathcal{F}'),$$

up to a cohomological shift by $2\dim(\mathrm{Bun}_T) - \dim(\mathrm{Bun}_B)$, the latter being different for each connected component Bun_B^λ.

Therefore, we have a monad $\Gamma := \mathrm{CT}_* \circ \mathrm{Eis}_!$ acting on $D(\mathrm{Bun}_T)$ such that there exists a canonical isomorphism

$$\mathrm{Hom}_{D(\mathrm{Bun}_G)}(\mathrm{Eis}_!(\mathcal{F}_1), \mathrm{Eis}_!(\mathcal{F}_2)) \simeq \mathrm{Hom}_{D(\mathrm{Bun}_T)}(\mathcal{F}_1, \Gamma(\mathcal{F}_2)) .$$

7.1.2. Recall, however, that according to [Eis], the functor

$$\mathrm{CT}_! : D(\mathrm{Bun}_G)_! \to D(\mathrm{Bun}_T)$$

given by $\mathfrak{q}_! \circ \mathfrak{p}^*$ is also well-defined, and there is a canonical isomorphism:

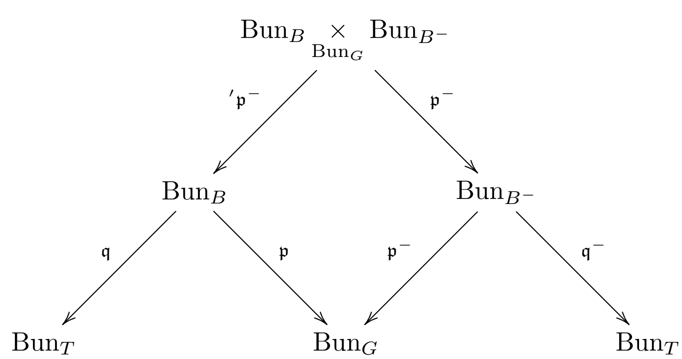
$$w_0 \circ \mathrm{CT}_! \simeq \mathrm{CT}_* . \tag{7.1}$$

The latter observation allows to calculate the composition Γ explicitly.

7.1.3. Consider the Cartesian product

$$\mathrm{Bun}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-} .$$

Let $'\mathfrak{p}^-$ and $'\mathfrak{p}$ denote its projections to Bun_B and Bun_{B^-}, respectively:



Let's denote the composed arrows

$$\mathfrak{r} := \mathfrak{q} \circ {}'\mathfrak{p}^- \text{ and } \mathfrak{r}^- := \mathfrak{q}^- \circ {}'\mathfrak{p}.$$

By base change and taking into account the fact that $\mathfrak{q}$ is smooth, we obtain:

$$\Gamma \simeq w_0 \circ \mathrm{CT}_! \circ \mathrm{Eis}_! \simeq (\mathfrak{q}^- \circ {}'\mathfrak{p})_! \circ ({}'\mathfrak{p}^- \circ \mathfrak{q})^*,$$

up to the cohomological shift by $\dim(\mathrm{Bun}_B) - \dim(\mathrm{Bun}_{B^-})$, which again depends on the connected component of Bun_B and Bun_{B^-}.

7.1.4. Bruhat decomposition defines a decomposition of this stack into locally closed substacks according to the relative position of the two reductions at the generic point of the curve. For $w \in W$ let

$$\left(\mathrm{Bun}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)_w$$

denote the corresponding locally closed substack; our conventions are such that $w = 1$ is the open stratum, which we also denote by $\mathcal{Z}$, the Zastava space. Note also that for $w = w_0$, we have

$$\left(\mathrm{Bun}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)^{w_0} \simeq \mathrm{Bun}_B .$$

Let $'\mathfrak{p}_w^-$ and $'\mathfrak{p}_w$ denote the projections from $\left(\mathrm{Bun}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)_w$ to Bun_B and Bun_{B^-}, respectively. Let

$$\mathfrak{r}_w := \mathfrak{q} \circ {}'\mathfrak{p}_w^- \text{ and } \mathfrak{r}_w^- := \mathfrak{q}^- \circ {}'\mathfrak{p}_w.$$

We obtain that the functor Γ admits a canonical filtration indexed by w, with the w-th subquotient denoted Γ_w given by

$$\Gamma_w := \mathfrak{r}^-_{w!} \circ \mathfrak{r}_w^*, \tag{7.2}$$

again up to the cohomological shift by $\dim(\mathrm{Bun}_B) - \dim(\mathrm{Bun}_{B^-})$.

7.2. **Relation to $\tilde{\Omega}$.** Our current goal is to establish the following:

Proposition 7.2.1. *The functor Γ_1 is canonically isomorphic to*

$$\mathcal{F} \mapsto \tilde{\Omega}(\check{\mathfrak{n}}_X^-) \star \mathcal{F}.$$

Remark. With a little extra work one can show that the resulting map

$$\tilde{\Omega}(\check{\mathfrak{n}}_X^-) \star \mathcal{F} \to \Gamma(\mathcal{F})$$

is a map of monads.

The rest of this section is devoted to the proof of this proposition.

7.2.2. For $\lambda \in \Lambda^{pos}$ let $\mathcal{Z}^\lambda$ be the union of connected components of $\mathcal{Z}$ equal to

$$\bigsqcup_{\lambda_1 - \lambda_2 = \lambda} \left(\mathrm{Bun}_B^{\lambda_1} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}^{\lambda_2}\right)_1 .$$

There exists a natural projection (defect of transversality):

$$p^\lambda : \mathcal{Z}^\lambda \to X^\lambda,$$

such that the composition

$$\mathcal{Z}^\lambda \overset{p^\lambda \times \mathfrak{r}_1}{\longrightarrow} X^\lambda \times \mathrm{Bun}_T \overset{\mathrm{mult}_\lambda}{\longrightarrow} \mathrm{Bun}_T$$

identifies with $\mathfrak{r}_1^-$.

Hence, by the projection formula, Γ_1 is given by

$$\mathcal{F} \mapsto \mathrm{mult}_{\lambda\,!}\left((p^\lambda \times \mathfrak{r}_1)_!(\mathbb{C}_{\mathcal{Z}^\lambda}) \otimes (\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T})^*(\mathcal{F})\right)[2|\lambda|],$$

where $\mathbb{C}_{\mathcal{Z}^\lambda}$ denotes the D-module corresponding to the constant sheaf on $\mathcal{Z}^\lambda$, and $2|\lambda|$ appears as the difference $\dim(\mathrm{Bun}_{B^-}^{\lambda_2}) - \dim(\mathrm{Bun}_B^{\lambda_1})$.

On the other hand, the functor $\mathcal{F} \mapsto \tilde{\Omega}(\check{\mathfrak{n}}_X^-)^\lambda \star \mathcal{F}$ can be written as

$$\mathcal{F} \mapsto \mathrm{mult}_{\lambda\,!}\left(\tilde{\Omega}(\check{\mathfrak{n}}_X^-)^\lambda \otimes (\pi^\lambda \times \mathrm{id}_{X^\lambda})^*(\mathcal{F})\right)[-2\dim(\mathrm{Bun}_T)].$$

This is due to the fact that $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$ is ULA with respect to the projection

$$\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T} : X^\lambda \times \mathrm{Bun}_T \to \mathrm{Bun}_T\,.$$

Thus, the required isomorphism would follow from the following one:

$$(p^\lambda \times \mathfrak{r}_1)_!(\mathbb{C}_{\mathcal{Z}^\lambda})[2|\lambda|] \simeq \tilde{\Omega}(\check{\mathfrak{n}}^-_X)^\lambda[-2\dim(\mathrm{Bun}_T)]. \tag{7.3}$$

7.2.3. The proof of (7.3) follows from the usual contraction picture for Zastava spaces:

Let

$$\overline{\mathcal{Z}}^\lambda \overset{\jmath_{\mathcal{Z}}}{\hookleftarrow} \mathcal{Z}^\lambda \tag{7.4}$$

be the partial compactification of $\mathcal{Z}^\lambda$, i.e., the open substack

$$\left(\overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\right)_1 \subset \overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-}\,.$$

The corresponding morphism

$$\overline{p}^\lambda : \overline{\mathcal{Z}}^\lambda \to X^\lambda \times \mathrm{Bun}_T$$

admits a section, denoted $\mathfrak{s}^\lambda$. Moreover, there is a $\mathbb{G}_m$-action on $\overline{\mathcal{Z}}^\lambda$ that contracts it on the image of $\mathfrak{s}^\lambda$. Hence,

$$(p^\lambda \times \mathfrak{r}_1)_!(\mathbb{C}_{\mathcal{Z}^\lambda}) \simeq (p^\lambda \times \mathfrak{r}_1)_! \circ \jmath_{\mathcal{Z}!}(\mathbb{C}_{\mathcal{Z}^\lambda}) \simeq \mathfrak{s}^{\lambda!} \circ \jmath_{\mathcal{Z}!}(\mathbb{C}_{\mathcal{Z}^\lambda}).$$

The assertion follows now from the fact that the pair in (7.4) is smoothly equivalent to the pair

$$\overline{\mathrm{Bun}}_B \overset{\jmath}{\hookleftarrow} \mathrm{Bun}_B,$$

and $\dim(\mathcal{Z}^\lambda) = 2|\lambda|$.

8. The space of rational reductions to B

The rest of the paper is devoted to addressing our goal 5.

8.1. **The category.**

8.1.1. What follows is an attempt to realize Drinfeld's idea of the space of G-bundles with a rational reduction to B. Unfortunately, we won't be able to construct the space itself, but rather the category of D-modules on it. Our approach will be very naive: we'll start with $\overline{\mathrm{Bun}}_B$ and we'll "contract" the strata

$$\iota^\lambda : X^\lambda \times \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$$

under $\pi^\lambda : X^\lambda \times \mathrm{Bun}_B \to \mathrm{Bun}_B$.

8.1.2. Let $\mathcal{H}$ be the closed substack of $\overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B$ corresponding to G-bundles with a pair of generalized B-reductions which agree at the generic point of X. This is a groupoid over $\overline{\mathrm{Bun}}_B$, and let

$$p_1, p_2 : \mathcal{H} \rightrightarrows \overline{\mathrm{Bun}}_B$$

denote its two projections to $\overline{\mathrm{Bun}}_B$. The maps p_i are proper, since $\overline{\mathrm{Bun}}_B$ is proper over Bun_G.

Let $D(\mathrm{Bun}_B^{rat})$ denote the category of $\mathcal{H}$-equivariant objects on $D(\overline{\mathrm{Bun}}_B)$. I.e., it consists of $\mathcal{F} \in D(\overline{\mathrm{Bun}}_B)$, endowed with an isomorphism:

$$p_1^!(\mathcal{F}) \simeq p_2^!(\mathcal{F}),$$

which satisfies a natural associativity condition.

