

You can use a wide range of wild, cultivated or supermarket greens in this recipe. Consider nettles, beet tops, turnip tops, spinach, or watercress in place of chard. The combination is also up to you, so choose the ones you like most.

Y. Ottolenghi. Plenty More.

## THE ATIYAH-BOTT FORMULA FOR THE COHOMOLOGY OF THE MODULI SPACE OF BUNDLES ON A CURVE

D. GAITSGORY

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## INTRODUCTION

0.1. **What is this text?** The present paper is a companion of [Main Text].<sup>1</sup> The goal of *loc.cit.* is to prove the Tamagawa number formula for function fields, namely that the volume of the automorphic space of a semi-simple simply connected group, with respect to the Tamagawa measure, equals 1. The present paper gives a different approach to some of the steps in the proof.

0.1.1. Let us recall the strategy of the proof in [Main Text]. We start with a (smooth, complete and geometrically connected) curve  $X$  over a finite field  $\mathbb{F}_q$  with the field of fractions  $K$ , and let  $G_K$  be a semi-simple simply connected group over  $K$ . We are interested in the quantity

$$\mu_{\text{Tam}}(G_K(\mathbb{A})/G_K(K)),$$

where  $\mathbb{A}$  is the ring of adèles of  $K$ , and  $\mu_{\text{Tam}}$  is the Tamagawa measure on  $G_K(\mathbb{A})$ .

First, we choose an integral model  $G$  of  $G_K$  over  $X$ . This is a smooth group-scheme over  $X$  with connected fibers. Let  $\text{Bun}_G$  denote the moduli space of  $G$ -bundles on  $X$ ; this is an algebraic stack locally of finite type and smooth over  $\mathbb{F}_q$ . We interpret the desired equality

$$\mu_{\text{Tam}}(G_K(\mathbb{A})/G_K(K)) = 1$$

as the following

$$(0.1) \quad |\text{Bun}_G(\mathbb{F}_q)| = q^{\dim(\text{Bun}_G)} \cdot \prod_x \frac{|k_x|^{\dim(G_x)}}{|G(k_x)|}.$$

In the above formula,  $|\text{Bun}_G(\mathbb{F}_q)|$  is the (infinite) sum over the set of isomorphism classes of  $G$ -bundles on  $X$ :

$$\sum_{\mathcal{P}} \frac{1}{|\text{Aut}(\mathcal{P})|}.$$

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<sup>1</sup>The contents of this paper are joint work with J. Lurie, who chose not to sign it as an author. It is made public with his consent, but the responsibility for any deficiency or undesired outcome of this paper lies entirely with the author.

In the right-hand side, the product is over the set of closed points of  $X$ ; for a point  $x$  we denote by  $k_x$  its residue field and by  $G_x$  the fiber of  $G$  at this point.

Note that both sides in (0.1) are infinite expressions, so part of the statement is that both sides are well-defined (i.e., are convergent).

0.1.2. The next step is the Grothendieck-Lefschetz trace formula that says that the left-hand side in (0.1) equals

$$(0.2) \quad \mathrm{Tr}(\mathrm{Frob}, H_c^*(\mathrm{Bun}_G)).$$

The Grothendieck-Lefschetz trace formula for *quasi-compact* algebraic stacks follows easily from the case of varieties. However,  $\mathrm{Bun}_G$  is not quasi-compact, so a separate argument is needed to justify it. In particular, one needs to show that the expression in (0.2) makes sense as a complex number. This is non-obvious because  $H_c^*(\mathrm{Bun}_G)$  lives in infinitely many cohomological degrees.

0.1.3. Assuming the Grothendieck-Lefschetz trace formula for  $\mathrm{Bun}_G$ , and using Verdier duality, we rewrite the desired equality as

$$(0.3) \quad \mathrm{Tr}(\mathrm{Frob}^{-1}, H^*(\mathrm{Bun}_G)) = \prod_x \frac{|k_x|^{\dim(G_x)}}{|G(k_x)|},$$

where the factor of  $q^{\dim(\mathrm{Bun}_G)}$  got subsumed in the dualizing sheaf on  $\mathrm{Bun}_G$ .

Along with the previous steps mentioned above, the equality (0.3) is established in [Main Text]. Our goal in this paper is to give a different proof of (0.3). In fact, there will be two points of difference, one big and the other small.

The key step in both proofs is the *cohomological product formula* for  $H^*(\mathrm{Bun}_G)$ , see (0.5).

0.1.4. The small point of difference is how we derive the numerical identity (0.3) from the formula for  $H^*(\mathrm{Bun}_G)$ .

In [Main Text] this is done by considering the filtration by powers of the maximal ideal on the algebra of cochains  $C^*(\mathrm{Bun}_G)$ . Since we live in the world of higher algebra, such a filtration is not something naive, but rather a certain canonical construction in homotopy theory.

In this paper we derive (0.3) by a more pedestrian method: we show that from the *cohomological product formula* (0.5) one can deduce a more explicit formula for  $H^*(\mathrm{Bun}_G)$ , called the Atiyah-Bott formula, see (0.7). Now, given the Atiyah-Bott formula, the derivation of (0.3) is a standard manipulation with L-functions, see Sect. 20.

So, in a sense, this part of the argument is more elementary in the present paper than in [Main Text].

0.1.5. The big point of difference is how we prove the cohomological product formula (0.5). This formula is a local-to-global result, and in both [Main Text] and the present paper we deduce it from another local-to-global result, namely, the non-abelian Poincaré duality, (0.8) (we do not reprove the non-abelian Poincaré duality here).

In both cases, the derivation

$$\text{Non-abelian Poincaré duality} \Rightarrow \text{Cohomological product formula}$$

morally amounts to performing Verdier duality on the *Ran space* of  $X$ .

Now, roughly speaking, the above derivation in [Main Text] does not explicitly use the words “Verdier duality”. Rather, it amounts to a giant commutative diagram of complexes of  $\mathbb{Z}_\ell$ -modules.

One can regard the bulk of the work in the present paper as breaking the giant diagram from [Main Text] into a series of several conceptually defined steps. Our primary tool is the extensive use of *sheaves*. I.e., most of the isomorphisms that we need to go through are isomorphisms between sheaves on some spaces (rather than complexes of  $\mathbb{Z}_\ell$ -modules). See Sect. 0.6, where we list the series of isomorphisms that eventually leads to the cohomological product formula.

However, there is a price that one needs to pay: for some of the most non-trivial steps, sheaves on varieties are not sufficient. And neither are sheaves on *prestacks* (the latter are geometric objects generalizing varieties, see Sect. 1.2). What we will need is the notion of  $\ell$ -adic sheaf on a (contravariant) functor from the category of schemes to that of  $\infty$ -categories; we will call the latter gadgets *lax prestacks*.

**0.2. The Ran space and the cohomological product formula.** Thus, the main thrust of this paper is the proof of the cohomological product formula. Here we shall explain what it says. To do so we shall need the language of sheaves on prestacks, see Sect. 1.

We change the notations from Sect. 0.1, and we take  $X$  to be a smooth and complete curve over an algebraically closed ground field  $k$ .

0.2.1. Let  $X$  be our curve. Both in [Main Text] and in the present paper, one of our main tools is the Ran space of  $X$ . This is a prestack that classifies *finite non-empty subsets* of points of  $X$ , see Sect. 4.1 for the precise definition.

Given the group-scheme  $G$  over  $X$ , we construct the following sheaf, denoted  $\mathcal{B}$ , on  $\text{Ran}$ . For a collection of (distinct) points  $x_1, \dots, x_n$ , the  $\mathbb{1}$ -fiber of  $\mathcal{B}$  at the point  $\{x_1, \dots, x_n\} \in \text{Ran}$  is

$$(0.4) \quad \bigotimes_{i=1, \dots, n} C^*(BG_{x_i}),$$

where  $G_{x_i}$  is the fiber of  $G$  at  $x_i$ , and  $BG_{x_i}$  is its classifying stack. Here and in the sequel,  $C^*(-)$  denotes the algebra of cochains on a given space.

For every  $x_1, \dots, x_n$  as above, we have a canonically defined restriction map

$$\text{Bun}_G \rightarrow BG_{x_1} \times \dots \times BG_{x_n},$$

and the corresponding pullback map on cohomology.

Integrating over the Ran space, we obtain a map

$$(0.5) \quad C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C^*(\text{Bun}_G)$$

The cohomological product formula says that the map (0.7) is an isomorphism (provided that the generic fiber of  $G$  is semi-simple and simply-connected).

0.2.2. The reader might wonder why we call the isomorphism (0.5) the ‘‘cohomological product formula’’. The reason is that, given the expression for the  $\mathbb{1}$ -fibers of  $\mathcal{B}$  in (0.4), one can think of  $C_c^*(\text{Ran}, \mathcal{B})$  as a kind of Euler product

$$(0.6) \quad \bigotimes_{x \in X} C^*(BG_x),$$

*understood informally.*

Note that when we compare the geometric picture to the classical one of automorphic functions, the latter is one categorical level lower: so a vector space in geometry corresponds to a number in the classical theory. So, the product (0.6) is a geometric analog of the special value of the corresponding L-function (rather than the restricted tensor product, akin to one in the definition of automorphic representations).

A possible of justification for thinking of  $C_c^*(\text{Ran}, \mathcal{B})$  as the ‘‘product’’ is the following. In the course of the proof of the numerical product formula (0.3) in Sect. 20, we will show that when working over the finite field, the trace of  $\text{Frob}^{-1}$  on  $H_c^*(\text{Ran}, \mathcal{B})$  will actually turn out to be equal to the product of

$$\prod_x \text{Tr}(\text{Frob}_x^{-1}, H^*(BG_x)).$$

We should emphasize, however, that the proof of this identity is specific to our situation (we prove it by first rewriting  $C_c^*(\text{Ran}, \mathcal{B})$  via the Atiyah-Bott formula, see below), i.e., we use some additional structure on our  $\mathcal{B}$ . It would be interesting to find a general statement applicable to  $C_c^*(\text{Ran}, \mathcal{F})$  for a general (*factorizable*) sheaf  $\mathcal{F}$  on  $\text{Ran}$ .

0.2.3. The formula (0.5) embodies a local-to-global principle: the cohomology of  $\mathrm{Bun}_G$  is assembled in a canonical way from the cohomologies of the classifying spaces of the fibers of  $G$ . However, it is not best adapted to explicit computations (recall that we need to compute the trace of Frobenius on  $H^*(\mathrm{Bun}_G)$ ).

In Sect. 19 we will derive from the formula for  $C^*(\mathrm{Bun}_G)$ , given by (0.5), another expression for  $C^*(\mathrm{Bun}_G)$ , known as the Atiyah-Bott formula. This is literally the formula from [AB] when the ground field is that of complex numbers and  $G$  is split.

Namely, let us assume that  $G$  is such that it is reductive over some open  $X' \subset X$ , and has unipotent fibers at points of  $X - X'$ . To the datum of  $G$  one attaches a lisse sheaf  $M$  over  $X'$ , whose  $!$ -fiber at  $x \in X'$  has a basis consisting of a set of homogeneous generators of  $H^*(BG_x)$ , i.e., the *exponents* of  $G_x$ .

In Sect. 19 we will show that the cohomological product formula implies the following (non-canonical) isomorphism:

$$(0.7) \quad C^*(\mathrm{Bun}_G) \simeq \mathrm{Sym}(C^*(X', M))$$

It is using this isomorphism that we will be able to calculate the trace of  $\mathrm{Frob}^{-1}$  on  $H^*(\mathrm{Bun}_G)$ .

**0.3. Non-abelian Poincaré duality.** We shall now try to outline the proof of the cohomological product formula, i.e., of the fact that the map (0.5) is an isomorphism. This is what the bulk of this paper is about.

0.3.1. Let  $\mathrm{Gr}_{\mathrm{Ran}}$  be the Ran version of the affine Grassmannian of  $G$ . I.e., this is the prestack that classifies the data of

$$(\mathcal{P}, I, \alpha),$$

where  $\mathcal{P}$  is a  $G$ -bundle on  $X$ ,  $I \in \mathrm{Ran}$  and  $\alpha$  is a trivialization of  $\mathcal{P}$  on  $X - I$ .

We have an evident forgetful map

$$\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Bun}_G, \quad (\mathcal{P}, I, \alpha) \mapsto \mathcal{P}.$$

The main geometric ingredient in the proof of the cohomological product formula is the fact that this map induces an isomorphism on homology. This is proved in [Main Text] under the name *non-abelian Poincaré duality*. We do not reprove this fact here.

So, we need to explain how the above isomorphism on homology implies the cohomological product formula. Note that the map  $\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Bun}_G$  used above, and the maps

$$\mathrm{Bun}_G \rightarrow BG_{x_1} \times \dots \times BG_{x_n},$$

used in (0.5) are of (seemingly) different nature.

0.3.2. Consider the forgetful map

$$\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Ran}, \quad (\mathcal{P}, I, \alpha) \mapsto I.$$

Let  $\mathcal{A}$  be the sheaf on  $\mathrm{Ran}$  equal to the pushforward of the dualizing sheaf under the above map. Explicitly, the  $!$ -fiber of  $\mathcal{A}$  at a point  $\{x_1, \dots, x_n\} \in \mathrm{Ran}$  is

$$\bigotimes_{i=1, \dots, n} C_*(\mathrm{Gr}_{x_i}),$$

where  $\mathrm{Gr}_{x_i}$  denotes the affine Grassmannian of  $G$  at  $x_i$ .

We can reformulate the non-abelian Poincaré duality as the fact that the map

$$(0.8) \quad C_c^*(\mathrm{Ran}, \mathcal{A}) \rightarrow C_*(\mathrm{Bun}_G)$$

is an isomorphism.

0.3.3. Comparing (0.5) with (0.8), we obtain that we need to construct an isomorphism

$$(0.9) \quad C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C_c^*(\text{Ran}, \mathcal{A})^\vee,$$

that makes the diagram

$$\begin{array}{ccc} C_c^*(\text{Ran}, \mathcal{B}) & \longrightarrow & C_c^*(\text{Ran}, \mathcal{A})^\vee \\ (0.5) \downarrow & & \sim \uparrow (0.8) \\ C^*(\text{Bun}_G) & \xrightarrow{\sim} & C_*(\text{Bun}_G)^\vee \end{array}$$

commute. We call the desired (but eventually, actual) isomorphism (0.9) the *global duality* statement.

#### 0.4. Verdier duality and the procedure of taking the units out.

0.4.1. In order to construct (0.9), it is very tempting to suppose that the sheaf  $\mathcal{B}$  is the Verdier dual of  $\mathcal{A}$ , thereby inducing a duality on their global cohomologies.

The first question is: what is Verdier duality on a geometric object such as  $\text{Ran}$ ? We do define what Verdier duality means in such a context in Part II of the paper and we prove a theorem to the effect that for a sheaf on  $\text{Ran}$  *satisfying certain cohomological estimates*, the (compactly supported) cohomology of its Verdier dual will map isomorphically to the dual of its (compactly supported) cohomology.

The problem is that our  $\mathcal{A}$  does *not* satisfy the above cohomological estimates. In fact, it violates them so badly that its Verdier dual equals 0.<sup>2</sup> So, the desired duality on global cohomology happens for a reason (seemingly) different from Verdier duality.

But it turns out that trying to push through the idea of Verdier duality is not a lost case: one needs to modify  $\mathcal{A}$  and  $\mathcal{B}$ . The corresponding modification is the procedure of *taking the units out*.

0.4.2. We shall explain how taking the units out allows to implement Verdier duality by analogy with associative algebras.<sup>3</sup>

Recall that on the category of *non-unital* associative algebras (say, over a ground field  $\Lambda$ ) there is a canonically defined contravariant self-functor, the Koszul duality. Namely, it sends an algebra  $A$  to

$$\text{KD}(A) := \ker(\mathcal{H}om_A(\Lambda, \Lambda) \rightarrow \mathcal{H}om_\Lambda(\Lambda, \Lambda)),$$

where  $\Lambda$  is equipped with the *trivial* (i.e., zero) action of  $A$ , and where  $\mathcal{H}om_\Lambda(\Lambda, \Lambda)$  is (obviously) isomorphic to  $\Lambda$ .

The functor  $\text{KD}$  is *not* an equivalence. In fact, it sends many objects to 0. For example, if  $A$  was obtained from a *unital* associative algebra  $A_{\text{unital}}$  by treating it as a non-unital algebra (i.e., by forgetting that it had a unit), then  $\text{KD}(A) = 0$ .

Now, suppose that we start with a unital augmented algebra  $A_{\text{unital, aug}}$  and take  $A_{\text{red}}$  to be its augmentation ideal. Then it is a sensible procedure to consider  $B_{\text{red}} := \text{KD}(A_{\text{red}})$  and then create a unital augmented algebra  $B_{\text{unital, aug}}$  by adjoining a unit to  $B_{\text{red}}$ , while at the background we also have  $A$  and  $B$ , obtained from  $A_{\text{unital, aug}}$  and  $B_{\text{unital, aug}}$ , respectively, by forgetting the unit and the augmentation.

This will turn out to be how our sheaves  $\mathcal{A}$  and  $\mathcal{B}$  are related to each other.

<sup>2</sup>In fact, the Verdier dual of  $\mathcal{A}$  is 0 even for  $G = \{1\}$ , in which case  $\mathcal{A}$  equals the dualizing sheaf on  $\text{Ran}$ . Moreover, the “constants” that come from the case  $G = \{1\}$  is what causes the problem for any  $G$ , and the way to get around it consists of taking these constants out, see Sect. 0.4.4 below.

<sup>3</sup>In fact, this is more than just an analogy, as sheaves on the  $\text{Ran}$  space and associative algebras are part of a common paradigm. Namely, in the context of topology with  $X$  being the real line  $\mathbb{R}$ , sheaves on its  $\text{Ran}$  space, equipped with a factorization structure, are equivalent to associative algebras.

0.4.3. We should think of sheaves on the Ran space as analogs of non-unital associative algebras (see footnote above). In Part I of this paper we introduce geometric gadgets, denoted  $\text{Ran}_{\text{unlt}}$  and  $\text{Ran}_{\text{unlt, aug}}$ , so that sheaves on them are analogs of unital and unital augmented associative algebras, respectively.

It is here that *lax prestacks* come into the picture, because such are  $\text{Ran}_{\text{unlt}}$  and  $\text{Ran}_{\text{unlt, aug}}$ .

The categories of sheaves on  $\text{Ran}$ ,  $\text{Ran}_{\text{unlt}}$  and  $\text{Ran}_{\text{unlt, aug}}$  are related to each other by several functors that mirror the corresponding functors for associative algebras:

$$\text{OblvUnit} : \text{Shv}(\text{Ran}_{\text{unlt}}) \rightarrow \text{Shv}(\text{Ran}), \quad \text{OblvAug} : \text{Shv}(\text{Ran}_{\text{unlt, aug}}) \rightarrow \text{Shv}(\text{Ran}_{\text{unlt}}),$$

$$\text{AddUnit} : \text{Shv}(\text{Ran}) \rightarrow \text{Shv}(\text{Ran}_{\text{unlt}}),$$

and a pair of adjoint functors

$$\text{AddUnit}_{\text{aug}} : \text{Shv}(\text{Ran}) \rightleftarrows \text{Shv}(\text{Ran}_{\text{unlt, aug}}) : \text{TakeOut},$$

while

$$\text{OblvAug} \circ \text{AddUnit}_{\text{aug}} \simeq \text{AddUnit}.$$

We will show that the sheaves  $\mathcal{A}$  and  $\mathcal{B}$  are obtained as

$$\mathcal{A} \simeq \text{OblvUnit} \circ \text{OblvAug}(\mathcal{A}_{\text{unlt, aug}}) \text{ and } \mathcal{B} \simeq \text{OblvUnit} \circ \text{OblvAug}(\mathcal{B}_{\text{unlt, aug}}),$$

respectively for canonically defined sheaves  $\mathcal{A}_{\text{unlt, aug}}$  and  $\mathcal{B}_{\text{unlt, aug}}$  on  $\text{Ran}_{\text{unlt, aug}}$ .

We then set

$$\mathcal{A}_{\text{red}} := \text{TakeOut}(\mathcal{A}_{\text{unlt, aug}}) \text{ and } \mathcal{B}_{\text{red}} := \text{TakeOut}(\mathcal{B}_{\text{unlt, aug}}).$$

0.4.4. In our case, the  $!$ -fibers of  $\mathcal{A}_{\text{red}}$  and  $\mathcal{B}_{\text{red}}$  at a point  $\{x_1, \dots, x_n\} \in \text{Ran}$  are

$$\bigotimes_{i=1, \dots, n} C_*^{\text{red}}(\text{Gr}_{x_i}) \text{ and } \bigotimes_{i=1, \dots, n} C_{\text{red}}^*(BG_{x_i}),$$

respectively. So the effect of replacing  $\mathcal{A}$  by  $\mathcal{A}_{\text{red}}$  and  $\mathcal{B}$  by  $\mathcal{B}_{\text{red}}$  consists at the level of  $!$ -fibers of replacing

$$C_*(\text{Gr}_{x_i}) \rightsquigarrow C_*^{\text{red}}(\text{Gr}_{x_i}) \text{ and } C^*(BG_{x_i}) \rightsquigarrow C_{\text{red}}^*(BG_{x_i})$$

*in each factor.*

0.4.5. But let us remember that we are interested in the (compactly supported) cohomology of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, on  $\text{Ran}$ . How is (compactly supported) cohomology affected by the passage

$$(0.10) \quad \mathcal{A} \rightsquigarrow \mathcal{A}_{\text{red}} \text{ and } \mathcal{B} \rightsquigarrow \mathcal{B}_{\text{red}}?$$

We will show that for any sheaf  $\mathcal{F}$  on the Ran space we have a canonical isomorphism

$$C_c^*(\text{Ran}, \mathcal{F}) \simeq C_c^*(\text{Ran}, \text{OblvUnit} \circ \text{OblvAug} \circ \text{AddUnit}_{\text{aug}}(\mathcal{F})).$$

So, the replacement (0.10) leaves the (compactly supported) cohomology on  $\text{Ran}$  unchanged (up to a direct summand equal to  $\mathbb{Q}_\ell$ ).

This is parallel to the situation in associative algebras, in which for a unital augmented algebra  $A_{\text{unlt, aug}}$ , the Hochschild homology of  $A$  (obtained from  $A_{\text{unlt, aug}}$  by forgetting the unit) and  $A_{\text{red}}$  (obtained from  $A_{\text{unlt, aug}}$  by taking the augmentation ideal) are isomorphic (up to a copy of the ground field).



0.4.6. Thus, we modify our attempt to deduce the duality between  $C_c^*(\text{Ran}, \mathcal{A})$  and  $C_c^*(\text{Ran}, \mathcal{B})$  from Verdier duality on the Ran space as follows:

We will show that the sheaves  $\mathcal{A}_{\text{red}}$  and  $\mathcal{B}_{\text{red}}$  on Ran are related by Verdier duality

$$(0.11) \quad \mathcal{B}_{\text{red}} \simeq \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$$

and that the resulting map

$$C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) \simeq C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})) \rightarrow C_c^*(\text{Ran}, \mathcal{A}_{\text{red}})^\vee$$

is an isomorphism, thereby producing (0.9).

The fact that

$$C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})) \rightarrow C_c^*(\text{Ran}, \mathcal{A}_{\text{red}})^\vee$$

is an isomorphism follows from the fact that the sheaf  $\mathcal{A}_{\text{red}}$  (unlike its precursor  $\mathcal{A}$ ) *does* satisfies the cohomological estimate mentioned in Sect. 0.3.3. It is actually here that the assumption that  $G$  be *semi-simple simply connected* is used. Indeed, the isomorphism (0.11) will take place for any reductive  $G$ , but the cohomological estimate would fail unless  $G$  is semi-simple simply connected.

*Remark 0.4.7.* The above phenomenon of the local duality implying the global one also has an analog for associative algebras: for a non-unital associative algebra that lives in cohomological degrees  $\leq -1$ , the Hochschild homology of its Koszul dual maps isomorphically to the dual of its Hochschild homology.

## 0.5. Local duality.

0.5.1. Above we have explained how to reduce the cohomological product formula to the isomorphism (0.11). Note, however, that (0.11) should not be just *some* isomorphism: we need it to make the diagram

$$\begin{array}{ccc} C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) & \xrightarrow{(0.11)} & C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})) \\ \sim \downarrow & & \downarrow \sim \\ C_c^*(\text{Ran}, \mathcal{B}) & & C_c^*(\text{Ran}, \mathcal{A}_{\text{red}})^\vee \\ & & \downarrow \sim \\ (0.5) \downarrow & & C_c^*(\text{Ran}, \mathcal{A})^\vee \\ & & \sim \uparrow (0.8) \\ C^*(\text{Bun}_G) & \xrightarrow{\sim} & C_*(\text{Bun}_G)^\vee \end{array}$$

commute.

In Part III of this paper we explain the general mechanism of what data on a pair of sheaves  $\mathcal{A}$  and  $\mathcal{B}$  on Ran (or, rather, their unital augmented enhancements  $\mathcal{A}_{\text{untl, aug}}$  and  $\mathcal{B}_{\text{untl, aug}}$ ) is needed in order to define a map in one direction

$$(0.12) \quad \mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}).$$

Not surprisingly, it turns out that in our case this data amounts to a geometric input, which is a local analog of one used in defining the maps (0.5) and (0.8).

0.5.2. Having produced the map in (0.12), we now have to show that it is an isomorphism. We call this the *local duality* statement. By definition, it says that a certain map of sheaves on Ran is an isomorphism.

Now, Ran is a huge space, but it has one piece that it easy to understand: we have a canonically defined map

$$X \rightarrow \text{Ran}.$$

We show that in order to prove that the map (0.12) is an isomorphism, it is sufficient to show that it induces an isomorphism between the respective restrictions of the two sides to  $X$ . This reduction

step is achieved via the *factorization* structure on the two sides. The relevant notion of factorization is developed in Part IV of the paper.

We then further explore the locality properties of the map (0.12) by showing that the question of this map being an isomorphism is enough to resolve in the case when  $G$  is a constant group scheme. We also show that we are free to replace the initial curve by any other curve, and it is sufficient to show that the map in question induces an isomorphism at the level of  $!$ -fibers at some/any point  $\{x\} \in \text{Ran}$ .

0.5.3. Thus, we are reduced to showing that the resulting map

$$(0.13) \quad (\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

is an isomorphism. We call this the *pointwise duality* statement.

We give two (ideologically similar, but technically different) proofs of this assertion, in Sects. 17 and 18, respectively. In both proofs we regard the map (0.13) as a local analog of the map (0.8).

Although the question of the map (0.13) being an isomorphism is a local one (it makes sense for non-complete curves), the method we use to prove it is global: we reduce it back to the fact that the map (0.8) is an isomorphism for a complete curve.

0.6. **Summary.** Let us summarize the steps involved in the derivation of the cohomological product formula (i.e., the isomorphism (0.5)) from non-abelian Poincaré duality (i.e., the isomorphism (0.8)).

0.6.1. Show that  $\mathcal{A}$  and  $\mathcal{B}$  can be equipped with a unital augmented structure; consider the corresponding reduced versions  $\mathcal{A}_{\text{red}}$  and  $\mathcal{B}_{\text{red}}$ ; show that the reduced versions have the same (compactly supported) cohomology on  $\text{Ran}$ .

The preparatory material for these constructions is Part I of the paper, and the construction itself is carried out in Sect. 15.

0.6.2. Show that  $\mathcal{A}_{\text{red}}$  is well-behaved for Verdier duality on the  $\text{Ran}$  space, i.e., that the map

$$\mathbb{C}_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})) \rightarrow \mathbb{C}_c^*(\text{Ran}, \mathcal{A}_{\text{red}})^\vee$$

is an isomorphism.

The preparatory material for this is Part II of the paper.

0.6.3. Construct a map

$$\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}).$$

The preparatory material for the construction is Part III of the paper, and the construction itself is carried out in Sect. 15.

0.6.4. Show that is enough to show that the above map  $\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$  becomes an isomorphism after restricting to  $X \subset \text{Ran}$ .

The preparatory material for this is Part IV of the paper.

0.6.5. Reduce to the pointwise assertion that the map

$$(\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

is an isomorphism (for the split form of  $G$ , and some/any  $x \in X$ ).

This is done in Sect. 16.

0.6.6. Prove that the map  $(\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$  is an isomorphism.

This is done in Sects. 17 and 18.

0.7. **Contents.**

0.7.1. After becoming familiar with the language of sheaves on prestacks and the Ran space (Sects. 1 and 4.1), the reader may want to start with Sect. 14, where the cohomological product formula is stated. The other sections in this paper essentially constitute the proof of the cohomological product formula.

After Sect. 14, the reader may try to proceed to Sect. 15, where the most essential part of the proof is contained. However, reading Sect. 15 without any idea of what is happening in Parts I, II and III of the paper will probably be impossible. Other than reading this paper in order, an alternative reasonable strategy would be to read Sect. 15 stage-by-stage and refer back to the relevant parts in the previous sections (we tried to point to the part of the background material needed for each step).

We shall now describe the contents of this paper section-by-section.

0.7.2. Part 0 of the paper is devoted to the discussion of sheaves on prestacks and lax prestacks.

In Sect. 1 we specify what we mean by a *sheaf theory* on schemes, explain how each such automatically extends to give rise to the notion of sheaf on a prestack. We then discuss the technically crucial notion of *pseudo-proper* map: these are maps for which the functor of pushforward of sheaves is well-behaved. The material in this section is necessary for the formulation of the main results of this paper.

In Sect. 2 we introduce *lax prestacks* and sheaves on them. Examples of lax prestacks are  $\text{Ran}_{\text{unl}}$  and  $\text{Ran}_{\text{unl},\text{aug}}$ , and they are needed in order to perform the manipulation of *taking the unit out*, discussed in Sect. 0.4 above. The reader needs to be familiar with lax prestacks if he wants to go in any depth into the proof of the cohomological product formula.

In Sect. 3 we explore the general question of when is a map of lax prestacks  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is such that pullback with respect to it defines a fully faithful functor. This section may be skipped on the first pass and returned to when necessary. (We should say, however, that the results in this section are (i) generally useful and (ii) not difficult to formulate and prove, so it may be fun to read in any case.)

0.7.3. Part I of the paper is devoted to the discussion of various versions of the Ran space.

In Sect. 4 we introduce the usual Ran space, as well as its unital version. We construct the functor of *adding the unit* from sheaves on Ran to sheaves on  $\text{Ran}_{\text{unl}}$ . We show that the forgetful functor from sheaves on  $\text{Ran}_{\text{unl}}$  to sheaves on Ran preserves compactly supported cohomology.

In Sect. 5 we introduce yet another version of the Ran space,  $\text{Ran}_{\text{unl},\text{aug}}$ . We show that the procedure of adding units, viewed now as a functor from sheaves on Ran to sheaves on  $\text{Ran}_{\text{unl},\text{aug}}$ , is fully faithful, with the right adjoint given by the functor of *taking the units out*.

In Sect. 6 we will present an alternative (but equivalent) point of view on the unital and unital augmented Ran space. The contents of this section will not be used elsewhere in the paper.

0.7.4. Part II of the paper is devoted to the discussion of Verdier duality on the (usual) Ran space.

In Sect. 7 we introduce the functor of Verdier duality on the category of sheaves on an arbitrary prestack (for which the diagonal morphism is pseudo-proper). We then single out a class of prestacks, called *pseudo-schemes with a finitary diagonal*, for which the functor of Verdier duality is manageable (=can be expressed in terms of Verdier duality on schemes).

In Sect. 8 we specialize the discussion of Verdier duality to the case of the Ran space. First, we show that Ran is a pseudo-scheme with a finitary diagonal, so we can explicitly describe Verdier duality on Ran. We then formulate a crucial result, Theorem 8.2.4, that gives sufficient conditions on a sheaf  $\mathcal{F}$  on Ran for the map

$$C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{F})) \rightarrow C_c^*(\text{Ran}, \mathcal{F})^\vee$$

to be an isomorphism.

In Sect. 9 we prove Theorem 8.2.4 and some other results of similar nature.

0.7.5. Part III is devoted to the discussion of the interaction of the procedure of inserting the unit (from Part I) with Verdier duality on  $\text{Ran}$ .

In Sect. 10 we discuss the following question: given a pair of sheaves  $\mathcal{F}_{\text{untl, aug}}$ ,  $\mathcal{G}_{\text{untl, aug}}$  on  $\text{Ran}_{\text{untl, aug}}$ , what kind of datum on them gives rise to a map

$$\mathcal{G}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{F}_{\text{red}})?$$

In the above formula,  $\mathcal{F}_{\text{red}}$  and  $\mathcal{G}_{\text{red}}$  are obtained from  $\mathcal{F}_{\text{untl, aug}}$  and  $\mathcal{G}_{\text{untl, aug}}$ , respectively, by *taking the units out*. It turns out that the corresponding datum on  $\mathcal{F}_{\text{untl, aug}}$ ,  $\mathcal{G}_{\text{untl, aug}}$  is a map

$$\mathcal{F}_{\text{untl, aug}} \boxtimes \mathcal{G}_{\text{untl, aug}} \rightarrow \omega_{\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}}},$$

defined over a certain open subset of  $\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}}$ . Note that this is very different from the usual Verdier duality on a prestack  $\mathcal{Y}$ , where the map goes to  $(\text{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}})$ .

In Sect. 11 we give an explicit construction of the  $!$ -fiber of the Verdier dual of a sheaf  $\mathcal{F}$  on the  $\text{Ran}$  space in terms of the sheaf  $\mathcal{F}_{\text{untl, aug}}$  on  $\text{Ran}_{\text{untl, aug}}$ , obtained from  $\mathcal{F}$  by inserting the unit.

0.7.6. Part IV is devoted to a structure that one can impose on sheaves on  $\text{Ran}$ , known as *factorization*.

In Sect. 12 we introduce the notions of commutative factorization algebra and cocommutative factorization coalgebra in sheaves on  $\text{Ran}$ . We show that, when a certain cohomological estimate is satisfied, Verdier duality maps cocommutative factorization coalgebras to commutative factorization algebras.

In Sect. 13 we study how the procedure of inserting the unit from Part I interacts with factorization. It turns out that for sheaves on  $\text{Ran}_{\text{untl, aug}}$  one can introduce its own notions of commutative factorization algebra and cocommutative factorization coalgebra (very different in spirit from those on the usual  $\text{Ran}$ ). Yet, we show that the two notions of factorization match up under the functor of inserting the unit.

0.7.7. Part V is the core of this paper, in which we assemble all the pieces of theory developed hitherto and prove the cohomological product formula.

In Sect. 14 we state the cohomological product formula. We also recall the statement of non-abelian Poincaré duality, from which we deduce that the cohomological product formula follows from the fact that a certain (specified) pairing between  $C_c^*(\text{Ran}, \mathcal{A})$  and  $C_c^*(\text{Ran}, \mathcal{B})$  defines an isomorphism

$$(0.14) \quad C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C_c^*(\text{Ran}, \mathcal{B})^\vee.$$

Sect. 15 is where everything happens. We use the material in Parts I and III of the paper to state the *local duality* theorem, and we use Part II of the paper to show the the local duality theorem implies the isomorphism (0.14).

In Sect. 16 we reduce the local duality theorem to the pointwise duality theorem, using the material from Part IV of the paper.

Finally, in Sects. 17 and 18 we give two proof of the pointwise duality theorem.

0.7.8. In Part VI we apply the cohomological product formula to deduce the numerical product formula (0.3).

In Sect. 19 we deduce from the cohomological product formula the Atiyah-Bott formula, which is an explicit (however, non-canonical) description of the algebra of cochains on  $\text{Bun}_G$ .

In Sect. 20 we use the the Atiyah-Bott formula to calculate the trace of  $\text{Frob}^{-1}$  on  $H^*(\text{Bun}_G)$ , thereby proving the numerical product formula (0.3).

## 0.8. Conventions and notation.

0.8.1. *Algebraic geometry.* Throughout the paper  $k$  will be an algebraically closed ground field.

We shall denote by  $\text{Sch}$  the category of *separated*  $k$ -schemes of finite type. In this paper we do *not* use derived algebraic geometry, so “schemes” are classical schemes.

We use the notation  $\text{pt}$  for  $\text{Spec}(k)$ .

0.8.2. *Higher category theory: why?* Higher category theory is indispensable in this paper.

Let us first point out the place where the use of higher categories is *not* essential: we define prestacks (resp., lax prestacks) to be functors from the category of schemes to that of  $\infty$ -groupoids (resp.,  $\infty$ -categories). This is not necessary: the prestacks (resp., lax prestacks) that we shall encounter in this paper take values in ordinary groupoids (resp., ordinary categories).<sup>4</sup>

The place where higher categories are essential is that we need the category  $\text{Shv}(S)$  of sheaves on a scheme  $S$  to be regarded as an  $\infty$ -category. In fact, we need more: we need the assignment that sends a scheme to the category of sheaves on it to be a functor:

$$\text{Shv} : (\text{Sch})^{\text{op}} \rightarrow \infty\text{-Cat}, \quad S \mapsto \text{Shv}(S)$$

where  $\text{Sch}$  is the (ordinary) category of schemes, perceived as an  $\infty$ -category, and  $\infty\text{-Cat}$  is *the*  $\infty$ -category of  $\infty$ -categories.

The reason we need this (as opposed to have sheaves form, say, a triangulated category) is that in order to define sheaves on a prestack  $\mathcal{Y}$ , we will be taking the *limit* of the categories  $\text{Shv}(S)$  over the index category of schemes  $S$  mapping to  $\mathcal{Y}$ . Now, in order for this limit to be well-behaved we need it to be taken in  $\infty\text{-Cat}$ .

We should say that our use of higher category theory is model-independent. I.e., one can perceive them as quasi-categories (i.e., a particular kind simplicial sets), but any other model (e.g., complete Segal spaces) will do as well. All the statements that we ever make are homotopy-invariant.

0.8.3. *Higher category theory: glossary.* For the most part, our use of higher category theory is limited to the material covered in [Lu1]. Here are some basic notions and pieces of notation that the reader should know:

We let  $\text{Spc}$  denote the  $\infty$ -category of  $\infty$ -groupoid.

An  $\infty$ -category  $\mathbf{C}$  is an  $\infty$ -groupoid of all 1-morphisms in  $\mathbf{C}$  are isomorphisms. For every  $\mathbf{C}$  there exists a universal  $\infty$ -groupoid that receives a map from  $\mathbf{C}$ ; we call it the enveloping groupoid of  $\mathbf{C}$  and denote it by  $\mathbf{C}_{\text{str}}$ ; it is obtained from  $\mathbf{C}$  by inverting all 1-morphisms. We shall say that  $\mathbf{C}$  is *contractible* or that it has a *trivial homotopy type* if  $\mathbf{C}_{\text{str}}$  is isomorphic, as an  $\infty$ -groupoid, to  $\{*\}$  (the one-point category).

For an  $\infty$ -category  $\mathbf{C}$  and objects  $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$  we write  $\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$  for the space ( $\infty$ -groupoid) of maps from  $\mathbf{c}_1$  to  $\mathbf{c}_2$ .

For a functor  $F : \mathbf{C}' \rightarrow \mathbf{C}$  and an object  $\mathbf{c} \in \mathbf{C}$  we denote by  $\mathbf{C}'_{/\mathbf{c}}$  (resp.,  $\mathbf{C}'_{/\mathbf{c}}$ ) the corresponding *under-category* (resp., *over-category*). By definition, this is the category of pairs  $(\mathbf{c}' \in \mathbf{C}', \mathbf{c} \rightarrow F(\mathbf{c}'))$  (resp.,  $(\mathbf{c}' \in \mathbf{C}', F(\mathbf{c}') \rightarrow \mathbf{c})$ ). We let  $\mathbf{C}'_{\mathbf{c}}$  to be the *fiber* of  $\mathbf{C}'$  over  $\mathbf{c}$ , i.e., the category of pairs  $(\mathbf{c}' \in \mathbf{C}', \mathbf{c} \simeq F(\mathbf{c}'))$ , where  $\simeq$  means a (specified) isomorphism.

Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor and let  $\alpha : \mathbf{c}_1 \rightarrow \mathbf{c}_2$  be a 1-morphism in  $\mathbf{C}$ . We shall say that  $\alpha$  is *coCartesian* if for any  $\mathbf{c} \in \mathbf{C}$  the map

$$\text{Maps}_{\mathbf{C}}(\mathbf{c}_2, \mathbf{c}) \rightarrow \text{Maps}_{\mathbf{D}}(F(\mathbf{c}_2), F(\mathbf{c})) \times_{\text{Maps}_{\mathbf{D}}(F(\mathbf{c}_1), F(\mathbf{c}))} \text{Maps}(\mathbf{c}_1, \mathbf{c})$$

is an isomorphism (of  $\infty$ -groupoids). A 1-morphism is said to be *Cartesian* if it is *coCartesian* for the functor  $F^{\text{op}} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ . A functor  $F$  is said to be a *locally coCartesian fibration* (resp., *locally Cartesian fibration*) if for every 1-morphism  $\mathbf{d}_1 \rightarrow \mathbf{d}_2$  in  $\mathbf{D}$  there exists a *coCartesian* (resp., *Cartesian*) 1-morphism  $\alpha : \mathbf{c}_1 \rightarrow \mathbf{c}_2$  equipped with a datum of commutativity for the diagram

$$\begin{array}{ccc} F(\mathbf{c}_1) & \xrightarrow{F(\alpha)} & F(\mathbf{c}_2) \\ \sim \downarrow & & \downarrow \sim \\ \mathbf{d}_1 & \longrightarrow & \mathbf{d}_2. \end{array}$$

<sup>4</sup>The latter is not surprising: if one stays within the realm of classical (as opposed to derived) algebraic geometry, higher prestacks or lax prestacks are far less ubiquitous.

A functor  $F$  is said to be a *coCartesian fibration* (resp., *Cartesian fibration*) if it is a *locally coCartesian fibration* (resp., *locally Cartesian fibration*) and the composition of two coCartesian (resp., Cartesian) 1-morphisms is coCartesian (resp., Cartesian). A functor  $F$  is said to be a *coCartesian fibration in groupoids* (resp., *Cartesian fibration in groupoids*) if, in addition, its fibers are  $\infty$ -groupoids. We refer the reader to [Lu1, Sect. 2.4] for more details.

If  $\mathbf{I}$  is an index  $\infty$ -category and  $\Phi : \mathbf{I} \rightarrow \mathbf{C}$  is a functor, where  $\mathbf{C}$  is another  $\infty$ -category, we can talk about its colimit or limit in  $\mathbf{C}$ , denoted,

$$\operatorname{colim}_{i \in \mathbf{I}} \Phi(i) \text{ and } \lim_{i \in \mathbf{I}} \Phi(i),$$

respectively, which, if they exist, are defined uniquely up to a canonical isomorphism (in proper language, up to a contractible set of choices). We shall say that  $\mathbf{C}$  is *cocomplete* if the colimits of all functors from all index categories  $\mathbf{I}$  (satisfying a certain cardinality condition that we ignore) exist. The reader is referred to [Lu1, Sect. 1.2.13] for more details and references for the actual definition.

Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor. Then for another  $\infty$ -category  $\mathbf{E}$ , precomposition with  $F$  defines a functor

$$\operatorname{Funct}(\mathbf{D}, \mathbf{E}) \rightarrow \operatorname{Funct}(\mathbf{C}, \mathbf{E}).$$

The (partially defined left (resp., right) adjoint of this functor is called the functor of left (resp., right) Kan extension. The left (resp., right) Kan extension of a given functor  $\Phi : \mathbf{C} \rightarrow \mathbf{E}$  can be calculated explicitly on objects:

$$\operatorname{LKE}(\Phi)(\mathbf{d}) = \operatorname{colim}_{\mathbf{c} \in \mathbf{C}_{/\mathbf{d}}} \Phi(\mathbf{c}) \text{ and } \operatorname{RKE}(\Phi)(\mathbf{d}) = \lim_{\mathbf{c} \in \mathbf{C}_{\mathbf{d}/}} \Phi(\mathbf{c}).$$

If  $F$  is a coCartesian (resp., Cartesian) fibration, then in the above colimit (resp., limit) the index category can be replaced by the fiber  $\mathbf{C}_{\mathbf{d}}$ . See [Lu1, Sect. 4.3] for more details.

0.8.4. *Higher algebra.* We shall need the following notions from higher algebra, developed in [Lu2]: stable  $\infty$ -category (Sect .1), symmetric monoidal  $\infty$ -category (Definition 2.0.0.7), and commutative algebra in a symmetric monoidal  $\infty$ -category (Sect. 2.1.3).

0.8.5. *A lemma on limits vs colimits.* In two places in the text we will encounter the following situation. Let  $\infty\text{-Cat}^{\operatorname{St}} \subset \infty\text{-Cat}$  be the non-full subcategory, where we restrict the objects to be cocomplete stable categories and 1-morphisms to be colimit-preserving. First, we note that by [Lu1, Proposition 5.5.3.13], the above inclusion preserves limits.

Let

$$\Phi : \mathbf{I} \rightarrow \infty\text{-Cat}^{\operatorname{St}}, \quad i \mapsto \mathbf{C}_i, \quad (i_1 \xrightarrow{\alpha} i_2) \mapsto \Phi_{\alpha} \in \operatorname{Funct}(\mathbf{C}_{i_1}, \mathbf{C}_{i_2})$$

be a functor. Denote

$$\mathbf{C} := \operatorname{colim}_{i \in \mathbf{I}} \mathbf{C}_i,$$

where the *colimit is taken in*  $\infty\text{-Cat}^{\operatorname{St}}$ . For every  $i \in \mathbf{I}$ , we have a tautologically defined functor  $\operatorname{ins}_i : \mathbf{C}_i \rightarrow \mathbf{C}$ .

By the Adjoint Functor Theorem (see [Lu1, Corollary 5.5.2.9(i)]), for every arrow  $i_1 \xrightarrow{\alpha} i_2$  in  $\mathbf{I}$ , the corresponding functor  $\Phi_{\alpha}$  admits a right adjoint  $\Psi_{\alpha}$  (which is not necessarily continuous). Consider the functor

$$\Psi : \mathbf{I}^{\operatorname{op}} \rightarrow \infty\text{-Cat},$$

which is the same as  $\Phi$  on objects, but which sends

$$(i_1 \xrightarrow{\alpha} i_2) \mapsto \Psi_{\alpha}.$$

The right adjoints to the functors  $\operatorname{ins}_i$ , denoted  $\operatorname{ev}_i$ , define a functor

$$(0.15) \quad \mathbf{C} \rightarrow \lim_{i \in \mathbf{I}^{\operatorname{op}}} \mathbf{C}_i,$$

where in the right-hand side we are taking the *limit of the functor*  $\Psi$  *in*  $\infty\text{-Cat}$ .

Suppose now that in the above situation the functors  $\Psi_\alpha$  are also colimit preserving. Then it follows that the functors  $ev_i$  are also colimit preserving. In this case it is easy to show that for every  $\mathbf{c}$  the map

$$\text{colim}_{i \in \mathbf{I}} \text{ins}_i \circ ev_i(\mathbf{c}) \rightarrow \mathbf{c}$$

is an isomorphism.

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## Part 0: Preliminaries on prestacks, lax prestacks and sheaves

### 1. SHEAVES AND PRESTACKS

The goal of this paper is to prove a certain isomorphism of vector spaces (the *Atiyah-Bott formula*, (0.7)). One of these vector spaces is defined by an explicit elementary procedure and another as a cohomology of something. So neither side explicitly mentions sheaves (that is to say, non-constant sheaves). However, our way of proving the desired isomorphism heavily uses sheaves—they already appear in the main step of the proof, namely, the *cohomological product formula*, (0.7).

We are used to considering sheaves on schemes. However, even to state the product formula, we need a slightly more general set-up: namely, we will need to consider sheaves on *prestacks*.

The goal of the present section is to define what prestacks are and what we mean by a sheaf on a prestack. These notions are necessary for understanding the core of this paper, namely Part V.

**1.1. Sheaves on schemes.** In this subsection we explain what we mean by a *theory of sheaves on schemes*. The discussion here is aimed at the technically-minded reader. So, the reader who is willing to understand sheaves on schemes intuitively can skip this subsection and proceed to Sect. 1.2.

1.1.1. In this paper we assume being given a *theory of sheaves*, which is a functor

$$\mathrm{Shv}^! : (\mathrm{Sch})^{\mathrm{op}} \rightarrow \infty\text{-Cat}.$$

I.e., to a scheme  $S$  we assign an  $\infty$ -category  $\mathrm{Shv}(S)$  and to a morphism  $f : S_1 \rightarrow S_2$  we assign a pullback functor

$$f^! : \mathrm{Shv}(S_2) \rightarrow \mathrm{Shv}(S_1).$$

These functors are endowed with a homotopy-coherent system of compatibilities for compositions of morphisms.

*Remark 1.1.2.* We should explain why we are using the  $!$ -pullback rather than the  $*$ -pullback. The reason is that the prestacks that we will consider are “ind-objects” (colimits of schemes under closed embeddings), and we want to think of a sheaf on such a prestack as a colimit of sheaves on schemes that comprise it. This dictates the choice of the  $!$ -pullback over the  $*$ -pullback, it being the *right adjoint* of the direct image functor, see Sect. 0.8.5.

1.1.3. *Technical assumptions.* We assume the following additional *properties* of the functor  $\mathrm{Shv}^!$ :

- (i) It takes values in the full subcategory of  $\infty$ -Cat formed by cocomplete stable categories.
- (ii) For every morphism  $f : S_1 \rightarrow S_2$  in Sch, the corresponding functor  $f^! : \mathrm{Shv}(S_2) \rightarrow \mathrm{Shv}(S_1)$  commutes with colimits.

In other words,  $\mathrm{Shv}^!$  (uniquely) factors through a non-full subcategory  $\infty\text{-Cat}^{\mathrm{St}} \subset \infty\text{-Cat}$ , where we restrict the objects to be cocomplete stable categories and 1-morphisms to be colimit-preserving.

1.1.4. *Symmetric monoidal structure.* Recall that the category  $\infty\text{-Cat}^{\mathrm{St}}$  has a natural symmetric monoidal structure given by tensor product. We shall assume that our functor  $\mathrm{Shv}^!$  is endowed with the following additional pieces of structure:

- (a)  $\mathrm{Shv}^!$  is endowed with a *right-lax symmetric monoidal structure*, where Sch is considered as equipped with the Cartesian symmetric monoidal structure.

The datum in (a) stipulates the existence of a functor

$$\mathrm{Shv}(S_1) \otimes \mathrm{Shv}(S_2) \rightarrow \mathrm{Shv}(S_1 \times S_2), \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \boxtimes \mathcal{F}_2,$$

equipped with a homotopy-coherent system of compatibilities.

In particular, for any  $S$ , pullback with respect to the diagonal morphism  $S \xrightarrow{\mathrm{diag}_S} S \times S$  defines on  $\mathrm{Shv}(S)$  a structure of symmetric monoidal category

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2 := \mathrm{diag}_S^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$



We shall refer to this structure as the *pointwise* symmetric monoidal structure.

The second additional piece of structure on the functor  $\text{Shv}^!$  is:

(b) The symmetric monoidal category  $\text{Shv}(\text{pt})$  is identified with  $\Lambda\text{-mod}$ , where  $\Lambda$  is our (fixed) commutative ring of coefficients.

1.1.5. Here are some examples of sheaf theories:

(1) When the ground field is  $\mathbb{C}$  and an arbitrary ring  $\Lambda$ , we can take  $\text{Shv}(S)$  to be the ind-completion of the category of constructible sheaves on  $S$  with  $\Lambda$ -coefficients.

(2) For any ground field  $k$  and  $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$  (where  $\ell$  is assumed to be invertible in  $k$ ), and we take  $\text{Shv}(S)$  to be the ind-completion of the category of constructible étale sheaves on  $S$  with  $\Lambda$ -coefficients, see [Main Text, Sect. 4].

(3) When the ground field has characteristic 0, we take  $\text{Shv}(S)$  to be the category of holonomic D-modules on  $S$ .

(4) When the ground field has characteristic 0, we take  $\text{Shv}(S)$  to be the category of all D-modules on  $S$ .

1.1.6. In what follows, we shall refer to Examples (1)-(3) above as the *context of constructible sheaves*. Its main feature is that for any morphism  $f : S_1 \rightarrow S_2$ , the functor  $f^!$  admits a left adjoint, denoted  $f_!$ . (We note that this property fails in general in example (4), unless the morphism  $f$  is proper.)

In particular, we obtain that in the constructible context, the functor  $f^!$  commutes with *limits*.

In order for a sheaf theory to work well in applications, one should impose several more conditions (e.g., proper base change and the Künneth formula).<sup>5</sup> Rather than listing these conditions, we shall assume that our sheaf theory is one of the examples (1)-(4) listed above.

Note that in each of these examples, for a scheme  $S$ , the category  $\text{Shv}(S)$  possesses a t-structure. In fact, in the constructible context there are two t-structures: the usual and the perverse one. In the context of D-modules, there is the usual D-module t-structure.

1.2. **Prestacks.** A prestack is an arbitrary contravariant functor from the category of affine schemes to that of groupoids. In this way prestacks are geometric objects that generalize algebraic stacks or ind-schemes. When working over the ground field of complex numbers, in many cases the behavior of a prestack is well approximated by that of the underlying topological space (for example, see Remark 1.4.6 below).

1.2.1. By definition, a prestack is an arbitrary functor

$$(\text{Sch})^{\text{op}} \rightarrow \text{Spc}.$$

We let  $\text{PreStk}$  denote the  $\infty$ -category of prestacks.

1.2.2. Yoneda embedding defines a fully-faithful functor

$$\text{Sch} \hookrightarrow \text{PreStk}.$$

1.3. **Sheaves on prestacks.** Sheaves on prestacks will be defined in a very straightforward way, see below. Of course, we will not be able to say much about the category of sheaves on a general prestack. But fortunately, the very formal properties that one gets for free from the definition will suffice for our purposes.

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<sup>5</sup>We emphasize that whatever these additional conditions are, they are *properties* (i.e., certain maps should be isomorphisms), but *not* additional pieces of structure.

1.3.1. For a prestack  $\mathcal{Y}$ , the category  $\mathrm{Shv}^!(\mathcal{Y})$  is defined by

$$\mathrm{Shv}^!(\mathcal{Y}) := \lim_{S \in \mathrm{Sch}/\mathcal{Y}} \mathrm{Shv}(S),$$

where for a morphism  $f : S_1 \rightarrow S_2$  in  $\mathrm{Sch}/\mathcal{Y}$  the corresponding transition functor  $\mathrm{Shv}(S_2) \rightarrow \mathrm{Shv}(S_1)$  is  $f^!$ .

1.3.2. In other words, an object  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y})$  is an assignment of

- $S \in \mathrm{Sch}, y \in \mathcal{Y}(S) \mapsto \mathcal{F}_{S,y} \in \mathrm{Shv}^!(S),$
- $(S' \xrightarrow{f} S) \in \mathrm{Sch}, y \in \mathcal{Y}(S_2) \mapsto (\mathcal{F}_{S',f(y)} \simeq f^!(\mathcal{F}_{S,y})) \in \mathrm{Shv}^!(S'),$

satisfying a homotopy-coherent system of compatibilities.

1.3.3. A bit more functorially, one can view the assignment  $\mathcal{Y} \mapsto \mathrm{Shv}^!(\mathcal{Y})$  as the *right Kan extension* of the functor

$$\mathrm{Shv}^! : (\mathrm{Sch})^{\mathrm{op}} \rightarrow \infty\text{-Cat}$$

along the Yoneda embedding

$$(\mathrm{Sch})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}.$$

In particular, for a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  between prestacks, we have a tautologically defined functor

$$f^! : \mathrm{Shv}^!(\mathcal{Y}_2) \rightarrow \mathrm{Shv}^!(\mathcal{Y}_1).$$

Notation: sometimes, for  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y}_2)$  we shall use a shorthand notation

$$\mathcal{F}|_{\mathcal{Y}_1} := f^!(\mathcal{F}).$$

For a prestack  $\mathcal{Y}$  we let  $\omega_{\mathcal{Y}} \in \mathrm{Shv}^!(\mathcal{Y})$  be the *dualizing sheaf*, i.e., the  $!$ -pullback of  $\Lambda \in \mathrm{Shv}(\mathrm{pt})$ .

1.3.4. The additional assumptions on the functor  $\mathrm{Shv}^!$  of Sects. 1.1.3 and 1.1.4 imply that the resulting functor

$$\mathrm{Shv}^! : (\mathrm{PreStk})^{\mathrm{op}} \rightarrow \infty\text{-Cat}$$

also factors via a functor to  $\infty\text{-Cat}^{\mathrm{St}}$  and as such is endowed with a symmetric monoidal structure. In particular, every  $\mathrm{Shv}^!(\mathcal{Y})$  acquires a symmetric monoidal structure, and a symmetric monoidal functor from  $\Lambda\text{-mod}$ .

**Lemma 1.3.5.** *In the context of constructible sheaves <sup>6</sup>, the functor  $f^!$  commutes with limits.*

*Proof.* Since the functor of  $!$ -pullback commutes with limits for morphisms between *schemes*, for a prestack  $\mathcal{Y}$ , limits in  $\mathrm{Shv}^!(\mathcal{Y})$  are computed value-wise, i.e., for a family  $a \mapsto \mathcal{F}^a$  and  $(S, y) \in \mathrm{Sch}/\mathcal{Y}$ , we have

$$(\lim_a \mathcal{F}^a)_{S,y} \simeq \lim_a (\mathcal{F}_{S,y}^a).$$

This makes the assertion of the lemma manifest. □

**1.4. Direct images?** We now have the theory of sheaves on prestacks, but the only functoriality so far is the  $!$ -pullback. One can wonder: what about the other functors, such as the  $*$ -pullback,  $*$ -pushforward or  $!$ -pushforward? The answer is that we will not even attempt to define them, but only grab whatever naturally comes our way.

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<sup>6</sup>See Sect. 1.1.6 for what we mean by that.

1.4.1. For a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  we can consider the *partially defined* left adjoint of  $f^!$

$$f_! : \text{Shv}^!(\mathcal{Y}_1) \rightarrow \text{Shv}^!(\mathcal{Y}_2).$$

We have:

**Corollary 1.4.2.** *In the context of constructible sheaves, the functor  $f_!$  is always defined.*

*Proof.* Follows from the Adjoint Functor Theorem (see [Lu1, Corollary 5.5.2.9(ii)]) and Lemma 1.3.5.  $\square$

1.4.3. In particular, taking  $\mathcal{Y}_1 = \mathcal{Y}$  and  $\mathcal{Y}_2 = \text{pt}$  we obtain a (partially defined) functor

$$\mathcal{F} \mapsto C_c^*(\mathcal{Y}, \mathcal{F}), \quad \text{Shv}^!(\mathcal{Y}) \rightarrow \Lambda\text{-mod},$$

left adjoint to the pullback functor (the latter being the same as  $- \otimes \omega_{\mathcal{Y}}$ ).

By Corollary 1.4.2, the functor  $C_c^*(\mathcal{Y}, -)$  is always defined in the context of constructible sheaves. In general, we have:

**Lemma 1.4.4.** *For  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$  we have*

$$C_c^*(\mathcal{Y}, \mathcal{F}) \simeq \text{colim}_{(S, y) \in \text{Sch}/\mathcal{Y}} C_c^*(S, \mathcal{F}_{S, y}),$$

whenever the right-hand side is defined.

**Corollary 1.4.5.** *In the context of  $D$ -modules, the functor  $C_c^*(\mathcal{Y}, -)$  is defined on any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$  for which for every  $(S, y)$ , the corresponding object  $\mathcal{F}_{S, y} \in \text{Shv}(S)$  is holonomic.*

In particular, in any context, the functor  $C_c^*(\mathcal{Y}, -)$  is defined on  $\omega_{\mathcal{Y}} \in \text{Shv}^!(\mathcal{Y})$ . We shall use the following notation:

$$C_c^*(\mathcal{Y}, \omega_{\mathcal{Y}}) =: C_*(\mathcal{Y}),$$

and we shall refer to  $C_*(\mathcal{Y})$  as the *homology* of  $\mathcal{Y}$ . In addition, we denote:

$$\text{Fib}\left((p_{\mathcal{Y}})_! \circ (p_{\mathcal{Y}})^!(\Lambda) \rightarrow \Lambda\right) = \text{Fib}(C_*(\mathcal{Y}) \rightarrow C_*(\text{pt})) =: C_*^{\text{red}}(\mathcal{Y});$$

this is the reduced homology of  $\mathcal{Y}$ .

*Remark 1.4.6.* When working over the ground field of complex numbers, the functor that associates to a scheme the underlying analytic space gives rise, by means of *left Kan extension*, to a functor from the category of prestacks to the  $\infty$ -category of topological spaces. Under this functor, the *homology of a prestack* introduced above is isomorphic to the homology of the corresponding topological space with coefficients in  $\Lambda$ .

1.4.7. For later use we also note the following:

**Lemma 1.4.8.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a schematic open embedding (i.e., the base change of  $f$  by any scheme yields an open embedding). Then the functor  $f_* : \text{Shv}^!(\mathcal{Y}_1) \rightarrow \text{Shv}^!(\mathcal{Y}_2)$ , right adjoint to  $f^!$ , is defined. Moreover, for a Cartesian diagram of prestacks*

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}'_2 & \xrightarrow{g_2} & \mathcal{Y}_2 \end{array}$$

the natural transformation

$$g_2^! \circ f_* \rightarrow f'_* \circ g_1^!,$$

that comes by adjunction from the isomorphism

$$f'^! \circ g_2^! \simeq g_1^! \circ f^!,$$

is an isomorphism.

*Proof.* Follows formally from the case when  $\mathcal{Y}_2$  (resp.,  $\mathcal{Y}_2$  and  $\mathcal{Y}'_2$ ) are schemes, in which case it is the usual base change isomorphism.  $\square$

**1.5. Pseudo-properness.** As we remarked above, for a morphism of prestacks  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  and  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y}_1)$  we often have a well-defined object

$$f_!(\mathcal{F}) \in \mathrm{Shv}^!(\mathcal{Y}_2).$$

However, in most cases this object is incalculable: we do not have an algorithm to say what the value of  $f_!(\mathcal{F})$  is on  $S \xrightarrow{y} \mathcal{Y}$ .

But there is one class of morphisms where  $f_!$  is given by a much more explicit procedure: these are maps that we call *pseudo-proper*, and which are ubiquitous in this paper.

1.5.1. Let  $f : \mathcal{Y} \rightarrow S$  be a map in  $\mathrm{PreStk}$ , where  $S \in \mathrm{Sch}$ . We shall say that  $\mathcal{Y}$  is *pseudo-proper* over  $S$  if  $\mathcal{Y}$  can be written as a colimit of objects representable by schemes proper over  $S$ .

We have:

**Proposition 1.5.2.** *For a pseudo-proper  $f : \mathcal{Y} \rightarrow S$ , the functor  $f_!$ , left adjoint to  $f^!$ , is defined. Furthermore, for a map of schemes  $g_S : S' \rightarrow S$  and  $\mathcal{Y}' := \mathcal{Y} \times_S S'$ , the natural transformation*

$$f'_! \circ g_Y^! \rightarrow g_S^! \circ f_!,$$

arising from the Cartesian diagram

$$(1.1) \quad \begin{array}{ccc} \mathcal{Y}' & \xrightarrow{g_Y} & \mathcal{Y} \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g_S} & S, \end{array}$$

and the identification

$$g_Y^! \circ f^! \simeq f'^! \circ g_S^!,$$

is an isomorphism.

*Proof.* Write

$$\mathcal{Y} = \mathrm{colim}_a Z_a,$$

where  $Z_a$  are schemes proper over  $S$ . By definition, we have:

$$\mathrm{Shv}^!(\mathcal{Y}) = \lim_a \mathrm{Shv}(Z_a),$$

where the limit is formed using the functors

$$f_{a_1, a_2}^! : \mathrm{Shv}(Z_{a_2}) \rightarrow \mathrm{Shv}(Z_{a_1}) \text{ for } f_{a_1, a_2} : Z_{a_1} \rightarrow Z_{a_2}.$$

Now, by Sect. 0.8.5, we have:

$$\mathrm{Shv}^!(\mathcal{Y}) = \mathrm{colim}_a \mathrm{Shv}(Z_a),$$

where the colimit is taken in the category of cocomplete categories and colimit-preserving functors. In the formation of the limit we use the functors

$$(f_{a_1, a_2})_! : \mathrm{Shv}(Z_{a_2}) \rightarrow \mathrm{Shv}(Z_{a_1}) \text{ for } f_{a_1, a_2} : Z_{a_1} \rightarrow Z_{a_2}.$$

Now, the sought-for left adjoint, viewed as a functor

$$\mathrm{colim}_a \mathrm{Shv}(Z_a) \rightarrow \mathrm{Shv}(S),$$

is given by the compatible family of functors

$$(f_a)_! : \mathrm{Shv}(Z_a) \rightarrow \mathrm{Shv}(S) \text{ for } f_a : Z_a \rightarrow S.$$

The commutation of the diagram (1.1) follows by base change, as the maps  $f_a$  are proper. <sup>7</sup> □

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<sup>7</sup>Note that in the context of constructible  $\ell$ -adic sheaves, the existence of  $f_!$  does not require the maps  $f_a$  to be proper, as the functors  $(f_a)_!$  are always defined (but this would not be the case in the context of arbitrary, i.e., not necessarily holonomic D-modules). However, the properness of the maps  $f_a$  is needed for the commutation of (1.1).

1.5.3. We shall say that a map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  between prestacks is pseudo-proper if its base change by any scheme yields a prestack over that scheme.

We shall make a repeated use of the following assertion:

**Corollary 1.5.4.** *If  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is pseudo-proper, then the functor  $f_!$ , left adjoint to  $f^!$ , is defined. Furthermore, for a map  $g_2 : \mathcal{Y}'_2 \rightarrow \mathcal{Y}_2$  and  $\mathcal{Y}'_1 := \mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{Y}'_2$ , the natural transformation*

$$f'_! \circ g_1^! \rightarrow g_2^! \circ f_!,$$

arising from the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}'_2 & \xrightarrow{g_2} & \mathcal{Y}_2, \end{array}$$

and the identification

$$g_1^! \circ f^! \simeq f'^! \circ g_2^!,$$

is an isomorphism.

*Proof.* Follows formally from Proposition 1.5.2. □

Note that the base change property in Corollary 1.5.4 in particular gives an expression to the value of  $f_!(\mathcal{F})$  on any  $S \xrightarrow{y} \mathcal{Y}_2$ : form the pullback

$$\mathcal{Y} := S \times_{\mathcal{Y}_2} \mathcal{Y}_1$$

and calculate the  $!$ -pushforward of  $\mathcal{F}|_{\mathcal{Y}}$  to  $S$ .