We have a natural forgetful functor $D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B)$, which admits a left adjoint denoted $\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}$ and given by

$$\mathcal{F} \mapsto p_{1*} \circ p_2^!(\mathcal{F}).$$

Our current goal is to describe the category $D(\mathrm{Bun}_B^{rat})$ more explicitly. Note that from the definitions we obtain:

Lemma 8.1.3. *The functor $\overline{\mathfrak{p}}^!$ factors as*

$$D(\mathrm{Bun}_G) \overset{\mathfrak{p}^{rat!}}{\longrightarrow} D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B).$$

By adjunction, the functor $\overline{\mathfrak{p}}_* : D(\overline{\mathrm{Bun}}_B) \to D(\mathrm{Bun}_G)$ naturally actors as

$$D(\overline{\mathrm{Bun}}_B) \overset{\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}}{\longrightarrow} D(\mathrm{Bun}_B^{rat}) \overset{\mathfrak{p}_*^{rat}}{\longrightarrow} D(\mathrm{Bun}_G).$$

8.2. A monoid(al) approach.

8.2.1. We define the category $'D(\mathrm{Bun}_B^{rat})$ to consist of the data of $\mathcal{F} \in D(\overline{\mathrm{Bun}}_B)$, endowed with isomorphisms

$$\alpha^\lambda : \pi^{\lambda !}(\mathcal{F}) \to \bar{\iota}^{\lambda !}(\mathcal{F}),$$

for $\lambda \in \Lambda^{pos}$, that are associative in the sense that for $\lambda = \lambda_1 + \lambda_2$ the two isomorpisms taking place on $X^{\lambda_1} \times X^{\lambda_2} \times \overline{\mathrm{Bun}}_B$:

$$\pi^{\lambda_1,\lambda_2 !}(\mathcal{F}) \simeq (\mathrm{id}_{X^{\lambda_1}} \times \pi^{\lambda_2})^! \circ \pi^{\lambda_1 !}(\mathcal{F}) \overset{\alpha^{\lambda_1}}{\simeq} (\mathrm{id}_{X^{\lambda_1}} \times \pi^{\lambda_2})^! \circ \bar{\iota}^{\lambda_1 !}(\mathcal{F}) \simeq$$
$$\simeq (\mathrm{id}_{X^{\lambda_2}} \times \bar{\iota}^{\lambda_1})^! \circ \pi^{\lambda_2 !}(\mathcal{F}) \overset{\alpha^{\lambda_2}}{\simeq} (\mathrm{id}_{X^{\lambda_2}} \times \bar{\iota}^{\lambda_1})^! \circ \bar{\iota}^{\lambda_2 !}(\mathcal{F}) \simeq \bar{\iota}^{\lambda_1,\lambda_2 !}(\mathcal{F}),$$

and

$$\pi^{\lambda_1,\lambda_2 !}(\mathcal{F}) \simeq (\mathrm{add}_{\lambda_1,\lambda_2} \times \mathrm{id}_{\overline{\mathrm{Bun}}_B})^! \circ \pi^{\lambda_1+\lambda_2 !}(\mathcal{F}) \overset{\alpha^{\alpha_1+\alpha_2}}{\simeq}$$
$$\simeq (\mathrm{add}_{\lambda_1,\lambda_2} \times \mathrm{id}_{\overline{\mathrm{Bun}}_B})^! \circ \bar{\iota}^{\lambda_1+\lambda_2 !}(\mathcal{F}) \simeq \bar{\iota}^{\lambda_1,\lambda_2 !}(\mathcal{F})$$

coincide, where

$$\pi^{\lambda_1,\lambda_2} : X^{\lambda_1} \times X^{\lambda_2} \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B \text{ and } \bar{\iota}^{\lambda_1,\lambda_2} : X^{\lambda_1} \times X^{\lambda_2} \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B$$

are the natural maps.

8.2.2. One can view the definition of $'D(\mathrm{Bun}_B^{rat})$ in the following framework.

Consider $D(\mathrm{Ran}(X,\Lambda^{pos}))$ as a monoidal category acting on $D(\overline{\mathrm{Bun}}_B)$ as in Sect. 3.2, and also as acting on Vect by

$$\{\mathcal{F}^\lambda\}\mapsto \underset{\lambda}{\oplus} H(X^\lambda,\mathcal{F}^\lambda).$$

Then

$$'D(\mathrm{Bun}_B^{rat})\simeq \mathrm{Vect}\underset{D(\mathrm{Ran}(X,\Lambda^{pos}))}{\otimes} D(\overline{\mathrm{Bun}}_B),$$

where the forgetful functor $'D(\mathrm{Bun}_B^{rat})\to D(\overline{\mathrm{Bun}}_B)$ is the right adjoint to the tautological functor

$$D(\overline{\mathrm{Bun}}_B)\simeq \mathrm{Vect}\otimes D(\overline{\mathrm{Bun}}_B)\to \mathrm{Vect}\underset{D(\mathrm{Ran}(X,\Lambda^{pos}))}{\otimes} D(\overline{\mathrm{Bun}}_B).$$

We'll denote the above left adjoint $D(\overline{\mathrm{Bun}}_B)\to {}'D(\mathrm{Bun}_B^{rat})$ by $'\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}$.

8.2.3. We shall now discuss two ways how to compute Hom's in the category $'D(\mathrm{Bun}_B^{rat})$.

Let $'D(\mathrm{Bun}_B^{rat})_{lax,1}$ denote the category consisting of pairs $(\mathcal{F},\{\alpha^\lambda\})$ as in the definition of $'D(\mathrm{Bun}_B^{rat})$, but without the requirement that α^λ's be isomorphisms. Let $'D(\mathrm{Bun}_B^{rat})_{lax,2}$ be a similarly defined category of pairs $(\mathcal{F},\{\beta^\lambda\})$, where

$$\beta^\lambda:\bar{\iota}^{\lambda !}(\mathcal{F})\to \pi^{\lambda !}(\mathcal{F}).$$

By definition, we have:

Lemma 8.2.4. *The functors*

$$'D(\mathrm{Bun}_B^{rat})\to' D(\mathrm{Bun}_B^{rat})_{lax,1} \text{ and } 'D(\mathrm{Bun}_B^{rat})\to' D(\mathrm{Bun}_B^{rat})_{lax,2}$$

are fully faithful.

Next, we claim that Hom's in the categories $'D(\mathrm{Bun}_B^{rat})_{lax,1}$ and $'D(\mathrm{Bun}_B^{rat})_{lax,2}$ can be computed "algorithmically" in terms of Hom's in $D(\overline{\mathrm{Bun}}_B)$.

In fact, we claim that $'D(\mathrm{Bun}_B^{rat})_{lax,1}$ (resp., $'D(\mathrm{Bun}_B^{rat})_{lax,2}$) is the category of modules for a certain monad M_1 (resp., M_2) acting on $D(\overline{\mathrm{Bun}}_B)$. Indeed, the monads in question are

$$\mathsf{M}_1(\mathcal{F})=\underset{\lambda\in\Lambda^{pos}}{\oplus}\bar{\iota}^\lambda_!\circ\pi^{\lambda !}(\mathcal{F}) \text{ and } \mathsf{M}_2(\mathcal{F})=\underset{\lambda\in\Lambda^{pos}}{\oplus}{\pi^\lambda}_*\circ\bar{\iota}^{\lambda !}(\mathcal{F}),$$

respectively.

So, Hom's in $'D(\mathrm{Bun}_B^{rat})_{lax,i}\simeq \mathsf{M}_i\text{-mod}(D(\overline{\mathrm{Bun}}_B))$ can be expressed in terms of Hom's in $D(\overline{\mathrm{Bun}}_B)$ via the corresponding bar-complex.

8.3. **A comparison.**

8.3.1. Note that the maps

$$\pi^\lambda : X^\lambda \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B \text{ and } \bar{\iota}^\lambda : X^\lambda \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B$$

combine to a map

$$X^\lambda \times \overline{\mathrm{Bun}}_B \to \mathcal{H}.$$

Hence, we obtain that the forgetful functor $D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B)$ factors through a functor

$$\Phi_1 : D(\mathrm{Bun}_B^{rat}) \to {}'D(\mathrm{Bun}_B^{rat}).$$

From the same geometric picture we obtain that the functor

$$\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}} : D(\overline{\mathrm{Bun}}_B) \to D(\mathrm{Bun}_B^{rat})$$

canonically factors as

$$D(\overline{\mathrm{Bun}}_B) \overset{{}'\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}}{\longrightarrow} {}'D(\mathrm{Bun}_B^{rat}) \to D(\mathrm{Bun}_B^{rat}),$$

by the definition of tensor product. We'll denote the resulting functor ${}'D(\mathrm{Bun}_B^{rat}) \to D(\mathrm{Bun}_B^{rat})$ by Φ_2. By construction, Φ_2 is the left adjoint to Φ_1.

Proposition 8.3.2. *The functors*

$$\Phi_2 : {}'D(\mathrm{Bun}_B^{rat}) \rightleftarrows D(\mathrm{Bun}_B^{rat}) : \Phi_1$$

are mutually inverse equivalences of categories.

The rest of this subsection is devoted to the proof of Proposition 8.3.2.

8.3.3. Let us denote the monads

$$D(\overline{\mathrm{Bun}}_B) \overset{\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}}{\longrightarrow} D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B)$$

and

$$D(\overline{\mathrm{Bun}}_B) \overset{{}'\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}}}{\longrightarrow} {}'D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B)$$

by Av and ${}'\mathrm{Av}$, respectively.

By construction, we have a natural transformation

$$ {}'\mathrm{Av}(\mathcal{F}) \to \mathrm{Av}(\mathcal{F}). \tag{8.1}$$

It is easy to see that the statement of Proposition 8.3.2 is equivalent to the assertion that the map (8.1) is an isomorphism. To prove the latter, it suffices to show that the map

$$\iota^{\lambda!} \circ {}'\mathrm{Av}(\mathcal{F}) \to \iota^{\lambda!} \circ \mathrm{Av}(\mathcal{F})$$

is an isomorphism for any $\lambda \in \Lambda^{pos}$. However, since both functors appearing in (8.1) factor through ${}'D(\mathrm{Bun}_B^{rat})$, it suffices to show that the map

$$\jmath^! \circ {}'\mathrm{Av}(\mathcal{F}) \to \jmath^! \circ \mathrm{Av}(\mathcal{F}) \tag{8.2}$$

is an isomorphism.

8.3.4. Let's first calculate the RHS of (8.2). We claim that

$$\jmath^! \circ \mathrm{Av}(\mathcal{F}) \simeq \underset{\lambda}{\oplus} \pi^\lambda_* \circ \iota^{\lambda !}(\mathcal{F}). \tag{8.3}$$

The above claim follows from the next geometric assertion: there exists a natural isomorphism

$$\bigcup_{\lambda\in\Lambda^{pos}} X^\lambda \times \mathrm{Bun}_B \to \mathrm{Bun}_B \underset{\overline{\mathrm{Bun}}_B}{\times} \mathcal{H}.$$

Note that (8.3) allows to give the following fairly explicit description of the category $D(\mathrm{Bun}_B^{rat})$:

Let $\mathcal{F}_i$, $i=1,2,3$ be compact objects of $D(\mathrm{Bun}_B)$, such that the objects $\jmath_!(\mathcal{F}_i) \in D(\overline{\mathrm{Bun}}_B)$ are well-defined. We obtain:

$$\mathrm{Hom}_{D(\mathrm{Bun}_B^{rat})}\left(\mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_1)), \mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_2))\right) \simeq$$
$$\simeq \underset{\lambda}{\oplus} \mathrm{Hom}_{D(\mathrm{Bun}_B)}\left(\mathcal{F}_1, \pi^\lambda_* \circ \iota^{\lambda !} \circ \jmath_!(\mathcal{F}_2)\right) \simeq$$
$$\simeq \underset{\lambda}{\oplus} \mathrm{Hom}_{D(\overline{\mathrm{Bun}}_B)}\left(\bar\iota^\lambda_*(\mathbb{C}_{X^\lambda} \boxtimes \jmath_!(\mathcal{F}_1)), \jmath_!(\mathcal{F}_2)\right).$$