## 2. LAX PRESTACKS AND SHEAVES

The contents of this section are needed for Parts I and III of the paper, which in turn will be used in Sect. 15 (the reduction of the cohomological product formula to a local statement).

It turns that even such general gadgets as prestacks will not be sufficient for a crucial manipulation that we will need to perform for the proof of the product formula. We will need to generalize them in a yet different direction, which in a sense is non-geometric.

We will need to consider *lax prestacks*, where those are functors that take values not in *groupoids*, but rather in *categories*. I.e., we will allow non-invertible morphisms between points. When dealing with lax prestacks, the conventional geometric intuition had better be abandoned for reasons of safety.

The idea of working with sheaves on lax prestacks was pioneered by S. Raskin, and to the best of our knowledge was first used in his PhD thesis (so far, unpublished).

**2.1. Lax prestacks.** In this subsection we define what we mean by a *lax prestack*.

2.1.1. We let  $\text{LaxPreStk}$  denote the category of functors<sup>8</sup>

$$(\text{Sch})^{\text{op}} \rightarrow \infty\text{-Cat}.$$

In other words, a prestack is an assignment for any  $S \in \text{Sch}$  of a category  $\mathcal{Y}(S)$  and for  $S' \xrightarrow{f} S$  of a functor

$$y \mapsto f(y), \quad \mathcal{Y}(S) \rightarrow \mathcal{Y}(S'),$$

plus a homotopy-coherent system of compatibilities for compositions.

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<sup>8</sup>Technically we need to consider functors that are *accessible* (see [Lu1, Sect. 5.4.2] for what this means) and that take values in essentially small categories. That said, from now on we shall ignore set-theoretical issues of this sort.

Alternatively, we can think of  $\text{LaxPreStk}$  as the category of Cartesian fibrations over  $\text{Sch}$ . For  $\mathcal{Y} \in \text{LaxPreStk}$  we let  $\text{Sch}/_{\mathcal{Y}}$  denote the corresponding category, fibered over  $\text{Sch}$ .

For  $\mathcal{Y} \in \text{LaxPreStk}$  let  $p_{\mathcal{Y}} : \mathcal{Y} \rightarrow \text{pt}$  denote the tautological map.

2.1.2. The category of lax prestacks contains as a full subcategory the category  $\text{PreStk}$  of prestacks: it consists of those functors that take values in  $\text{Spc} \subset \infty\text{-Cat}$ .

2.1.3. For  $\mathcal{Y} \in \text{LaxPreStk}$  we let  $\mathcal{Y}^{\text{op}}$  be the lax prestack obtained by passing to fiberwise opposite categories in

$$\text{Sch}/_{\mathcal{Y}} \rightarrow \text{Sch}.$$

Evidently, if  $\mathcal{Y} \in \text{PreStk}$ , then  $\mathcal{Y}^{\text{op}} \simeq \mathcal{Y}$ .

2.1.4. To a lax prestack  $\mathcal{Y}$  we attach a prestack  $\mathcal{Y}_{\text{str}}$  defined by

$$\mathcal{Y}_{\text{str}}(S) = \text{colim}_{y \in \mathcal{Y}(S)} \{*\}.$$

I.e.,  $\mathcal{Y}_{\text{str}}(S)$  is the groupoid, universal with respect to the property of receiving a functor from  $\mathcal{Y}(S)$ ; explicitly, it is obtained by inverting all the morphisms in  $\mathcal{Y}(S)$ .

We have a map  $\mathcal{Y} \rightarrow \mathcal{Y}_{\text{str}}$ , which is universal with respect to maps from  $\mathcal{Y}$  to prestacks.

We have:

$$\mathcal{Y}_{\text{str}} \simeq (\mathcal{Y}^{\text{op}})_{\text{str}}.$$

2.2. **Sheaves on lax prestacks.** By now we are used to working with sheaves on prestacks: a sheaf on a prestack is a compatible family of sheaves on schemes that map to it. When working over  $\mathbb{C}$ , sheaves on a prestack are closely related to sheaves on the underlying topological space.

Sheaves on a lax prestack are a very different animal.

2.2.1. Let  $\text{Shv}_{\text{Sch}}^{\dagger}$  denote the Cartesian fibration over  $\text{Sch}$  that corresponds to the functor

$$\text{Shv}^{\dagger} : (\text{Sch})^{\text{op}} \rightarrow \infty\text{-Cat}, \quad S \mapsto \text{Shv}^{\dagger}(S), \quad (S_1 \xrightarrow{f} S_2) \mapsto f^{\dagger}.$$

For  $\mathcal{Y} \in \text{LaxPreStk}$  we let  $\text{Shv}^{\dagger}(\mathcal{Y})$  denote the category of functors

$$\text{Sch}/_{\mathcal{Y}} \rightarrow \text{Shv}_{\text{Sch}}^{\dagger}$$

that take Cartesian arrows to Cartesian arrows.

I.e., an object  $\mathcal{F} \in \text{Shv}^{\dagger}(\mathcal{Y})$  is an assignment of

- $S \in \text{Sch}, y \in \mathcal{Y}(S) \mapsto \mathcal{F}_{S,y} \in \text{Shv}^{\dagger}(S)$ ,
- $(y_1 \rightarrow y_2) \in \mathcal{Y}(S) \mapsto (\mathcal{F}_{S,y_1} \rightarrow \mathcal{F}_{S,y_2}) \in \text{Shv}^{\dagger}(S)$ ,
- $(S' \xrightarrow{f} S) \in \text{Sch}, y \in \mathcal{Y}(S) \mapsto (\mathcal{F}_{S',f(y)} \simeq f^{\dagger}(\mathcal{F}_{S,y})) \in \text{Shv}^{\dagger}(S')$ ,

satisfying a homotopy-coherent system of compatibilities.

2.2.2. The assignment

$$\mathcal{Y} \mapsto \text{Shv}^{\dagger}(\mathcal{Y})$$

extends to a functor

$$(\text{LaxPreStk})^{\text{op}} \rightarrow \infty\text{-Cat}, \quad (\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2) \mapsto f^{\dagger}.$$

2.2.3. For a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , the functor

$$f^{\dagger} : \text{Shv}^{\dagger}(\mathcal{Y}_2) \rightarrow \text{Shv}^{\dagger}(\mathcal{Y}_1)$$

commutes with colimits, by construction.

In addition, we have the following assertion, proved in the same way as Lemma 1.3.5:

**Lemma 2.2.4.** *If we are working in the context of constructible sheaves <sup>9</sup>, the functor  $f^{\dagger}$  commutes with limits.*

<sup>9</sup>Here and in the sequel, we include here the case of holonomic D-modules.

2.2.5. Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be two lax prestacks, and let

$$f : \mathcal{Y}_1 \rightrightarrows \mathcal{Y}_2 : g$$

be maps equipped with natural transformations

$$f \circ g \rightarrow \text{id}_{\mathcal{Y}_2} \quad \text{and} \quad \text{id}_{\mathcal{Y}_1} \rightarrow g \circ f$$

making them into an adjoint pair.

The following observation will be useful:

**Lemma 2.2.6.** *Under the above circumstances, the functors*

$$g^! : \text{Shv}^!(\mathcal{Y}_2) \rightrightarrows \text{Shv}^!(\mathcal{Y}_1) : f^!$$

*form an adjoint pair.*

2.2.7. For a lax prestack  $\mathcal{Y}$ , we let  $\text{Shv}_{\text{str}}^!(\mathcal{Y}) \subset \text{Shv}^!(\mathcal{Y})$  denote the full subcategory of objects for which the arrows  $\mathcal{F}_{S, y_1} \rightarrow \mathcal{F}_{S, y_2}$  are isomorphisms for any  $S \in \text{Sch}$  and  $(y_1 \rightarrow y_2) \in \mathcal{Y}(S)$ .

Pullback along  $\mathcal{Y} \rightarrow \mathcal{Y}_{\text{str}}$  defines an equivalence

$$\text{Shv}^!(\mathcal{Y}_{\text{str}}) \rightarrow \text{Shv}_{\text{str}}^!(\mathcal{Y}).$$

In particular, if  $\mathcal{Z}$  is a prestack equipped with a map  $g : \mathcal{Y} \rightarrow \mathcal{Z}$ , then for any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Z})$ , the object  $\mathcal{F}|_{\mathcal{Y}}$  belongs to  $\text{Shv}_{\text{str}}^!(\mathcal{Y})$ .

In particular,  $\omega_{\mathcal{Y}} \in \text{Shv}_{\text{str}}^!(\mathcal{Y})$ .

2.3. **Existence of left adjoints.** We now want to address the issue of direct images for morphisms between lax prestacks. Note that this is a whole zoo: there is the \*- vs !- image. But also, at the purely categorical level with no geometry, there are the left Kan extension vs the right Kan extensions.<sup>10</sup>

In this subsection we will set for ourselves a limited goal: we will investigate some instances when the functor

$$f^! : \text{Shv}^!(\mathcal{Y}_2) \rightarrow \text{Shv}^!(\mathcal{Y}_1)$$

admits a left adjoint.

2.3.1. We will denote by  $f_!$  the (partially defined) functor, left adjoint to  $f^!$ . As in Corollary 1.4.2, the functor  $f_!$  is always defined in the context of constructible sheaves.

2.3.2. When  $\mathcal{Y}_2 = \text{pt}$  and  $\mathcal{Y}_1 = \mathcal{Y}$ , we shall denote the corresponding (partially defined) functor  $\text{Shv}^!(\mathcal{Y}) \rightarrow \Lambda\text{-mod}$  by

$$\mathcal{F} \mapsto C_c^*(\mathcal{Y}, \mathcal{F}).$$

As was already mentioned, the functor  $C_c^*(\mathcal{Y}, -)$  is always defined in the context of  $\ell$ -adic sheaves. As in Sect. 1.4.3, we have:

**Lemma 2.3.3.** *For  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$  we have*

$$C_c^*(\mathcal{Y}, \mathcal{F}) \simeq \text{colim}_{(S, y) \in \text{Sch}/\mathcal{Y}} C_c^*(S, \mathcal{F}_{S, y}),$$

*whenever the right-hand side is defined.*

In particular, in the context of D-modules, the functor  $C_c^*(\mathcal{Y}, -)$  is defined on any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$  for which for every  $(S, y)$ , the corresponding object  $\mathcal{F}_{S, y} \in \text{Shv}(S)$  is holonomic. In particular, it is always defined for the object  $\omega_{\mathcal{Y}}$ .

For  $\mathcal{F} = \omega_{\mathcal{Y}}$  we shall also use the notations

$$C_c^*(\mathcal{Y}, \omega_{\mathcal{Y}}) =: C_*(\mathcal{Y}) \quad \text{and} \quad \text{Fib}\left((p_{\mathcal{Y}})_! \circ (p_{\mathcal{Y}})^!(\Lambda) \rightarrow \Lambda\right) = \text{Fib}(C_*(\mathcal{Y}) \rightarrow C_*(\text{pt})) =: C_*^{\text{red}}(\mathcal{Y}).$$

We will refer to  $C_*(\mathcal{Y})$  (reps.,  $C_*^{\text{red}}(\mathcal{Y})$ ) as the *homology* (resp., *reduced homology*) of  $\mathcal{Y}$ .

<sup>10</sup>As was mentioned earlier, it was the idea of Sam Raskin that one ought to be brave and consider direct images for morphisms between lax prestacks when they seem to be the natural thing to do.

2.3.4. *Pseudo-properness for lax prestacks.* We will now discuss a generalization of Corollary 1.5.4 for maps in  $\text{LaxPreStk}$ .

Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map in  $\text{LaxPreStk}$ . We shall say that  $f$  is pseudo-proper if:

- For every  $S \in \text{Sch}$ , the functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  is a coCartesian fibration in groupoids;
- For every  $(S \in \text{Sch}, y_2 \in \mathcal{Y}_2(S))$ , the fiber product  $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$  is a pseudo-proper prestack over  $S$ .

As a formal corollary of Proposition 1.5.2 we obtain:

**Corollary 2.3.5.** *If  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is pseudo-proper, then the functor  $f_!$ , left adjoint to  $f^!$ , is defined. Furthermore, for a map  $g_2 : \mathcal{Y}'_2 \rightarrow \mathcal{Y}_2$  and  $\mathcal{Y}'_1 := \mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{Y}'_2$ , the natural transformation*

$$f'_! \circ g_1^! \rightarrow g_2^! \circ f_!,$$

arising from the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}'_2 & \xrightarrow{g_2} & \mathcal{Y}_2, \end{array}$$

and the identification

$$g_1^! \circ f^! \simeq f'^! \circ g_2^!,$$

is an isomorphism.

2.4. **Right adjoints.** The contents of subsection will be needed for the proof of Proposition 5.3.2. It can be skipped on the first pass.

We now turn to the discussion of the right adjoint to the functor  $f^!$  (note that this right adjoint will be a pretty weird functor even for a morphism between schemes).

First, we know that the right adjoint to  $f^!$  is always defined, because  $f^!$  commutes with colimits (by the Adjoint Functor Theorem, [Lu1, Corollary 5.5.2.9(i)]).

So, the question we want to address is that of a more explicit description of  $(f^!)^R$ .

2.4.1. For a general morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  it is difficult to describe the functor  $(f^!)^R$  in any explicit way. Yet, it is not completely “wild”:

Let

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}'_2 & \xrightarrow{g_2} & \mathcal{Y}_2 \end{array}$$

be a Cartesian diagram of lax prestacks. The isomorphism

$$f'^! \circ g_2^! \simeq g_1^! \circ f^!$$

gives rise to a natural transformation:

$$(2.1) \quad g_2^! \circ (f^!)^R \rightarrow (f'^!)^R \circ g_1^!$$

We claim:

**Lemma 2.4.2.** *Suppose that  $g_2$  is pseudo-proper. Then the natural transformation (2.1) is an isomorphism.*



*Proof.* By passing to left adjoints, we obtain a natural transformation

$$(g_1)! \circ f'^! \rightarrow f^! \circ (g_2)!,$$

which is an isomorphism by Corollary 2.3.5.  $\square$

2.4.3. We shall now describe a situation in which we can get a handle on the functor  $(f^!)^R$ .

Let  $\mathcal{F}$  be an object of  $\text{Shv}^!(\mathcal{Y}_1)$ . Fix  $S \in \text{Sch}$ ,  $y_2 \in \mathcal{Y}_2(S)$ , and a map of schemes  $g_S : S' \rightarrow S$ . Let us denote  $y'_2 = g_S(y_2) \in \mathcal{Y}_2(S')$ . Pullback defines a functor

$$g_Y : \mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\} \rightarrow \mathcal{Y}_1(S') \times_{\mathcal{Y}_2(S')} \{y'_2\}.$$

The datum of  $\mathcal{F}$  defines functors

$$\mathcal{F}(y_2) : \mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\} \rightarrow \text{Shv}^!(S) \text{ and } \mathcal{F}(y'_2) : \mathcal{Y}_1(S') \times_{\mathcal{Y}_2(S')} \{y'_2\} \rightarrow \text{Shv}^!(S'),$$

so that the diagram

$$\begin{array}{ccc} \mathcal{F}(y_2) : \mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\} & \xrightarrow{\mathcal{F}(y_2)} & \text{Shv}^!(S) \\ g_Y \downarrow & & \downarrow (g_S)^! \\ \mathcal{F}(y'_2) : \mathcal{Y}_1(S') \times_{\mathcal{Y}_2(S')} \{y'_2\} & \xrightarrow{\mathcal{F}(y'_2)} & \text{Shv}^!(S') \end{array}$$

commutes.

Hence, we obtain the following maps in  $\text{Shv}^!(S')$ :

$$(2.2) \quad (g_S)^! \left( \lim_{\mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\}} \mathcal{F}(y_2) \right) \rightarrow \lim_{\mathcal{Y}_1(S') \times_{\mathcal{Y}_2(S')} \{y'_2\}} (g_S)^! (\mathcal{F}(y_2)) \simeq \\ \simeq \lim_{\mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\}} \mathcal{F}(y'_2) \circ g_Y \leftarrow \lim_{\mathcal{Y}_1(S') \times_{\mathcal{Y}_2(S')} \{y'_2\}} \mathcal{F}(y'_2).$$

Note that in the context of constructible sheaves, the map  $\rightarrow$  in (2.2) is automatically an isomorphism.

2.4.4. Let us now assume that:

- For every  $S \in \text{Sch}$ , the functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  is a Cartesian fibration;
- For any  $g : S' \rightarrow S$  the pullback functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_1(S')$  maps arrows that are  $f(S)$ -Cartesian to arrows that are  $f(S')$ -Cartesian.

Together these conditions can be combined into saying that the functor  $\text{Sch}/_{\mathcal{Y}_1} \rightarrow \text{Sch}/_{\mathcal{Y}_2}$  is a Cartesian fibration.

**Lemma 2.4.5.** *Assume that for a given  $\mathcal{F}$  and any  $(S' \xrightarrow{g_S} S, y_2 \in \mathcal{Y}_2(S))$  as above, both maps in (2.2) are isomorphisms. Then the natural map*

$$(f^!)^R(\mathcal{F})_{S, y_2} \rightarrow \lim_{\mathcal{Y}_1(S) \times_{\mathcal{Y}_2(S)} \{y_2\}} \mathcal{F}(y_2)$$

*is an isomorphism.*

### 2.5. The notion of universally homological contractibility over a prestack.

Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between  $\infty$ -categories, where  $\mathbf{D}$  is an  $\infty$ -groupoid. In this case it is easy to give a criterion when the functor

$$\mathrm{Funct}(\mathbf{D}, \mathbf{E}) \rightarrow \mathrm{Funct}(\mathbf{C}, \mathbf{E}),$$

given by precomposition with  $F$ , is fully faithful for any  $\mathbf{E}$ . Namely, this happens if and only if all the fibers of  $F$  are contractible (i.e., have a trivial homotopy type).

In this subsection we will study a generalization of this with the geometry mixed-in: i.e., when instead of  $\infty$ -categories we have lax prestacks.

2.5.1. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks, where  $\mathcal{Y}_2 \in \mathrm{PreStk}$ . We shall say that  $f$  is *universally homologically contractible* if for any  $S \in \mathrm{Sch}/_{\mathcal{Y}_2}$ , for the induced map

$$f_S : S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \rightarrow S,$$

the functor

$$(f_S)^! : \mathrm{Shv}^!(S) \rightarrow \mathrm{Shv}^!(S \times_{\mathcal{Y}_2} \mathcal{Y}_1)$$

is fully faithful.

In this section we will study what geometric properties of  $f$  guarantee universally homological contractibility, and generalizations to the case when  $\mathcal{Y}_2$  is a lax prestack.

2.5.2. The following is easy:

**Lemma 2.5.3.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map, where  $\mathcal{Y}_2$  is a prestack. Suppose that for any  $S \in \mathrm{Sch}$ , the functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  has contractible fibers. Then the functor  $f$  is universally homologically contractible.*

2.5.4. The following results easily from the definitions:

**Lemma 2.5.5.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks, where  $\mathcal{Y}_2 \in \mathrm{PreStk}$ . Suppose that  $f$  is universally homologically contractible. Then the functor*

$$f^! : \mathrm{Shv}(\mathcal{Y}_2) \rightarrow \mathrm{Shv}^!(\mathcal{Y}_1)$$

*is fully faithful.*

2.5.6. Assume for a moment that  $\mathcal{Y}_2 = \mathrm{pt}$ . Then we shall say that  $\mathcal{Y} = \mathcal{Y}_1$  is universally homologically contractible if its map to  $\mathrm{pt}$  is.

**Lemma 2.5.7.** *A lax prestack is universally homologically contractible if and only if the trace map*

$$C_*(\mathcal{Y}) \rightarrow \Lambda$$

*is an isomorphism.*

*Proof.* Follows from the fact that for a pair of schemes  $S$  and  $S'$  and  $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}(S)$ , the map

$$\mathrm{Maps}_{\mathrm{Shv}(S)}(\mathcal{F}_1 \otimes C_*(S'), \mathcal{F}_2) \rightarrow \mathrm{Maps}_{\mathrm{Shv}(S \times S')}(\mathcal{F}_1 \boxtimes \omega_{S'}, \mathcal{F}_2 \boxtimes \omega_{S'})$$

is an isomorphism. □

## 3. UNIVERSALLY HOMOLOGICALLY CONTRACTIBLE MAPS AND RELATED NOTIONS

In this section we address the following question: under what conditions is a morphism between *lax prestacks*<sup>11</sup> such the corresponding pullback functor is fully faithful? When does it induce an isomorphism on homology of any sheaf? These questions arise in many situations that come up in practice.

The material in this section could be skipped on the first pass, and returned to when necessary.

<sup>11</sup>In Sect. 2.5 this question was addressed in the case when the target is a prestack.

**3.1. Contractible functors (a digression).** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between  $\infty$ -categories. In this section we will give a criterion for when the functor

$$\text{Funct}(\mathbf{D}, \mathbf{E}) \rightarrow \text{Funct}(\mathbf{C}, \mathbf{E}),$$

given by precomposition with  $F$  is a fully faithful embedding, for any  $\infty$ -category  $\mathbf{E}$ .

3.1.1. We shall say that a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is *contractible* if for any arrow  $d \xrightarrow{\alpha} d'$  in  $\mathbf{D}$ , the category  $\text{Factor}_F(\alpha)$  of its factorizations as

$$d \rightarrow F(c) \rightarrow d', \quad c \in \mathbf{C}$$

is contractible (i.e., has a trivial homotopy type), see Sect. 0.8.3 for what this means.

3.1.2. Suppose for a moment that  $\mathbf{D}$  is a groupoid. Then the following conditions are equivalent: (i)  $F$  is contractible; (ii)  $F$  is left or right cofinal; (iii)  $F$  has contractible fibers.

For a general  $\mathbf{D}$  we have:

**Lemma 3.1.3.** *Assume that  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a Cartesian or coCartesian fibration. Then  $F$  is contractible if and only if it has contractible fibers.*

3.1.4. We now claim:

**Proposition 3.1.5.** *A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is contractible if and only if for any  $\infty$ -category  $\mathbf{E}$ , the functor*

$$\text{Funct}(\mathbf{D}, \mathbf{E}) \rightarrow \text{Funct}(\mathbf{C}, \mathbf{E}),$$

*given by precomposition with  $F$  is a fully faithful embedding.*

*Proof.* It is easy to see that the fully-faithfulness condition appearing in the statement of the proposition holds if and only if it holds for  $\mathbf{E} = \text{Spc}$ , i.e., if and only if the functor

$$\text{Funct}(\mathbf{D}, \text{Spc}) \rightarrow \text{Funct}(\mathbf{C}, \text{Spc}),$$

is fully faithful.

Furthermore, the latter is equivalent to the fact that for any  $d \in \mathbf{D}$  and  $\Phi : \mathbf{D} \rightarrow \text{Spc}$  the map

$$\text{Maps}_{\text{Funct}(\mathbf{D}, \text{Spc})}(\tilde{h}_d, \Phi) \rightarrow \text{Maps}_{\text{Funct}(\mathbf{C}, \text{Spc})}(\tilde{h}_d \circ F, \Phi \circ F)$$

should be isomorphism, where  $\tilde{h}_d$  is the covariant Yoneda functor corresponding to the object  $d \in \mathbf{D}$ .

I.e.,  $F$  is contractible if and only if the counit of the adjunction

$$\text{LKE}_F(\tilde{h}_d \circ F) \rightarrow \tilde{h}_d$$

is an isomorphism.

Now, we calculate the value of  $\text{LKE}_F(\tilde{h}_d \circ F)$  on  $d' \in \mathbf{D}$  as

$$\text{colim}_{c \in \mathbf{C}, c \rightarrow d'} \tilde{h}_d(F(c)).$$

Thus, the fiber of  $\text{LKE}_F(\tilde{h}_d \circ F)$  over a given point  $d \rightarrow d'$  of  $\tilde{h}_d(d')$  is the homotopy type of the category  $\text{Factor}_F(\alpha)$ , as required.  $\square$

**3.2. Value-wise contractibility.** We shall now begin to explore how to transport the notion of contractible functor to the context of lax prestacks, i.e., where we have category theory coupled with geometry.

But before we do that, we will define a notion which is an overkill (but which is often useful in practice).

3.2.1. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks. We shall say that  $\mathcal{Y}$  is *value-wise contractible* if the corresponding functor

$$\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$$

is contractible.

3.2.2. From Lemma 3.1.3 we obtain:

**Corollary 3.2.3.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks. Suppose that for any  $S \in \text{Sch}$ , the functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  is a Cartesian or coCartesian fibration with contractible fibers. Then  $f$  is value-wise contractible.*

3.2.4. From Proposition 3.1.5 we obtain:

**Corollary 3.2.5.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be value-wise contractible. Then the functor*

$$f^! : \text{Shv}^!(\mathcal{Y}_1) \rightarrow \text{Shv}^!(\mathcal{Y}_2)$$

*is fully faithful. In particular:*

(i) *For any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$ , the map*

$$C_*(\mathcal{Y}_1, f^!(\mathcal{F})) \rightarrow C_*(\mathcal{Y}_2, \mathcal{F})$$

*is an isomorphism whenever either side is defined.*

(ii) *The map  $C_*(\mathcal{Y}_1) \rightarrow C_*(\mathcal{Y}_2)$  is an isomorphism.*

3.3. **Some weaker value-wise notions.** In Sect. 3.1 we have an explicit criterion for when a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is such that for any  $\mathbf{E}$  and  $\Phi_1, \Phi_2 : \mathbf{D} \rightarrow \mathbf{E}$ , the map

$$(3.1) \quad \text{Maps}_{\text{Funct}(\mathbf{D}, \mathbf{E})}(\Phi_1, \Phi_2) \rightarrow \text{Maps}_{\text{Funct}(\mathbf{C}, \mathbf{E})}(\Phi_1 \circ F, \Phi_2 \circ F)$$

is an isomorphism.

One can relax this condition as follows. One can ask that (3.1) be an isomorphism when  $\Phi_2$  (resp.,  $\Phi_1$ ) maps all arrows in  $\mathbf{D}$  to isomorphisms in  $\mathbf{E}$  (i.e., when it factors through the maximal sub-groupoid in  $\mathbf{E}$ ). One can relax it even further by requiring that both  $\Phi_1$  and  $\Phi_2$  have this property.

The latter (i.e., the weakest) condition is equivalent to  $F$  inducing an equivalence of homotopy types. The former condition is equivalent to  $F$  being left (resp., right) cofinal.<sup>12</sup> The left (resp., right) cofinality condition can be rewritten as saying that for every object  $\mathbf{d} \in \mathbf{D}$  the category  $\mathbf{C}_{\mathbf{d}/}$  (resp.,  $\mathbf{C}_{/\mathbf{d}}$ ) should be contractible.

We will now consider the analogous value-wise notions for maps between lax prestacks.

3.3.1. We shall say that a map of lax prestacks  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is *value-wise left cofinal* if for every  $S \in \text{Sch}$ , the functor

$$\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$$

is left cofinal.

**Lemma 3.3.2.** *Suppose that  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is value-wise left cofinal. Then for any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y}_2)$ , and  $\mathcal{G} \in \text{Shv}_{\text{str}}^!(\mathcal{Y}_2)$ , the map*

$$\text{Maps}_{\text{Shv}(\mathcal{Y}_2)}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Maps}_{\text{Shv}(\mathcal{Y}_1)}(f^!(\mathcal{F}), f^!(\mathcal{G}))$$

*is an isomorphism. In particular:*

(i) *For any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$ , the map*

$$C_*(\mathcal{Y}_1, f^!(\mathcal{F})) \rightarrow C_*(\mathcal{Y}_2, \mathcal{F})$$

*is an isomorphism whenever either side is defined.*

(ii) *The map  $C_*(\mathcal{Y}_1) \rightarrow C_*(\mathcal{Y}_2)$  is an isomorphism.*

<sup>12</sup>The traditional, but equivalent, definition of cofinality uses functors  $\Phi_2$  (resp.,  $\Phi_1$ ) that take a *constant* value in  $\mathbf{E}$ . This is tautologically equivalent to the map  $\text{colim}_{\mathbf{C}} \Phi \circ F \rightarrow \text{colim}_{\mathbf{D}} \Phi$  (resp.,  $\lim_{\mathbf{D}} \Phi \rightarrow \lim_{\mathbf{C}} \Phi \circ F$ ) being an isomorphism for any  $\mathbf{E}$  and  $\Phi : \mathbf{D} \rightarrow \mathbf{E}$ .

3.3.3. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks. We shall say that  $f$  is a *value-wise homotopy-type equivalence* if for any  $S \in \text{Sch}$ , the functor

$$\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$$

induces an equivalence of homotopy types.

We have:

**Lemma 3.3.4.** *Suppose that  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is a value-wise homotopy-type equivalence. Then for any  $\mathcal{F}, \mathcal{G} \in \text{Shv}_{\text{str}}^!(\mathcal{Y}_2)$ , the induced map*

$$\text{Maps}_{\text{Shv}(\mathcal{Y}_2)}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Maps}_{\text{Shv}(\mathcal{Y}_1)}(f^!(\mathcal{F}), f^!(\mathcal{G}))$$

is an isomorphism. In particular,

(i) For any  $\mathcal{F} \in \text{Shv}_{\text{str}}^!(\mathcal{Y}_2)$ , the map

$$C_c^*(\mathcal{Y}_1, f^!(\mathcal{F})) \rightarrow C_c^*(\mathcal{Y}_2, \mathcal{F})$$

is an isomorphism whenever either side is defined.

(ii) The map  $C_*(\mathcal{Y}_1) \rightarrow C_*(\mathcal{Y}_2)$  is an isomorphism.

**3.4. Universal homological contractibility for lax prestacks.** We now come to the less naive notion of homological contractibility for a map between lax prestacks.

3.4.1. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks. For  $S \in \text{Sch}$  and a map  $\alpha : y'_2 \rightarrow y''_2$  in  $\mathcal{Y}_2(S)$ , let  $\text{Factor}_f(\alpha)$  denote the following lax prestack over  $S$ :

For  $\tilde{S} \in \text{Sch}/_S$ , the category  $\text{Factor}_f(\alpha)(\tilde{S})$  is that of factorizations of  $\alpha|_{\tilde{S}}$  as

$$y'_2|_{\tilde{S}} \rightarrow f(y_1) \rightarrow y''_2|_{\tilde{S}}, \quad y_1 \in \mathcal{Y}_1(\tilde{S}).$$

We shall say that  $f$  is *universally homologically contractible* if for any  $(S, \alpha)$  as above, the map

$$\text{Factor}_f(\alpha) \rightarrow S$$

is universally homologically contractible.

3.4.2. Note that if  $\mathcal{Y}_2$  is a prestack, the two notions of universal homological contractibility (one defined above and another in Sect. 2.5.1) coincide.

3.4.3. We have:

**Lemma 3.4.4.** *If  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is value-wise contractible then it is universally homologically contractible.*

*Proof.* Follows from Lemma 2.5.3. □

3.4.5. The following assertion is parallel to Lemma 3.1.3:

**Proposition 3.4.6.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be such that for any  $S \in \text{Sch}$ , the functor  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  is a Cartesian or coCartesian fibration. Then  $f$  is universally homologically contractible if and only if it has a universally homologically contractible fiber over any  $S \in \text{Sch}/_{\mathcal{Y}_2}$ .*

*Proof.* We will give a proof for Cartesian fibrations; the case of coCartesian fibrations is similar. For a given  $(S, \alpha)$  consider the prestack  $\text{Factor}'_f(\alpha)$  over  $S$ , that attaches to  $\tilde{S} \in \text{Sch}/_S$  the category of

$$y'_2|_{\tilde{S}} \xrightarrow{\sim} f(y'_1) \rightarrow f(y_1) \rightarrow y''_2|_{\tilde{S}}, \quad (y'_1 \rightarrow y_1) \in \mathcal{Y}_1(\tilde{S}).$$

We have a natural forgetful map

$$\text{Factor}'_f(\alpha) \rightarrow \text{Factor}_f(\alpha),$$

and we claim that it is value-wise contractible. Indeed, for a given  $\tilde{S}$ , the functor

$$\text{Factor}'_f(\alpha)(\tilde{S}) \rightarrow \text{Factor}_f(\alpha)(\tilde{S})$$

is a coCartesian fibration, and the assumption that

$$\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$$

is a Cartesian fibration implies that it has contractible fibers.

Hence, by Corollary 3.2.5, the pullback functor

$$\mathrm{Shv}(S) \rightarrow \mathrm{Shv}^1(\mathrm{Factor}_f(\alpha))$$

is fully faithful if and only if the functor

$$\mathrm{Shv}(S) \rightarrow \mathrm{Shv}^1(\mathrm{Factor}'_f(\alpha))$$

is fully faithful.

Consider now the forgetful map

$$\mathrm{Factor}'_f(\alpha) \rightarrow S \times_{y'_2, \mathcal{Y}_2} \mathcal{Y}_1, \quad (y'_2|_{\tilde{S}} \xrightarrow{\sim} f(y'_1) \rightarrow f(y_1) \rightarrow y''_2|_{\tilde{S}}) \mapsto y'_1.$$

We claim that it is value-wise contractible. Indeed, for a given  $\tilde{S}$ , the corresponding map

$$\mathrm{Factor}'_f(\alpha)(\tilde{S}) \rightarrow (S \times_{y'_2, \mathcal{Y}_2} \mathcal{Y}_1)(\tilde{S})$$

is a Cartesian fibration with contractible fibers (each fiber has an initial point).