The composition

$$\mathrm{Hom}_{D(\mathrm{Bun}_B^{rat})}\left(\mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_1)), \mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_2))\right) \times$$
$$\times \mathrm{Hom}_{D(\mathrm{Bun}_B^{rat})}\left(\mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_2)), \mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_3))\right)$$
$$\to \mathrm{Hom}_{D(\mathrm{Bun}_B^{rat})}\left(\mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_1)), \mathbf{ind}^{\mathrm{Bun}_B^{rat}}_{\overline{\mathrm{Bun}}_B}(\jmath_!(\mathcal{F}_3))\right)$$

can be described as follows. For

$$\bar\iota^\lambda_*(\mathbb{C}_{X^\lambda} \boxtimes \jmath_!(\mathcal{F}_1)) \to \jmath_!(\mathcal{F}_2) \text{ and } \bar\iota^\mu_*(\mathbb{C}_{X^\mu} \boxtimes \jmath_!(\mathcal{F}_2)) \to \jmath_!(\mathcal{F}_3),$$

the resulting morphism

$$\bar\iota^{\lambda+\mu}_*(\mathbb{C}_{X^{\lambda+\mu}} \boxtimes \jmath_!(\mathcal{F}_1)) \to \jmath_!(\mathcal{F}_3)$$

equals the composition

$$\bar\iota^{\lambda+\mu}_*(\mathbb{C}_{X^{\lambda+\mu}} \boxtimes \jmath_!(\mathcal{F}_1)) \to \bar\iota^{\lambda+\mu}_*\left(\mathrm{add}_{\lambda,\mu\,*}(\mathbb{C}_{X^\lambda} \boxtimes \mathbb{C}_{X^\mu}) \boxtimes \jmath_!(\mathcal{F}_1)\right) \simeq$$
$$\simeq \bar\iota^{\lambda,\mu}_*\left(\mathbb{C}_{X^\lambda} \boxtimes \mathbb{C}_{X^\mu} \boxtimes \jmath_!(\mathcal{F}_1)\right) \simeq \bar\iota^\mu_* \circ (\mathrm{id}_{X^\mu} \times \bar\iota^\lambda)_*\left(\mathbb{C}_{X^\mu} \boxtimes (\mathbb{C}_{X^\lambda} \boxtimes \jmath_!(\mathcal{F}_1))\right) \to$$
$$\to \bar\iota^\mu_*\left(\mathbb{C}_{X^\mu} \boxtimes \jmath_!(\mathcal{F}_2)\right) \to \jmath_!(\mathcal{F}_3).$$

8.3.5. Next, we'll show that the functor $\mathcal{F} \mapsto \jmath^! \circ {}' \mathrm{Av}(\mathcal{F})$ is also isomorphic to

$$\mathcal{F} \mapsto \underset{\lambda}{\oplus} \pi^\lambda_* \circ \iota^{\lambda !}(\mathcal{F}). \tag{8.4}$$

(It will be clear from the construction that the map in (8.2) corresponds to the identity map on (8.4)).

For a coweight μ, consider the following category equipped with a natural action of $D(\mathrm{Ran}(X, \Lambda^{pos}))$:

$$\mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu)) := D\left(\bigcup_{\lambda\in\Lambda^{pos}} X^\lambda \times \mathrm{Bun}_B^\mu\right) \simeq D(\mathrm{Ran}(X, \Lambda^{pos})) \otimes D(\mathrm{Bun}_B).$$

Note that

$$\mathrm{Vect} \underset{D(\mathrm{Ran}(X,\Lambda^{pos}))}{\otimes} \mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu)) \simeq D(\mathrm{Bun}_B^\mu),$$

such that the functors

$$\mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu)) \rightleftarrows D(\mathrm{Bun}_B^\mu)$$

identify with

$$\{\mathcal{F}^\lambda\} \mapsto \underset{\lambda\in\Lambda^{pos}}{\oplus} \pi_*^\lambda(\mathcal{F}^\lambda)$$

and

$$\mathcal{F} \mapsto \{\pi^{\lambda !}(\mathcal{F})\},$$

respectively. Let $\mathrm{Av}_{\mathbf{ind}}$ denote the composition

$$\mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu)) \to D(\mathrm{Bun}_B^\mu) \to \mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu)).$$

Consider the functor

$$\phi : D(\overline{\mathrm{Bun}}_B) \to \mathbf{ind}^{\mathrm{Ran}}(D(\mathrm{Bun}_B^\mu))$$

given by

$$\mathcal{F} \mapsto \{\iota^{\lambda !}(\mathcal{F})\}.$$

We claim that ϕ commutes with the action of $D(\mathrm{Ran}(X,\Lambda^{pos}))$. Indeed, this follows from the fact that for $\lambda,\mu \in \Lambda^{pos}$, the diagram

$$\begin{array}{ccc} X^{\lambda+\mu}\times \mathrm{Bun}_B & \xrightarrow{\iota^{\lambda+\mu}} & \overline{\mathrm{Bun}}_B \\ {\scriptstyle \mathrm{add}_{\lambda+\mu}}\uparrow & & \uparrow{\scriptstyle \bar\iota^{\lambda}} \\ X^\lambda \times X^\mu \times \mathrm{Bun}_B & \xrightarrow{\mathrm{id}_{X^\lambda}\times \iota^\mu} & X^\lambda \times \overline{\mathrm{Bun}}_B \end{array}$$

is Cartesian.

The required result regarding the functor $j^! \circ {}'\mathrm{Av}$ follows from the next assertion:

Proposition 8.3.6. *There exists a natural isomorphism of functors*

$$\mathrm{Av}_{\mathbf{ind}} \circ \phi \simeq \phi \circ {}'\mathrm{Av}.$$

8.3.7. Recall the following general pattern. Let $\mathbf{A}$ be a monoidal category acting on the left on $\mathbf{C}_2$ and on the right on $\mathbf{C}_1$. We assume that the multiplication and action functors

$$\mathbf{A}\otimes\mathbf{A}\to\mathbf{A} \text{ and } \mathbf{A}\otimes\mathbf{C}_2\to\mathbf{C}_2,\ \mathbf{C}_1\otimes\mathbf{A}\to\mathbf{C}_1$$

have right adjoints that commute with direct sums. In this case the natural functor

$$\mathbf{C}_1\otimes\mathbf{C}_2 \to \mathbf{C}_1\underset{\mathbf{A}}{\otimes}\mathbf{C}_2$$

admits a right adjoint that commutes with direct sums. Let now $\mathbf{C}_1'$ and $\mathbf{C}_2'$ be another such pair, and $\phi_1 : \mathbf{C}_1 \to \mathbf{C}_1'$ and $\phi_2 : \mathbf{C}_2 \to \mathbf{C}_2'$ be functors that commute with the $\mathbf{A}$-actions. In this case, we have a commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{C}_1\otimes\mathbf{C}_2 & \longrightarrow & \mathbf{C}_1'\otimes\mathbf{C}_2' \\ \downarrow & & \downarrow \\ \mathbf{C}_1\underset{\mathbf{A}}{\otimes}\mathbf{C}_2 & \longrightarrow & \mathbf{C}_1'\underset{\mathbf{A}}{\otimes}\mathbf{C}_2'. \end{array} \tag{8.5}$$

Consider the right adjoint functors

$$\mathbf{C}_1\underset{\mathbf{A}}{\otimes}\mathbf{C}_2 \to \mathbf{C}_1\otimes\mathbf{C}_2 \text{ and } \mathbf{C}_1'\underset{\mathbf{A}}{\otimes}\mathbf{C}_2' \to \mathbf{C}_1'\otimes\mathbf{C}_2'.$$

Assume now that the functors ϕ_1 and ϕ_2 have the property that they admit left adjoints that also commute with the actions of $\mathbf{A}$. In this case the following diagram

$$\begin{array}{ccc} \mathbf{C}_1 \otimes \mathbf{C}_2 & \longrightarrow & \mathbf{C}'_1 \otimes \mathbf{C}'_2 \\ \uparrow & & \uparrow \\ \mathbf{C}_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}_2 & \longrightarrow & \mathbf{C}'_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}'_2. \end{array} \tag{8.6}$$

commutes as well. Hence, if we denote by N and N' the monads

$$\mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{C}_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}_2 \to \mathbf{C}_1 \otimes \mathbf{C}_2 \text{ and } \mathbf{C}'_1 \otimes \mathbf{C}'_2 \to \mathbf{C}'_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}'_2 \to \mathbf{C}'_1 \otimes \mathbf{C}'_2,$$

respectively, we obtain a natural isomorphism

$$(\phi_1 \otimes \phi_2) \circ N \simeq N' \circ (\phi_1 \otimes \phi_2), \tag{8.7}$$

as functors

$$\mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{C}'_1 \otimes \mathbf{C}'_2.$$

In addition, we note the diagram (8.6) commutes and isomorphism (8.3.6) holds also if ϕ_1 and ϕ_2 admit fully faithful right adjoints (i.e., $\mathbf{C}'_i$ is a localization of $\mathbf{C}_i$ for $i = 1, 2$).

8.3.8. We apply the above general discussion in the following circumstances. Throughout, we'll take $\mathbf{A} = D(\operatorname{Ran}(X, \Lambda^{pos}))$ and $\mathbf{C}_1 = \mathbf{C}'_1 = \operatorname{Vect}$, and ϕ_1 to be the identity map.

Consider now the following category:

$$D(\overline{\operatorname{Bun}}_B^{\mu,\leq}) := D\left(\bigcup_{\lambda \in \Lambda^{pos}} \operatorname{Bun}_B^{\mu-\lambda,\leq\lambda}\right),$$

where

$$\operatorname{Bun}_B^{\mu-\lambda,\leq\lambda} \overset{\jmath^\lambda}{\hookrightarrow} \overline{\operatorname{Bun}}_B^{\mu-\lambda}$$

is the open substack corresponding to the condition that the total degeneration is $\leq \lambda$. For every λ, the map ι^λ defines a closed embedding:

$$X^\lambda \times \operatorname{Bun}_B^\mu \to \operatorname{Bun}_B^{\mu-\lambda,\leq\lambda}.$$

We have a natural action of $D(\operatorname{Ran}(X, \Lambda^{pos}))$ on $D(\overline{\operatorname{Bun}}_B^{\mu,\leq})$. Let $\operatorname{Av}^{\leq}$ denote the corresponding functor

$$D(\overline{\operatorname{Bun}}_B^{\mu,\leq}) \to \operatorname{Vect} \underset{D(\operatorname{Ran}(X,\Lambda^{pos}))}{\otimes} D(\overline{\operatorname{Bun}}_B^{\mu,\leq}) \to D(\overline{\operatorname{Bun}}_B^{\mu,\leq}).$$

In the setting of Sect. 8.3.7, let's first take $\mathbf{C}_2 := D(\overline{\operatorname{Bun}}_B^{\mu,\leq})$, $\mathbf{C}'_2 := \mathbf{ind}^{\operatorname{Ran}}(D(\operatorname{Bun}_B^\mu))$ and ϕ_2 to be the functor $\phi^{\leq}$ given by

$$\{\mathcal{F}^{\mu-\lambda}\} \mapsto \{\iota^{\lambda !}(\mathcal{F})\}.$$

As in the case of the functor ϕ, it is easy to see that the functor $\phi^{\leq}$ commutes with the action of $D(\operatorname{Ran}(X, \Lambda^{pos}))$. However, unlike ϕ, the functor $\phi^{\leq}$ admits a left adjoint, given by

$$\{\mathcal{F}^\lambda\} \mapsto \{\iota^\lambda_!(\mathcal{F}^\lambda)\},$$

which is easily seen to commute with the action of $D(\operatorname{Ran}(X, \Lambda^{pos}))$. Hence, we obtain that the corresponding isomorphism (8.7) holds, i.e.,

$$\phi^{\leq} \circ \operatorname{Av}^{\leq} \simeq \operatorname{Av}_{\mathbf{ind}} \circ \phi^{\leq}.$$

We shall now apply the formalism of Sect. 8.3.7 again, this time to

$$\mathbf{C}_2 := D(\overline{\mathrm{Bun}}_B) \text{ and } \mathbf{C}'_2 := D(\overline{\mathrm{Bun}}_B^{\mu,\leq}),$$

with the functor ϕ_2 being $\jmath^{\leq !}$ given by definition by

$$\mathcal{F} \mapsto \{\jmath^{\lambda !}(\mathcal{F})\}.$$

In this case, the isomorphism (8.7) holds again, since $D(\overline{\mathrm{Bun}}_B^{\mu,\leq})$ is a localization of $D(\overline{\mathrm{Bun}}_B)$. So,

$$\jmath^! \circ {}' \mathrm{Av} \simeq \mathrm{Av}^{\leq} \circ \jmath^{\leq !}.$$

Together, we obtain that

$$\phi^{\leq} \circ \jmath^{\leq !} \circ {}' \mathrm{Av} \simeq \mathrm{Av}_{\mathbf{ind}} \circ \phi^{\leq} \circ \jmath^{\leq !}.$$

Finally, it remains to observe that $\phi \simeq \phi^{\leq} \circ \jmath^!$.