Hence, the pullback functor

$$\mathrm{Shv}(S) \rightarrow \mathrm{Shv}^1(\mathrm{Factor}'_f(\alpha))$$

is fully faithful if and only if the functor

$$\mathrm{Shv}(S) \rightarrow \mathrm{Shv}^1(S \times_{y'_2, \mathcal{Y}_2} \mathcal{Y}_1)$$

is fully faithful, as required.  $\square$

*Remark 3.4.7.* Note, however, that the notion of universally homologically contractibility over a lax prestack (unlike a usual prestack) is *not* stable under base change.

3.4.8. We have:

**Proposition 3.4.9.** *If  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is universally homologically contractible, then the functor*

$$f^! : \mathrm{Shv}^1(\mathcal{Y}_1) \rightarrow \mathrm{Shv}^1(\mathcal{Y}_2)$$

*is fully faithful.*

Note that Proposition 3.4.9 generalizes Corollary 3.2.5.

*Sketch of the proof.* Given  $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^1(\mathcal{Y}_2)$ , let us construct the map

$$\mathrm{Maps}_{\mathrm{Shv}^1(\mathcal{Y}_1)}(f^!(\mathcal{F}), f^!(\mathcal{G})) \rightarrow \mathrm{Maps}_{\mathrm{Shv}^1(\mathcal{Y}_2)}(\mathcal{F}, \mathcal{G}),$$

inverse to the map given by the functor  $f^!$ .

Given a map

$$(3.2) \quad f^!(\mathcal{F}) \rightarrow f^!(\mathcal{G}),$$

to specify the corresponding map  $\mathcal{F} \rightarrow \mathcal{G}$ , we need to give for every  $S \in \mathrm{Sch}$  and every arrow  $\alpha : y'_2 \rightarrow y''_2$  in  $\mathcal{Y}_2(S)$  a map

$$(3.3) \quad \mathcal{F}_{S, y'_2} \rightarrow \mathcal{G}_{S, y''_2}$$

in  $\mathrm{Shv}(S)$ .

Consider the lax prestack  $\mathrm{Factor}_f(\alpha)$ , and note that it is endowed with the following maps

$$\mathrm{ev}', \mathrm{ev}'' : \mathrm{Factor}_f(\alpha) \rightarrow \mathcal{Y}_2 \text{ and } \mathrm{ev}^m : \mathrm{Factor}_f(\alpha) \rightarrow \mathcal{Y}_1,$$

that send an  $\tilde{S}$ -point of  $\mathrm{Factor}_f(\alpha)$ , given by

$$y'_2|_{\tilde{S}} \rightarrow f(y_1) \rightarrow y''_2|_{\tilde{S}}$$

to

$$y'_2|_{\tilde{S}}, y''_2|_{\tilde{S}} \text{ and } y_1,$$

respectively.

Consider the following four objects in  $\text{Shv}^1(\text{Factor}_f(\alpha))$ :

$$(\text{ev}')^!(\mathcal{F}), (\text{ev}^m)^! \circ f^!(\mathcal{F}), (\text{ev}^m)^! \circ f^!(\mathcal{G}), (\text{ev}'')^!(\mathcal{G}).$$

Note that

$$(\text{ev}')^!(\mathcal{F}) \simeq \mathcal{F}_{S, y'_2}|_{\text{Factor}_f(\alpha)} \text{ and } (\text{ev}'')^!(\mathcal{G}) \simeq \mathcal{G}_{S, y''_2}|_{\text{Factor}_f(\alpha)},$$

and that we have the natural maps

$$(\text{ev}')^!(\mathcal{F}) \rightarrow (\text{ev}^m)^! \circ f^!(\mathcal{F}) \text{ and } (\text{ev}^m)^! \circ f^!(\mathcal{G}) \rightarrow (\text{ev}'')^!(\mathcal{G}).$$

Composing, we obtain a map

$$\mathcal{F}_{S, y'_2}|_{\text{Factor}_f(\alpha)} \simeq (\text{ev}')^!(\mathcal{F}) \rightarrow (\text{ev}^m)^! \circ f^!(\mathcal{F}) \rightarrow (\text{ev}^m)^! \circ f^!(\mathcal{G}) \rightarrow (\text{ev}'')^!(\mathcal{G}) \simeq \mathcal{G}_{S, y''_2}|_{\text{Factor}_f(\alpha)},$$

where the third arrow is induced by (3.2).

Thus, we have obtained a map  $\mathcal{F}_{S, y'_2}|_{\text{Factor}_f(\alpha)} \rightarrow \mathcal{G}_{S, y''_2}|_{\text{Factor}_f(\alpha)}$  in  $\text{Shv}^1(\text{Factor}_f(\alpha))$ .

Now, the assumption that the projection  $\text{Factor}_f(\alpha) \rightarrow S$  is universally homologically contractible implies that the latter map comes from a uniquely defined map (3.3).  $\square$

**Corollary 3.4.10.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be universally homologically contractible. Then:*

(i) *For any  $\mathcal{F} \in \text{Shv}^1(\mathcal{Y})$ , the map*

$$C_*(\mathcal{Y}_1, f^!(\mathcal{F})) \rightarrow C_*(\mathcal{Y}_2, \mathcal{F})$$

*is an isomorphism whenever either side is defined.*

(ii) *The map  $C_*(\mathcal{Y}_1) \rightarrow C_*(\mathcal{Y}_2)$  is an isomorphism.*

**3.5. The notion of universal homological left cofinality.** In this subsection we will replace the notion of value-wise left cofinality by a less naive one.

3.5.1. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map of lax prestacks. For  $S \in \text{Sch}$  and  $y_2 \in \mathcal{Y}_2(S)$ , consider the lax prestack  $(\mathcal{Y}_1)_{y_2/}$  that assigns to  $\tilde{S} \rightarrow S$  the category of

$$y_1 \in \mathcal{Y}_1(\tilde{S}), y_2|_{\tilde{S}} \rightarrow f(y_1).$$

We shall say that  $f$  is *universally homologically left cofinal* if for all  $(S, y_2)$  as above, the lax prestack  $(\mathcal{Y}_1)_{y_2/}$  be universally homologically contractible over  $S$ .

3.5.2. Note that if  $\mathcal{Y}_2$  is a *prestack* (as opposed to a lax prestack), then the notions of universally homological contractibility and universally homological left cofinality coincide.

3.5.3. We have:

**Proposition 3.5.4.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be universally homologically contractible. Then it is universally homologically left cofinal.*

*Proof.* Fix  $S \in \text{Sch}$  and  $y_2 \in \mathcal{Y}_2(S)$ . Consider the following lax prestacks  $(\mathcal{Y}_2)_{y_2/}$  and  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/}$  over  $S$ . The lax prestack  $(\mathcal{Y}_2)_{y_2/}$  attaches to  $\tilde{S} \rightarrow S$  the category  $\mathcal{Y}_2(\tilde{S})_{y_2|_{\tilde{S}}/}$ . The lax prestack  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/}$  attaches to  $\tilde{S} \rightarrow S$  the category of

$$y'_2 \in \mathcal{Y}_2(\tilde{S}), y_1 \in \mathcal{Y}_1(\tilde{S}), y_2|_{\tilde{S}} \rightarrow f(y_1) \rightarrow y'_2.$$

We have naturally defined morphisms over  $S$

$$(\mathcal{Y}_2)_{y_2/} \leftarrow (\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/} \rightarrow (\mathcal{Y}_1)_{y_2/}.$$

The map  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/} \rightarrow (\mathcal{Y}_1)_{y_2/}$  is value-wise contractible, and hence universally homologically contractible. Therefore, to prove the proposition, it suffices to show that the map  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/} \rightarrow S$  is universally homologically contractible.

The map  $(\mathcal{Y}_2)_{y_2/} \rightarrow S$  is value-wise contractible, and hence is universally homologically contractible. Therefore, it remains to show that the map  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/} \rightarrow (\mathcal{Y}_2)_{y_2/}$  is universally homologically contractible.

We note that the map  $(\mathcal{Y}_1, \mathcal{Y}_2)_{y_2/} \rightarrow (\mathcal{Y}_2)_{y_2/}$  is a value-wise Cartesian fibration. Hence, by Proposition 3.4.6, it suffices to show that its fibers are universally homologically contractible. However, the latter follows from the assumption that  $f$  is universally homologically contractible.  $\square$

3.5.5. We observe:

**Lemma 3.5.6.** *Let  $f$  be value-wise left cofinal. Then it is universally homologically left cofinal.*

*Proof.* If  $f$  is value-wise left cofinal, then for any  $S \in \text{Sch}$  and  $y_2 \in \mathcal{Y}_2(S)$ , the map

$$(\mathcal{Y}_1)_{y_2/} \rightarrow S$$

has value-wise contractible fibers, and hence is universally homologically contractible, by Lemma 2.5.3.  $\square$

**Lemma 3.5.7.** *Assume that  $f$  is such that for any  $S \in \text{Sch}$ , the map*

$$\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$$

*is a Cartesian fibration. Then  $f$  is universally homologically left cofinal if and only if all of its fibers are universally homologically contractible.*

*Proof.* If  $f$  is a value-wise Cartesian fibration, then for any  $S \in \text{Sch}$  and  $y_2 \in \mathcal{Y}_2(S)$ , the inclusion

$$S \times_{y_2, \mathcal{Y}_2} \mathcal{Y}_1 \rightarrow (\mathcal{Y}_1)_{y_2/}$$

is a value-wise homotopy-type equivalence. Hence, the assertion of the lemma follows from Lemma 3.3.4.  $\square$

3.5.8. We will need the following:

**Proposition 3.5.9.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be universally homologically left cofinal and let  $g : \mathcal{Y}'_2 \rightarrow \mathcal{Y}_2$  be a value-wise coCartesian fibration. Then the base-changed map*

$$f' : \mathcal{Y}'_1 := \mathcal{Y}'_2 \times_{\mathcal{Y}_2} \mathcal{Y}_1 \rightarrow \mathcal{Y}'_2$$

*is also universally homologically left cofinal.*

*Proof.* For  $S \in \text{Sch}$  and  $y'_2 \in \mathcal{Y}'_2(S)$  we have a tautological map of lax prestacks over  $S$ :

$$(\mathcal{Y}'_1)_{y'_2/} \rightarrow (\mathcal{Y}_1)_{g(y'_2)/}$$

By Lemma 3.3.4, it suffices to show that the above map is a value-wise homotopy-type equivalence. However, the latter follows from the fact that for any  $\tilde{S} \rightarrow S$ , the resulting functor

$$(\mathcal{Y}'_1)_{y'_2/}(\tilde{S}) \rightarrow (\mathcal{Y}_1)_{g(y'_2)/}(\tilde{S})$$

admits a left adjoint.  $\square$



3.5.10. Finally, we claim:

**Proposition 3.5.11.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be universally homologically left cofinal. Then for any  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y}_2)$  and  $\mathcal{G} \in \text{Shv}_{\text{str}}^!(\mathcal{Y}_2)$ , the map*

$$\text{Maps}_{\text{Shv}^!(\mathcal{Y}_2)}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Maps}_{\text{Shv}^!(\mathcal{Y}_1)}(f^!(\mathcal{F}), f^!(\mathcal{G}))$$

*is an isomorphism.*

*Sketch of proof.* Given  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y}_2)$  and  $\mathcal{G} \in \text{Shv}_{\text{str}}^!(\mathcal{Y}_2)$ , let us construct the map

$$\text{Maps}_{\text{Shv}^!(\mathcal{Y}_1)}(f^!(\mathcal{F}), f^!(\mathcal{G})) \rightarrow \text{Maps}_{\text{Shv}^!(\mathcal{Y}_2)}(\mathcal{F}, \mathcal{G}),$$

inverse to the map given by the functor  $f^!$ .

Given a map

$$(3.4) \quad f^!(\mathcal{F}) \rightarrow f^!(\mathcal{G}),$$

to specify the corresponding map  $\mathcal{F} \rightarrow \mathcal{G}$ , we need to give for every  $S \in \text{Sch}$  and every  $y_2 \in \mathcal{Y}_2(S)$  a map

$$(3.5) \quad \mathcal{F}_{S, y_2} \rightarrow \mathcal{G}_{S, y_2}$$

in  $\text{Shv}(S)$ .

Consider the prestack  $(\mathcal{Y}_1)_{y_2/}$ , and consider the evaluation map

$$\text{ev} : (\mathcal{Y}_1)_{y_2/} \rightarrow \mathcal{Y}_1,$$

that sends an  $\tilde{S}$ -point of  $(\mathcal{Y}_1)_{y_2/}$ , given by  $y_2|_{\tilde{S}} \rightarrow y_1$ , to  $y_1 \in \mathcal{Y}_1(\tilde{S})$ .

Note that we have a canonical map

$$\mathcal{F}_{S, y_2}|_{(\mathcal{Y}_1)_{y_2/}} \rightarrow \text{ev}^! \circ f^!(\mathcal{F})$$

and an isomorphism

$$(\mathcal{G}_{S, y_2})|_{(\mathcal{Y}_1)_{y_2/}} \simeq \text{ev}^! \circ f^!(\mathcal{G}).$$

Composing with (3.4), we obtain a map

$$\mathcal{F}_{S, y_2}|_{(\mathcal{Y}_1)_{y_2/}} \rightarrow \text{ev}^! \circ f^!(\mathcal{F}) \rightarrow \text{ev}^! \circ f^!(\mathcal{G}) \simeq (\mathcal{G}_{S, y_2})|_{(\mathcal{Y}_1)_{y_2/}}.$$

Using the assumption on the morphism  $(\mathcal{Y}_1)_{y_2/} \rightarrow S$ , we obtain that the latter map comes from a uniquely defined map in (3.5). □

**Corollary 3.5.12.** *Let  $f$  be universally homologically left cofinal. Then:*

(i) *For  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y}_2)$ , the map*

$$C_*(\mathcal{Y}_1, f^!(\mathcal{F})) \rightarrow C_*(\mathcal{Y}_2, \mathcal{F})$$

*is an isomorphism, whenever either side is defined.*

(ii) *The map  $C_*(\mathcal{Y}_1) \rightarrow C_*(\mathcal{Y}_2)$  is an isomorphism.*

## Part I: Various incarnations of the Ran space

### 4. THE NON-UNITAL AND UNITAL VERSIONS OF THE RAN SPACE

Let  $X$  be a separated scheme (in most applications so far one takes  $X$  to be a curve). The Ran space of  $X$  is a geometric object that incarnates the idea that it classifies finite collections of points of  $X$ , but where the cardinality of the collection is not fixed (in this it has an advantage over symmetric powers of  $X$  that count points with multiplicities).

When taken literally, the Ran space is a *prestack*, and when working over the ground field of complex numbers it corresponds to a reasonably behaved topological space. However, for our purposes we will also need the *unital* and *unital augmented* versions of the Ran space, and those will be *lax prestacks*.

The contents of Sect. 4.1 are necessary for the statement of the cohomological product formula Theorem 14.1.6, i.e., the core of this paper. As a prerequisite for Sect. 4.1 one only needs Sect. 1. The rest of the present section is needed for the material in Part III of the paper, which in turn is used in Sect. 15 (the reduction of the cohomological product formula to a local duality statement).

**4.1. The usual (=non-unital) Ran space.** In this subsection we will introduce the usual Ran space; this is the same object as one that was originally studied in [BD].

4.1.1. Let  $X$  be a (separated) scheme. We let  $\text{Ran}$  denote the following object of  $\text{PreStk}$ :

For  $S \in \text{Sch}$  we let  $\text{Ran}(S)$  be the (discrete) groupoid of finite non-empty subsets of  $\text{Maps}(S, X)$ .

We shall refer to  $\text{Ran}$  as the *non-unital* version of the Ran space of  $X$ .

4.1.2. The following give a description of  $\text{Ran}$  as colimit of schemes:

**Proposition 4.1.3.** *The prestack  $\text{Ran}$  is canonically isomorphic to*

$$\text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}} X^{\mathcal{J}},$$

where  $\text{Fin}^s$  is the category of finite non-empty sets and surjective maps.

*Remark 4.1.4.* In what follows we shall denote by straight letters  $I, J, K$  finite subsets of  $\text{Maps}(S, X)$  (for a given  $S$ ) and by script characters  $\mathcal{J}, \mathcal{J}, \mathcal{K}$  abstract finite sets.

Note that Proposition 4.1.3 gives an explicit description of the category  $\text{Shv}^!(\text{Ran})$  in terms of categories of sheaves on schemes:

$$\text{Shv}^!(\text{Ran}) \simeq \lim_{\mathcal{J} \in \text{Fin}^s} \text{Shv}(X^{\mathcal{J}}).$$

*Proof of Proposition 4.1.3.* This follows from the fact that, given a set  $A$ , the set of its finite non-empty subsets is canonically isomorphic to  $\text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}} \text{Maps}(\mathcal{J}, A)$ .

Indeed, the above colimit is the disjoint union over finite subsets  $I \subset A$  of

$$\text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}} \text{Maps}(\mathcal{J}, A)_{\text{image equals } I}.$$

Each such expression is the same as

$$\text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}, \mathcal{J} \rightarrow I} \{*\},$$

whereas the latter index category has the final point, given by  $\mathcal{J} = I$ . □

In what follows, for a finite set  $\mathcal{J}$  we shall denote by  $\text{ins}_{\mathcal{J}}$  the corresponding map

$$X^{\mathcal{J}} \rightarrow \text{Ran}.$$

Also, for future use we introduce the notation

$$\overset{\circ}{X}^{\mathcal{J}} \subset X^{\mathcal{J}}$$

for the open locus, whose  $S$ -points are  $J$ -tuples of elements of  $\text{Maps}(S, X)$  that have *pairwise non-intersecting images*. I.e.,  $k$ -points of  $\overset{\circ}{X}^J$  are  $J$ -tuples of pairwise distinct  $k$ -points of  $X$ .

4.1.5. From Proposition 1.5.2, we obtain:

**Corollary 4.1.6.** *Assume that  $X$  is:*

–*proper if we are working in the context of  $D$ -mdoules, and*

–*arbitrary for the context of constructible sheaves.*

*Then the functor  $C_c^*(\text{Ran}, -) : \text{Shv}^!(\text{Ran}) \rightarrow \Lambda\text{-mod}$  is defined.*

The functor  $C_c^*(\text{Ran}, -)$  from Corollary 4.1.6 is the functor of (non-unital) chiral homology.

4.1.7. A basic feature of the Ran space is the following theorem, due to Beilinson and Drinfeld (see [Main Text, Theorem 2.4.5] for a proof):

**Theorem 4.1.8.** *Suppose  $X$  is connected. Then  $\text{Ran}$  is universally homologically contractible.*

According to Lemma 2.5.7, an equivalent way to state this theorem is that the trace map

$$C_*(\text{Ran}) \rightarrow \Lambda$$

is an isomorphism.

**4.2. The unital version of the Ran space.** As is explained in [Lu2, Sect. 5.5], in the topological setting for  $X = \mathbb{R}^1$ , *non-unital* associative algebras give rise to objects of  $\text{Shv}^!(\text{Ran})$ . Namely, for an associative algebra  $A$  and a subset  $I \subset \mathbb{R}$ , the  $!$ -fiber of the corresponding sheaf  $\mathcal{A}$  at  $I$  will be

$$\mathcal{A}_I := \bigotimes_{i \in I} A.$$

The unital Ran space for  $\mathbb{R}^1$  will be the natural recipient of a functor from *unital* associative algebras: the unit in  $A$  allows to map

$$\bigotimes_{i \in I_1} A \rightarrow \bigotimes_{i \in I_2} A$$

each time we have an inclusion  $I_1 \subseteq I_2$ , so we have a map  $\mathcal{A}_{I_1} \rightarrow \mathcal{A}_{I_2}$ .

In this subsection we introduce the unital version of the Ran space. The difference between it and the usual Ran space is that now we will be able to account for the fact that one finite subset of  $X$  is contained in another by means of maps  $\mathcal{F}_{S, I_1} \rightarrow \mathcal{F}_{S, I_2}$  for  $I_1 \subseteq I_2 \subset \text{Maps}(S, X)$ .

4.2.1. We let  $\text{Ran}_{\text{unital}}$  denote the following object of  $\text{LaxPreStk}$ :

For  $S \in \text{Sch}$  we let  $\text{Ran}_{\text{unital}}(S)$  be the (ordinary) category whose objects are finite non-empty subsets of  $\text{Maps}(S, X)$ , and where the morphisms are given by inclusion of finite subsets.

We shall refer to  $\text{Ran}_{\text{unital}}$  as the *unital* version of the Ran space of  $X$ .

*Remark 4.2.2.* One can introduce a version of  $\text{Ran}_{\text{unital}}$ , where one also allows the empty set. In the topological context this is a good idea if we want to have a closer contact with associative algebras.

*Remark 4.2.3.* In the case of the usual Ran space, we had its explicit expression as a colimit of schemes, given by Proposition 4.1.3. A similar description is possible also for  $\text{Ran}_{\text{unital}}$ , see Sect. 6.2.

4.2.4. We have the tautological map

$$(4.1) \quad \phi : \text{Ran} \rightarrow \text{Ran}_{\text{unital}}.$$

Denote by

$$\text{OblvUnit} := \phi^! : \text{Shv}^!(\text{Ran}_{\text{unital}}) \rightarrow \text{Shv}^!(\text{Ran})$$

the corresponding pullback functor.

*Remark 4.2.5.* In terms of the analogy with associative algebras the functor  $\text{OblvUnit}$  corresponds to the obvious forgetful functor from the category of unital algebras to that of non-unital ones.

4.2.6. One can show that the functor

$$C_c^*(\text{Ran}_{\text{unital}}, -) : \text{Shv}^!(\text{Ran}_{\text{unital}}) \rightarrow \Lambda\text{-mod},$$

is well-defined (in the context of D-modules, under the additional assumption that  $X$  be proper). When  $X$  is connected, this will be done in the course of the proof of Theorem 4.2.7. This is the functor of unital chiral homology.

**Theorem 4.2.7.** *Assume that  $X$  is connected. Then the functor  $C_c^*(\text{Ran}_{\text{unital}}, -)$  is well-defined and the natural transformation*

$$C_c^*(\text{Ran}, -) \circ \text{OblvUnit} \rightarrow C_c^*(\text{Ran}_{\text{unital}}, -)$$

is an isomorphism.

The proof is given in Sect. 4.6.

4.3. **Adding the unit.** Continuing to draw on the analogy from topology, there is a naturally defined functor from the category non-unital algebras to that of unital algebras, given by adjoining the unit. If  $A$  is a non-unital algebra and  $\mathcal{A} \in \text{Shv}^!(\text{Ran})$  the corresponding object, then the object of  $\text{Shv}^!(\text{Ran}_{\text{unital}})$ , corresponding to the algebra  $\text{AddUnit}(A)$ , is given as follows

$$\text{AddUnit}(\mathcal{A}_I) = \bigoplus_{J \subseteq I} \mathcal{A}_J.$$

The corresponding functor for sheaves on the Ran space will be introduced in this subsection. I.e., we will construct a functor

$$\text{AddUnit} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{unital}}).$$

Ultimately, we will show (see Sect. 4.4) that the functor  $\text{AddUnit}$  is the left adjoint of  $\text{OblvUnit}$ .

The construction of this functor was explained to us by S. Raskin. To the best of our knowledge, it was the first usage of pushforward for a morphism between lax prestacks.

4.3.1. Consider the following object of  $\text{LaxPreStk}$ , denoted  $\text{Ran}^{\rightarrow}$ :

For  $S \in \text{Sch}$  we let  $\text{Ran}^{\rightarrow}(S)$  be the (ordinary) category whose objects are pairs  $(J \subseteq I)$  of finite subsets of  $\text{Maps}(S, X)$  with  $J \neq \emptyset$ . Morphisms from  $(J \subseteq I)$  to  $(J_1 \subseteq I_1)$  are inclusions  $I \subseteq I_1$  such that  $J = J_1$ .

We have the maps

$$\begin{aligned} \psi : \text{Ran}^{\rightarrow} &\rightarrow \text{Ran}_{\text{unital}}, & (J \subseteq I) &\mapsto I, \\ \xi : \text{Ran}^{\rightarrow} &\rightarrow \text{Ran}, & (J \subseteq I) &\mapsto J \\ v : \text{Ran} &\rightarrow \text{Ran}^{\rightarrow}, & J &\mapsto (J \subseteq J). \end{aligned}$$

We claim:

**Lemma 4.3.2.** *The map  $\psi$  is pseudo-proper.*

*Proof.* For  $S \in \text{Sch}$  and  $I$  an object of  $\text{Ran}_{\text{unital}}(S)$ , the fiber of  $\text{Ran}^{\rightarrow}(S)$  over it is the groupoid of non-empty subsets  $J \subseteq I$ . For a morphism  $I_1 \subseteq I_2$  in  $\text{Ran}_{\text{unital}}(S)$ , we have a map between the corresponding fibers, given by

$$(J \subseteq I_1) \mapsto (J \subseteq I_2).$$

This shows that  $\text{Ran}^{\rightarrow}(S) \rightarrow \text{Ran}_{\text{unital}}(S)$  is a coCartesian fibration in groupoids.

Let us now show that for a given  $I \subset \text{Maps}(S, X)$ , the fiber product

$$(4.2) \quad S \times_{\text{Ran}_{\text{unital}}} \text{Ran}^{\rightarrow}$$

is pseudo-proper over  $S$ .

Consider the category whose objects are the data of  $\mathcal{J} \rightarrow \mathcal{J}' \leftarrow I$ , where  $\mathcal{J}$  is a finite non-empty set. Morphisms in this category are commutative diagrams that induce surjections on the  $\mathcal{J}$ 's. Now, the fiber product (4.2) equals

$$\text{colim}_{\mathcal{J} \rightarrow \mathcal{J}' \leftarrow I} X^{\mathcal{J}'} \times_{X^I} S,$$

which is proved similarly to Proposition 4.1.3. □

4.3.3. Thus, it follows from Corollary 2.3.5 that the functor

$$\psi_! : \text{Shv}^!(\text{Ran}^{\rightarrow}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{unital}}),$$

left adjoint to  $\psi^!$ , is defined.

We define the functor

$$\text{AddUnit} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{unital}})$$

to be the composition  $\psi_! \circ \xi^!$ .

*Warning:* We emphasize that in our definition the unital Ran space does not allow the empty set. Related to this is the fact that  $\text{AddUnit}(0) = 0$ .

*Remark 4.3.4.* In Sect. 6.2.6 we will give an explicit description of the functor  $\text{AddUnit}$  as a colimit of direct image functors for schemes.

4.3.5. The functor  $\text{AddUnit}$  can be described explicitly as follows:

For  $S \in \text{Sch}$  fix an object  $I \subset \text{Ran}_{\text{unital}}(S)$  corresponding to a finite set  $I$  of maps  $S \rightarrow X$  with *pairwise non-intersecting images*. I.e., the map  $S \rightarrow \text{Ran}_{\text{unital}}$  factors as

$$S \rightarrow X^{\circ I} \rightarrow X^I \xrightarrow{\text{ins}_I} \text{Ran} \xrightarrow{\phi} \text{Ran}_{\text{unital}}.$$

**Proposition 4.3.6.** *For  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , the corresponding object  $\text{AddUnit}(\mathcal{F})_{S,I} \in \text{Shv}^!(S)$  is given by*

$$\bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_{S,J},$$

where we regard  $J$  as an  $S$ -point of  $\text{Ran}$  via  $J \subseteq I \subset \text{Maps}(S, X)$ .

*Proof.* Follows from Corollary 2.3.5. □

#### 4.4. Another interpretation.

4.4.1. Consider the morphism

$$\phi : \text{Ran} \rightarrow \text{Ran}_{\text{unital}}.$$

This morphism is *not* pseudo-proper. However, we claim:

**Proposition 4.4.2.** *The functor*

$$\phi_! : \text{Shv}(\text{Ran}) \rightarrow \text{Shv}(\text{Ran}_{\text{unital}})$$

*exists and identifies canonically with  $\text{AddUnit}$ .*

This proposition can be reformulated as follows:

**Corollary 4.4.3.** *The functor  $\text{AddUnit}$  is the left adjoint of  $\text{ObvUnit}$ .*

4.4.4. For the proof of Proposition 4.4.2 we note:

**Lemma 4.4.5.** *The (iso)morphism*

$$\mathrm{Id}_{\mathrm{Shv}^!(\mathrm{Ran})} \xrightarrow{\sim} (\xi \circ v)^! \simeq v^! \circ \xi^!$$

*is unit of an adjunction.*

*Proof.* Follows from the fact that the (iso)morphism

$$\xi \circ v \xrightarrow{\sim} \mathrm{id}$$

defines the counit of an adjunction when evaluated on any  $S \in \mathrm{Sch}$ .  $\square$

**Corollary 4.4.6.** *The functor  $v_! : \mathrm{Shv}^!(\mathrm{Ran}) \rightarrow \mathrm{Shv}^!(\mathrm{Ran}^{\rightarrow})$ , left adjoint to  $v^!$ , is well-defined and identifies with  $\xi^!$ .*

4.4.7. *Proof of Proposition 4.4.2.* We have:

$$\mathrm{AddUnit} = \psi_! \circ \xi^! \simeq \psi_! \circ v_! \simeq (\psi \circ v)_! \simeq \phi_!.$$

$\square$

4.4.8. Consider the unit of the adjunction

$$\mathrm{Id}_{\mathrm{Shv}^!(\mathrm{Ran})} \rightarrow \mathrm{OblvUnit} \circ \mathrm{AddUnit}.$$

It follows from the construction that this natural transformation can be described as the composition

$$(4.3) \quad \mathrm{Id}_{\mathrm{Shv}^!(\mathrm{Ran})} \simeq v^! \circ \xi^! \rightarrow v^! \circ \psi^! \circ \psi_! \circ \xi^! = \phi^! \circ \psi_! \circ \xi^! = \mathrm{OblvUnit} \circ \mathrm{AddUnit}.$$

4.5. **Comparing chiral homology.** Assume that  $X$  is proper (or arbitrary if we are working in the context of constructible sheaves).

4.5.1. Note that by interpreting  $\mathrm{AddUnit}$  as  $\phi_!$ , we obtain a tautological isomorphism

$$(4.4) \quad \mathrm{C}_c^*(\mathrm{Ran}_{\mathrm{untl}}, -) \circ \mathrm{AddUnit} \simeq \mathrm{C}_c^*(\mathrm{Ran}, -).$$

**Corollary 4.5.2.** *Assume that  $X$  is connected. Then for  $\mathcal{F} \in \mathrm{Shv}^!(\mathrm{Ran})$ , the map*

$$\mathrm{C}_c^*(\mathrm{Ran}, \mathcal{F}) \rightarrow \mathrm{C}_c^*(\mathrm{Ran}, \mathrm{OblvUnit} \circ \mathrm{AddUnit}(\mathcal{F})),$$

*induced by (4.3), is an isomorphism.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \mathrm{C}_c^*(\mathrm{Ran}, \mathcal{F}) & \xrightarrow{\sim} & \mathrm{C}_c^*(\mathrm{Ran}_{\mathrm{untl}}, \mathrm{AddUnit}(\mathcal{F})) \\ \downarrow & & \downarrow \mathrm{id} \\ \mathrm{C}_c^*(\mathrm{Ran}, \mathrm{OblvUnit} \circ \mathrm{AddUnit}(\mathcal{F})) & \longrightarrow & \mathrm{C}_c^*(\mathrm{Ran}_{\mathrm{untl}}, \mathrm{AddUnit}(\mathcal{F})), \end{array}$$

where the bottom horizontal arrow is an isomorphism by Theorem 4.2.7. This implies that the left vertical map is an isomorphism as well.  $\square$

#### 4.6. Proof of Theorem 4.2.7.

4.6.1. We will prove a stronger assertion:

**Theorem 4.6.2.** *Suppose that  $X$  is connected. Then the map  $\phi : \mathrm{Ran} \rightarrow \mathrm{Ran}_{\mathrm{untl}}$  is universally homologically left cofinal.*

See Sect. 3.5 for the notion of universal homological left cofinality. This theorem implies Theorem 4.2.7 by Corollary 3.5.12.

4.6.3. The proof of Theorem 4.6.2 is essentially given in [Main Text] in the guise of the proof of Proposition 2.5.23 from *loc. cit.* Let us repeat it for completeness.

Given an  $S$ -point  $I$  of  $\text{Ran}_{\text{untl}}$ , consider the corresponding lax prestack (which turns out to actually be a prestack)  $(\text{Ran})_{I/}$  over  $S$ , see Sect. 3.5 for the notation. For  $S' \in \text{Sch}$  the corresponding category (actually, a groupoid)  $(\text{Ran})_{I/}(S')$  consists of

$$(S' \rightarrow S, I' \subset \text{Maps}(S', X) \text{ such that } I|_{S'} \subseteq I').$$

Note that this groupoid is a retract of  $\text{Maps}(S', S \times \text{Ran})$ . Indeed, the inverse map sends  $I' \mapsto I' \cup I|_{S'}$ .

We need to show that  $(\text{Ran})_{I/}$  is universally homologically contractible over  $S$ . However, the property of being universally homologically contractible survives retracts. Hence, it suffices to show that  $S \times \text{Ran}$  is universally homologically contractible over  $S$ , which follows from Theorem 4.1.8.

4.6.4. We shall now prove another assertion in the spirit of Theorem 4.6.2, to be used later (concretely, in the proof of Theorem 16.4.7).

Fix a point  $x \in X$ , and consider the maps

$$(4.5) \quad \mathbf{union}_x : \text{Ran} \rightarrow \text{Ran}, \quad I \mapsto I \cup x.$$

and

$$(4.6) \quad \mathbf{union}_{x, \text{untl}} : \text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl}}, \quad I \mapsto I \cup x.$$

We will prove:

**Proposition 4.6.5.** *The map (4.6) is universally homologically left cofinal.*

Combining with Corollary 3.5.12 we obtain:

**Corollary 4.6.6.**

(a) *For  $\mathcal{F} \in \text{Shv}^!(\text{Ran}_{\text{untl}})$ , the map*

$$C_c^*(\text{Ran}_{\text{untl}}, \mathbf{union}_{x, \text{untl}}^!(\mathcal{F})) \rightarrow C_c^*(\text{Ran}_{\text{untl}}, \mathcal{F})$$

*is an isomorphism.*

(b) *If  $X$  is connected, for  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  that lies in the essential image of the functor  $\text{OblvUnit}$ , the map*

$$C_c^*(\text{Ran}, \mathbf{union}_x^!(\mathcal{F})) \rightarrow C_c^*(\text{Ran}, \mathcal{F})$$

*is an isomorphism.*

*Remark 4.6.7.* In the theory of chiral algebras, the assertion of the above corollary is known under the motto “inserting of the vacuum does not change chiral homology”.

*Proof of Proposition 4.6.5.* Given an  $S$ -point  $I$  of  $\text{Ran}_{\text{untl}}$ , consider the corresponding lax prestack  $(\text{Ran}_{\text{untl}})_{I/}$ . We claim that the map  $(\text{Ran}_{\text{untl}})_{I/} \rightarrow S$  is value-wise contractible, which would imply the required assertion by Lemma 2.5.3.

For  $S' \in \text{Sch}_{/S}$  the category of lifts of  $I$  to an  $S'$ -point of  $(\text{Ran}_{\text{untl}})_{I/}$  is that of

$$(I' \subset \text{Maps}(S', X) \text{ such that } I|_{S'} \subseteq I' \cup x).$$

We wish to show that this category is contractible.

The above category contains a left cofinal full subcategory consisting of  $I'$  for which  $I|_{S'} \subset I'$ . Now, the latter subcategory is contractible because it has an initial point, namely, one with  $I' = I|_{S'}$ .  $\square$

## 5. THE AUGMENTED VERSION OF THE RAN SPACE AND TAKING THE UNITS OUT

The material in this section is needed for the material in Part III of the paper, which in turn is used in Sect. 15 (the reduction of the cohomological product formula to a local statement).

In this section we will introduce yet one more version of the Ran space—the augmented one. Continuing the analogy with associative algebras from Sect. 4.2, in the topological context and when  $X = \mathbb{R}^1$ , the augmented Ran space will be the natural recipient of the functor from *unital augmented* associative algebras.

Note, however, that the category of unital augmented associative algebras is naturally equivalent to the category of non-unital associative algebras. Therefore, it is natural to expect a parallel phenomenon for sheaves on the corresponding Ran spaces: this will be reflected by Theorem 5.4.3.

**5.1. The augmented Ran space.** To motivate the definition of the unital augmented Ran space we shall once again appeal to the analogy with associative algebras. For an associative algebra  $A$  consider the corresponding  $\mathcal{A} \in \text{Shv}^!(\text{Ran})$  so that for a finite subset  $I$  of points of  $\mathbb{R}$  we have

$$\mathcal{A}_I = \bigotimes_{i \in I} A.$$

As was mentioned above, if  $A$  is unital, whenever  $I_1 \subseteq I_2$  we have a map  $\mathcal{A}_{I_1} \rightarrow \mathcal{A}_{I_2}$ . Assume now that  $A$  is augmented, and let  $A^+$  denote its augmentation ideal. Then, for a finite set  $I$  and its (possibly empty) subset  $K$ , we set

$$\mathcal{A}_{K \subseteq I} := \left( \bigotimes_{i \in K} A/A^+ \right) \otimes \left( \bigotimes_{i \in I-K} A \right),$$

where  $A/A^+$  is the ground ring, so tensoring by it is the identity functor.

Now, whenever we have an inclusion  $I_1 \subseteq I_2$  such that  $K_1 \subseteq K_2$ , we have a map

$$\mathcal{A}_{K_1 \subseteq I_1} \rightarrow \mathcal{A}_{K_2 \subseteq I_2}.$$

This is the kind of structure that will be encoded by sheaves on the augmented Ran space.

**5.1.1.** We define the unital augmented version of the Ran space, denoted  $\text{Ran}_{\text{unl}, \text{aug}} \in \text{LaxPreStk}$ , as follows:

For  $S \in \text{Sch}$  we let  $\text{Ran}_{\text{unl}, \text{aug}}(S)$  be the (ordinary) category whose objects are pairs  $(K \subseteq I)$  of finite subsets of  $\text{Maps}(S, X)$  with  $I \neq \emptyset$ . Morphisms from  $(K \subseteq I)$  to  $(K_1 \subseteq I_1)$  are inclusions  $I \subseteq I_1$  such that  $K \subseteq K_1$ .

**5.1.2.** We have the tautological map

$$\iota : \text{Ran}_{\text{unl}} \rightarrow \text{Ran}_{\text{unl}, \text{aug}}, \quad I \mapsto (\emptyset \subset I)$$

and its left inverse

$$\pi : \text{Ran}_{\text{unl}, \text{aug}} \rightarrow \text{Ran}_{\text{unl}}, \quad (K \subseteq I) \mapsto I.$$

We let

$$\text{OblvAug} := \iota^! : \text{Shv}^!(\text{Ran}_{\text{unl}, \text{aug}}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{unl}})$$

denote the corresponding pullback functor.

*Remark 5.1.3.* In terms of the analogy with associative algebras, the functor  $\text{OblvAug}$  corresponds to the forgetful functor from the category of unital augmented algebras to that of just unital algebras.

**5.2. Adding the unit and augmentation.** We now claim that the functor

$$\text{AddUnit} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{unl}}),$$

constructed in Sect. 4.3, canonically factors as

$$\text{Shv}^!(\text{Ran}) \xrightarrow{\text{AddUnit}_{\text{aug}}} \text{Shv}^!(\text{Ran}_{\text{unl}, \text{aug}}) \xrightarrow{\text{OblvAug}} \text{Shv}^!(\text{Ran}_{\text{unl}}).$$

Note that this is in line with the situation with associative algebras: the algebra obtained by adjoining a unit to a non-unital algebra is naturally augmented.



5.2.1. Consider the object  $\text{Ran}_{\text{aug}}^{\rightarrow} \in \text{LaxPreStk}$ , defined as follows:

For  $S \in \text{Sch}$  we let  $\text{Ran}_{\text{aug}}^{\rightarrow}(S)$  be the (ordinary) category whose objects are triples

$$(K \subseteq I \supseteq J), \quad K \cap J \neq \emptyset$$

of finite subsets of  $\text{Maps}(S, X)$ . Morphisms from  $(K \subseteq I \supseteq J)$  to  $(K_1 \subseteq I_1 \supseteq J_1)$  are inclusions  $I \subseteq I_1$ , such that  $K \subseteq K_1$  and  $J = J_1$ .

We have the natural maps

$$\psi_{\text{aug}} : \text{Ran}_{\text{aug}}^{\rightarrow} \rightarrow \text{Ran}_{\text{untl, aug}}, \quad (K \subseteq I \supseteq J) \mapsto (K \subseteq I)$$

and

$$\xi_{\text{aug}} : \text{Ran}_{\text{aug}}^{\rightarrow} \rightarrow \text{Ran}, \quad (K \subseteq I \supseteq J) \mapsto J.$$

**Lemma 5.2.2.** *The map  $\psi_{\text{aug}}$  is pseudo-proper.*

*Proof.* The proof is similar to that of Lemma 4.3.2 except for the expression for the fiber product

$$S \times_{\text{Ran}_{\text{untl, aug}}} \text{Ran}_{\text{aug}}^{\rightarrow}$$

for a given  $S$ -point  $K \subseteq I$  of  $\text{Ran}_{\text{untl, aug}}$ .

The fiber product in question is

$$\text{colim}_{\mathcal{J} \rightarrow \mathcal{J}' \leftarrow I} X^{\mathcal{J}'} \times_{X^I} S,$$

where the colimit is taken over the full subcategory of that appearing in the proof of Lemma 4.3.2, consisting of those  $\mathcal{J} \rightarrow \mathcal{J}' \leftarrow I$ , for which the images of  $\mathcal{J}$  and  $K$  in  $\mathcal{J}'$  have a non-empty intersection.  $\square$

5.2.3. We claim that there is a canonically defined natural transformation

$$(5.1) \quad (\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^! \rightarrow \pi^! \circ \text{AddUnit}.$$

as functors

$$\text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl, aug}}).$$

Indeed, it comes by adjunction using the commutative diagram

$$\begin{array}{ccccc} \text{Ran}_{\text{aug}}^{\rightarrow} & \xrightarrow{\pi^{\rightarrow}} & \text{Ran}^{\rightarrow} & \xrightarrow{\xi} & \text{Ran} \\ \psi_{\text{aug}} \downarrow & & \downarrow \psi & & \\ \text{Ran}_{\text{untl, aug}} & \xrightarrow{\pi} & \text{Ran}_{\text{untl}}, & & \end{array}$$

where

$$\pi^{\rightarrow}(K \subseteq I \supseteq J) = (J \subseteq I),$$

and using the fact that

$$\xi_{\text{aug}} = \xi \circ \pi^{\rightarrow}.$$

5.2.4. We define the functor

$$\text{AddUnit}_{\text{aug}} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$$

as the cofiber of the natural transformation (5.1).

5.2.5. The functor  $\text{AddUnit}_{\text{aug}}$  can be described explicitly as follows:

For  $S \in \text{Sch}$  fix an object  $I \in \text{Ran}_{\text{untl}}(S)$  corresponding to a finite set  $I$  of maps  $S \rightarrow X$  with pairwise non-intersecting images. Let  $K \subseteq I$  be a subset, regarded as an object of  $\text{Ran}_{\text{untl, aug}}(S)$ .

**Proposition 5.2.6.** *For  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , the object  $\text{AddUnit}_{\text{aug}}(\mathcal{F})_{S, K \subseteq I} \in \text{Shv}^!(S)$  is given by*

$$\bigoplus_{\emptyset \neq J \subseteq (I-K)} \mathcal{F}_{S, J},$$

where  $J$  is regarded as an  $S$ -point of  $\text{Ran}$  via  $J \subseteq I \subset \text{Maps}(S, X)$ .

*Proof.* Follows from Corollary 2.3.5. □

5.2.7. By construction, the functor  $\text{AddUnit}_{\text{aug}}$  comes equipped with a natural transformation

$$\pi^! \circ \text{AddUnit} \rightarrow \text{AddUnit}_{\text{aug}}, \quad \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl, aug}}).$$

Applying the functor

$$\text{OblvAug} : \text{Shv}^!(\text{Ran}_{\text{untl, aug}}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl}})$$

we obtain a natural transformation

$$(5.2) \quad \text{AddUnit} \rightarrow \text{OblvAug} \circ \text{AddUnit}_{\text{aug}}.$$

**Lemma 5.2.8.** *The natural transformation (5.2) is an isomorphism.*

*Proof.* Follows from Proposition 5.2.6, using the next lemma. □

**Lemma 5.2.9.** *In order to check whether a map between two objects in the category  $\text{Shv}^!(\text{Ran})$  or  $\text{Shv}^!(\text{Ran}_{\text{untl}})$  (resp.,  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ ) is an isomorphism, it is enough to check that it induces on  $S$ -points corresponding to finite subsets of  $I \subset \text{Maps}(S, X)$  (resp.,  $K \subset I \subset \text{Maps}(S, X)$ ) such that the maps  $S \rightarrow X$  that comprise  $I$  have pairwise non-intersecting images.*

*Proof.* Use the diagonal stratification of  $X^I$ . □

5.3. **The functor of taking the unit out.** As was mentioned in the beginning of this chapter, we expect the procedure of inserting the unit, viewed as taking values in the unital augmented category, to be invertible. We will realize this in the present subsection.

Namely, we will now consider the functor

$$\text{TakeOut} : \text{Shv}^!(\text{Ran}_{\text{untl, aug}}) \rightarrow \text{Shv}^!(\text{Ran}),$$

right adjoint to  $\text{AddUnit}_{\text{aug}}$  (this right adjoint exists because  $\text{AddUnit}_{\text{aug}}$  is colimit-preserving). We will show that the functor  $\text{TakeOut}$  is the left inverse of the functor  $\text{AddUnit}_{\text{aug}}$ .

5.3.1. The functor  $\text{TakeOut}$  can be explicitly described as follows.

**Proposition 5.3.2.** *For  $\tilde{\mathcal{F}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ , an object  $S \in \text{Sch}$  and an object  $J \in \text{Ran}(S)$ , the object*

$$\text{TakeOut}(\tilde{\mathcal{F}})_{S, J} \in \text{Shv}^!(S)$$

is given by

$$\text{Fib} \left( \tilde{\mathcal{F}}_{S, \emptyset \subset J} \rightarrow \lim_{\emptyset \neq K \subseteq J} \tilde{\mathcal{F}}_{S, K \subseteq J} \right).$$

*Remark 5.3.3.* Although we will not need this, note that Proposition 5.3.2 implies that the functor  $\text{TakeOut}$  is colimit-preserving.

*Proof.* We first show:

**Lemma 5.3.4.** *The functor  $\text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}}) \rightarrow \text{Shv}^!(\text{Ran})$ , right adjoint to the functor  $\pi^! \circ \text{AddUnit}$ , is given by*

$$\text{OblvUnit} \circ \text{OblvAug} = \phi^! \circ \iota^!.$$

*Proof.* By Corollary 2.3.5, the functor  $\pi^! \circ \text{AddUnit}$  is given by pull-push along the following diagram

$$\begin{array}{ccccc} \text{Ran}_{\text{untl},\text{aug}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} & \xrightarrow{\pi'} & \text{Ran}^{\rightarrow} & \xrightarrow{\xi} & \text{Ran} \\ \tilde{\psi} \downarrow & & & & \downarrow \psi \\ \text{Ran}_{\text{untl},\text{aug}} & \xrightarrow{\pi} & \text{Ran}_{\text{untl}} & & \end{array}$$

Hence, its right adjoint is given by

$$((\xi \circ \pi')^!)^R \circ \tilde{\psi}^!.$$

Note there is a canonically defined map of lax prestacks

$$v' : \text{Ran} \rightarrow \text{Ran}_{\text{untl},\text{aug}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow}, \quad J \mapsto (\emptyset \subset J, J \subseteq J),$$

such that for every  $S \in \text{Sch}$ , the functors  $(v'(S), (\xi \circ \pi')(S))$  form an adjoint pair. Hence, by Lemma 2.2.6, the functor  $(v')^!$  identifies with the right adjoint of  $(\xi \circ \pi')^!$ .

Therefore, the right adjoint to  $\pi^! \circ \text{AddUnit}$  is given by  $(\tilde{\psi} \circ v')^!$ , while  $\tilde{\psi} \circ v' = \iota \circ \phi$ .  $\square$

We will now show that the right adjoint to the functor  $(\psi_{\text{aug}})^! \circ (\xi_{\text{aug}})^!$  is calculated by the formula

$$(5.3) \quad (\tilde{\mathcal{F}} \in \text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}}), S \in \text{Sch}, J \in \text{Ran}(S)) \mapsto \left( \lim_{\emptyset \neq K \subseteq J} \tilde{\mathcal{F}}_{S, K \subseteq J} \right) \in \text{Shv}^!(S).$$

We shall do so by applying Lemma 2.4.5 to the morphism  $\xi_{\text{aug}} : \text{Ran}_{\text{aug}}^{\rightarrow} \rightarrow \text{Ran}$ . For a given  $(S, J)$  as above, note that the limit appearing in Lemma 2.4.5 is

$$\lim_{K \subseteq I} \tilde{\mathcal{F}}_{S, K \subseteq I},$$

where the limit is taken over the category of pairs of finite subsets  $K \subseteq I$  of  $\text{Maps}(S, X)$ , such that  $I$  contains  $J$  and  $K \cap J \neq \emptyset$ .

However, right cofinal in this category is the full subcategory consisting of those  $I$ , for which the inclusion  $J \subseteq I$  is an equality. Hence, the limit in question maps isomorphically to that in formula (5.3).

To finish the proof we need to show that the maps (2.2) are isomorphisms in our case. The fact that the map  $\rightarrow$  is an isomorphism is automatic, because the index category of  $\emptyset \neq K \subseteq J$  is finite. To show that the map  $\leftarrow$  is an isomorphism, we argue as follows:

For a given  $g : S' \rightarrow S$  and the corresponding set  $J' = g(J) \subset \text{Maps}(S', X)$ , we need to show that the limit over  $\emptyset \neq K' \subseteq J'$  of a certain functor with values in  $\text{Shv}(S')$  maps isomorphically to the limit of the restriction of this functor to the category  $\emptyset \neq K \subseteq J$ , where the map between the index categories is given by  $K \mapsto g(K)$ .

However, we claim that the above map of index categories is right cofinal (which would imply that the maps of the limits is an isomorphism). Namely, it is easy to see that this map is a Cartesian fibration, and each fiber is contractible (it has a final object).  $\square$

*Remark 5.3.5.* The description of the functor  $\text{TakeOut}$  given by Proposition 5.3.2 admits the following reformulation (we will not need it in the sequel, except for the optional Sect. 6.4):

Let  $\text{Ran}^{\leftarrow}$  denote the lax prestack, whose category of  $S$ -points is that of  $\mathcal{K} \subseteq \mathcal{J} \subset \text{Maps}(S, X)$ , where both  $\mathcal{K}$  and  $\mathcal{J}$  are non-empty, and morphisms  $(\mathcal{K} \subseteq \mathcal{J}) \rightarrow (\mathcal{K}' \subseteq \mathcal{J}')$  are inclusions  $\mathcal{K} \subseteq \mathcal{K}'$  with  $\mathcal{J} = \mathcal{J}'$ .

Let  $\xi'_{\text{aug}}$  and  $\psi'_{\text{aug}}$  denote the maps from  $\text{Ran}^{\leftarrow}$  to  $\text{Ran}$  and  $\text{Ran}_{\text{untl, aug}}$  that send

$$(\mathcal{K} \subseteq \mathcal{J}) \mapsto \mathcal{J} \text{ and } (\mathcal{K} \subseteq \mathcal{J}) \mapsto (\mathcal{K} \subseteq \mathcal{J}),$$

respectively.

Then we obtain functor  $\text{TakeOut}$  is canonically isomorphic to

$$\text{Fib} \left( \text{OblvUnit} \circ \text{OblvAug} \rightarrow ((\xi'_{\text{aug}})^!)^R \circ (\psi'_{\text{aug}})^! \right).$$

**5.4. Fully faithfulness of the functor  $\text{AddUnit}_{\text{aug}}$ .** In this subsection we will establish the following crucial result: the functor  $\text{AddUnit}_{\text{aug}}$  is fully faithful.

5.4.1. For  $S \in \text{Sch}$  and  $I_1, I_2 \subset \text{Maps}(S, X)$ , we shall say that the  $S$ -points  $I_1$  and  $I_2$  of  $\text{Ran}$  have *disjoint images* if for every  $i_1 \in I_1$  and every  $i_2 \in I_2$  the resulting two maps  $S \rightrightarrows X$  have non-intersecting images.

5.4.2. We now claim:

**Theorem 5.4.3.** *The functor*

$$\text{AddUnit}_{\text{aug}} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$$

*is fully faithful. Its essential image consists of those objects  $\tilde{\mathcal{F}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  that satisfy the following two conditions:*

- (\*) *For every  $S \in \text{Sch}$  and  $I \in \text{Ran}_{\text{untl}}(S)$ , for  $K = I \subseteq I$ , the object  $\tilde{\mathcal{F}}_{S, I \subseteq I}$  is zero.*
- (\*\*) *For  $S \in \text{Sch}$  and  $L \in \text{Ran}(S)$  and  $(K \subseteq I) \in \text{Ran}_{\text{untl, aug}}(S)$ , such that  $L$  and  $I$  have disjoint images, the map*

$$(K \subseteq I) \rightarrow (K \cup L \subseteq I \cup L)$$

*in  $\text{Ran}_{\text{untl, aug}}(S)$  gives rise to an isomorphism  $\tilde{\mathcal{F}}_{S, K \subseteq I} \rightarrow \tilde{\mathcal{F}}_{S, K \cup L \subseteq I \cup L}$ .*

*Proof.* To prove that the functor  $\text{AddUnit}_{\text{aug}}$  is fully faithful, we have to check that the unit of the adjunction

$$\mathcal{F} \rightarrow \text{TakeOut} \circ \text{AddUnit}_{\text{aug}}(\mathcal{F}), \quad \mathcal{F} \in \text{Shv}^!(\text{Ran})$$

is an isomorphism. By Lemma 5.2.9, it is enough to check that the above map induces an isomorphism at  $S$ -valued points  $I$  that correspond to finite subsets of  $\text{Maps}(S, X)$  with pairwise non-intersecting images.

By Propositions 5.2.6 and 5.3.2, the value of the right-hand side at  $I$  is the fiber of the map

$$(5.4) \quad \bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_{S, J} \rightarrow \lim_{\emptyset \neq K \subseteq I} \bigoplus_{\emptyset \neq J \subseteq I - K} \mathcal{F}_{S, J},$$

defined as follows:

For a given  $\emptyset \neq K \subseteq I$ , the corresponding map

$$\bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_{S, J} \rightarrow \bigoplus_{\emptyset \neq J \subseteq I - K} \mathcal{F}_{S, J}$$

is the projection onto those direct summands for which  $J \cap K = \emptyset$ .

In terms of this description, the unit of the adjunction is given by the map

$$\mathcal{F}_{S, I} \rightarrow \bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_{S, J}$$

corresponding to the direct summand  $J = I$ .

Now, the limit in the right-hand side of (5.4) can be computed as

$$\bigoplus_{\emptyset \neq J \subseteq I} \lim_{\emptyset \neq K \subseteq I - J} \mathcal{F}_{S, J},$$

which is isomorphic to

$$\bigoplus_{\emptyset \neq J \subseteq I} \mathcal{F}_{S, J},$$

because for each  $J \neq I$  the category of indices  $\emptyset \neq K \subseteq I - J$  is contractible. Under this identification, the map (5.4) is the projection on the summands with  $J \neq I$ .

This makes the isomorphism assertion manifest.

It is clear from Proposition 5.2.6 and Lemma 5.2.9 that any object in the essential image of  $\text{AddUnit}_{\text{aug}}$  satisfies conditions (\*) and (\*\*). To finish the proof, we need to show that if  $\tilde{\mathcal{F}}$  is an object of  $\tilde{\mathcal{F}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  that satisfies conditions (\*) and (\*\*) such that  $\text{TakeOut}(\tilde{\mathcal{F}}) = 0$ , then  $\tilde{\mathcal{F}} = 0$ .

By Lemma 5.2.9, it is enough to show that  $\tilde{\mathcal{F}}_{S, K \subseteq I} = 0$  for  $I$  corresponding to a finite subset of  $\text{Maps}(S, X)$  consisting of maps with pairwise non-intersecting images. By induction on the cardinality of  $I$  and conditions (\*) and (\*\*), we can assume that  $\tilde{\mathcal{F}}_{S, K' \subseteq I'} = 0$ , whenever  $|I'| - |K'| < |I|$ . In particular, it remains to prove that  $\tilde{\mathcal{F}}_{S, \emptyset \subseteq I} = 0$  we can assume that  $K = \emptyset$ .

By Proposition 5.3.2, we have:

$$\text{TakeOut}(\tilde{\mathcal{F}})_{S, I} = \text{Fib} \left( \tilde{\mathcal{F}}_{S, \emptyset \subseteq I} \rightarrow \lim_{\emptyset \neq K \subseteq I} \tilde{\mathcal{F}}_{S, K \subseteq I} \right).$$

However, by the induction hypothesis, all the terms in the above limit vanish. Hence,

$$0 = \text{TakeOut}(\tilde{\mathcal{F}})_{S, I} \simeq \tilde{\mathcal{F}}_{S, \emptyset \subseteq I}.$$

□

5.4.4. By Lemma 5.3.4, we have a natural transformation of functors

$$(5.5) \quad \text{TakeOut} \rightarrow \text{OblvUnit} \circ \text{OblvAug}, \quad \text{Shv}^!(\text{Ran}_{\text{untl, aug}}) \rightarrow \text{Shv}^!(\text{Ran}).$$

We claim:

**Corollary 5.4.5.** *Assume that  $X$  is connected<sup>13</sup>. Let  $\mathcal{G}$  be an object of  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  satisfying conditions (\*) and (\*\*). Then the natural transformation (5.5) induces an isomorphism*

$$\text{C}_c^*(\text{Ran}, -) \circ \text{TakeOut}(\mathcal{G}) \rightarrow \text{C}_c^*(\text{Ran}, -) \circ \text{OblvUnit} \circ \text{OblvAug}(\mathcal{G}).$$

*Proof.* Note that if we precompose the natural transformation (5.5) with the functor  $\text{AddUnit}_{\text{aug}}$  we obtain a natural transformation

$$\text{Id}_{\text{Shv}^!(\text{Ran})} \xrightarrow{\sim} \text{TakeOut} \circ \text{AddUnit}_{\text{aug}} \rightarrow \text{OblvUnit} \circ \text{OblvAug} \circ \text{AddUnit}_{\text{aug}} \simeq \text{OblvUnit} \circ \text{AddUnit},$$

which equals the natural transformation (4.3).

Hence, the assertion of the corollary follows from Corollary 4.5.2.

□

## 6. AN ALTERNATIVE POINT OF VIEW ON SHEAVES ON THE UNITAL RAN SPACE

The contents of this section will not be used in the sequel; it is added for the sake for completeness.

Recall the presentation of  $\text{Ran}$  as a colimit of schemes, given by Proposition 4.1.3. So, a datum of a sheaf on  $\text{Ran}$  amounts to an assignment

$$(J \in \text{Fin}^s) \rightsquigarrow \mathcal{F}_J \in \text{Shv}(X^J),$$

equipped with a homotopy-compatible system of isomorphisms

$$(J \xrightarrow{\alpha} \mathcal{J}) \rightsquigarrow (\text{diag}_\alpha)^!(\mathcal{F}_J) \simeq \mathcal{F}_\mathcal{J},$$

where  $\text{diag}_\alpha$  denotes the map  $X^J \rightarrow X^\mathcal{J}$ , induced by  $\alpha$ .

In this section we will give a similar description of the categories  $\text{Shv}^!(\text{Ran}_{\text{untl}})$ ,  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  and the functors  $\text{AddUnit}$ ,  $\text{AddUnit}_{\text{aug}}$  and  $\text{TakeOut}$ .

<sup>13</sup>And proper, if we are working in the context of D-modules.

**6.1. The case of the usual Ran space.** First, we will give a different interpretation of the presentation of  $\text{Ran}$  as a colimit, given by Proposition 4.1.3.

6.1.1. For a set  $A$  the functor

$$\mathcal{J} \mapsto A^{\mathcal{J}}, \quad (\text{Fin}^s)^{\text{op}} \rightarrow \text{Sets}$$

defines a Cartesian fibration in groupoids  $A^{\text{Fin}^s} \rightarrow \text{Fin}^s$ .

Let  $X^{\text{Fin}^s}$  denote the lax prestack that assigns to  $S \in \text{Sch}$  the category

$$\text{Maps}(S, X)^{\text{Fin}^s}.$$

In other words, the category  $\text{Sch}/_{X^{\text{Fin}^s}}$  is a Cartesian fibration in groupoids over  $\text{Sch} \times \text{Fin}^s$  with the fiber over  $S \times \mathcal{J}$  being  $\text{Maps}(S, X)^{\mathcal{J}}$ .

Note that we can think of an object  $\mathcal{F} \in \text{Shv}^!(X^{\text{Fin}^s})$  as an assignment

$$(\mathcal{J} \in \text{Fin}^s) \rightsquigarrow \mathcal{F}_{\mathcal{J}} \in \text{Shv}(X^{\mathcal{J}}),$$

equipped with a homotopy-compatible system of *morphisms*

$$(6.1) \quad (\mathcal{J} \xrightarrow{\alpha} \mathcal{J}') \rightsquigarrow (\text{diag}_{\alpha})^!(\mathcal{F}_{\mathcal{J}}) \rightarrow \mathcal{F}_{\mathcal{J}'}$$

6.1.2. Let  $A^{\text{Fin}^s, \sim}$  denote the following the *localization* of  $A^{\text{Fin}^s}$ : we invert all arrows that are Cartesian over  $\text{Fin}^s$ . Since the fibers of  $A^{\text{Fin}^s}$  over objects of  $\text{Fin}^s$  are groupoids, we obtain that  $A^{\text{Fin}^s, \sim}$  is itself a groupoid.

Let  $X^{\text{Fin}^s, \sim}$  denote the corresponding prestack.

The category  $\text{Shv}^!(X^{\text{Fin}^s, \sim})$  is a full subcategory of  $\text{Shv}^!(X^{\text{Fin}^s})$  consisting of those objects for which the maps (6.1) are *isomorphisms*.

6.1.3. Note now that we have a canonical identification:

$$X^{\text{Fin}^s, \sim} \simeq \text{colim}_{\mathcal{J} \in \text{Fin}^s} X^{\mathcal{J}}.$$

Combining with Proposition 4.1.3, we obtain an identification

$$(6.2) \quad X^{\text{Fin}^s, \sim} \simeq \text{Ran}.$$

In the rest of this section we will develop an analog of the interpretation of  $\text{Ran}$ , given by (6.2), to the case of  $\text{Ran}_{\text{untl}}$  and  $\text{Ran}_{\text{untl}, \text{aug}}$ .

**6.2. The case of the unital Ran space.**

6.2.1. Let  $\text{Fin}$  denote the category of finite non-empty sets (and all maps). For a set  $A$  the functor

$$\mathcal{J} \mapsto A^{\mathcal{J}}, \quad (\text{Fin})^{\text{op}} \rightarrow \text{Sets}$$

defines a Cartesian fibration in groupoids  $A^{\text{Fin}} \rightarrow \text{Fin}$ .

Let  $X^{\text{Fin}}$  denote the lax prestack that assigns to  $S \in \text{Sch}$  the category

$$\text{Maps}(S, X)^{\text{Fin}}.$$

In other words, the category  $\text{Sch}/_{X^{\text{Fin}}}$  is a Cartesian fibration in groupoids over  $\text{Sch} \times \text{Fin}$  with the fiber over  $S \times \mathcal{J}$  being  $\text{Maps}(S, X)^{\mathcal{J}}$ .

By definition, an object  $\mathcal{F} \in \text{Shv}^!(X^{\text{Fin}})$  is an assignment

$$(\mathcal{J} \in \text{Fin}) \rightsquigarrow \mathcal{F}_{\mathcal{J}} \in \text{Shv}(X^{\mathcal{J}}),$$

equipped with a homotopy-compatible system of *morphisms*

$$(6.3) \quad (\mathcal{J} \xrightarrow{\alpha} \mathcal{J}') \rightsquigarrow (\text{diag}_{\alpha})^!(\mathcal{F}_{\mathcal{J}}) \rightarrow \mathcal{F}_{\mathcal{J}'}$$

6.2.2. For a set  $A$ , let  $A^{\text{Fin}, \sim}$  denote the following the *localization* of  $A^{\text{Fin}}$ : we invert all *Cartesian* arrows that lie above that maps in  $\text{Fin}$  given by *surjective* maps of finite sets.

Let  $X^{\text{Fin}, \sim}$  denote the corresponding lax prestack.

The category  $\text{Shv}^!(X^{\text{Fin}, \sim})$  is a full subcategory of  $\text{Shv}^!(X^{\text{Fin}})$  consisting of those objects for which the maps (6.3) are *isomorphisms for those  $\alpha$  that are surjective*.

6.2.3. We claim:

**Proposition 6.2.4.** *There exists a canonical isomorphism*

$$X^{\text{Fin}, \sim} \simeq \text{Ran}_{\text{untl}}.$$

*Proof.* We need to show that for a set  $A$ , the category  $A^{\text{Fin}, \sim}$  is (canonically) equivalent to the category  $\text{Subsets}_f(A)$  of finite non-empty subsets of  $A$ .

We have a naturally defined functor

$$(6.4) \quad A^{\text{Fin}} \rightarrow \text{Subsets}_f(A)$$

that assigns to an object  $(\mathcal{J} \rightarrow A) \in A^{\text{Fin}}$  its image in  $A$ . We claim that this functor defines an equivalence from  $A^{\text{Fin}, \sim}$  to  $\text{Subsets}_f(A)$ .

First, it is easy to see that (6.4) is a Cartesian fibration, and the arrows in  $A^{\text{Fin}}$  that are Cartesian over *surjective* maps in  $\text{Fin}^{\text{op}}$  get sent to isomorphisms in  $\text{Subsets}_f(A)$ . Hence, it remains to show that that the localization of every fiber of (6.4) with respect to the same class of morphisms is contractible.