□(Proposition 8.3.6 and Theorem 8.3.2)

8.4. **Verdier duality on** Bun_B^{rat}**.**

8.4.1. Let's fix an element $\lambda \in \Lambda$, and consider a closed substack of $\overline{\mathrm{Bun}}_B$, denoted by ${}^{\geq\lambda}\overline{\mathrm{Bun}}_B$, whose intersection with every $\overline{\mathrm{Bun}}_B^{\mu}$ equals

$$\bigcup_{\lambda' \geq \lambda, \mu} \mathrm{Im}\left(\bar{\imath}^{\lambda'-\mu}(\overline{\mathrm{Bun}}_B^{\lambda'})\right).$$

In other words, the intersection

$${}^{\geq\lambda}\overline{\mathrm{Bun}}_B \cap \overline{\mathrm{Bun}}_B^{\mu}$$

is empty unless $\mu - \lambda$ belongs to the coroot lattice, and in the latter case is the union of the strata

$$X^{\nu} \times \mathrm{Bun}_B^{\lambda'}, \;\; \lambda' \geq \lambda, \nu \in \Lambda^{pos}.$$

It is clear that the substack ${}^{\geq\lambda}\overline{\mathrm{Bun}}_B$ is stable with respect to the correspondence $\mathcal{H}$. Therefore, the category $D({}^{\geq\lambda}\mathrm{Bun}_B^{rat})$ makes sense. This is a full subcategory of $D(\mathrm{Bun}_B^{rat})$, and its embedding admits a right adjoint. We have:

$$D({}^{\geq\lambda'}\mathrm{Bun}_B^{rat}) \subset D({}^{\geq\lambda}\mathrm{Bun}_B^{rat})$$

whenever $\lambda' \geq \lambda$. It is clear that the entire $D(\mathrm{Bun}_B^{rat})$ is the direct limit of $D({}^{\geq\lambda}\mathrm{Bun}_B^{rat})$'s as λ ranges over Λ.

8.4.2. For two elements $\lambda_1 \leq \lambda_2$, consider the open substack

$${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \subset {}^{\geq\lambda_1}\overline{\mathrm{Bun}}_B$$

equal to

$${}^{\geq\lambda_1}\overline{\mathrm{Bun}}_B - \bigcup_{\lambda \notin \lambda_2 - \Lambda^{pos}} {}^{\geq\lambda}\overline{\mathrm{Bun}}_B.$$

In other words, for a coweight μ, the intersection

$${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \cap \overline{\mathrm{Bun}}_B^{\mu}$$

is the union of the strata

$$X^{\nu} \times \mathrm{Bun}_B^{\lambda'}, \;\; \lambda_2 \geq \lambda' \geq \lambda_1, \nu \in \Lambda^{pos}.$$

The stack ${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B$ is stable with respect to $\mathcal{H}$, and we can consider the corresponding category $D({}^{\leq\lambda_2,\geq\lambda_1}\mathrm{Bun}_B^{rat})$. In fact, $D({}^{\leq\lambda_2,\geq\lambda_1}\mathrm{Bun}_B^{rat})$ identifies with a localization of $D({}^{\geq\lambda_1}\mathrm{Bun}_B^{rat})$ with respect to the subcategory generated by

$$D({}^{\geq\lambda}\mathrm{Bun}_B^{rat}) \text{ with } \lambda \geq \lambda_2, \text{ but } \lambda \notin \lambda_2 - \Lambda^{pos}.$$

8.4.3. The stack ${}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B$ is a disjoint union of stacks of finite type. Therefore, Verdier duality identifies $D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B)$ with its own dual. Let

$$\mathrm{ev}_{\overline{\mathrm{Bun}}_B} : D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \otimes D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \to \mathrm{Vect}$$

denote the corresponding pairing.

Proposition-Construction 8.4.4.

(1) *The functor*

$$D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \otimes D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat}) \to D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \otimes D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \overset{\mathrm{ev}_{\overline{\mathrm{Bun}}_B}}{\longrightarrow} \mathrm{Vect}$$

canonically factors through a functor

$$\mathrm{ev}_{\mathrm{Bun}_B^{rat}} : D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat}) \otimes D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat}) \to \mathrm{Vect}$$

via

$$\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}} : D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \to D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat}).$$

(2) *The resulting pairing* $\mathrm{ev}_{\mathrm{Bun}_B^{rat}}$ *identifies* $D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat})$ *with its own dual in such a way that the diagram*

$$D({}^{\leq \lambda_2, \geq \lambda_1}\overline{\mathrm{Bun}}_B) \rightleftarrows D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat})$$

is self-dual.

Proof. Consider the following framework. Let $\mathbf{A}$ be an augmented monoidal category acting on $\mathbf{C}$. We assume that the functors

$$\mathbf{A} \otimes \mathbf{A} \to \mathbf{A} \text{ and } \mathbf{A} \otimes \mathbf{C} \to \mathbf{C}$$

admit right adjoints. Assume now have an identification $\mathbf{A}^\vee \simeq \mathbf{A}$, such that the dual of the map $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ identifies with its right adjoint. Let $\mathbf{C}^\vee$ be the dual category of $\mathbf{C}$. We define the action $\mathbf{A} \otimes \mathbf{C}^\vee \to \mathbf{C}^\vee$ to be dual map to the right adjoint of $\mathbf{A} \otimes \mathbf{C} \to \mathbf{C}$. Under such circumstances, it is easy to see that the dual of $\mathbf{C}_{\mathbf{A}}$ identifies with $\mathbf{C}^\vee_{\mathbf{A}}$.

We apply the above discussion to $\mathbf{A} := D(\mathrm{Ran}(X, \Lambda^{pos}))$ and $\mathbf{C} := D({}^{\leq \lambda_2, \geq \lambda_1}\,\mathrm{Bun}_B^{rat})$. Verdier duality defines identifications $\mathbf{A} \simeq \mathbf{A}^\vee$ and $\mathbf{C} \simeq \mathbf{C}^\vee$, such that the resulting $\mathbf{A}$-action on $\mathbf{C}^\vee$ coincides with the original action on $\mathbf{C}$.

Hence, the assertion of the proposition follows from the above general principle.

□

9. Eisenstein series, take III

9.1. **The "stack" $\overline{\mathrm{Bun}}_T$.** The projection $\mathfrak{q} : \mathrm{Bun}_B \to \mathrm{Bun}_T$ has contractible fibers. Hence the functor $\mathfrak{q}^*$ is a fully faithful embedding. In this subsection we'll define the category $D(\overline{\mathrm{Bun}}_T)$ which will play a similar role vis-a-vis $D(\overline{\mathrm{Bun}}_B)$.

9.1.1. We let $D(\overline{\mathrm{Bun}}_T)$ be the full subcategory of $D(\overline{\mathrm{Bun}}_B)$ consisting of objects $\mathcal{F}$ satisfying the following: for each $\lambda \in \Lambda^{pos}$,

$$(\iota^\lambda)^!(\mathcal{F}) \in D(X^\lambda \times \mathrm{Bun}_B)$$

belongs to the essential image of

$$(\mathrm{id} \times \mathfrak{p})^! : D(X^\lambda \times \mathrm{Bun}_T) \to D(X^\lambda \times \mathrm{Bun}_B).$$

We claim:

Proposition 9.1.2.
(1) *For* $\mathcal{F} \in D(X^\lambda \times \mathrm{Bun}_T)$ *the object*

$$(\iota^\lambda)_! \left((\mathrm{id} \times \mathfrak{q})^!(\mathcal{F})\right) \in D(\overline{\mathrm{Bun}}_B)$$

is defined, and belongs to $D(\overline{\mathrm{Bun}}_T)$.
(2) *Objects as above generate* $D(\overline{\mathrm{Bun}}_T)$.

Proof. First, we need to show that the object as in (1) is indeed defined. Since $\iota^\lambda = \bar{\iota}^\lambda \circ (\mathrm{id} \times \jmath)$, it suffices to show that

$$(\mathrm{id} \times \jmath)_! \circ (\mathrm{id} \times \mathfrak{q})^! : D(X^\lambda \times \mathrm{Bun}_T) \to D(X^\lambda \times \overline{\mathrm{Bun}}_B)$$

is well-defined. However, since the latter categories are obtained by tensoring up with $D(X^\lambda)$, it is sufficient to show that

$$\jmath_! \circ \mathfrak{q}^! : D(\mathrm{Bun}_T) \to D(\overline{\mathrm{Bun}}_B)$$

is well-defined. However, this follows from the ULA property of $\jmath_!(K_{\mathrm{Bun}_B})$ with respect to $\mathfrak{q}$. Indeed, we have:

$$\jmath_! \circ \mathfrak{q}^!(\mathcal{F}) \simeq \jmath_!(K_{\mathrm{Bun}_B}) \overset{!}{\otimes} \bar{\mathfrak{q}}^!(\mathcal{F}).$$

To prove that $(\iota^\lambda)_! \left((\mathrm{id} \times \mathfrak{q})^!(\mathcal{F})\right)$ belongs to $D(\overline{\mathrm{Bun}}_T)$ we need to calculate the composition

$$(\iota^\mu)^! \circ (\iota^\lambda)_! \left((\mathrm{id} \times \mathfrak{q})^!(\mathcal{F})\right) \simeq (\iota^\mu)^! \circ (\bar{\iota}^\lambda)_* \left((\mathrm{id} \times \bar{\mathfrak{q}})^!(\mathcal{F}) \overset{!}{\otimes} (\mathrm{id} \times \jmath)_!(K_{X^\lambda \times \mathrm{Bun}_B})\right). \tag{9.1}$$

Note that we have a Cartesian square:

$$\begin{array}{ccc} X^\mu \times \mathrm{Bun}_B & \xrightarrow{\iota^\mu} & \overline{\mathrm{Bun}}_B \\ {\scriptstyle \mathrm{add}_{\lambda,\lambda-\mu} \times \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \bar{\iota}^\lambda} \\ X^\lambda \times X^{\mu-\lambda} \times \mathrm{Bun}_B & \xrightarrow{\mathrm{id} \times \iota^{\mu-\lambda}} & X^\lambda \times \overline{\mathrm{Bun}}_B \end{array}$$

The expression in (9.1) can thus be rewritten as:

$$(\mathrm{add}_{\lambda,\lambda-\mu} \times \mathrm{id})_* \circ (\mathrm{id} \times \iota^{\mu-\lambda})^! \left((\mathrm{id} \times \bar{\mathfrak{q}})^!(\mathcal{F}) \overset{!}{\otimes} (\mathrm{id} \times \jmath)_!(K_{X^\lambda \times \mathrm{Bun}_B})\right).$$

It would be sufficient to show that

$$(\mathrm{id} \times \iota^{\mu-\lambda})^! \left((\mathrm{id} \times \bar{\mathfrak{q}})^!(\mathcal{F}) \overset{!}{\otimes} (\mathrm{id} \times \jmath)_!(K_{X^\lambda \times \mathrm{Bun}_B})\right)$$

belongs to the essential image of $D(X^\lambda \times X^{\mu-\lambda} \times \mathrm{Bun}_T)$ under. $(\mathrm{id}_{X^\lambda \times X^{\mu-\lambda}} \times \mathfrak{q})^!$. Again, the factor X^λ comes in along for the ride, so we have to show that for $\nu \in \Lambda^{pos}$,

$$(\iota^\nu)^! \left(\bar{\mathfrak{q}}^!(\mathcal{F}) \overset{!}{\otimes} \jmath_!(K_{\mathrm{Bun}_B})\right)$$

belongs to the essential image of $(\mathrm{id}_{X^\nu} \times \mathfrak{q})^!$. However, the latter expression is isomorphic to

$$(\mathrm{id}_{X^\nu} \times \mathfrak{q})^! \left(\tilde{\Omega}(\check{\mathfrak{n}}_X^-)^\nu \overset{!}{\otimes} \mathrm{mult}_\nu^!(\mathcal{F})\right).$$

The second point of the proposition is tautological from the first one.

□

9.2. **The "stack"** Bun_T^{rat}**.**

9.2.1. Let $D(\mathrm{Bun}_T^{rat})$ denote the full subcategory of $D(\mathrm{Bun}_B^{rat})$ equal to the preimage of $D(\mathrm{Bun}) \subset D(\overline{\mathrm{Bun}}_B)$ under the forgetful functor.

Lemma 9.2.2. *The functor*

$$\mathbf{ind}_{\overline{\mathrm{Bun}}_B}^{\mathrm{Bun}_B^{rat}} : D(\overline{\mathrm{Bun}}_B) \to D(\mathrm{Bun}_B^{rat})$$

sends $D(\overline{\mathrm{Bun}}_T)$ *to* $D(\mathrm{Bun}_T^{rat})$.

Proof. By definition, we need to show that the monad Av sends $D(\overline{\mathrm{Bun}}_T)$ to itself, i.e., that for an object $\mathcal{F} \in D(\overline{\mathrm{Bun}}_T)$, the object $(\iota^\lambda)^! \circ \mathrm{Av}(\mathcal{F})$ belongs to the essential image of $(\mathrm{id} \times \mathfrak{q})^!$. However,

$$(\iota^\lambda)^! \circ \mathrm{Av}(\mathcal{F}) \simeq (\pi^\lambda \times \mathrm{id})^! \circ \jmath^! \circ \mathrm{Av}(\mathcal{F}),$$

so it is enough to show that $\jmath^! \circ \mathrm{Av}(\mathcal{F})$ belongs to the essential image of $\mathfrak{q}^!$. Now, the result follows from the explicit description of $\jmath^! \circ \mathrm{Av}$ given in Sect. 8.3.4.

□

We'll denote by $\mathbf{ind}_{\overline{\mathrm{Bun}}_T}^{\mathrm{Bun}_T^{rat}}$ the resulting functor

$$D(\overline{\mathrm{Bun}}_T) \to D(\mathrm{Bun}_T^{rat}).$$

9.2.3. One can view the category $D(\mathrm{Bun}_T^{rat})$ as the universal source of the Eisenstein series functor by means of

$$D(\mathrm{Bun}_T^{rat}) \hookrightarrow D(\mathrm{Bun}_B^{rat}) \xrightarrow{\mathfrak{p}_*^{rat}} D(\mathrm{Bun}_G).$$

We shall now investigate its relationship with the functor $\mathrm{Eis}_!^{ult}$. Namely, we'll establish:

Proposition 9.2.4. *The functor*

$$\mathcal{F} \mapsto \mathbf{ind}_{\overline{\mathrm{Bun}}_T}^{\mathrm{Bun}_T^{rat}} (\jmath_! \circ \mathfrak{q}^!(\mathcal{F})) : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T^{rat})$$

canonically extends to a functor

$$(\mathfrak{q}^{rat})^! : \tilde{\Omega}(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T^{rat}),$$

and the latter is an equivalence of categories. Moreover:

(1) *The forgetful functor*

$$\tilde{\Omega}(\check{\mathfrak{n}}_X^-)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_T)$$

identifies with

$$D(\mathrm{Bun}_T^{rat}) \to D(\overline{\mathrm{Bun}}_T) \xrightarrow{\jmath^!} D(\mathrm{Bun}_T).$$

(2) *The functor* $\mathfrak{p}_*^{rat}$ *identifies with* $\mathrm{Eis}_!^{ult}$.

9.2.5. *Proof of Proposition 9.2.4.* All we need to show is that the monad on $D(\mathrm{Bun}_T)$ given by

$$\mathcal{F} \mapsto \jmath^! \circ \mathrm{Av} \circ \jmath_! \circ \mathfrak{q}^!(\mathcal{F})$$

identifies with

$$\mathcal{F} \mapsto \mathfrak{q}^! \circ (\tilde{\Omega}(\check{\mathfrak{n}}_X^-) \circ \mathcal{F}).$$

However, this again follows immediately from the description of $\jmath^! \circ \mathrm{Av}$ in Sect. 8.3.4.

□

9.3. An equivalence with Bun_G.

9.3.1. Let $\lambda_1 \leq \lambda_2$ be a pair of weights. Consider the corresponding categories

$$D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B) \text{ and } D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_B^{rat}).$$

By the same taken as in Sect. 9.1 and Sect. 9.2, we define the corresponding full subcategories

$$D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_T) \text{ and } D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_T^{rat}).$$

It is clear that the Verdier duality functor (defined on the subcategory of compact objects of $D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B)$) sends $D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_T)$ to itself. Indeed, it's enough to check this fact on compact generators provided by Proposition 9.1.2. Hence, the same is true for the subcategory

$$\left(D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_T^{rat})\right)^c \subset \left(D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_B^{rat})\right)^c.$$

9.3.2. Recall that for every regular dominant weight λ, there exists a canonically defined locally closed substack

$$\mathrm{Bun}_G^\lambda \subset \mathrm{Bun}_G\,.$$

In fact, the projection $\mathfrak{p}^\lambda$ maps Bun_B^λ isomorphically onto Bun_G^λ.

Let now $\lambda_1 \leq \lambda_2$ be as above, and assume that they are deep enough in the dominant chamber so that every weight λ satisfying

$$\lambda_1 \leq \lambda \leq \lambda_2$$

is dominant and regular. In this case, the union of strata

$${}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G := \underset{\lambda_1\leq\lambda\leq\lambda_2}{\cup}\ \mathrm{Bun}_G^\lambda$$

is also a locally closed substack of Bun_G.

The preimage of ${}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G$ under $\overline{\mathfrak{p}}$ equals the substack

$${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \subset \overline{\mathrm{Bun}}_B.$$

Hence, the functor $\overline{\mathfrak{p}}_!$ gives rise to a well-defined functor

$$D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B) \to D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G),$$

which canonically factors through a functor

$$\mathfrak{p}_*^{rat} : D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_B^{rat}) \to D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G).$$

9.3.3. We are going to prove:

Theorem 9.3.4. *Assume that λ_1, λ_2 are deep enough the dominant chamber so that every weight λ satisfying*

$$\lambda_1 \leq \lambda \leq \lambda_2$$

satisfies also

$$\langle \lambda, \check{\alpha}_i \rangle > 2g-2,$$

for every simple root α_i. In this case the composed functor

$$D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_T^{rat}) \hookrightarrow D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_B^{rat}) \overset{\mathfrak{p}_*^{rat}}{\to} D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G)$$

is an equivalence.

The rest of this section is devoted to the proof of this theorem.

9.3.5. First, we note that under the assumptions on λ_1, λ_2 for every λ between them the map

$$\mathfrak{q}^! : D(\mathrm{Bun}_T^\lambda) \to D(\mathrm{Bun}_B^\lambda)$$

is an equivalence. Hence, the functors

$$D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_T) \hookrightarrow D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B)$$

and

$$D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_T^{rat}) \hookrightarrow D({}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_B^{rat})$$

are equivalences as well.

This implies that the essential image of the functor in the theorem generates the target category. Therefore, it remains to establish fully faithfulness. The latter is, in turn, equivalent to the fact that the monad

$$\overline{\mathfrak{p}}^! \circ \overline{\mathfrak{p}}_! : D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B) \to D({}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B) \tag{9.2}$$

is canonically isomorphic to the monad Av.

The monad in the LHS of (9.2) is given by pull-push along the diagram

$${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \leftarrow {}^{\leq\lambda_2,\geq\lambda_1}\,\overline{\mathrm{Bun}}_B \underset{{}^{\leq\lambda_2,\geq\lambda_1}\,\mathrm{Bun}_G}{\times} {}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \to {}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B.$$

The monad in the RHS of (9.2) is given by pull-push along the diagram

$${}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \leftarrow {}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \underset{\overline{\mathrm{Bun}}_B}{\times} \mathcal{H} \underset{\overline{\mathrm{Bun}}_B}{\times} {}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B \to {}^{\leq\lambda_2,\geq\lambda_1}\overline{\mathrm{Bun}}_B.$$

We have a canonical closed embedding from the latter diagram to the former, and it remains to show that this embedding is in fact an isomorphism. This is equivalent to the following well-known assertion:

Lemma 9.3.6. *Let P_G be an unstable G-bundle, equipped with two B-reductions both of which have dominant Chern classes. Then these two reductions coincide.*

10. The local nature of $\tilde{\Omega}(\check{\mathfrak{n}}_X^-)$

10.1. **The Hecke stack.**

10.1.1. For a reductive group G let Sat_G be the Satake category, i.e., the (slightly renormalized version of) category $D(\mathrm{Hecke}_{G,x,loc})$, [3] where $\mathrm{Hecke}_{G,x,loc}$ is the local Hecke stack at x, i.e.,

$$\mathrm{Hecke}_{G,x,loc} := \mathrm{Bun}_G(\mathcal{D}_x) \underset{\mathrm{Bun}_G(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathrm{Bun}_G(\mathcal{D}_x),$$

where $\mathcal{D}_x$ and $\overset{\circ}{\mathcal{D}}_x$ are the formal and formal punctured discs around x, respectively.

[3] It's crucial for what follows that we consider the "full" i.e., derived category and not the heart of its t-structure.

We can also consider the global Hecke stack

$$\begin{array}{ccccc} & & \mathrm{Hecke}_{G,x,glob} & & \\ & \swarrow^{\overleftarrow{h}_{glob}} & \downarrow & \searrow^{\overrightarrow{h}_{glob}} & \\ \mathrm{Bun}_G & & \downarrow & & \mathrm{Bun}_G \\ \downarrow & & \mathrm{Hecke}_{G,x,loc} & & \downarrow \\ \downarrow & \swarrow^{\overleftarrow{h}_{loc}} & & \searrow^{\overrightarrow{h}_{loc}} & \downarrow \\ \mathrm{Bun}_G(\mathcal{D}_x) & & & & \mathrm{Bun}_G(\mathcal{D}_x) \end{array}$$

There is a natural functor

$$\mathrm{Sat}_{G,x} \to D(\mathrm{Hecke}_{G,x,glob}).$$

10.1.2. We regard the assignment $x \mapsto \mathrm{Sat}_{G,x}$ as a chiral monoidal category, which we denote by Sat_G. In particular, the category $\mathrm{Sat}_G(\mathrm{Ran}(X))$ makes sense, and there is a functor

$$\mathrm{Sat}_G(\mathrm{Ran}(X)) \to D(\mathrm{Hecke}_{G,glob,\mathrm{Ran}(X)}), \tag{10.1}$$

where $\mathrm{Hecke}_{G,glob,\mathrm{Ran}(X)}$ is the Ran version of the global Hecke stack.

10.1.3. We apply this to the group being T. For $\lambda \in \Lambda^{pos}$ we can regard the product $X^\lambda \times \mathrm{Bun}_T$ with its two projections

$$\mathrm{Bun}_T \overset{\mathrm{mult}_\lambda}{\longleftarrow} X^\lambda \times \mathrm{Bun}_T \overset{\pi^\lambda \times \mathrm{id}_{\mathrm{Bun}_T}}{\longrightarrow} \mathrm{Bun}_T$$

as a closed substack of the λ-graded component of $\mathrm{Hecke}_{T,glob,\mathrm{Ran}(X)}$.

In what follows, we'll describe a chiral algebra in Sat_T, such that the corresponding object of $\mathrm{Sat}_G(\mathrm{Ran}(X))$ conjecturally gives rise to $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$ via (10.1).

10.2. **Geometric Satake.**

10.2.1. For a group G_1, let us denote by $\mathrm{Hecke}_{G_1,x,loc,spec}$ the local spectral Hecke (DG)-stack, i.e.,

$$\mathrm{Hecke}_{G_1,x,loc,spec} := \mathrm{LocSys}_{G_1}(\mathcal{D}_x) \underset{\mathrm{LocSys}_{G_1}(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{G_1}(\mathcal{D}_x),$$

and let

$$\mathrm{Sat}_{G_1,x,spec} := \mathrm{Ind}\,\mathsf{Coh}(\mathrm{Hecke}_{G_1,x,loc,spec}).$$

The assignment $x \mapsto \mathrm{Sat}_{G_1,x,spec}$ forms a monoidal chiral category that we denote $\mathrm{Sat}_{G_1,spec}$.