For a subset  $B \subset A$ , the fiber in question is the category of pairs  $(\mathcal{J}, \mathcal{J} \xrightarrow{f} B)$ . We localize it with respect to the morphisms that are surjective on the  $\mathcal{J}$ 's. However, the above category has a final object, namely  $\mathcal{J} = B$ , and the canonical map from any object to it belongs to the class of morphisms with respect to which we localize. The assertion follows now from the next general lemma:

**Lemma 6.2.5.** *Let  $\mathbf{C}$  be a category with a final object  $\mathbf{c}_f$ . Then any localization of  $\mathbf{C}$  with respect to a class of morphisms that contains the canonical morphisms  $\mathbf{c} \rightarrow \mathbf{c}_f$  for all  $\mathbf{c} \in \mathbf{C}$ , is contractible.*

□

6.2.6. Thus, from Proposition 6.2.4 we obtain a fully faithful embedding

$$\text{Shv}^!(\text{Ran}_{\text{untl}}) \hookrightarrow \text{Shv}^!(X^{\text{Fin}}).$$

Let us now describe the functor

$$(6.5) \quad \text{Shv}^!(\text{Ran}) \xrightarrow{\text{AddUnit}} \text{Shv}^!(\text{Ran}_{\text{untl}}) \hookrightarrow \text{Shv}^!(X^{\text{Fin}}).$$

As we shall see in Sect. 8.1.1, for every finite non-empty set  $\mathcal{J}$ , the morphism  $\text{ins}_{\mathcal{J}} : X^{\mathcal{J}} \rightarrow \text{Ran}$  is pseudo-proper, and for any  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , the canonical map

$$\text{colim}_{\mathcal{J} \in \text{Fin}^{\text{op}}} (\text{ins}_{\mathcal{J}})_! \circ (\text{ins}_{\mathcal{J}})^!(\mathcal{F}) \rightarrow \mathcal{F}$$

is an isomorphism.

So, it would suffice to describe the composition of (6.5) with the functor  $(\text{ins}_{\mathcal{J}})_!$  for every  $\mathcal{J}$ . What we will actually do is write explicitly the composed functor

$$(6.6) \quad \text{Shv}(X^{\mathcal{J}}) \xrightarrow{(\text{ins}_{\mathcal{J}})_!} \text{Shv}^!(\text{Ran}) \xrightarrow{\text{AddUnit}} \text{Shv}^!(\text{Ran}_{\text{untl}}) \hookrightarrow \text{Shv}^!(X^{\text{Fin}}) \rightarrow \text{Shv}(X^{\mathcal{J}})$$

the last arrow is the evaluation functor corresponding to a finite non-empty set  $\mathcal{J}$ .

Since the maps  $\psi : \text{Ran}^\rightarrow \rightarrow \text{Ran}_{\text{untl}}$  and  $\text{ins}_{\mathcal{J}}$  are pseudo-proper, the composition (6.6) is computed algorithmically via base change: it is given by pull-push along

$$(6.7) \quad \begin{array}{ccc} (X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}^{\sim} & := & X^{\mathcal{J}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^\rightarrow \times_{\text{Ran}} X^{\mathcal{J}} \longrightarrow X^{\mathcal{J}} \\ & & \downarrow \\ & & X^{\mathcal{J}}. \end{array}$$

So, it remains to describe explicitly the above prestack  $(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}^{\sim}$ . As in Proposition 4.1.3 one shows that it is given as

$$\text{colim}_{\mathcal{J} \rightarrow \mathcal{K} \leftarrow \mathcal{J}} X^{\mathcal{K}},$$

where the colimit is taken over the category of finite non-empty sets  $\mathcal{K}$ , equipped with a surjection  $\mathcal{J} \twoheadrightarrow \mathcal{K}$  and a map  $\mathcal{J} \rightarrow \mathcal{K}$ .

*Remark 6.2.7.* Let  $(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}$  be the closed reduced subscheme of  $X^{\mathcal{J}} \times X^{\mathcal{J}}$ , whose  $k$ -points are those pairs of an  $\mathcal{J}$ -tuple and a  $\mathcal{J}$ -tuple of  $k$ -points of  $X$ , for which the  $\mathcal{J}$ -tuple is contained in the  $\mathcal{J}$ -tuple as a subset of  $X(k)$ . In other words,  $(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}$  is the reduced subscheme underlying the union of images of the maps  $X^{\mathcal{K}} \rightarrow X^{\mathcal{J}} \times X^{\mathcal{J}}$  for all  $\mathcal{J} \twoheadrightarrow \mathcal{K} \leftarrow \mathcal{J}$ .

It follows from Lemma 7.4.11(d) that pullback along the morphism

$$(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}^{\sim} \rightarrow (X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}$$

is an equivalence of categories. Hence, as in Corollary 8.1.5, when computing the functor (6.6) as pull-push, we can replace the diagram (6.7) that involves the prestack  $(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq}^{\sim}$  by the diagram

$$\begin{array}{ccc} (X^{\mathcal{J}} \times X^{\mathcal{J}})_{\supseteq} & \longrightarrow & X^{\mathcal{J}} \\ \downarrow & & \\ X^{\mathcal{J}} & & \end{array}$$

which only involves schemes.

**6.3. The case of the unital augmented Ran space.** In this subsection we shall describe the category  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  in terms similar to those in Sect. 6.2.

6.3.1. Let  $\text{Fin}_{\text{aug}}$  denote the category of pairs  $(\mathcal{K} \subseteq \mathcal{J})$ , where  $\mathcal{J}$  is a finite non-empty set and  $\mathcal{K}$  is its (possibly empty) subset; morphisms in the category are defined naturally.

For a set  $A$  we define the Cartesian fibration  $A^{\text{Fin}_{\text{aug}}} \rightarrow \text{Fin}_{\text{aug}}$  to correspond to the functor

$$(\text{Fin}_{\text{aug}})^{\text{op}} \rightarrow \text{Sets}, \quad (\mathcal{K} \subseteq \mathcal{J}) \mapsto A^{\mathcal{J}}.$$

Let  $A^{\text{Fin}_{\text{aug}}, \sim}$  be the localization of  $A^{\text{Fin}_{\text{aug}}}$ , where we invert the Cartesian arrows that lie over the arrows  $(\mathcal{K}_1 \subseteq \mathcal{J}_1) \rightarrow (\mathcal{K}_2 \subseteq \mathcal{J}_2)$  for which both  $\mathcal{K}_1 \rightarrow \mathcal{K}_2$  and  $\mathcal{J}_1 \rightarrow \mathcal{J}_2$  are surjective.

6.3.2. Let

$$X^{\text{Fin}_{\text{aug}}} \text{ and } X^{\text{Fin}_{\text{aug}}, \sim}$$

denote the corresponding lax prestacks.

In other words, the category  $\text{Sch}/X^{\text{Fin}_{\text{aug}}}$  is a Cartesian fibration over  $\text{Sch} \times \text{Fin}_{\text{aug}}$ , with the fiber over  $S \times (\mathcal{K} \subseteq \mathcal{J})$  being  $\text{Maps}(S, X)^{\mathcal{J}}$ .

As in Proposition 6.2.4 one shows that the natural map

$$X^{\text{Fin}_{\text{aug}}, \sim} \rightarrow \text{Ran}_{\text{untl, aug}}$$

is an isomorphism.



6.3.3. The category  $\text{Shv}^!(X^{\text{Fin}_{\text{aug}}})$  can be (tautologically) described as follows. It consists of assignments

$$(6.8) \quad (\mathcal{K} \subseteq \mathcal{J}) \rightsquigarrow \mathcal{F}_{\mathcal{K} \subseteq \mathcal{J}} \in \text{Shv}(X^{\mathcal{J}}),$$

equipped with a homotopy-compatible set of morphisms

$$(6.9) \quad \mathcal{F}_{\mathcal{K}_1 \subseteq \mathcal{J}_1} |_{X^{\mathcal{J}_2}} \rightarrow \mathcal{F}_{\mathcal{K}_2 \subseteq \mathcal{J}_2}$$

for every arrow  $(\mathcal{K}_1 \subseteq \mathcal{J}_1) \rightarrow (\mathcal{K}_2 \subseteq \mathcal{J}_2)$  in  $\text{Fin}_{\text{aug}}$ .

The category  $\text{Shv}^!(X^{\text{Fin}_{\text{aug}, \sim}})$  is a full subcategory of  $\text{Shv}^!(X^{\text{Fin}_{\text{aug}}})$  that consists of those objects for which the maps (6.9) are isomorphisms for those maps  $(\mathcal{K}_1 \subseteq \mathcal{J}_1) \rightarrow (\mathcal{K}_2 \subseteq \mathcal{J}_2)$  for which both  $\mathcal{K}_1 \rightarrow \mathcal{K}_2$  and  $\mathcal{J}_1 \rightarrow \mathcal{J}_2$  are surjective.

6.3.4. Let us now describe the functor  $\text{AddUnit}_{\text{aug}}$ . As in Sect. 6.2.6, we shall describe the corresponding functor

$$\text{Shv}(X^{\mathcal{J}}) \xrightarrow{(\text{ins}_{\mathcal{J}})^!} \text{Shv}^!(\text{Ran}) \xrightarrow{\text{AddUnit}_{\text{aug}}} \text{Shv}^!(\text{Ran}_{\text{untl}, \text{aug}}) \hookrightarrow \text{Shv}^!(X^{\text{Fin}_{\text{aug}}}) \rightarrow \text{Shv}(X^{\mathcal{J}})$$

for  $X^{\mathcal{J}} \rightarrow X^{\text{Fin}_{\text{aug}}}$ , corresponding to any given  $(\mathcal{K} \subseteq \mathcal{J}) \in \text{Fin}_{\text{aug}}$ .

As in Sect. 6.2.6, this amounts to a description of the fiber product

$$(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\cap \neq \emptyset}^{\sim} := X^{\mathcal{J}} \times_{\text{Ran}_{\text{untl}, \text{aug}}} \text{Ran}_{\text{aug}}^{\rightarrow} \times_{\text{Ran}} X^{\mathcal{J}}$$

as a prestack over  $X^{\mathcal{J}} \times X^{\mathcal{J}}$ . This prestack is described as the colimit

$$\text{colim}_{\mathcal{J} \supseteq \mathcal{K} \supseteq \mathcal{K}' \rightarrow \mathcal{L} \leftarrow \mathcal{J}' \subseteq \mathcal{J}} X^{\mathcal{J}} \times_{X^{\mathcal{K}'}} X^{\mathcal{L}} \times_{X^{\mathcal{J}'}} X^{\mathcal{J}},$$

where the colimit is taken over the category of diagrams

$$\mathcal{J} \supseteq \mathcal{K} \supseteq \mathcal{K}' \rightarrow \mathcal{L} \leftarrow \mathcal{J}' \subseteq \mathcal{J}, \quad \mathcal{L} \neq \emptyset,$$

and where the morphisms are surjections on the  $\mathcal{L}$ 's.

*Remark 6.3.5.* The above index category is quite complicated. This is the reason that for some applications, it is better to use the definition of the functor  $\text{AddUnit}_{\text{aug}}$  as it is given in Sect. 5.2 rather than try to express it in terms of schemes as in Sect. 6.3.2. <sup>14</sup>

*Remark 6.3.6.* As in Remark 6.2.7, we can replace the prestack  $(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\cap \neq \emptyset}^{\sim}$  by the corresponding closed subscheme

$$(X^{\mathcal{J}} \times X^{\mathcal{J}})_{\cap \neq \emptyset} \subset (X^{\mathcal{J}} \times X^{\mathcal{J}}).$$

6.4. **A description of the functor TakeOut.** In this subsection we will describe the functor TakeOut in terms of the presentation of  $\text{Ran}_{\text{untl}, \text{aug}}$  as  $X^{\text{Fin}_{\text{aug}, \sim}}$ .

6.4.1. Specifically, we would like to describe the composition

$$(6.10) \quad \text{Shv}^!(\text{Ran}_{\text{untl}, \text{aug}}) \xrightarrow{(\psi_{\text{aug}})^!} \text{Shv}^!(\text{Ran}_{\text{aug}}^{\rightarrow}) \xrightarrow{((\xi_{\text{aug}})^!)^R} \text{Shv}^!(\text{Ran}) \xrightarrow{\text{ins}_{\mathcal{J}}^!} \text{Shv}(X^{\mathcal{J}}).$$

On the one hand, Proposition 5.3.2 readily provides the description of this functor. Namely, it sends an object  $\mathcal{F} \in \text{Shv}^!(\text{Ran}_{\text{untl}, \text{aug}})$ , thought of as in (6.8), to

$$\lim_{\emptyset \neq \mathcal{K} \subseteq \mathcal{J}} \mathcal{F}_{\mathcal{K} \subseteq \mathcal{J}} \in \text{Shv}^!(X^{\mathcal{J}}).$$

Just for fun, we will now rederive this formula by a different method. Namely, by Remark 5.3.5, the functor (6.10) identifies with the composition

$$\text{Shv}^!(\text{Ran}_{\text{untl}, \text{aug}}) \xrightarrow{(\psi'_{\text{aug}})^!} \text{Shv}^!(\text{Ran}^{\leftarrow}) \xrightarrow{((\xi'_{\text{aug}})^!)^R} \text{Shv}^!(\text{Ran}) \xrightarrow{\text{ins}_{\mathcal{J}}^!} \text{Shv}(X^{\mathcal{J}}),$$

where  $\text{Ran}^{\leftarrow}$  is as in Remark 5.3.5.

<sup>14</sup>Exhibiting a lax prestack as a *lax* colimit of schemes is analogous to choosing coordinates on a manifold. This is convenient for some purposes, but is a nuisance for others.

By Lemma 2.4.2, the above functor is given by !-pull and (right adjoint of !-pullback)-push along the diagram

$$\begin{array}{ccc} X^{\mathcal{J}} \times_{\text{Ran}} \text{Ran}^{\leftarrow} & \xrightarrow{\psi'_{\mathcal{J}}} & \text{Ran}_{\text{untl, aug}} \\ \xi'_{\mathcal{J}} \downarrow & & \\ X^{\mathcal{J}} & & \end{array}$$

We claim:

**Proposition 6.4.2.** *The functor*

$$((\xi'_{\mathcal{J}})^!)^R \circ (\psi'_{\mathcal{J}})^! : \text{Shv}^!(\text{Ran}_{\text{untl, aug}}) \rightarrow \text{Shv}(X^{\mathcal{J}})$$

is canonically isomorphic to the functor

$$\mathcal{F} \mapsto \lim_{\emptyset \neq \mathcal{K} \subseteq \mathcal{J}} \mathcal{F}_{\mathcal{K} \subseteq \mathcal{J}}.$$

The rest of this subsection is devoted to the proof of Proposition 6.4.2.

6.4.3. Let  $\text{Fin}^{\leftarrow}$  denote the category, whose objects are pairs of non-empty finite sets  $\mathcal{K} \subseteq \mathcal{L}$ , and whose morphisms are *surjections*  $\mathcal{L}_1 \rightarrow \mathcal{L}_2$  that map  $\mathcal{K}_1 \rightarrow \mathcal{K}_2$  (not necessarily surjectively).

For a set  $A$ , let  $A^{\text{Fin}^{\leftarrow}}$  denote the Cartesian fibration in groupoids over  $\text{Fin}^{\leftarrow}$  that assigns to  $\mathcal{K} \subseteq \mathcal{L}$  the set  $A^{\mathcal{L}}$ .

Let  $A^{\text{Fin}^{\leftarrow}, \sim}$  denote the localization of  $A^{\text{Fin}^{\leftarrow}}$ , where we invert Cartesian arrows lying over those morphisms in  $\text{Fin}^{\leftarrow}$ , for which the map  $\mathcal{K}_1 \rightarrow \mathcal{K}_2$  is also surjective.

Let  $X^{\text{Fin}^{\leftarrow}}$  and  $X^{\text{Fin}^{\leftarrow}, \sim}$  denote the corresponding lax prestacks. As in Proposition 6.2.4 one shows that  $X^{\text{Fin}^{\leftarrow}, \sim}$  is canonically isomorphic to  $\text{Ran}^{\leftarrow}$ .

The composed map

$$X^{\text{Fin}^{\leftarrow}} \rightarrow X^{\text{Fin}^{\leftarrow}, \sim} \simeq \text{Ran}^{\leftarrow} \rightarrow \text{Ran}_{\text{untl, aug}}$$

factors as

$$X^{\text{Fin}^{\leftarrow}} \rightarrow X^{\text{Fin}_{\text{aug}}} \rightarrow X^{\text{Fin}_{\text{aug}}, \sim} \simeq \text{Ran}_{\text{untl, aug}}.$$

Since the maps

$$X^{\text{Fin}_{\text{aug}}} \rightarrow \text{Ran}_{\text{untl, aug}} \quad \text{and} \quad X^{\text{Fin}^{\leftarrow}} \rightarrow \text{Ran}^{\leftarrow}$$

are *localizations* when evaluated on every  $S \in \text{Sch}$ , functor  $((\xi'_{\mathcal{J}})^!)^R \circ (\psi'_{\mathcal{J}})^!$  is canonically isomorphic to !-pull and (right adjoint of !-pullback)-push along the diagram

$$(6.11) \quad \begin{array}{ccccc} X^{\mathcal{J}} \times_{\text{Ran}} X^{\text{Fin}^{\leftarrow}} & \longrightarrow & X^{\text{Fin}_{\text{aug}}} & \longrightarrow & \text{Ran}_{\text{untl, aug}} \\ \downarrow & & & & \\ X^{\mathcal{J}} & & & & \end{array}$$

6.4.4. The lax prestack  $X^{\mathcal{J}} \times_{\text{Ran}} X^{\text{Fin}^{\leftarrow}}$  attaches to  $S \in \text{Sch}$  the Cartesian fibration in groupoids over  $\text{Fin}_{\text{aug}}$  with the fiber over  $\mathcal{K} \subseteq \mathcal{L}$  being

$$\text{Maps}(S, X^{\mathcal{J}} \times_{\text{Ran}} X^{\mathcal{L}}) \simeq \text{colim}_{\mathcal{L} \rightarrow \mathcal{M} \leftarrow \mathcal{J}} \text{Maps}(S, X)^{\mathcal{M}}.$$

Let  $\mathcal{Q}$  denote the category of

$$(6.12) \quad \mathcal{K} \subseteq \mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xleftarrow{\beta} \mathcal{J},$$

where the morphisms are commutative diagrams that induce surjective maps between the  $\mathcal{L}$ 's. For a set  $A$ , let  $A^{\mathcal{Q}}$  denote the Cartesian fibration in groupoids over  $\mathcal{Q}$ , with the fiber over an object (6.12) being  $A^{\mathcal{M}}$ .

Let  $X^\mathcal{Q}$  be the corresponding lax prestack. We have a canonically defined map

$$X^\mathcal{Q} \rightarrow X^\mathcal{J} \times_{\text{Ran}} X^{\text{Fin}^{\leftarrow}},$$

which is a localization when evaluated on every  $S \in \text{Sch}$  (we invert the Cartesian arrows that lie over those maps in  $\mathcal{Q}$  that induce isomorphisms on the  $\mathcal{K}$ 's and the  $\mathcal{L}$ 's).

Hence, we obtain that when computing the  $!$ -pull and (right adjoint of  $!$ -pullback)-push along the diagram (6.11), we can replace this diagram by

$$(6.13) \quad \begin{array}{ccc} X^\mathcal{Q} & \longrightarrow & \text{Ran}_{\text{untl, aug}} \\ \downarrow & & \\ X^\mathcal{J} & & \end{array}$$

6.4.5. Note now that right-cofinal in the category  $\mathcal{Q}$  there is a full subcategory  $\mathcal{Q}'$  consisting of those objects (6.12) for which the map  $\beta$  is an isomorphism.

Hence, we can further replace the diagram (6.13) by the diagram

$$(6.14) \quad \begin{array}{ccc} X^{\mathcal{Q}'} & \longrightarrow & \text{Ran}_{\text{untl, aug}} \\ \downarrow & & \\ X^\mathcal{J} & & \end{array}$$

Note, however, that the lax prestack  $X^{\mathcal{Q}'}$  identifies with

$$X^\mathcal{J} \times \mathcal{Q}'.$$

Hence, we rewrite the functor in the proposition as

$$\mathcal{F} \mapsto \lim_{\mathcal{K} \subseteq \mathcal{L} \xrightarrow{\alpha} \mathcal{J}} \text{diag}_\alpha^!(\mathcal{F}_{\mathcal{K} \subseteq \mathcal{L}}).$$

6.4.6. Consider now the category  $\mathcal{Q}''$  of finite non-empty subsets of  $\mathcal{J}$  (i.e., this is the category appearing in the statement of the proposition). We have a canonically defined functor

$$(6.15) \quad \mathcal{Q}' \rightarrow \mathcal{Q}'', \quad (\mathcal{K} \subseteq \mathcal{L} \xrightarrow{\alpha} \mathcal{J}) \mapsto (\alpha(\mathcal{K}) \subset \mathcal{J}).$$

The functor

$$\mathcal{Q}' \rightarrow \text{Shv}(X^\mathcal{J}), \quad (\mathcal{K} \subseteq \mathcal{L} \xrightarrow{\alpha} \mathcal{J}) \mapsto \text{diag}_\alpha^!(\mathcal{F}_{\mathcal{K} \subseteq \mathcal{L}})$$

factors through the above functor (6.15) and the functor

$$\mathcal{Q}'' \rightarrow \text{Shv}(X^\mathcal{J}), \quad (\mathcal{K} \subseteq \mathcal{J}) \mapsto \mathcal{F}_{\mathcal{K} \subseteq \mathcal{J}}.$$

Hence, it remains to show that the functor (6.15) is right cofinal. However, the latter is clear as it is a Cartesian fibration, with contractible fibers (each fiber has a final object).

## Part II: Verdier duality on the Ran space

### 7. VERDIER DUALITY ON PRESTACKS

The functor of Verdier duality (for schemes, or topological spaces) is not something mysterious, see Sect. 7.1.3 below.

Our proof of the *cohomological product formula* (0.5), is based on considering the operation of Verdier duality on sheaves on the Ran space. The trouble is that the Ran space is not a scheme, and here the difference between schemes and prestacks (even the nice one such as Ran) will be significant.

In this section we will define what we mean by the Verdier duality functor on a prestack and describe it rather explicitly for a class of prestacks (called pseudo-schemes with a finitary diagonal), for which it is reasonable to expect such a description.

The material in Sects. 7.1 and 7.2 is needed for the statements of the results in Sect. 15. The material in the rest of this section is needed for the proofs of the theorems stated in Sect. 8. As a prerequisite for the present section, as well as the rest of Part II, one only needs Sect. 1 (i.e., lax prestacks will not appear).

**7.1. Verdier duality.** In this subsection we introduce the functor of Verdier duality in the context of prestacks.

7.1.1. Let  $\mathcal{Y}$  be a prestack such that the diagonal map

$$\mathrm{diag}_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$$

is pseudo-proper (see Sect. 1.5.3 for what this means). In particular, the functor

$$(\mathrm{diag}_{\mathcal{Y}})_! : \mathrm{Shv}^!(\mathcal{Y}) \rightarrow \mathrm{Shv}^!(\mathcal{Y} \times \mathcal{Y})$$

is defined and satisfies base change.

Given two objects  $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^!(\mathcal{Y})$ , by a *pairing* between them we shall mean a map

$$\mathcal{F} \boxtimes \mathcal{G} \rightarrow (\mathrm{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}}).$$

*Remark 7.1.2.* The pseudo-properness assumption on  $\mathrm{diag}_{\mathcal{Y}}$  does not merely appear for technical reasons, i.e., to ensure the existence of  $(\mathrm{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}})$ . More importantly, in actual Verdier duality, we want to use  $(\mathrm{diag}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})$  rather than  $(\mathrm{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}})$ , and while we do not necessarily know what  $(\mathrm{diag}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})$  is supposed to be for an arbitrary prestack  $\mathcal{Y}$ , the pseudo-properness is supposed to imply that it is isomorphic to  $(\mathrm{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}})$ .

7.1.3. For a given  $\mathcal{G} \in \mathrm{Shv}^!(\mathcal{Y})$ , we define its Verdier dual  $\mathbb{D}_{\mathcal{Y}}(\mathcal{G}) \in \mathrm{Shv}^!(\mathcal{Y})$  to represent the functor that sends  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y})$  to the space of pairings between  $\mathcal{F}$  and  $\mathcal{G}$ . The above contravariant functor is representable, because it sends colimits in  $\mathrm{Shv}^!(\mathcal{Y})$  to limits in  $\mathrm{Spc}$ .

In particular, for a given  $\mathcal{F}$ , we have a canonically defined pairing

$$(7.1) \quad \mathcal{F} \boxtimes \mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \rightarrow (\mathrm{diag}_{\mathcal{Y}})_!(\omega_{\mathcal{Y}}).$$

**7.2. Interaction of Verdier duality with pushforwards.** The basic property of Verdier duality on schemes is *commutation with direct images* for proper morphisms. In this subsection we shall start exploring what can be said about the extension of this property for maps between prestacks.

7.2.1. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map which is itself pseudo-proper, so that the functor  $f_!$ , left adjoint to  $f^!$  is defined. If  $\mathcal{F}, \mathcal{G}$  are objects of  $\mathrm{Shv}^!(\mathcal{Y}_1)$ , then a datum of a pairing between them gives rise to a pairing between  $f_!(\mathcal{F})$  and  $f_!(\mathcal{G})$  via

$$f_!(\mathcal{F}) \boxtimes f_!(\mathcal{G}) \xrightarrow[\sim]{\text{pseudo-properness}} (f \times f)_!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow (f \times f)_! \circ (\mathrm{diag}_{\mathcal{Y}_1})_!(\omega_{\mathcal{Y}_1}) \rightarrow (\mathrm{diag}_{\mathcal{Y}_2})_!(\omega_{\mathcal{Y}_2}).$$

In particular, if  $\mathcal{Y}$  is pseudo-proper, a datum of a pairing between  $\mathcal{F}$  and  $\mathcal{G}$  gives rise to a pairing

$$(7.2) \quad C_c^*(\mathcal{Y}, \mathcal{F}) \otimes C_c^*(\mathcal{Y}, \mathcal{G}) \rightarrow \Lambda.$$

7.2.2. We obtain that a pseudo-proper map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , the construction from Sect. 7.2.1 gives rise to a canonically defined map

$$(7.3) \quad f_!(\mathbb{D}_{\mathcal{Y}_1}(\mathcal{F})) \rightarrow \mathbb{D}_{\mathcal{Y}_2}(f_!(\mathcal{F})).$$

We have the following fundamental fact:

**Lemma 7.2.3.** *Let  $f : Y_1 \rightarrow Y_2$  be a proper map between schemes. Then the map (7.3) is an isomorphism.*

7.2.4. In particular, we obtain that if  $\mathcal{Y}$  is a pseudo-proper prestack and  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$ , the pairings (7.1) and (7.2) give rise to a pairing:

$$C_c^*(\mathcal{Y}, \mathcal{F}) \otimes C_c^*(\mathcal{Y}, \mathbb{D}_{\mathcal{Y}}(\mathcal{F})) \rightarrow \Lambda,$$

and thus to a map

$$(7.4) \quad C_c^*(\mathcal{Y}, \mathbb{D}_{\mathcal{Y}}(\mathcal{F})) \rightarrow (C_c^*(\mathcal{Y}, \mathcal{F}))^\vee,$$

(which is the map (7.3) for  $\mathcal{Y}_2 = \text{pt}$ ).

If  $\mathcal{Y} = Y$  is a proper scheme, then Lemma 7.2.3 says that the map (7.4) is an isomorphism.

*Remark 7.2.5.* Note that the map (7.4) is not an isomorphism for a general pseudo-proper prestack. For example, take  $\mathcal{Y}$  to be the disjoint union of infinitely many copies of  $\text{pt}$ .

**7.3. Interaction of Verdier duality with other functors.** The material in the rest of this section is of technical nature and can be skipped on the first pass.

In this subsection we will start exploring how Verdier duality interacts with the operations of external product and pullback under an étale morphism.

7.3.1. *Products.* Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be two prestacks and  $\mathcal{F}_i \in \text{Shv}^!(\mathcal{Y}_i)$ . We have a tautologically defined map

$$(7.5) \quad \mathbb{D}_{\mathcal{Y}_1}(\mathcal{F}_1) \boxtimes \mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}_2) \rightarrow \mathbb{D}_{\mathcal{Y}_1 \times \mathcal{Y}_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

The following is well-known:

**Lemma 7.3.2.** *Assume that the ring of coefficients  $\Lambda$  has a bounded Tor-dimension. Let  $\mathcal{Y}_i = Y_i$  be schemes and  $\mathcal{F}_i \in \text{Shv}(Y_i)$  be bounded above with compact cohomology sheaves. Then the map (7.5) is an isomorphism.*

7.3.3. Let now  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be an étale map between prestacks. We claim that for  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y}_2)$ , there exists a canonically defined map

$$(7.6) \quad f^! \circ \mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \rightarrow \mathbb{D}_{\mathcal{Y}_1} \circ f^!(\mathcal{F}).$$

Indeed, to specify such a map is equivalent to specifying a map

$$(7.7) \quad f^! \circ \mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \boxtimes f^!(\mathcal{F}) \rightarrow (\text{diag}_{\mathcal{Y}_1})_!(\omega_{\mathcal{Y}_1}).$$

The latter map is constructed as follows. The fact that  $f$  is étale implies that the diagonal map

$$\mathcal{Y}_1 \rightarrow \mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{Y}_1 = (\mathcal{Y}_1 \times \mathcal{Y}_1) \times_{\mathcal{Y}_2 \times \mathcal{Y}_2} \mathcal{Y}_2$$

is an embedding of a connected component.

In particular, we have a canonically defined map

$$(f \times f)^! \circ (\text{diag}_{\mathcal{Y}_2})_!(\omega_{\mathcal{Y}_2}) \rightarrow (\text{diag}_{\mathcal{Y}_1})_!(\omega_{\mathcal{Y}_1}).$$

Now, the map in (7.7) is the composition

$$f^! \circ \mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \boxtimes f^!(\mathcal{F}) = (f \times f)^!(\mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \boxtimes \mathcal{F}) \rightarrow (f \times f)^! \circ (\text{diag}_{\mathcal{Y}_2})_!(\omega_{\mathcal{Y}_2}) \rightarrow (\text{diag}_{\mathcal{Y}_1})_!(\omega_{\mathcal{Y}_1}),$$

where the second arrow is the pullback by means of  $f \times f$  of the tautological map

$$\mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \boxtimes \mathcal{F} \rightarrow (\text{diag}_{\mathcal{Y}_2})_!(\omega_{\mathcal{Y}_2}).$$

We have:

**Lemma 7.3.4.** *Let us be working in the context of constructible sheaves, and let  $\mathcal{Y}_1 = Y_1$  and  $\mathcal{Y}_2 = Y_2$  be schemes. Then the map (7.6) is an isomorphism.*

*Proof.* The assertion is well-known if  $\mathcal{F}$  is compact. In the general case, it follows from the fact that in the context of constructible sheaves, the functor  $f^!$  commutes with limits (because it admits a left adjoint).  $\square$

**7.4. Digression: properties of prestacks.** In this subsection we will introduce several notions that will help describe the category of sheaves on a prestack and the Verdier duality functor.

The key notion is that of *pseudo-scheme*. In a sense, pseudo-schemes are “as good as schemes”, as far as sheaves on them are concerned.

7.4.1. Let  $\mathcal{Y}$  be a prestack. We shall say that  $\mathcal{Y}$  is a *pseudo-scheme* if  $\mathcal{Y}$  can be written as a colimit

$$(7.8) \quad \mathcal{Y} = \operatorname{colim}_{a \in A} Z_a, \quad Z_a \in \operatorname{Sch},$$

where the transition maps  $f_{a_1, a_2} : Z_{a_1} \rightarrow Z_{a_2}$  are proper.<sup>15</sup>

For  $a \in A$ , we let  $\operatorname{ins}_a : Z_a \rightarrow \mathcal{Y}$  denote the corresponding map. We claim:

**Proposition 7.4.2.** *In the above notations, the maps  $\operatorname{ins}_a$  are pseudo-proper.*

As an immediate corollary of this proposition, we obtain that the diagonal morphism of a pseudo-scheme is pseudo-proper (we use here the fact that all our schemes are assumed separated).

*Proof.* Let  $\operatorname{Sch}_{\text{proper}}$  be the non-full subcategory of  $\operatorname{Sch}$ , where we restrict 1-morphisms to be proper. Let  $\operatorname{PreStk}_{\text{proper}}$  be the category of functors<sup>16</sup>

$$(\operatorname{Sch}_{\text{proper}})^{\text{op}} \rightarrow \operatorname{Spc}.$$

Restriction and left Kan extension define an adjoint pair of functors

$$\operatorname{LKE} : \operatorname{PreStk}_{\text{proper}} \rightleftarrows \operatorname{PreStk} : \operatorname{Res}.$$

Note that for  $\mathcal{Y} \in \operatorname{PreStk}_{\text{proper}}$ , written as a colimit  $\operatorname{colim}_{a \in A} Z_a$ , the value of  $\operatorname{LKE}(\mathcal{Y})$  is the same colimit, but taken in  $\operatorname{PreStk}$ . Hence, we obtain that pseudo-schemes are exactly those objects of  $\operatorname{PreStk}$  that lie in the essential image of the functor  $\operatorname{LKE}$ .