Recall that for a reductive group G we have a canonical equivalence of monoidal chiral categories.

$$\mathrm{Sat}_G \simeq \mathrm{Sat}_{\check{G},spec}\,. \tag{10.2}$$

10.2.2. The following general construction will be used in the sequel. Let $G_1 \to G_2$ be a group homomorphism. Consider the following relative version of the local spectral Hecke stack:

$$\mathrm{Hecke}_{G_1,G_2,x,loc,spec} := \mathrm{LocSys}_{G_1}(\mathcal{D}_x) \underset{\left(\mathrm{LocSys}_{G_2}(\mathcal{D}_x) \underset{\mathrm{LocSys}_{G_2}(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{G_1}(\overset{\circ}{\mathcal{D}}_x)\right)}{\times} \mathrm{LocSys}_{G_1}(\mathcal{D}_x).$$

endowed with the corresponding maps

$$\begin{array}{ccc} & \mathrm{Hecke}_{G_1,G_2,x,loc,spec} & \\ {\scriptstyle \overleftarrow{h}_{rel,loc}} \swarrow & & \searrow {\scriptstyle \overrightarrow{h}_{rel,loc}} \\ \mathrm{LocSys}_{G_1}(\mathcal{D}_x) & & \mathrm{LocSys}_{G_1}(\mathcal{D}_x). \end{array}$$

The assignment

$$x \mapsto \mathrm{Ind}\,\mathsf{Coh}(\mathrm{Hecke}_{G_1,G_2,x,loc,spec})$$

also forms a monoidal chiral category, which we denote $\mathrm{Sat}_{G_1,G_2,spec}$.

We have a natural forgetful morphism

$$\mathrm{Hecke}_{G_1,G_2,x,loc,spec} \to \mathrm{Hecke}_{G_1,x,loc,spec},$$

which defines a monoidal chiral functor

$$\mathrm{Sat}_{G_1,G_2,spec} \to \mathrm{Sat}_{G_1,spec}\,.$$

We let

$$\mathcal{A}_{G_1,G_2} \in \mathrm{Sat}_{G_1,spec}$$

be the chiral algebra equal to image under the above functor of the chiral algebra in $\mathrm{Sat}_{G_1,G_2,spec}$, whose value on each $\mathrm{Hecke}_{G_1,G_2,x,loc,spec}$ is the (relative with respect to $\overleftarrow{h}_{rel,loc}$) of the dualizing sheaf on this (DG)-stack. By construction, this chiral algebra carries an associative algebra structure with respect to the monoidal structure on $\mathrm{Sat}_{G_1,spec}$. To disambiguate the notation, we'll denote by $\mathcal{A}_{G_1,G_2}(\mathrm{Ran}(X))$ the corresponding factorizable object of $\mathrm{Sat}_{G_1,spec}(\mathrm{Ran}(X))$.

10.3. **The conjecture.**

10.3.1. We apply the discussion of Sect. 10.2.2 to $G_1 = \check{B}$ and $\check{G}_2 = G$. We obtain the chiral algebra

$$\mathcal{A}_{\check{B},\check{G}} \in \mathrm{Sat}_{\check{B},spec}\,.$$

Note that the (DG)-stack

$$\mathrm{LocSys}_{\check{G}}(\mathcal{D}_x) \underset{\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{B}}(\overset{\circ}{\mathcal{D}}_x)$$

identifies with the derived Springer fiber over $0 \in \check{\mathfrak{g}}$, denoted Spr_0.

10.3.2. Consider now the canonical map

$$\mathfrak{q}_{\mathrm{Hecke},x,spec} : \mathrm{Hecke}_{\check{B},x,loc,spec} \to \mathrm{Hecke}_{\check{T},x,loc,spec},$$

induced by the projection $\check{B} \to \check{T}$. Direct image defines a chiral functor

$$(\mathfrak{q}_{\mathrm{Hecke},spec})_* : \mathrm{Sat}_{\check{B},spec} \to \mathrm{Sat}_{\check{T},spec}. \tag{10.3}$$

The above functor (10.3) has an additional structure of lax monoidal functor.

We let

$$\mathcal{A}_{\check{T},\check{B},\check{G}} \in \mathrm{Sat}_{\check{T},spec}$$

be the image of $\mathcal{A}_{\check{B},\check{G}}$ under the above functor $(\mathfrak{q}_{\mathrm{Hecke},spec})_*$.

By construction, this is a chiral algebra in $\mathrm{Sat}_{\check{T},spec}$ with an additional structure of associative algebra with respect to the monoidal structure on $\mathrm{Sat}_{\check{T},spec}$.

10.3.3. We propose:

Conjecture 10.3.4. *The pull-back by means of*

$$(\mathrm{id}_{X^\lambda} \boxtimes -\rho(\omega_X)) : X^\lambda \times \mathrm{Bun}_T \to X^\lambda \times \mathrm{Bun}_T$$

of $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$*, when viewed as an associative algebra in* $D(\mathrm{Hecke}_{T,glob,\mathrm{Ran}(X)})$ *arises via* (10.1) *from* $\mathcal{A}_{\check{T},\check{B},\check{G}}(\mathrm{Ran}(X))$ *via* (10.2).

This conjecture doesn't seem very far-fetched. We expect that it results from the description of the local category

$$D(G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/N^{\rho(\omega_{\mathcal{D}_x})}(\mathcal{K}_x))$$

as $\mathsf{QCoh}(\mathrm{Spr}_0)$ a la Bezrukavnikov. Here $N^{\rho(\omega_{\mathcal{D}_x})}$ is the twist of N, veiwed as a groop-scheme over $\mathcal{D}_x$ by the T-torsor $\rho(\omega_X)$.

In the next two sections we'll explain the role that Conjecture 10.3.4 plays in viewing geometric Eisenstein series within geometric Langlands correspondence.

11. Eisenstein series and Langlands correspondence

This section is devoted to our goal 1, namely the analysis of Eisenstein series in the framework of Langlands correspondence.

11.1. Spectral Eisenstein series.

11.1.1. Let $\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{Ind}\,\mathsf{Coh}(\mathrm{LocSys}_{\check{G}})$ be the category introduced in [Sum]. Consider the diagram

$$\begin{array}{ccc} & \mathrm{LocSys}_{\check{G}} & \\ {\scriptstyle \mathfrak{p}_{spec}} \swarrow & & \searrow {\scriptstyle \mathfrak{q}_{spec}} \\ \mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{B}} \end{array}$$

As was explained in *loc. cit.*, the functor

$$\mathrm{Eis}_{spec} := \mathfrak{p}_{spec*} \circ \mathfrak{q}^*_{spec}$$

is well-defined as a functor

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{Coh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}),$$

and hence gives rise to (the same named functor)

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}).$$

The functor Eis_{spec} admits a right adjoint CT_{spec} given by

$$\mathrm{CT}_{spec} := \mathfrak{q}_{spec*} \circ \mathfrak{p}^!_{spec}.$$

Let Γ_{spec} denote the monad on $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$ given by $\mathrm{CT}_{spec} \circ \mathrm{Eis}_{spec}$.

11.1.2. The morphism

$$\mathfrak{p}_{spec} : \mathrm{LocSys}_{\check{B}} \to \mathrm{LocSys}_{\check{G}}$$

is easily seen to be proper. Let

$$\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}) \subset \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \tag{11.1}$$

be the full subcategory consisting of objects that vanish when localized on the complement to the image of $\mathfrak{p}_{spec}$. I.e., the functor Eis_{spec} factors through a functor

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}),$$

which we'll denote by the same symbol by a slight abuse of notation. The adjoint functor CT_{spec} factors through the right adjoint

$$\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}) \leftarrow \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}).$$

The following has been recently established by D. Arinkin:

Theorem 11.1.3. *The image of the functor* $(\mathfrak{p}_{spec})_*$ *(or, equivalently,* Eis_{spec}*) generates* $\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}})$.

11.1.4. Recall that our goal 5 stated in the introduction was to construct an equivalence

$$\Psi_{G,\mathrm{Eis}} : \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}) \to D(\mathrm{Bun}_G)_{\mathrm{Eis}},$$

which would make the following diagram

$$\begin{array}{ccc} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}) & \xrightarrow{\Psi_{G,\mathrm{Eis}}} & D(\mathrm{Bun}_G)_{\mathrm{Eis}} \\ {\scriptstyle \mathrm{Eis}_{spec}} \uparrow & & \uparrow {\scriptstyle \mathrm{Eis}_! \circ -\rho(\omega_X)} \\ \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) & \xrightarrow{\Psi_T} & D(\mathrm{Bun}_T) \end{array}$$

commute, where $-\rho(\omega_X)$ is the functor of shift by $-\rho(\omega_X) \in \mathrm{Bun}_T$. Equivalently, we'd like the following diagram to commute:

$$\begin{array}{ccc} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G},\mathrm{Eis}}) & \xrightarrow{\Psi_{G,\mathrm{Eis}}} & D(\mathrm{Bun}_G)_{\mathrm{Eis}} \\ {\scriptstyle (\mathfrak{p}_{spec})_*} \uparrow & & \uparrow {\scriptstyle \mathrm{Eis}^{int}_! \circ -\rho(\omega_X)} \\ \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) & \xrightarrow{\Psi_B} & \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)), \end{array}$$

where Ψ_B is the enhancement of the Fourier-Mukai transform Ψ_T obtained by considering modules over the corresponding monads.

Taking into account Theorem 11.1.3, the construction of such a functor amounts to establishing the following plausible conjecture:

Conjecture 11.1.5. *The Fourier-Mukai equivalence Ψ_T intertwines the two monads:*

$$\rho(\omega_X) \circ \Gamma \circ -\rho(\omega_X) : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T) \tag{11.2}$$

and

$$\Gamma_{spec} : \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}). \tag{11.3}$$

Unfortunately, we can't prove this conjecture at the moment. However, we'll able able to match certain parts of both sides.

11.2. Bruhat decomposition of Γ_{spec}.

11.2.1. Consider the diagram

$$\begin{array}{ccccccccc} & & & & \mathrm{LocSys}_{\check{B}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{B}} & & & & \\ & & & \swarrow^{\mathfrak{p}^l_{spec}} & & \searrow^{\mathfrak{p}^r_{spec}} & & & \\ & & \mathrm{LocSys}_{\check{B}} & & & & \mathrm{LocSys}_{\check{B}} & & \\ & \swarrow^{\mathfrak{q}_{spec}} & & \searrow^{\mathfrak{p}_{spec}} & & \swarrow^{\mathfrak{p}_{spec}} & & \searrow^{\mathfrak{q}_{spec}} & \\ \mathrm{LocSys}_{\check{T}} & & & & \mathrm{LocSys}_{\check{G}} & & & & \mathrm{LocSys}_{\check{T}} \end{array}$$

The functor Γ_{spec} is isomorphic to the composition

$$(\mathfrak{q}_{spec})_* \circ (\mathfrak{p}^r_{spec})_* \circ (\mathfrak{p}^l_{spec})^! \circ (\mathfrak{p}_{spec})^*.$$

11.2.2. For each $w \in W$ let

$$\left(\mathrm{LocSys}_{\check{B}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{B}}\right)_w \subset \left(\mathrm{LocSys}_{\check{B}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{B}}\right)^{\sim}_w$$

be the corresponding locally closed substack and its completion. (Our normalization is such that for $w = 1$, the substack $\left(\mathrm{LocSys}_{\check{B}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{B}}\right)_1$ is closed. Let us denote its two projections as in the diagram below:

$$\begin{array}{ccccc} & & \left(\mathrm{LocSys}_{\check{B}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{B}}\right)^{\sim}_w & & \\ & \swarrow^{\mathfrak{p}^l_{w,spec}} & & \searrow^{\mathfrak{p}^r_{w,spec}} & \\ \mathrm{LocSys}_{\check{B}} & & & & \mathrm{LocSys}_{\check{B}}\,. \end{array}$$

We define the corresponding functor

$$\Gamma_{w,spec} : \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$$

by

$$(\mathfrak{q}_{spec})_* \circ (\mathfrak{p}^r_{w,spec})_* \circ (\mathfrak{p}^l_{w,spec})^! \circ (\mathfrak{q}_{spec})^*. \tag{11.4}$$

11.2.3. The following seems to be a local (and hence, more tractable) case of Conjecture 11.1.5:

Conjecture 11.2.4. *Fourier-Mukai equivalence Ψ_T intertwines the functors*

$$\rho(\omega_X) \circ \Gamma_{w,spec} \circ -\rho(\omega_X) \text{ and } \Gamma_w.$$

Let us note that one case of this conjecture is easy, namely $w = w_0$. Indeed, in this case both functors are given by the action of w_0 on Bun_T.