To prove the proposition, it suffices to show that for  $\mathcal{Y}$  as above and a pair of indices  $a, b$ , the fiber product  $Z_a \times_{\mathcal{Y}} Z_b$  is a pseudo-scheme and that its map to  $Z_b$  comes from a morphism in  $\operatorname{PreStk}_{\text{proper}}$ . For that it suffices to show that the functor  $\operatorname{LKE}$  commutes with finite limits. This follows from the next lemma:

**Lemma 7.4.3.** *Let  $F : \mathbf{C}' \rightarrow \mathbf{C}$  be a functor that commutes with finite limits. Then so does the functor*

$$\operatorname{LKE} : \operatorname{Funct}((\mathbf{C}')^{\text{op}}, \operatorname{Spc}) \rightarrow \operatorname{Funct}(\mathbf{C}^{\text{op}}, \operatorname{Spc}).$$

$\square$

*Proof of Lemma 7.4.3.* It is enough to show that for any  $\mathbf{c} \in \mathbf{C}$ , the functor

$$\Phi \mapsto \operatorname{LKE}(\Phi)(\mathbf{c}), \quad \operatorname{Funct}((\mathbf{C}')^{\text{op}}, \operatorname{Spc}) \rightarrow \operatorname{Spc}$$

is pro-representable. However, it is pro-represented by the object of  $\operatorname{Pro}(\mathbf{C}')$  (thought of as an object of  $\operatorname{Pro}(\operatorname{Funct}((\mathbf{C}')^{\text{op}}, \operatorname{Spc}))$ ) via the pro-extension of the Yoneda embedding, corresponding to the functor

$$\mathbf{c}' \mapsto \operatorname{Maps}_{\mathbf{C}}(\mathbf{c}, F(\mathbf{c}')).$$

$\square$

<sup>15</sup>Pseudo-schemes are different from ind-schemes in two respects: for the latter one requires that the transition maps  $f_{a_1, a_2}$  be *closed embeddings*, and (which is also crucial) that the index category  $A$  be *filtered*.

<sup>16</sup>The proof given below works for the category  $\operatorname{Sch}$  replaced by any  $\infty$ -category  $\mathbf{C}$  and  $\operatorname{PreStk}_{\text{proper}}$  replaced by any non-full subcategory  $\mathbf{C}'$  of  $\mathbf{C}$  that has the same objects, but which is obtained by restricting 1-morphisms to certain isomorphism classes.

*Remark 7.4.4.* The proof of Proposition 7.4.2 shows that any morphism between pseudo-schemes that comes from a morphism in  $\text{PreStk}_{\text{proper}}$  is pseudo-proper.

More generally, for  $\mathcal{Y}_1, \mathcal{Y}_2 \in \text{PreStk}_{\text{proper}}$ , the map

$$\text{Maps}_{\text{PreStk}_{\text{proper}}}(\mathcal{Y}_1, \mathcal{Y}_2) \rightarrow \text{Maps}_{\text{PreStk}}(\mathcal{Y}_1, \mathcal{Y}_2)$$

is a monomorphism in  $\text{Spc}$ , i.e., the embedding of a union of connected components, and its image consists of those maps of prestacks  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  that are pseudo-proper. In particular, any *isomorphism*  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  as prestacks comes from an isomorphism in  $\text{PreStk}_{\text{proper}}$ .

In addition, if  $\mathcal{Y}$  is a pseudo-scheme presented as in (7.8), and  $S$  is a scheme equipped with a *pseudo-proper* map  $S \rightarrow \mathcal{Y}$ , then for any index  $i$  such that the above map

$$S \rightarrow Z_i \rightarrow \mathcal{Y}$$

factors as  $S \rightarrow Z_i \rightarrow \mathcal{Y}$ , the resulting map  $S \rightarrow Z_i$  is proper (see end of proof of Corollary 7.5.6).

7.4.5. Since the functor  $\text{Shv}^!$  takes colimits to limits, we have:

$$\text{Shv}^!(\mathcal{Y}) \simeq \lim_{a \in A} \text{Shv}(Z_a),$$

where the transition functors in the formation of the limit are  $f_{a_1, a_2}^!$ . The corresponding functors

$$\text{Shv}^!(\mathcal{Y}) \rightarrow \text{Shv}(Z_a)$$

are given by  $(\text{ins}_a)^!$ .

In addition, as was already noted in the proof of Proposition 1.5.2, we also have:

$$\text{Shv}^!(\mathcal{Y}) \simeq \text{colim}_{a \in A} \text{Shv}(Z_a),$$

where the transition functors are  $(f_{a_1, a_2})_!$ , and the colimit is taken in  $\infty$ -category of cocomplete categories and colimit-preserving functors. The corresponding functors

$$\text{Shv}(Z_a) \rightarrow \text{Shv}^!(\mathcal{Y})$$

are given by  $(\text{ins}_a)_!$ .

Furthermore, as the functors  $\text{ins}_a^!$  are colimit-preserving, for  $\mathcal{F} \in \text{Shv}^!(\mathcal{Y})$  we have a canonical isomorphism:

$$(7.9) \quad \text{colim}_{a \in A} (\text{ins}_a)_! \circ (\text{ins}_a)^!(\mathcal{F}) \simeq \mathcal{F}$$

(more precisely, the tautological map  $\rightarrow$  is an isomorphism).

7.4.6. We will now add several adjectives to the notion of pseudo-scheme.

We shall say that  $\mathcal{Y}$  is a *finitary pseudo-scheme* if it admits a presentation as in (7.8) with the category  $A$  being finite.

We recall that  $\mathcal{Y}$  is said to be *pseudo-proper* over a given scheme  $S$ , if it admits a presentation as in (7.8) with the schemes  $Z_\alpha$  being proper over  $S$ .

We shall say that  $\mathcal{Y}$  is a *finitary pseudo-proper* over  $S$  if it admits a presentation as in (7.8) with the category  $A$  being finite *and* with the schemes  $Z_\alpha$  being proper over  $S$ .

The above notions have evident relative versions. So, we obtain the notion of a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  between prestacks to be: (i) pseudo-schematic; (ii) finitary pseudo-schematic; (iii) pseudo-proper; (iv) finitary pseudo-proper.

Recall that Corollary 1.5.4 implies that for a pseudo-proper map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , the functor  $f^!$  admits a left adjoint  $f_!$ . The following results from the proof of Proposition 1.5.2:

**Lemma 7.4.7.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be finitary pseudo-proper, where  $\mathcal{Y}_2$  is assumed to be (i) a pseudo-scheme if we are working in the context of  $D$ -modules or (ii) arbitrary if we are working in the context of constructible sheaves. Then the functor  $f_!$  commutes with limits.*

7.4.8. As was shown in Proposition 7.4.2, if  $\mathcal{Y}$  is a pseudo-scheme presented as in (7.8), then the maps  $\text{ins}_a : Z_a \rightarrow \mathcal{Y}$  and hence the diagonal map  $\text{diag}_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$  are pseudo-proper.

The following is easy:

**Lemma 7.4.9.** *For a pseudo-scheme presented as in (7.8), the following conditions are equivalent:*

- (i) *For every  $a \in A$ , the map  $\text{ins}_a : Z_a \rightarrow \mathcal{Y}$  is finitary pseudo-proper.*
- (ii) *The diagonal map  $\text{diag}_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$  is finitary pseudo-proper.*

We shall say that pseudo-scheme has a *finitary diagonal* if it satisfies the equivalent conditions of the above lemma.

It will turn out that pseudo-schemes with a finitary diagonal are well adapted to the operation of Verdier duality.

7.4.10. We proceed with several more observations.

**Lemma 7.4.11.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a pseudo-proper map between prestacks.*

- (a) *If  $f$  is injective (i.e., the map of  $\infty$ -groupoids  $\mathcal{Y}_1(S) \rightarrow \mathcal{Y}_2(S)$  is a monomorphism for any  $S \in \text{Sch}$ ), then the functor  $f_!$  is fully faithful.*
- (b) *If  $f$  is finitary pseudo-proper, there exists a uniquely defined reduced closed sub-prestack  $\mathcal{Y}'_2 \subset \mathcal{Y}_2$  (to be thought of as the reduced scheme-theoretic image of  $f$ ), such that the map  $f|_{(\mathcal{Y}_1)_{\text{red}}}$  factors through a map  $f' : (\mathcal{Y}_1)_{\text{red}} \rightarrow \mathcal{Y}'_2$ , the latter being surjective on  $k$ -points.*
- (c) *If  $f$  is finitary pseudo-proper and surjective on  $k$ -points, then the functor  $f^!$  is conservative.*
- (d) *If  $f$  is injective, finitary pseudo-proper and surjective on  $k$ -points, then the functors  $(f_!, f^!)$  are mutually inverse equivalences.*

*Proof.* Point (i) follows from base change, since the assumption that  $f$  be injective is equivalent to the fact that the diagonal map

$$\mathcal{Y}_1 \rightarrow \mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{Y}_1$$

be an isomorphism.

For point (ii) we can assume that  $\mathcal{Y}_2 = S \in \text{Sch}$ . If  $\mathcal{Y}_1$  is the (finite) colimit of the schemes  $Z_a$ , we take  $\mathcal{Y}'_2 =: S'$  to be the union of the images of the schemes  $Z_a$  in  $S$ , equipped with the reduced structure.

For point (iii) we can again take  $\mathcal{Y}_2 = S \in \text{Sch}$ . Then the assumption implies that  $S$  can be written as a union of locally closed subschemes  $S_\alpha$ , for each of which there exists a scheme  $S'_\alpha$ , equipped with a map  $S'_\alpha \rightarrow \mathcal{Y}_1$ , such that the composed map

$$S'_\alpha \rightarrow \mathcal{Y}_1 \rightarrow S$$

factors through a map  $S'_\alpha \rightarrow S_\alpha$ , the latter being surjective on  $k$ -points. Now, the assertion follows from the fact that pullback along a map of schemes that is surjective on  $k$ -points is a conservative functor on  $\text{Shv}(-)$ .

Point (iv) is a combination of (i) and (iii). □

**7.5. Consequences for Verdier duality.** In this subsection we will assume that our prestacks are *pseudo-schemes with a finitary diagonal* (see Sect. 7.4.8 for what this means). We will show that the Verdier duality functor on such prestacks can be described explicitly in terms of Verdier duality on schemes.

In addition, we will show that Verdier duality commutes with direct images under finitary pseudo-proper maps.



7.5.1. Let  $\mathcal{Y}$  be written as in (7.8). We claim:

**Proposition 7.5.2.** *In the above notations, for every  $a \in A$ , the natural transformation*

$$(\text{ins}_a)_! \circ \mathbb{D}_{Z_a} \rightarrow \mathbb{D}_{\mathcal{Y}} \circ (\text{ins}_a)_!$$

of (7.3) is an isomorphism.

Note that this proposition gives an expression for what the Verdier duality functor *actually does* on some particular objects of  $\text{Shv}^!(\mathcal{Y})$ .

*Proof.* Let  $\mathcal{F}$  be an object of  $\text{Shv}(Z_a)$ , and let  $\mathcal{G}$  be an object of  $\text{Shv}^!(\mathcal{Y})$ . It suffices to show that for any index  $b$ , the map

$$\begin{aligned} \text{Maps}_{\text{Shv}(Z_b)}(\text{ins}_b^!(\mathcal{G}), \text{ins}_b^! \circ (\text{ins}_a)_! \circ \mathbb{D}_{Z_a}(\mathcal{F})) &\rightarrow \\ &\rightarrow \text{Maps}_{\text{Shv}^!(\mathcal{Y} \times Z_b)}\left(\text{ins}_a)_!(\mathcal{F}) \boxtimes \text{ins}_b^!(\mathcal{G}), (\text{Graph}_{\text{ins}_b})_!(\omega_{Z_b})\right) \end{aligned}$$

is an isomorphism. We will prove more generally that for any  $\mathcal{H} \in \text{Shv}(Z_b)$ , the map

$$(7.10) \quad \begin{aligned} \text{Maps}_{\text{Shv}(Z_b)}(\mathcal{H}, \text{ins}_b^! \circ (\text{ins}_a)_! \circ \mathbb{D}_{Z_a}(\mathcal{F})) &\rightarrow \\ &\rightarrow \text{Maps}_{\text{Shv}^!(\mathcal{Y} \times Z_b)}\left((\text{ins}_a)_!(\mathcal{F}) \boxtimes \mathcal{H}, (\text{Graph}_{\text{ins}_b})_!(\omega_{Z_b})\right) \end{aligned}$$

is an isomorphism.

Consider the prestack  $\tilde{Z}_{a,b} := Z_a \times_{\mathcal{Y}} Z_b$ ; let  $\tilde{q}_a$ ,  $\tilde{q}_b$  and  $\tilde{q}_{a,b}$  denote its maps to  $Z_a$ ,  $Z_b$  and  $Z_a \times Z_b$ , respectively. By base change (i.e., Corollary 1.5.4, due to the fact that  $\text{ins}_a$  is pseudo-proper), we can rewrite the left-hand side in (7.10) as

$$(7.11) \quad \text{Maps}_{\text{Shv}(Z_b)}(\mathcal{H}, (\tilde{q}_b)_! \circ \tilde{q}_a^! \circ \mathbb{D}_{Z_a}(\mathcal{F}))$$

and the right-hand side as

$$(7.12) \quad \text{Maps}_{\text{Shv}^!(Z_a \times Z_b)}\left(\mathcal{F} \boxtimes \mathcal{H}, (\tilde{q}_{a,b})_!(\omega_{\tilde{Z}_{a,b}})\right),$$

respectively.

The map from (7.11) to (7.12) comes from the map in  $\text{Shv}(Z_a \times Z_b)$

$$\mathcal{F} \boxtimes \left((\tilde{q}_b)_! \circ \tilde{q}_a^! \circ \mathbb{D}_{Z_a}(\mathcal{F})\right) \rightarrow (\tilde{q}_{a,b})_!(\omega_{\tilde{Z}_{a,b}}),$$

which is obtained by applying the functor  $(\text{id} \times \tilde{q}_b)_!$  to the map in  $\text{Shv}(Z_a \times \tilde{Z}_{a,b})$

$$\mathcal{F} \boxtimes \left(\tilde{q}_a^! \circ \mathbb{D}_{Z_a}(\mathcal{F})\right) \rightarrow (\text{Graph}_{\tilde{q}_a})_!(\omega_{\tilde{Z}_{a,b}}),$$

and which is in turned obtained by applying  $(\text{id} \times \tilde{q}_a)^!$  to the map

$$\mathcal{F} \boxtimes \mathbb{D}_{Z_a}(\mathcal{F}) \rightarrow (\text{diag}_{Z_a})_!(\omega_{Z_a})$$

of (7.1).

Write

$$\tilde{Z}_{a,b} = \text{colim}_i Z_{a,b}^i,$$

where the index  $i$  runs over a finite category and where  $Z_{a,b}^i$  are schemes proper over  $Z_a$  and  $Z_b$ . Let  $q_a^i, q_b^i, q_{a,b}^i$  denote the maps from  $Z_{a,b}^i$  to  $Z_a$ ,  $Z_b$  and  $Z_a \times Z_b$ , respectively.

We have:

$$(\tilde{q}_b)_! \circ \tilde{q}_a^! \circ \mathbb{D}_{Z_a}(\mathcal{F}) \simeq \text{colim}_i (q_b^i)_! \circ (q_a^i)^! \circ \mathbb{D}_{Z_a}(\mathcal{F})$$

and

$$(\tilde{q}_{a,b})_!(\omega_{\tilde{Z}_{a,b}}) \simeq \text{colim}_i (q_{a,b}^i)_!(\omega_{Z_{a,b}^i}).$$

The map from (7.11) to (7.12) fits into a commutative diagram

$$\begin{array}{ccc}
\mathrm{Maps}_{\mathrm{Shv}(Z_b)}(\mathcal{H}, (\tilde{q}_b)_! \circ \tilde{q}_a^! \circ \mathbb{D}_{Z_a}(\mathcal{F})) & \longrightarrow & \mathrm{Maps}_{\mathrm{Shv}^!(Z_a \times Z_b)}(\mathcal{F} \boxtimes \mathcal{H}, (\tilde{q}_{a,b})!(\omega_{\bar{Z}_{a,b}})) \\
\sim \uparrow & & \uparrow \sim \\
\mathrm{Maps}_{\mathrm{Shv}(Z_b)}(\mathcal{H}, \mathrm{colim}_i (q_b^i)_! \circ (q_a^i)_! \circ \mathbb{D}_{Z_a}(\mathcal{F})) & & \mathrm{Maps}_{\mathrm{Shv}^!(Z_a \times Z_b)}(\mathcal{F} \boxtimes \mathcal{H}, \mathrm{colim}_i (q_{a,b}^i)_!(\omega_{Z_{a,b}^i})) \\
\uparrow & & \uparrow \\
\mathrm{colim}_i \mathrm{Maps}_{\mathrm{Shv}(Z_b)}(\mathcal{H}, (q_b^i)_! \circ (q_a^i)_! \circ \mathbb{D}_{Z_a}(\mathcal{F})) & \longrightarrow & \mathrm{colim}_i \mathrm{Maps}_{\mathrm{Shv}^!(Z_a \times Z_b)}(\mathcal{F} \boxtimes \mathcal{H}, (q_{a,b}^i)_!(\omega_{Z_{a,b}^i})),
\end{array}$$

where the bottom horizontal arrow comes from maps

$$(7.13) \quad \mathrm{Maps}_{\mathrm{Shv}(Z_b)}(\mathcal{H}, (q_b^i)_! \circ (q_a^i)_! \circ \mathbb{D}_{Z_a}(\mathcal{F})) \rightarrow \mathrm{Maps}_{\mathrm{Shv}^!(Z_a \times Z_b)}(\mathcal{F} \boxtimes \mathcal{H}, (q_{a,b}^i)_!(\omega_{Z_{a,b}^i})),$$

defined in the same way as the map from (7.11) to (7.12).

Now, in the above diagram the vertical maps are isomorphisms because the index category was finite. The bottom horizontal arrow is an isomorphism because the maps (7.13) are isomorphisms, due to the properness of  $q_b^i$ .

Hence, the top horizontal arrow is an isomorphism, as required.  $\square$

7.5.3. As a consequence of Proposition 7.5.2, we obtain the following expression for the Verdier duality functor on the prestack  $\mathcal{Y}$  in terms of the schemes  $Z_a$ :

**Proposition-Construction 7.5.4.** *In the above notations, for  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y})$  we have a canonical isomorphism*

$$\mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \simeq \lim_{a \in A^{\mathrm{op}}} (\mathrm{ins}_a)_! \circ \mathbb{D}_{Z_a} \circ (\mathrm{ins}_a)^!(\mathcal{F}).$$

*Proof.* By (7.9), we have:

$$\mathcal{F} \simeq \mathrm{colim}_{a \in A} (\mathrm{ins}_a)_! \circ (\mathrm{ins}_a)^!(\mathcal{F}).$$

The functor of Verdier duality maps colimits into limits. Hence,

$$\mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \simeq \lim_{a \in A} \mathbb{D}_{\mathcal{Y}} \circ (\mathrm{ins}_a)_! \circ \mathrm{ins}_a^!(\mathcal{F}).$$

Now, the assertion of the proposition follows from Proposition 7.5.2.  $\square$

7.5.5. The next assertion shows that, when dealing with maps between pseudo-schemes, each with a finitary diagonal, Verdier duality commutes with direct images under finitary pseudo-proper maps:

**Corollary 7.5.6.** *Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a finitary pseudo-proper map between pseudo-schemes, each having a finitary diagonal. Then the natural transformation*

$$f_! \circ \mathbb{D}_{\mathcal{Y}_2} \rightarrow \mathbb{D}_{\mathcal{Y}_1} \circ f_!$$

*of (7.3) is an isomorphism.*

*Proof.* Fix presentations

$$\mathcal{Y}_1 = \mathrm{colim}_{a_1 \in A_1} Z_{1,a_1}, \quad Z_{1,a_1} \in \mathrm{Sch}$$

and

$$\mathcal{Y}_2 = \mathrm{colim}_{a_2 \in A_2} Z_{2,a_2}, \quad Z_{2,a_2} \in \mathrm{Sch}$$

as in (7.8).

For  $\mathcal{F} \in \mathrm{Shv}^!(\mathcal{Y}_1)$  we have

$$\mathcal{F} \simeq \mathrm{colim}_{a_1 \in A_1} (\mathrm{ins}_{a_1})_! \circ (\mathrm{ins}_{a_1})^!(\mathcal{F}),$$

and hence

$$f_!(\mathcal{F}) \simeq \text{colim}_{a_1 \in A_1} (f \circ \text{ins}_{a_1})_! \circ (\text{ins}_{a_1})^!(\mathcal{F}).$$

By Proposition 7.5.4, we have

$$\mathbb{D}_{\mathcal{Y}_1}(\mathcal{F}) \simeq \lim_{a_1 \in A_1} (\text{ins}_{a_1})_! \circ \mathbb{D}_{Z_{a_1}} \circ (\text{ins}_{a_1})^!(\mathcal{F}).$$

By Lemma 7.4.7, we have

$$f_! \circ \mathbb{D}_{\mathcal{Y}_1}(\mathcal{F}) \simeq \lim_{a_1 \in A_1} (f \circ \text{ins}_{a_1})_! \circ \mathbb{D}_{Z_{a_1}} \circ (\text{ins}_{a_1})^!(\mathcal{F}),$$

and since the functor  $\mathbb{D}_{\mathcal{Y}_2}$  takes colimits to limits, we also have:

$$\mathbb{D}_{\mathcal{Y}_2} \circ f_!(\mathcal{F}) \simeq \lim_{a_1 \in A_1} \mathbb{D}_{\mathcal{Y}_2} \circ (f \circ \text{ins}_{a_1})_! \circ (\text{ins}_{a_1})^!(\mathcal{F}).$$

Thus, we obtain that it suffices to show that for a given index  $a_1 \in A_1$  and  $\mathcal{F}_1 \in \text{Shv}(Z_{a_1})$ , the map

$$(f \circ \text{ins}_{a_1})_! \circ \mathbb{D}_{Z_{a_1}}(\mathcal{F}_1) \rightarrow \mathbb{D}_{\mathcal{Y}_2} \circ (f \circ \text{ins}_{a_1})_!(\mathcal{F}_1)$$

(which is the map (7.3) for the morphism  $f \circ \text{ins}_{a_1} : Z_{a_1} \rightarrow \mathcal{Y}_2$ ) is an isomorphism.

Let  $a_2 \in A_2$  be some index so that the morphism  $f \circ \text{ins}_{a_1}$  factors as

$$Z_{a_1} \xrightarrow{g} Z_{a_2} \xrightarrow{\text{ins}_{a_2}} \mathcal{Y}_2.$$

By Proposition 7.5.2 applied to the morphism  $\text{ins}_{a_2}$ , it suffices to show that the map  $g : Z_{a_1} \rightarrow Z_{a_2}$  is proper.

Write

$$Z_{a_2} \times_{\mathcal{Y}_2} \mathcal{Y}_1 \simeq \text{colim}_b W_b, \quad W_b \in \text{Sch}$$

where the schemes  $W_b$  are proper over  $Z_{a_2}$ . Let  $b$  be an index such that the map

$$Z_{a_1} \rightarrow Z_{a_2} \times_{\mathcal{Y}_2} \mathcal{Y}_1$$

factors through a map  $h : Z_{a_1} \rightarrow W_b$ . It suffices to show that the map  $h$  is proper.

Write

$$W_b \times_{\mathcal{Y}_1} Z_{a_1} \simeq \text{colim}_c V_c, \quad V_c \in \text{Sch},$$

where the schemes  $V_c$  are proper over  $W_b$ . Let  $c$  be an index such that the map

$$Z_{a_1} \rightarrow W_b \times_{\mathcal{Y}_1} Z_{a_1}$$

factors through a map  $Z_{a_1} \rightarrow V_c$ .

It suffices to show that the latter map  $Z_{a_1} \rightarrow V_c$  is proper. But this is so because it admits a left inverse, namely,

$$V_c \rightarrow W_b \times_{\mathcal{Y}_1} Z_{a_1} \rightarrow Z_{a_1}.$$

□

7.5.7. Finally, we have the following assertion:

**Lemma 7.5.8.** *Let us be working in the context of constructible sheaves. Let  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be an étale map. Then the natural transformation (7.6) is an isomorphism.*

*Proof.* Fix a presentation

$$\mathcal{Y}_2 = \operatorname{colim}_{a \in A} Z_{2,a}, \quad Z_{2,a} \in \operatorname{Sch}$$

as in (7.8), and set

$$Z_{1,a} := \mathcal{Y}_1 \times_{\mathcal{Y}_2} Z_{2,a},$$

so that

$$\mathcal{Y}_1 \simeq \operatorname{colim}_{a \in A} Z_{1,a}.$$

Let

$$\operatorname{ins}_{1,a} : Z_{1,a} \rightarrow \mathcal{Y}_1 \text{ and } \operatorname{ins}_{2,a} : Z_{2,a} \rightarrow \mathcal{Y}_2$$

denote the resulting maps. Let  $f_a$  denote the map  $Z_{1,a} \rightarrow Z_{2,a}$ ; by assumption these maps are étale.

For  $\mathcal{F} \in \operatorname{Shv}^!(\mathcal{Y}_2)$ , we have

$$\mathbb{D}_{\mathcal{Y}_2}(\mathcal{F}) \simeq \lim_a (\operatorname{ins}_{2,a})_! \circ \mathbb{D}_{Z_{2,a}} \circ (\operatorname{ins}_{2,a})^!(\mathcal{F})$$

and

$$\mathbb{D}_{\mathcal{Y}_1} \circ f^!(\mathcal{F}) \simeq \lim_a (\operatorname{ins}_{1,a})_! \circ \mathbb{D}_{Z_{1,a}} \circ (\operatorname{ins}_{1,a})^! \circ f^!(\mathcal{F}).$$

In terms of these identification, the map (7.6) equals the composition

$$\begin{aligned} (7.14) \quad f^! \left( \lim_a (\operatorname{ins}_{2,a})_! \circ \mathbb{D}_{Z_{2,a}} \circ (\operatorname{ins}_{2,a})^!(\mathcal{F}) \right) &\rightarrow \lim_a f^! \circ (\operatorname{ins}_{2,a})_! \circ \mathbb{D}_{Z_{2,a}} \circ (\operatorname{ins}_{2,a})^!(\mathcal{F}) \simeq \\ &\simeq \lim_a (\operatorname{ins}_{1,a})_! \circ f_a^! \circ \mathbb{D}_{Z_{2,a}} \circ (\operatorname{ins}_{2,a})^!(\mathcal{F}) \rightarrow \lim_a (\operatorname{ins}_{1,a})_! \circ \mathbb{D}_{Z_{1,a}} \circ f_a^! \circ (\operatorname{ins}_{2,a})^!(\mathcal{F}) = \\ &= \lim_a (\operatorname{ins}_{1,a})_! \circ \mathbb{D}_{Z_{1,a}} \circ (\operatorname{ins}_{1,a})^! \circ f^!(\mathcal{F}), \end{aligned}$$

where the third arrow is the natural transformation (7.6) for the morphism  $f_a$ , and hence is an isomorphism by Lemma 7.3.4.

Hence, it remains to show that the first arrow in (7.14) is an isomorphism. However, this follows from the fact that in the context of constructible sheaves, the functor of pullback commutes with limits, see Lemma 2.2.4. □

## 8. THE CASE OF THE RAN SPACE, STATEMENTS OF VERDIER DUALITY RESULTS

As was mentioned in the preamble to Sect. 7, for the proof of the cohomological product formula, we will need to perform Verdier duality on the Ran space. More precisely, we will need to show that for some specific object in  $\operatorname{Shv}^!(\operatorname{Ran})$ , the compactly supported cohomology of its Verdier dual maps isomorphically to the dual of its own compactly supported cohomology. Now, such an isomorphism would fail in general (not surprisingly so, because Ran is “infinite”, which technically means *not finitary*). Our task in this section we will be to give sufficient conditions on an object of  $\operatorname{Shv}^!(\operatorname{Ran})$  so that the isomorphism in question does hold.

The main results of this section are stated in Sects. 8.2 and 8.3; the prerequisites for these subsections are Sect. 1 and Sects. 7.1 and 7.2. The material in the rest of the present section is needed for the proofs of the results stated in Sects. 8.2 and 8.3; in order to be able to understand it, the reader needs to be familiar with the entirety of Sect. 7.

**8.1. The category of sheaves on the Ran space.** In this subsection we will give a more explicit description of the category of sheaves on the Ran space. Namely, for a finite non-empty set  $\mathcal{J}$ , we have a pair of adjoint functors

$$(\text{ins}_{\mathcal{J}})_! : \text{Shv}^!(X^{\mathcal{J}}) \rightleftarrows \text{Shv}^!(\text{Ran}) : (\text{ins}_{\mathcal{J}})^!$$

We will give a more explicit description of the composed functor  $(\text{ins}_{\mathcal{J}})^! \circ (\text{ins}_{\mathcal{J}})_!$  for a pair of finite sets  $\mathcal{J}$  and  $\mathcal{J}'$ .

The material in this subsection is needed for the proof of Theorems 8.2.4, 8.3.6 and 8.2.6. But it is not necessary for the statement of these theorems, and so can be skipped on the first pass.

8.1.1. Recall that we can represent Ran as

$$\text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}} X^{\mathcal{J}},$$

where  $\text{Fin}^s$  is the category of finite non-empty sets and surjective maps and that  $\text{ins}_{\mathcal{J}}$  denotes the resulting map  $X^{\mathcal{J}} \rightarrow \text{Ran}$ .

Thus, we obtain that Ran is a pseudo-scheme (see Sect. 7.4.1 for what this means). In particular, by Proposition 7.4.2, the morphisms  $\text{ins}_{\mathcal{J}}$  are pseudo-proper (for an alternative proof of this result in the case of Ran, see Sect. 8.1.2 below). By Sect. 7.4.5, the category  $\text{Shv}^!(\text{Ran})$  can be described as

$$\text{Shv}^!(\text{Ran}) \simeq \lim_{\mathcal{J} \in \text{Fin}^s} \text{Shv}^!(X^{\mathcal{J}}),$$

and also

$$\text{Shv}^!(\text{Ran}) \simeq \text{colim}_{\mathcal{J} \in \text{Fin}^s} \text{Shv}(X^{\mathcal{J}}).$$

In particular, for  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , we have a canonical isomorphism

$$(8.1) \quad \text{colim}_{\mathcal{J} \in \text{Fin}^s} (\text{ins}_{\mathcal{J}})_! \circ (\text{ins}_{\mathcal{J}})^!(\mathcal{F}) \rightarrow \mathcal{F}$$

(more precisely, the tautological map  $\rightarrow$  is an isomorphism).

8.1.2. We claim that Ran is a pseudo-scheme with a finitary diagonal. Indeed, for a pair of finite sets  $\mathcal{J}$  and  $\mathcal{J}'$ , the fiber product

$$\tilde{X}^{\mathcal{J}, \mathcal{J}'} := X^{\mathcal{J}} \times_{\text{Ran}} X^{\mathcal{J}'} \simeq (X^{\mathcal{J}} \times X^{\mathcal{J}'}) \times_{\text{Ran} \times \text{Ran}} \text{Ran}$$

can be described as follows:

$$(8.2) \quad \tilde{X}^{\mathcal{J}, \mathcal{J}'} \simeq \text{colim}_{\mathcal{K} \leftarrow \mathcal{J} \leftarrow \mathcal{J}'} X^{\mathcal{K}}.$$

Let  $\tilde{q}_{\mathcal{J}}$  and  $\tilde{q}_{\mathcal{J}'}$  denote the maps from  $\tilde{X}^{\mathcal{J}, \mathcal{J}'}$  to  $X^{\mathcal{J}}$  and  $X^{\mathcal{J}'}$ , respectively. By Corollary 1.5.4 we have

$$(\text{ins}_{\mathcal{J}'}^!) \circ (\text{ins}_{\mathcal{J}})_! \simeq (\tilde{q}_{\mathcal{J}'})_! \circ (\tilde{q}_{\mathcal{J}})^!, \quad \text{Shv}(X^{\mathcal{J}}) \rightarrow \text{Shv}(X^{\mathcal{J}'}).$$

However, in order to have a better understanding of the category  $\text{Shv}^!(\text{Ran})$ , one should describe the above functors more explicitly, as we shall presently do.

8.1.3. Let  $X^{\mathcal{J}, \mathcal{J}'} \subset X^{\mathcal{J}} \times X^{\mathcal{J}'}$  be the reduced closed subscheme, whose set of  $k$ -points corresponds to those pairs of an  $\mathcal{J}$ -tuple and a  $\mathcal{J}'$ -tuple of  $k$ -points of  $X$ , for which the corresponding subsets of  $X(k)$  coincide.

In other words,  $X^{\mathcal{J}, \mathcal{J}'}$  is the reduced subscheme underlying the union of the closed subsets

$$\text{Graph}_{\text{diag}_{\alpha, \beta}},$$

where the union runs over the set of isomorphism classes of surjections

$$\mathcal{J} \xrightarrow{\alpha} \mathcal{K} \xleftarrow{\beta} \mathcal{J}',$$

and  $\text{diag}_{\alpha, \beta}$  denotes the resulting map  $X^{\mathcal{K}} \rightarrow X^{\mathcal{J}} \times X^{\mathcal{J}'}$ .

We have an evident map

$$g : \tilde{X}^{\mathcal{J}, \mathcal{J}'} \rightarrow X^{\mathcal{J}, \mathcal{J}'}$$

**Lemma 8.1.4.** *The functors  $(g_!, g^!)$  are mutually inverse equivalences of categories.*

*Proof.* Follows from Lemma 7.4.11(d). □

Let

$$X^{\mathcal{J}} \xleftarrow{q_{\mathcal{J}}} X^{\mathcal{J}, \mathcal{J}} \xrightarrow{q_{\mathcal{J}}} X^{\mathcal{J}}$$

denote the two maps. We obtain:

**Corollary 8.1.5.** *The functor*

$$(\text{ins}_{\mathcal{J}})^! \circ (\text{ins}_{\mathcal{J}})_! \simeq (\tilde{q}_{\mathcal{J}})_! \circ (\tilde{q}_{\mathcal{J}})^!$$

*is canonically isomorphic to  $(q_{\mathcal{J}})_! \circ (q_{\mathcal{J}})^!$ .*

**8.2. Verdier duality on the Ran space.** In this subsection we will formulate two main results pertaining to Verdier duality on the Ran space, Theorems 8.2.4 and 8.2.6. Theorem 8.2.4 (or rather its generalization Theorem 8.3.6) will be used in Sect. 15 to deduce the cohomological product formula from a *local duality* statement. Theorem 8.2.6 will be used in Sect. 16 to establish a crucial factorization property.