In what follows we'll concentrate on "the most interesting" case of Conjecture 11.2.4, namely for $w = 1$, i.e.:

Conjecture 11.2.5. *Fourier-Mukai equivalence Ψ_T intertwines the monads*

$$\rho(\omega_X) \circ \Gamma_{1,spec} \circ -\rho(\omega_X) \text{ and } \Gamma_1.$$

We'll show that Conjecture 11.2.5 follows from Conjecture 10.3.4. It's plausible that a similar argument will show that Conjecture 11.2.4 also follows from Conjecture 10.3.4.

11.3. **"Proof" of Conjecture 11.2.5.**

11.3.1. Let's note that the monad

$$\mathfrak{p}^r_{1,spec})_* \circ (\mathfrak{p}^l_{1,spec})^! : \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}})$$

falls into the following general paradigm.

Let $\mathfrak{f} : Z_1 \to Z_2$ be a map between Artin (DG)-stacks of finite type. Consider the completion of the diagonal:

$$\begin{array}{ccccc} & & \left(Z_1 \underset{Z_2}{\times} Z_1\right)^{\sim} & & \\ & {}^{\prime}\mathfrak{f}^l \swarrow & & \searrow {}^{\prime}\mathfrak{f}^r & \\ Z_1 & & & & Z_1 \\ & \mathfrak{f} \searrow & & \swarrow \mathfrak{f} & \\ & & Z_2. & & \end{array}$$

Then we can consider the monad

$$\mathrm{Ind}\,\mathsf{Coh}(Z_1) \xrightarrow{({}^{\prime}\mathfrak{f}^l)^!} \mathrm{Ind}\,\mathsf{Coh}\left(\left(Z_1 \underset{Z_2}{\times} Z_1\right)^{\sim}\right) \xrightarrow{({}^{\prime}\mathfrak{f}^r)_*} \mathrm{Ind}\,\mathsf{Coh}(Z_1). \tag{11.5}$$

We'll denote it by $U(\Theta_{Z_1/Z_2})$ for the following reason: this monad is given by tensoring with the unversal enveloping D_{Z_1}-algebra of the relative tangent Lie algebroid Θ_{Z_1/Z_2}.

By construction, the functor $\mathfrak{f}_* : \mathrm{Ind}\,\mathsf{Coh}(Z_1) \to \mathrm{Ind}\,\mathsf{Coh}(Z_2)$ factors via the category $U(\Theta_{Z_1/Z_2})\text{-mod}(\mathrm{Ind}\,\mathsf{Coh}(Z_1))$.

Note that when $\mathfrak{f}$ is a closed embedding (or, more generally, radicial) the category of modules over this monad is, by the Barr-Beck theorem, equivalent to the full subcategory of $\mathrm{Ind}\,\mathsf{Coh}(Z_2)$ consisting of modules supported set-theoretically on Z_1.

In another extreme case, when $Z_2 = \mathrm{pt}$ and Z_1 is l.c.i., the category of modules over this monad is equivalent to that of right D-modules on Z_1.

Note that the composition

$$\operatorname{Ind}\mathsf{Coh}(Z_1) \overset{U(\Theta_{Z_1/Z_2})}{\longrightarrow} \operatorname{Ind}\mathsf{Coh}(Z_1) \to \mathsf{QCoh}(Z_1)$$

factors through the localization $\operatorname{Ind}\mathsf{Coh}(Z_1) \to \mathsf{QCoh}(Z_1)$. We'll abuse the notation slightly and use the same symbol $U(\Theta_{Z_1/Z_2})$ to denote the resulting monad $\mathsf{QCoh}(Z_1) \to \mathsf{QCoh}(Z_1)$, i.e., we have a commutative diagram:

$$\begin{array}{ccc} \operatorname{Ind}\mathsf{Coh}(Z_1) & \xrightarrow{U(\Theta_{Z_1/Z_2})} & \operatorname{Ind}\mathsf{Coh}(Z_1) \\ \downarrow & & \downarrow \\ \mathsf{QCoh}(Z_1) & \xrightarrow{U(\Theta_{Z_1/Z_2})} & \mathsf{QCoh}(Z_1). \end{array}$$

11.3.2. Let $\mathcal{Y}$ be a crystal of formally smooth schemes or Artin stacks over X. Along with $\mathcal{Y}$, we can consider the crystal of formal schemes/Artin stacks, denoted $\mathcal{Y}^{mer,form}$, whose fiber over $x \in X$ is the formal neighborhood of $\mathcal{Y}_x = \mathcal{Y}(\mathcal{D}_x)$ inside the functor $\mathcal{Y}(\overset{\circ}{\mathcal{D}}_x)$.

Consider the ind-stack

$$\operatorname{Hecke}_{\mathcal{Y},x,loc} := \mathcal{Y}(\mathcal{D}_x) \underset{\mathcal{Y}(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathcal{Y}(\mathcal{D}_x),$$

and category

$$\operatorname{Sat}_{\mathcal{Y},x} := \operatorname{Ind}\mathsf{Coh}\left(\operatorname{Hecke}_{\mathcal{Y},x,loc}\right).$$

The assignment $x \mapsto \operatorname{Sat}_{\mathcal{Y},x}$ forms a chiral category; moreover convolution defines on it a structure of monoidal chiral category, which we denote $\operatorname{Sat}_{\mathcal{Y}}$.

Note that the projections

$$\mathcal{Y}(\mathcal{D}_x) \overset{\overleftarrow{h}_{loc}}{\longleftarrow} \operatorname{Hecke}_{\mathcal{Y},x,loc} \overset{\overrightarrow{h}_{loc}}{\longrightarrow} \mathcal{Y}(\mathcal{D}_x)$$

and ind-finite. Let $\mathcal{A}_{\mathcal{Y}}$ denote an associative factorization algebra in $\operatorname{Sat}_{\mathcal{Y}}$ equal to the dualizing sheaf of $\operatorname{Hecke}_{\mathcal{Y},x,loc}$ with respect to $\overleftarrow{h}_{loc}$.

Let $\mathcal{Y}(X)$ be the scheme/stack of global sections of $\mathcal{Y}$. For every x we have a diagram

$$\begin{array}{ccccc} & & \operatorname{Hecke}_{\mathcal{Y},x,glob} & & \\ & \overset{\overleftarrow{h}_{glob}}{\swarrow} & \downarrow & \overset{\overrightarrow{h}_{glob}}{\searrow} & \\ \mathcal{Y}(X) & & \downarrow & & \mathcal{Y}(X \\ \downarrow & & \operatorname{Hecke}_{\mathcal{Y},x,loc} & & \downarrow \\ \downarrow & \overset{\overleftarrow{h}_{loc}}{\swarrow} & & \overset{\overrightarrow{h}_{loc}}{\searrow} & \downarrow \\ \mathcal{Y}(\mathcal{D}_x) & & & & \mathcal{Y}(\mathcal{D}_x) \end{array}$$

with both diamonds being Cartesian.

In particular, objects of $\operatorname{Sat}_{\mathcal{Y}}(\operatorname{Ran})$ define functors

$$\operatorname{Ind}\mathsf{Coh}(\mathcal{Y}(X)) \to \operatorname{Ind}\mathsf{Coh}(\mathcal{Y}(X)),$$

which descent do well-defined functors on the localization

$$\mathsf{QCoh}(\mathcal{Y}(X)) \to \mathsf{QCoh}(\mathcal{Y}(X)).$$

Associative algebras in $\mathrm{Sat}_{\mathcal{Y}}(\mathrm{Ran})$ give rise to monads.

The following is due to N. Rozenblyum:

Theorem 11.3.3. *The monad* $\mathrm{Ind}\,\mathsf{Coh}(\mathcal{Y}(X)) \to \mathrm{Ind}\,\mathsf{Coh}(\mathcal{Y}(X))$ *(resp.,* $\mathsf{QCoh}(\mathcal{Y}(X)) \to \mathsf{QCoh}(\mathcal{Y}(X))$*) given by* $\mathcal{A}_{\mathcal{Y}}(\mathrm{Ran})$ *is canonically isomorphic to* $U(\Theta_{\mathcal{Y}(X)/\,\mathrm{pt}})$.

In what follows, we'll need a relative version of the above situation.

11.3.4. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two crystals of schemes/stacks as above, and let $\mathfrak{f}_{loc} : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism.

For a point $x \in X$ consider the ind-stack

$$\mathrm{Hecke}_{\mathcal{Y}_1,\mathcal{Y}_2,x,loc} := \mathcal{Y}_1(\mathcal{D}_x) \underset{\mathcal{Y}_1(\overset{\circ}{\mathcal{D}}_x) \underset{\mathcal{Y}_2(\overset{\circ}{\mathcal{D}}_x)}{\times} \mathcal{Y}_2(\mathcal{D}_x)}{\times} \mathcal{Y}_1(\mathcal{D}_x),$$

equipped with the two projections

$$\mathcal{Y}_1(\mathcal{D}_x) \overset{\overleftarrow{h}_{rel,loc}}{\longleftarrow} \mathrm{Hecke}_{\mathcal{Y}_1,\mathcal{Y}_2,x,loc} \overset{\overrightarrow{h}_{rel,loc}}{\longrightarrow} \mathcal{Y}_1(\mathcal{D}_x).$$

These projections are ind-finite.

In addition, we have a natural map

$$\mathrm{Hecke}_{\mathcal{Y}_1,\mathcal{Y}_2,x,loc} \to \mathrm{Hecke}_{\mathcal{Y}_1,x,loc}\,.$$

We let $\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2,x}$ be the associative algebra in $\mathrm{Sat}_{\mathcal{Y}_1,x}$ equal to the direct image under the above map of the relative (w.r.t. to $\overleftarrow{h}_{rel,loc}$) of the dualizing sheaf of $\mathrm{Hecke}_{\mathcal{Y}_1,\mathcal{Y}_2,x,loc}$.

The assignment $x \mapsto \mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2,x}$ defines an associative factorization algebra in $\mathrm{Sat}_{\mathcal{Y}_1}$ that we denote by $\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2}$.

Theorem 11.3.5. *The monad* $\mathrm{Ind}\,\mathsf{Coh}(\mathcal{Y}(X)) \to \mathrm{Ind}\,\mathsf{Coh}(\mathcal{Y}(X))$ *(resp.,* $\mathsf{QCoh}(\mathcal{Y}(X)) \to \mathsf{QCoh}(\mathcal{Y}(X))$*) given by* $\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2}(\mathrm{Ran})$ *is canonically isomorphic to* $U(\Theta_{\mathcal{Y}_1(X)/\mathcal{Y}_2(X)})$.

11.3.6. We apply the above discussion to the case $\mathcal{Y}_1 = \mathrm{pt}\,/G_1$, and $\mathcal{Y}_2 = \mathrm{pt}\,/G_2$. Note the slight discrepancy of notation

$$\mathrm{Hecke}_{\mathrm{pt}\,/G_1,x,loc} = \mathrm{Hecke}_{G_1,x,loc,spec}$$

$$\mathcal{A}_{\mathrm{pt}\,/G_1,\mathrm{pt}\,/G_2} = \mathcal{A}_{G_1,G_2,spec}.$$

From Theorem 11.3.5, we obtain a description of the monad

$$U(\Theta_{\mathrm{LocSys}_{G_1}\,/\,\mathrm{LocSys}_{G_2}}) : \mathsf{QCoh}(\mathrm{LocSys}_{G_1}) \to \mathsf{QCoh}(\mathrm{LocSys}_{G_1}). \tag{11.6}$$

Namely:

Lemma 11.3.7. *The monad* (11.6) *is given by the action of the chiral algebra* $\mathcal{A}_{G_1,G_2,spec}(\mathrm{Ran})$.

11.3.8. We apply the discussion of Sect. 11.3.6 to $G_1 = \check{B}$, $G_2 = \check{G}$.

Lemma 11.3.7 provides an answer for the monad

$$(\mathfrak{p}^r_{1,spec})_* \circ (\mathfrak{p}^l_{1,spec})^! = U(\Theta_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}).$$

We now wish to describe the monad

$$(\mathfrak{q}_{spec})_* \circ U(\Theta_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}) \circ (\mathfrak{p}_{spec})^* : \mathsf{QCoh}(\mathrm{Bun}_T) \to \mathsf{QCoh}(\mathrm{Bun}_T)$$

of (11.4).

Lemma 11.3.9. *For a functor $S : \mathsf{QCoh}(\mathrm{Bun}_B) \to \mathsf{QCoh}(\mathrm{Bun}_B)$ given by the action of $\mathcal{A}(\mathrm{Ran}(X))$ for a unital chiral algebra $\mathcal{A} \in \mathrm{Sat}_{B,spec}$, the composed functor*

$$\mathsf{QCoh}(\mathrm{Bun}_T) \overset{(\mathfrak{q}_{spec})^*}{\longrightarrow} \mathsf{QCoh}(\mathrm{Bun}_B) \overset{S}{\to} \mathsf{QCoh}(\mathrm{Bun}_B) \overset{(\mathfrak{q}_{spec})_*}{\longrightarrow} \mathsf{QCoh}(\mathrm{Bun}_T)$$

is given by $\mathcal{A}'(\mathrm{Ran}(X))$, where $\mathcal{A}' \in \mathrm{Sat}_{T,spec}$ is the chiral algebra $(\mathfrak{q}_{\mathrm{Hecke},spec})_*(\mathcal{A})$.

11.3.10. Lemmas 11.3.7, 11.3.9 monad show that the statement of Conjecture 11.2.5 follows from that of Conjecture 10.3.4, using the fact that the Fourier-Mukai transform Ψ_T is compatible with the geometric Satake isomorphism (10.2) via the action of Hecke functors on both sides.

12. What acts compactified Eisenstein series

In this subsection we'll assume Conjecture 10.3.4 and pursue our goal 3.

12.1. **The local monad.**

12.1.1. In Corollary 3.5.11 we've seen that the functor $\mathrm{Eis}_{!*} : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$ canonically extends to a functor

$$\mathrm{Eis}^{int}_{!*} : U(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G),$$

where $U(\check{\mathfrak{n}}^-_X) \star -$ was a certain explicit monad acting on $D(\mathrm{Bun}_T)$.

The goal of this section is to extend this constructing further. We'll construct a monad that we'll denote $U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$ acting on $D(\mathrm{Bun}_T)$, which receives a map from $U(\check{\mathfrak{n}}^-_X)$, and a functor

$$\mathrm{Eis}^{ult}_{!*} : U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G),$$

such that

$$\mathrm{Eis}^{int}_{!*} \simeq \mathrm{Eis}^{ult}_{!*} \circ \mathbf{ind}^{U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)}_{U(\check{\mathfrak{n}}^-_X)}.$$

12.1.2. Namely, consider the chiral algebra in $\mathrm{Sat}_{\check{T},spec}$ equal to $\mathcal{A}_{\check{T},\check{G}}$. Let $U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$ denote the corresponding object of $\mathrm{Sat}_T(\mathrm{Ran})$ obtained via (10.2). By construction, there exists a canonical algebra homomorphism $U(\check{\mathfrak{n}}^-_X) \to U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$. By a slight abuse of notation, we'll denote by the same symbol $U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$ the resulting monad acting on $D(\mathrm{Bun}_T)$.

In what follows, we'll show that Conjecture 10.3.4 implies the following:

Conjecture 12.1.3. *The functor*

$$\mathrm{Eis}^{int}_{!*} : U(\check{\mathfrak{n}}^-_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G)$$

canonically extends to a functor

$$\mathrm{Eis}^{ult}_{!*} : U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)\text{-mod}(D(\mathrm{Bun}_T)) \to D(\mathrm{Bun}_G),$$

such that the functor

$$\mathrm{Eis}^{int}_{!*} \simeq \mathrm{Eis}^{ult}_{!*} \circ \mathbf{ind}^{U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)}_{U(\check{\mathfrak{n}}^-_X)}.$$

Remark. Recall $\Omega(\check{\mathfrak{n}}^-_X)$, viewed as an object of $D(\mathrm{Ran}(X,\Lambda^{pos})\times \mathrm{Bun}_T)$ was initially defined through the geometry of the stack $\overline{\mathrm{Bun}}_B$ (as a certain local cohomology). In Sect. **??**, we'll see a similar interpretation for $U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$.

Note that from Theorem 11.3.5 we obtain:

Corollary 12.1.4. *The monad acting on* $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}})$ *that corresponds to* $U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X)$ *via the Fourier-Mukai equivalence* Ψ_T *identifies canonically with* $U(\Theta_{\mathrm{LocSys}_{\check{T}}\,/\,\mathrm{LocSys}_{\check{G}}})$.

12.2. **"Proof" of Conjecture 12.1.3.**

12.2.1. We'll work inside the monoidal chiral category Sat_T; let $\mathbf{1}_{\mathrm{Sat}_T}\in \mathrm{Sat}_T(\mathrm{Ran})$ be the unit. The Koszul resolution

$$\mathbf{1}_{\mathrm{Sat}_T}\simeq \Omega(\check{\mathfrak{n}}^-_X)\star\otimes U^\vee(\check{\mathfrak{n}}^-_X)$$

makes it a bi-module with respect to $\Omega(\check{\mathfrak{n}}^-_X)$ and $U^\vee(\check{\mathfrak{n}}^-_X)$. Convolution with this bi-module defines a functor

$$U^\vee(\check{\mathfrak{n}}^-_X)\text{-mod}\to \Omega(\check{\mathfrak{n}}^-_X)\text{-mod}$$

within any category acted on monoidally by $\mathrm{Sat}_T(\mathrm{Ran})$.

We obtain that Conjecture 12.1.3 would follow from the following isomorphism:

$$\tilde{\Omega}(\check{\mathfrak{n}}^-_X)\underset{\Omega(\check{\mathfrak{n}}^-_X)}{\otimes}\mathbf{1}_{\mathrm{Sat}_T}\simeq \mathbf{1}_{\mathrm{Sat}_T}\underset{U(\check{\mathfrak{n}}^-_X)}{\otimes}U(\check{\mathfrak{g}}/\check{\mathfrak{t}}_X), \tag{12.1}$$

as bi-modules with respect to $\Omega(\check{\mathfrak{n}}^-_X)$ and $U^\vee(\check{\mathfrak{n}}^-_X)$, in a way compatible with multiplication.

We'll establish (12.1) by transferring it to a (tautological) isomorphism that takes place in $\mathrm{Sat}_{\check{T},spec}$, and then applying Conjecture 10.3.4.

12.2.2. First, we recall that $\Omega(\check{\mathfrak{n}}^-_X)$ identifies with direct image of the structure sheaf under the morphism of factorization stacks $\mathrm{pt}/\check{B}\to \mathrm{pt}/\check{T}$. Here we view $\mathsf{QCoh}(\mathrm{pt}/\check{T})$ as mapping to $\mathrm{Sat}_{\check{T},spec}$ via the diagonal map

$$\mathrm{pt}/\check{T}\simeq \mathrm{pt}/\check{T}(\mathcal{D}_x)\to \mathrm{pt}/\check{T}(\mathcal{D}_x)\underset{\mathrm{pt}/\check{T}(\overset{\circ}{\mathcal{D}}_x)}{\times}\mathrm{pt}/\check{T}(\mathcal{D}_x).$$

According to Conjecture 10.3.4, $\tilde{\Omega}(\check{\mathfrak{n}}^-_X)$ corresponds to the associative algebra $\mathcal{A}_{\check{T},\check{B},\check{G},spec}$.

Finally, we observe that $U(\check{\mathfrak{n}}^-_X)$ corresponds to the associative chital algebra $\mathcal{A}_{\check{T},\check{B},spec}$.

Hence, we need to establish an isomorphism

$$\mathcal{A}_{\check{T},\check{B},\check{G},spec}\underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes}\mathbf{1}_{\mathrm{Sat}_{\check{T},spec}}\simeq \mathbf{1}_{\mathrm{Sat}_{\check{T},spec}}\underset{\mathcal{A}_{\check{T},\check{B},spec}}{\otimes}\mathcal{A}_{\check{T},\check{G}}. \tag{12.2}$$

This isomorphism needs to take place in $\mathrm{Sat}_{\check{T},spec}(\mathrm{Ran})$, i.e., pointwise in $\mathrm{Ind}\,\mathsf{Coh}(\mathrm{Hecke}_{\check{T},spec})$. We will establish that it takes place in a localization of this category, namely in $\mathsf{QCoh}(\mathrm{Hecke}_{\check{T},spec})$. However, since both sides are co-connective, the isomorphism in the localization implies the original one.

12.2.3. Let us return to the context of Sect. 11.3.2. We'll use the following result of J. Lurie:

Theorem 12.2.4. *The action of* $\mathsf{QCoh}(\mathrm{Hecke}_{\mathcal{Y},x,loc})$ *on* $\mathsf{QCoh}(\mathcal{Y}_x)$ *identifies the former with the category of endomorphisms of the latter as a chiral module category over itself.*

Thus, in order to establish (12.2) we need to identify both sides as endomorphisms of $\mathsf{QCoh}(\mathrm{pt}\,\check{B}(\mathcal{D}_x))$, viewed as a category over $\mathrm{pt}\,\check{B}(\overset{\circ}{\mathcal{D}}_x)$.

12.2.5. Recall the general context of Sect. 11.3.4, and suppose now that we have three stacks $\mathcal{Y}_1 \to \mathcal{Y}_2 \to \mathcal{Y}_3$. Consider the forgetful functor

$$\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2,x}\text{-mod}(\mathsf{QCoh}(\mathcal{Y}_{1,x})) \to \mathsf{QCoh}(\mathcal{Y}_{2,x}).$$

Tautologically, we have:

Lemma 12.2.6. *The above functor intertwines the monad*

$$\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_3,x} \underset{\mathcal{A}_{\mathcal{Y}_1,\mathcal{Y}_2,x}}{\times} -$$

acting on the LHS, with the monad $\mathcal{A}_{\mathcal{Y}_2,\mathcal{Y}_3,x}$ acting on the RHS. The isomorphism of functors in question holds as an isomorphism over the stack $\mathcal{Y}_2(\overset{\circ}{\mathcal{D}}_x)$.

This proves the required isomorphism (12.2), when we view $\text{pt}/\check{T} \to \text{pt}/\check{B}$ as a map of stacks over $\text{pt}/\check{T}$ via the projection $\check{B} \to \check{T}$.

References

[BG1] A. Braverman, D Gaitsgory, "Geometric Eisenstein Series", arXiv:math/9912097.
[BG2] A. Braverman, D Gaitsgory, "Deformations of local systems and Eisenstein series", arXiv:math/0605139.
[BFGM] A. Braverman, M. Finkelberg, D. Gaitsgory and I. Mirkovic, "Intersection cohomology of Drinfeld's compactifications", arXiv:math/0012129.
[Eis] *Weird adjunction for Eisenstein series*
[CHA] "Chiraral algebras".
[FFKM] B. Feigin, M. Finkelberg, A. Kuznetsov and I. Mirkovic, "Semi-infinite flags-II", arXiv:alg-geom/9711009.
[JNKF] John Francis, in preparation.
[Sum] *An attempt to formulate the conjecture*