8.2.1. We will apply the discussion of Sect. 7.1 to the case  $\mathcal{Y} = \text{Ran}$ , i.e., the Ran space of  $X$ , where  $X$  is a (separated) scheme.

Thus, we obtain:

- For two objects  $\mathcal{F}, \mathcal{G} \in \text{Shv}^!(\text{Ran})$ , there is a notion of pairing between map, by which we mean a map

$$\mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})_!(\omega_{\text{Ran}});$$

- Given a pairing, we obtain a pairing between  $\mathcal{F}$  and  $\mathcal{G}$ ,

$$C_c^*(\text{Ran}, \mathcal{F}) \otimes C_c^*(\text{Ran}, \mathcal{G}) \rightarrow k$$

(under the additional assumption that  $X$  be proper if we are in the context of arbitrary D-modules).

- For  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , we have a well-defined object  $\mathbb{D}_{\text{Ran}}(\mathcal{F}) \in \text{Shv}^!(\text{Ran})$ ;
- There is a canonically defined pairing

$$C_c^*(\text{Ran}, \mathcal{F}) \otimes C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{F})) \rightarrow k.$$

In addition, from Proposition 7.5.4 and Sect. 8.1.2, we obtain:

**Corollary 8.2.2.** *For a given  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , there exists a canonical isomorphism*

$$\mathbb{D}_{\text{Ran}}(\mathcal{F}) \simeq \lim_{\mathcal{J} \in \text{Fin}^s} (\text{ins}_{\mathcal{J}})_! \circ \mathbb{D}_{X^{\mathcal{J}}} \circ (\text{ins}_{\mathcal{J}})^!(\mathcal{F}).$$

8.2.3. The functor  $\mathbb{D}_{\text{Ran}}$  is not in general very well-behaved. For example, one can show that for  $X$  smooth of positive dimension, we have:

$$\mathbb{D}_{\text{Ran}}(\omega_{\text{Ran}}) = 0.$$

Therefore, it is not true that (for  $X$  proper) the functor of compactly supported cohomology commutes with Verdier duality. We will now formulate a connectivity assumption that ensures that the corresponding isomorphism holds for a given object.

Namely, the following result gives a sufficient condition for that map (7.4) to be an isomorphism (i.e., commutation of Verdier duality with the functor of compactly supported cohomology):

**Theorem 8.2.4.** *Let  $X \in \text{Sch}$  be a proper smooth curve. Let  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  have the property that for every integer  $k \geq 0$  there exists an integer  $n_k \geq 0$ , such that the object  $\text{ins}_{\mathcal{J}}^!(\mathcal{F})|_{X^{\mathcal{J}}}$  is concentrated in perverse<sup>17</sup> cohomological degrees  $\leq -k - |\mathcal{J}|$  whenever  $|\mathcal{J}| > n_k$ . Then the map (7.4) is an isomorphism.*

<sup>17</sup>In the context of D-modules, this is the standard t-structure on the category of D-modules.

8.2.5. The next theorem gives a sufficient condition for the map (7.5) to be an isomorphism. It will play a crucial role in the discussion of the behavior of *factorization algebras* under Verdier duality, see Sect. 12.2.

**Theorem 8.2.6.** *Let  $X$  be a smooth curve, and let the ring of coefficients  $\Lambda$  be of finite cohomological dimension. Let  $\mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}^!(\text{Ran})$  have the property appearing in Theorem 8.2.4. Assume, in addition, that every  $\mathcal{J}$ , the object  $\text{ins}_{\mathcal{J}}^!(\mathcal{F}_i) \in \text{Shv}(X^{\mathcal{J}})$  is bounded above and has compact cohomology sheaves. Then the map*

$$\mathbb{D}_{\text{Ran}}(\mathcal{F}_1) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{F}_2) \rightarrow \mathbb{D}_{\text{Ran} \times \text{Ran}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

of (7.5) is an isomorphism.

8.2.7. The proofs of the above two theorems rely on considering truncated Ran spaces  $\text{Ran}^{\leq n}$  and a crucial stabilization assertion, namely Proposition 8.4.9.

8.3. **A variant.** In this subsection we will be working in the context of constructible sheaves. We will state a version of Theorem 8.2.4 that involves non-proper curves. This theorem will be used in the derivation of the cohomological product formula from the local duality statement in Sect. 15.

8.3.1. Let  $X' \xrightarrow{\mathbf{j}} X$  be an open subset, and let

$$\text{Ran}' \xrightarrow{\mathbf{j}_{\text{Ran}'}} \text{Ran}$$

be the corresponding open sub-prestack.

Note that the pullback functor  $\mathbf{j}_{\text{Ran}'}^! : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}')$  admits both a left and a right adjoints, to be denoted  $(\mathbf{j}_{\text{Ran}'})_!$  and  $(\mathbf{j}_{\text{Ran}'})_*$ , respectively,

Explicitly, for  $\mathcal{F}' \in \text{Shv}^!(\text{Ran}')$ , we have:

$$(8.3) \quad (\mathbf{j}_{\text{Ran}'})_!(\mathcal{F}') = \text{colim}_{\mathcal{J} \in \text{Fin}^s} (\text{ins}_{\mathcal{J}})_! \circ (\mathbf{j}_{\mathcal{J}})_! \circ (\text{ins}'_{\mathcal{J}})^!(\mathcal{F}'),$$

where

$$\text{ins}'_{\mathcal{J}} : X'^{\mathcal{J}} \rightarrow \text{Ran}' \quad \text{and} \quad \mathbf{j}_{\mathcal{J}} : X^{\mathcal{J}} \rightarrow X^{\mathcal{J}}$$

denote the corresponding maps.

For  $\mathcal{F}' \in \text{Shv}^!(\text{Ran}')$ , the object  $(\mathbf{j}_{\text{Ran}'})_*(\mathcal{F}')$  is uniquely characterized by the property that

$$(8.4) \quad (\text{ins}_{\mathcal{J}})^! \circ (\mathbf{j}_{\text{Ran}'})_*(\mathcal{F}') = (\mathbf{j}_{\mathcal{J}})_* \circ (\text{ins}'_{\mathcal{J}})^!(\mathcal{F}').$$

In fact, the above formula for  $(\mathbf{j}_{\text{Ran}'})_*(\mathcal{F}')$  implies that

$$(\mathbf{j}_{\text{Ran}'})_*(\mathcal{F}') = \text{colim}_{\mathcal{J} \in \text{Fin}^s} (\text{ins}_{\mathcal{J}})_! \circ (\mathbf{j}_{\mathcal{J}})_* \circ (\text{ins}'_{\mathcal{J}})^!(\mathcal{F}').$$

8.3.2. The above description of  $(\mathbf{j}_{\text{Ran}'})_*$  implies that the counit of the adjunction

$$(\mathbf{j}_{\text{Ran}'})^! \circ (\mathbf{j}_{\text{Ran}'})_* \rightarrow \text{Id}_{\text{Shv}^!(\text{Ran}')}$$

is an isomorphism, so the functor  $(\mathbf{j}_{\text{Ran}'})_*$  is fully faithful.

As a formal consequence, we obtain:

**Corollary 8.3.3.** *The functor  $(\mathbf{j}_{\text{Ran}'})_!$  is fully faithful.*

Finally, we have:

**Lemma 8.3.4.** *For  $\mathcal{F}' \in \text{Shv}^!(\text{Ran}')$ , we have a canonical isomorphism*

$$\mathbb{D}_{\text{Ran}} \circ (\mathbf{j}_{\text{Ran}'})_!(\mathcal{F}') \simeq (\mathbf{j}_{\text{Ran}'})_* \circ \mathbb{D}_{\text{Ran}'}(\mathcal{F}').$$

*Proof.* Let  $\mathcal{G}$  be an object of  $\mathrm{Shv}^!(\mathrm{Ran})$ . We have

$$\begin{aligned} \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran})}(\mathcal{G}, (\mathbf{j}_{\mathrm{Ran}})_* \circ \mathbb{D}_{\mathrm{Ran}}(\mathcal{F}')) &\simeq \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran}')}((\mathbf{j}_{\mathrm{Ran}})^!(\mathcal{G}), \mathbb{D}_{\mathrm{Ran}}(\mathcal{F}')) = \\ &= \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran}' \times \mathrm{Ran}')} \left( (\mathbf{j}_{\mathrm{Ran}})^!(\mathcal{G}) \boxtimes \mathcal{F}', (\mathrm{diag}_{\mathrm{Ran}'}^!)_!(\omega_{\mathrm{Ran}'}) \right) \end{aligned}$$

and

$$(8.5) \quad \begin{aligned} \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran})}(\mathcal{G}, \mathbb{D}_{\mathrm{Ran}} \circ (\mathbf{j}_{\mathrm{Ran}})_!(\mathcal{F}')) &= \\ &= \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran} \times \mathrm{Ran})}(\mathcal{G} \boxtimes (\mathbf{j}_{\mathrm{Ran}})_!(\mathcal{F}'), (\mathrm{diag}_{\mathrm{Ran}}^!)_!(\omega_{\mathrm{Ran}})). \end{aligned}$$

Now, the description of the functor  $(\mathbf{j}_{\mathrm{Ran}})_!$  implies that the canonically defined map

$$(\mathrm{id}_{\mathrm{Ran}} \times \mathbf{j}_{\mathrm{Ran}})_!(\mathcal{G} \boxtimes \mathcal{F}') \rightarrow \mathcal{G} \boxtimes (\mathbf{j}_{\mathrm{Ran}})_!(\mathcal{F}')$$

is an isomorphism. Hence, the expression in (8.5) can be rewritten as

$$\begin{aligned} \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran} \times \mathrm{Ran}')} \left( \mathcal{G} \boxtimes \mathcal{F}', (\mathrm{id}_{\mathrm{Ran}} \times \mathbf{j}_{\mathrm{Ran}})^!(\mathrm{diag}_{\mathrm{Ran}'}^!)_!(\omega_{\mathrm{Ran}'}) \right) &\simeq \\ &\simeq \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran} \times \mathrm{Ran}')} \left( \mathcal{G} \boxtimes \mathcal{F}', (\mathrm{Graph}_{\mathbf{j}_{\mathrm{Ran}}})_!(\omega_{\mathrm{Ran}'}) \right) \simeq \\ &\simeq \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran} \times \mathrm{Ran}')} \left( \mathcal{G} \boxtimes \mathcal{F}', (\mathbf{j}_{\mathrm{Ran}} \times \mathrm{id}_{\mathrm{Ran}'})_* \circ (\mathrm{diag}_{\mathrm{Ran}'}^!)_!(\omega_{\mathrm{Ran}'}) \right) \simeq \\ &\simeq \mathrm{Maps}_{\mathrm{Shv}^!(\mathrm{Ran}' \times \mathrm{Ran}')} \left( (\mathbf{j}_{\mathrm{Ran}})^!(\mathcal{G}) \boxtimes \mathcal{F}', (\mathrm{diag}_{\mathrm{Ran}'}^!)_!(\omega_{\mathrm{Ran}'}) \right), \end{aligned}$$

as desired. □

8.3.5. We are now going to state a variant of Theorem 8.2.4 that involves  $\mathrm{Shv}^!(\mathrm{Ran}')$ :

**Theorem 8.3.6.** *Let  $X$  be a proper smooth curve and  $X' \subset X$  an open subscheme. Let  $\mathcal{F}' \in \mathrm{Shv}^!(\mathrm{Ran}')$  have the property as in Theorem 8.2.4, for the curve  $X'$ . Then for  $\mathcal{F} := (\mathbf{j}_{\mathrm{Ran}})_!(\mathcal{F}')$ , the map (7.4) is an isomorphism.*

This theorem will be proved in Sect. 9.3.

8.4. **The truncated Ran space.** In this subsection we will perform the first step towards the proofs of the results stated above. Namely, we will reduce these theorems to statements of the sort that some cohomology stabilizes in the limit. The idea is the following:

The reason for the non-commutation of the functor of compactly supported cohomology on the Ran space with Verdier duality is that  $\mathrm{Ran}$  is not finitary. In this subsection we will introduce a truncated version of the Ran space, denoted  $\mathrm{Ran}^{\leq n}$ , which will be a finitary pseudo-scheme. The idea of  $\mathrm{Ran}^{\leq n}$  is very simple: whereas  $\mathrm{Ran}$  parameterizes finite non-empty collections of points of  $X$ , its sub-prestack  $\mathrm{Ran}^{\leq n}$  parameterizes those collections that have cardinality  $\leq n$ .

The stabilization referred to above says that for a given range of cohomological degrees we can replace all of  $\mathrm{Ran}$  by  $\mathrm{Ran}^{\leq n}$ .

8.4.1. For an integer  $n$ , let  $\mathrm{Ran}^{\leq n}$  be the following prestack: for  $S \in \mathrm{Sch}$  we let  $\mathrm{Ran}^{\leq n}(S)$  be the (discrete) groupoid of finite non-empty sets of  $\mathrm{Maps}(S, X)$  of cardinality  $\leq n$ . Thus,

$$\mathrm{Ran}^{\leq n} \simeq \mathrm{colim}_{\mathcal{J} \in (\mathrm{Fin}^{s, \leq n})^{\mathrm{op}}} X^{\mathcal{J}},$$

where  $\mathrm{Fin}^{s, \leq n} \subset \mathrm{Fin}^s$  is the full subcategory consisting of sets of cardinality  $\leq n$ . In particular, if  $X$  is proper, then the prestack  $\mathrm{Ran}^{\leq n}$  is finitary pseudo-proper. Moreover, the discussion in Sect. 8.1.2 shows that  $\mathrm{Ran}^{\leq n}$  is a pseudo-scheme with a finitary diagonal.



8.4.2. Let  $\text{ins}^{\leq n}$  denote the tautological map  $\text{Ran}^{\leq n} \hookrightarrow \text{Ran}$ . It is easy to see as in Sect. 8.1.2 that the map  $\text{ins}^{\leq n}$  is finitary pseudo-proper. In particular, we have an adjoint pair

$$(\text{ins}^{\leq n})_! : \text{Shv}^!(\text{Ran}^{\leq n}) \rightleftarrows \text{Shv}^!(\text{Ran}) : (\text{ins}^{\leq n})^!.$$

We have

$$\text{Ran} \simeq \text{colim}_n \text{Ran}^{\leq n},$$

where the maps  $\text{ins}^{\leq n}$  are pseudo-proper. Hence, by Sect. 0.8.5, we have

$$(8.6) \quad \mathcal{F} \simeq \text{colim}_n (\text{ins}^{\leq n})_! \circ (\text{ins}^{\leq n})^!(\mathcal{F}).$$

In particular, we have:

$$(8.7) \quad C_c^*(\text{Ran}, \mathcal{F}) \simeq \text{colim}_n C_c^*\left(\text{Ran}^{\leq n}, (\text{ins}^{\leq n})^!(\mathcal{F})\right).$$

8.4.3. The following assertion, proved in Sect. 9.1, is one stabilization statement that goes into the proof of Theorem 8.2.4:

**Proposition 8.4.4.** *Let  $X$  be a smooth curve. Let  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  be such that there exists an integer  $m \geq 0$  such that the object  $\text{ins}_j^!(\mathcal{F})|_{X^{\circ j}}$  is concentrated in perverse cohomological degrees  $\leq -3$  whenever  $|j| > m$ . Then the map*

$$C_c^*\left(\text{Ran}^{\leq n}, (\text{ins}^{\leq n})^!(\mathcal{F})\right) \rightarrow C_c^*(\text{Ran}, \mathcal{F})$$

*induces an isomorphism in cohomological degrees  $\geq 0$  for  $n \geq m$ .*

*Remark 8.4.5.* Note that the requirement on the cohomological degree in Proposition 8.4.4 is weaker than that in Theorem 8.2.4. We need the bound on the cohomological degrees to be  $\leq -C_0$  rather than  $-|j| - C_1$ , where  $C_0$  and  $C_1$  are constants.

8.4.6. *Duality via the truncated Ran space.* Since the map  $\text{ins}^{\leq n}$  is finitary pseudo-proper, the functor  $(\text{ins}^{\leq n})_!$  intertwines Verdier duality on  $\text{Ran}^{\leq n}$  with Verdier duality on  $\text{Ran}$ , by Corollary 7.5.6. Hence, as in Proposition 7.5.4, we obtain

$$(8.8) \quad \mathbb{D}_{\text{Ran}}(\mathcal{F}) \simeq \lim_n (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}).$$

Observe also that by Proposition 7.5.4, applied to  $\text{Ran}^{\leq n}$  and Corollary 7.5.6, we obtain:

**Corollary 8.4.7.** *For  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  we have a canonical isomorphism*

$$(\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \simeq \lim_{j \in \mathbb{F}_{\text{ins}^{\leq n}}} (\text{ins}_j)_! \circ \mathbb{D}_{X^{\circ j}} \circ (\text{ins}_j)^!(\mathcal{F}).$$

8.4.8. In Sect. 9.2 and 9.4 we will prove the following crucial ingredient in the proof of Theorems 8.2.4 and 8.2.6:

**Proposition 8.4.9.** *Let  $X$  be a smooth curve, and let  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  be such that there exists an integer  $m \geq 0$  such that the object  $\text{ins}_j^!(\mathcal{F})|_{X^{\circ j}}$  is concentrated in perverse cohomological degrees  $\leq -|j| - 2$  whenever  $|j| > m$ .*

(i) *Assume that  $X$  is proper. Then the map*

$$C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}}(\mathcal{F})) \rightarrow C_c^*\left(\text{Ran}^{\leq n}, \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F})\right)$$

*induces an isomorphism in cohomological degrees  $\leq 0$  for  $n \geq m$ .*

(ii) *For any  $j$ , the map*

$$\text{ins}_j^! \circ \mathbb{D}_{\text{Ran}}(\mathcal{F}) \rightarrow \text{ins}_j^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F})$$

*induces an isomorphism in perverse cohomological degrees  $\leq 0$  for  $n \geq m$ .*

8.4.10. *Proof of Theorem 8.2.4.* Let us assume Proposition 8.4.4 and Proposition 8.4.9 and deduce Theorem 8.2.4. The idea is to reduce to the situation where we can apply Corollary 7.5.6 (we cannot apply this proposition to  $\mathrm{Ran}$  because it is not finitary).

We need to prove that for every cohomological degree  $k \geq 0$ , the map

$$(C_c^*(\mathrm{Ran}, \mathbb{D}_y(\mathcal{F})))^{\leq k} \rightarrow (C_c^*(\mathrm{Ran}, \mathcal{F})^\vee)^{\leq k}$$

is an isomorphism. With no restriction of generality, we can take  $k = 0$  (if an object  $\mathcal{F} \in \mathrm{Shv}^!(\mathrm{Ran})$  satisfies the assumption of Theorem 8.2.4, then so does any of its shifts).

We have a commutative diagram

$$\begin{array}{ccc} (C_c^*(\mathrm{Ran}, \mathbb{D}_y(\mathcal{F})))^{\leq 0} & \longrightarrow & (C_c^*(\mathrm{Ran}, \mathcal{F})^\vee)^{\leq 0} \\ \downarrow & & \downarrow \\ (C_c^*(\mathrm{Ran}^{\leq n}, \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F})))^{\leq 0} & \longrightarrow & (C_c^*(\mathrm{Ran}^{\leq n}, (\mathrm{ins}^{\leq n})^!(\mathcal{F})^\vee))^{\leq 0}. \end{array}$$

Note that the bottom horizontal arrow in the diagram is an isomorphism for any  $n$  because the map

$$C_c^*(\mathrm{Ran}^{\leq n}, \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F})) \rightarrow C_c^*(\mathrm{Ran}^{\leq n}, (\mathrm{ins}^{\leq n})^!(\mathcal{F}))^\vee$$

is (by Corollary 7.5.6).

Hence, it remains to show that the vertical maps are isomorphisms for some  $n$ . Now, the left vertical map is an isomorphism for  $n \gg 0$  by Proposition 8.4.9(i). For the right vertical map, it suffices to show that the map

$$(C_c^*(\mathrm{Ran}^{\leq n}, (\mathrm{ins}^{\leq n})^!(\mathcal{F})))^{\geq 0} \rightarrow (C_c^*(\mathrm{Ran}, \mathcal{F}))^{\geq 0}$$

is an isomorphism for  $n \gg 0$ , and that is given by Proposition 8.4.4.

## 9. PROOFS OF THE STABILIZATION AND VERDIER DUALITY RESULTS

### 9.1. Proof of Proposition 8.4.4.

9.1.1. Consider the  $n$ -th Cartesian power  $X^n$  and the *prestack-theoretic quotient*  $X^n/\Sigma_n$ . Note that the map  $\mathrm{ins}_n : X^n \rightarrow \mathrm{Ran}$  canonically factors through a map

$$\mathrm{ins}_n/\Sigma_n : X^n/\Sigma_n \rightarrow \mathrm{Ran}.$$

Let  $\overset{\circ}{X}^n \xrightarrow{j_n} X^n$  denote the complement in  $X^n$  to the diagonal divisor, and consider the corresponding open embedding

$$j_n/\Sigma_n : \overset{\circ}{X}^n/\Sigma_n \hookrightarrow X^n/\Sigma_n.$$

Note that the symmetric power  $\overset{\circ}{X}^{(n)}$  is the étale sheafification of  $\overset{\circ}{X}^n/\Sigma_n$ , so the pullback functor

$$(9.1) \quad \mathrm{Shv}(\overset{\circ}{X}^{(n)}) \rightarrow \mathrm{Shv}^!(\overset{\circ}{X}^n/\Sigma_n)$$

is an equivalence of categories.

9.1.2. We will use the following assertion:

**Lemma 9.1.3.** *For  $\mathcal{F} \in \mathrm{Shv}^!(\mathrm{Ran})$ , the object*

$$\mathrm{coFib}\left((\mathrm{ins}^{\leq n-1})_! \circ (\mathrm{ins}^{\leq n-1})^!(\mathcal{F}) \rightarrow ((\mathrm{ins}^{\leq n})_! \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}))\right) \in \Lambda\text{-mod}$$

*is canonically isomorphic to*

$$(\mathrm{ins}_n/\Sigma_n)_! \circ (j_n/\Sigma_n)_* \circ (j_n/\Sigma_n)^! \circ (\mathrm{ins}_n/\Sigma_n)^!(\mathcal{F}).$$

*Proof.* The map

$$\text{ins}^{n-1,n} : \text{Ran}^{\leq n-1} \rightarrow \text{Ran}^{\leq n}$$

is finitary pseudo-proper and injective. Let

$$(\text{Ran}^{\leq n} - \text{Ran}^{\leq n-1}) \subset \text{Ran}^{\leq n}$$

denote the open sub-prestack equal the complement of its scheme-theoretic image (see Lemma 7.4.11(b) for what this means). Now, the assertion of the lemma follows from Lemma 7.4.11(d) and the fact that the map  $\text{ins}_n : X^n \rightarrow \text{Ran}^{\leq n}$  defines an isomorphism

$$\overset{\circ}{X}^n / \Sigma_n \rightarrow (\text{Ran}^{\leq n} - \text{Ran}^{\leq n-1}).$$

□

**Corollary 9.1.4.** *For  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , the object*

$$\text{coFib} \left( C_c^* \left( \text{Ran}^{\leq n-1}, (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right) \rightarrow C_c^* \left( \text{Ran}^{\leq n}, (\text{ins}^{\leq n})^!(\mathcal{F}) \right) \right) \in \Lambda\text{-mod}$$

*is canonically isomorphic to*

$$C^* \left( \overset{\circ}{X}^n, (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}) \right)_{\Sigma_n}.$$

*Remark 9.1.5.* Note that in Corollary 9.1.4, we are using the functor  $C^*(\overset{\circ}{X}^n, -)$ , and not  $C_c^*(\overset{\circ}{X}^n, -)$ .

9.1.6. We now claim:

**Lemma 9.1.7.** *Let  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  and  $m \in \mathbb{N}$  have the property that  $\text{ins}_J^!(\mathcal{F})|_{\overset{\circ}{X}^J}$  is concentrated in perverse cohomological degrees  $\leq -3$  whenever  $|J| > m$ . Then for any  $m \leq n_1 \leq n_2$ , the map*

$$C_c^* \left( \text{Ran}^{\leq n_1}, (\text{ins}^{\leq n_1})^!(\mathcal{F}) \right) \rightarrow C_c^* \left( \text{Ran}^{\leq n_2}, (\text{ins}^{\leq n_2})^!(\mathcal{F}) \right)$$

*induces an isomorphism in cohomological degrees  $\geq 0$ .*

*Proof.* By Corollary 9.1.4, it suffices to show that for  $n > m$ , the object

$$C^* \left( \overset{\circ}{X}^n, (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}) \right)_{\Sigma_n}$$

lives in cohomological degrees  $< -1$  (i.e.,  $\leq -2$ ). For that, it suffices to show that the object  $C^*(\overset{\circ}{X}^n, (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}))$  lives in cohomological degrees  $\leq -2$ .

By assumption,  $j_n^! \circ \text{ins}_n^!(\mathcal{F})$  is concentrated in perverse cohomological degrees  $\leq -3$ . Thus, it remains to show that the variety  $\overset{\circ}{X}^n$  has the property that the functor  $C^*(\overset{\circ}{X}^n, -)$  has cohomological dimension bounded on the right by 1. This follows from the fact that the projection on, say, the first coordinate

$$\overset{\circ}{X}^n \rightarrow X$$

is an affine morphism.

□

9.1.8. Finally, we note that Lemma 9.1.7 readily implies Proposition 8.4.4: take  $n_1 = n$  and pass to the colimit with respect to  $n_2$ .

**9.2. Proof of Proposition 8.4.9(i).** The proof of Proposition 8.4.9(i) is more subtle than that of Proposition 8.4.4, because it is not true that the functor  $C_c^*(\text{Ran}, -)$  commutes with limits. So, the crux of the argument will be to commute a limit with a colimit.

9.2.1. By (8.7), it suffices to show that for all  $k \geq 0$ , the map

$$(9.2) \quad C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ \mathbb{D}_{\text{Ran}}(\mathcal{F}) \right) \rightarrow C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \right)$$

induces an isomorphism in cohomological degrees  $\leq 0$  whenever  $n \geq m$ .

Note that the functor  $(\text{ins}^{\leq k})^!$  commutes with limits (because it admits a left adjoint). In addition, the functor  $C_c^*(\text{Ran}^{\leq k}, -)$  *does* commute with limits by Lemma 7.4.7.

Therefore, in order to prove that (9.2) induces an isomorphism in cohomological degrees  $\leq 0$ , it suffices to show that under the assumptions of the proposition, for  $m \leq n_1 \leq n_2$ , the map

$$\begin{aligned} C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ (\text{ins}^{\leq n_2})_! \circ \mathbb{D}_{\text{Ran}^{\leq n_2}} \circ (\text{ins}^{\leq n_2})^!(\mathcal{F}) \right) \rightarrow \\ \rightarrow C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ (\text{ins}^{\leq n_1})_! \circ \mathbb{D}_{\text{Ran}^{\leq n_1}} \circ (\text{ins}^{\leq n_1})^!(\mathcal{F}) \right) \end{aligned}$$

induces an isomorphism in cohomological degrees  $\leq 0$ .

Thus, it remains to show:

**Lemma 9.2.2.** *Under the assumptions of Proposition 8.4.9, for  $n > m$  and any  $k$  the map*

$$(9.3) \quad C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \right) \rightarrow \\ \rightarrow C_c^* \left( \text{Ran}^{\leq k}, (\text{ins}^{\leq k})^! \circ (\text{ins}^{\leq n-1})_! \circ \mathbb{D}_{\text{Ran}^{\leq n-1}} \circ (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right)$$

*induces an isomorphism in cohomological degrees  $\leq 0$ .*

*Remark 9.2.3.* Note that the fact that the map

$$C_c^* \left( \text{Ran}, (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \right) \rightarrow C_c^* \left( \text{Ran}, (\text{ins}^{\leq n-1})_! \circ \mathbb{D}_{\text{Ran}^{\leq n-1}} \circ (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right)$$

induces an isomorphism in cohomological degrees  $\leq 0$  follows by duality from Lemma 9.1.7. In particular, this statement needs a weaker assumption on the cohomological degrees in which  $\text{ins}_j^!(\mathcal{F})|_{\overset{\circ}{X}^j}$  lives.

9.2.4. The rest of this subsection is devoted to the proof of Lemma 9.2.2.

By Corollary 9.1.4, it suffices to show that for any  $k' \leq k$ , the object

$$\begin{aligned} \left( \text{Fib} \left( C^* \left( \overset{\circ}{X}^{k'}, (j_{k'})^! \circ (\text{ins}_{k'})^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \right) \rightarrow \right. \\ \left. \rightarrow C^* \left( \overset{\circ}{X}^{k'}, (j_{k'})^! \circ (\text{ins}_{k'})^! \circ (\text{ins}^{\leq n-1})_! \circ \mathbb{D}_{\text{Ran}^{\leq n-1}} \circ (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right) \right) \Big|_{\Sigma_{k'}} \end{aligned}$$

is concentrated in cohomological degrees  $> 1$  (i.e.,  $\geq 2$ ). Since  $k$  was arbitrary, we can take  $k' = k$ .

Note that for  $\mathcal{G} \in \text{Shv}^!(\overset{\circ}{X}^k/\Sigma_k)$ , the norm map

$$C^*(\overset{\circ}{X}^k, \mathcal{G})_{\Sigma_k} \rightarrow C^*(\overset{\circ}{X}^k, \mathcal{G})^{\Sigma_k}$$

is an isomorphism.

Hence, it suffices to show that the object

$$(9.4) \quad \text{Fib} \left( C^* \left( \overset{\circ}{X}^k, j_k^! \circ (\text{ins}_k)^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})^!(\mathcal{F}) \right) \rightarrow \right. \\ \left. C^* \left( \overset{\circ}{X}^k, (j_k)^! \circ (\text{ins}_k)^! \circ (\text{ins}^{\leq n-1})_! \circ \mathbb{D}_{\text{Ran}^{\leq n-1}} \circ (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right) \right)$$

is concentrated in cohomological degrees  $\geq 2$ .

9.2.5. By Lemmas 9.1.3 and 7.4.7, the object (9.4) identifies with

$$\left( \mathbb{C}^* \left( \overset{\circ}{X}^k, j_k^! \circ (\text{ins}_k)^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n} \circ (j_n)_* \circ (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}) \right) \right)^{\Sigma_n}.$$

Thus, it suffices that the object

$$\mathbb{C}^* \left( \overset{\circ}{X}^k, j_k^! \circ (\text{ins}_k)^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n} \circ (j_n)_* \circ (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}) \right)$$

is concentrated in cohomological degrees  $\geq 2$ .

By assumption, the object  $(j_n)^! \circ (\text{ins}_n)^!(\mathcal{F}) \in \text{Shv}(X^n)$  is concentrated in perverse cohomological degrees  $\leq -n - 2$ . Hence,  $(j_n)_* \circ (j_n)^! \circ (\text{ins}_n)^!(\mathcal{F})$  lives in perverse cohomological degrees  $\leq -n - 2$ , as the morphism  $(j_n)_*$  is affine.

We will show that for any  $\mathcal{G} \in \text{Shv}(X^n)$  that lives in perverse cohomological degrees  $\leq -n - 2$ , the object

$$(9.5) \quad \mathbb{C}^* \left( \overset{\circ}{X}^k, j_k^! \circ (\text{ins}_k)^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n}(\mathcal{G}) \right)$$

is concentrated in cohomological degrees  $\geq 2$ .

9.2.6. Let  $X^{k,n} \subset X^k \times X^n$  be the closed subset, corresponding to the condition that the  $k$ -tuple and the  $n$ -tuple coincide set-theoretically (see Sect. 8.1.3). Let  $q_k$  and  $q_n$  denote the two projections

$$X^k \xleftarrow{q_k} X^{k,n} \xrightarrow{q_n} X^n.$$

By Corollary 8.1.5, the functor

$$(\text{ins}_k)^! \circ (\text{ins}_n)_! : \text{Shv}^!(X^n) \rightarrow \text{Shv}^!(X^m)$$

is canonically isomorphic to

$$(q_k)_! \circ (q_n)^!,$$

where  $q_k$  and  $q_n$  are the two projections

$$X^k \xleftarrow{q_k} X^{k,n} \xrightarrow{q_n} X^n.$$

Hence, we obtain that the expression in (9.5) identifies canonically with

$$(9.6) \quad \mathbb{C}^* \left( \overset{\circ}{X}^k, j_k^! \circ (q_k)_! \circ (q_n)^! \circ \mathbb{D}_{X^n}(\mathcal{G}) \right),$$

and the latter expression only involves schemes (rather than prestacks).

9.2.7. Note that both maps  $q_k$  and  $q_n$  are finite. Let  $\overset{\circ}{X}^{k,n}$  denote the preimage of  $\overset{\circ}{X}^k$  under  $q_k$ . Let  $\overset{\circ}{q}^n$  denote the resulting (quasi-finite) map  $\overset{\circ}{X}^{k,n} \rightarrow X^n$ .

By base change (and using the fact that  $q_k$  is proper), we rewrite the expression in (9.6) as

$$\mathbb{C}^* \left( X^n, \left( (\overset{\circ}{q}_n)_* (\omega_{\overset{\circ}{X}^{k,n}}^! \otimes \mathbb{D}_{X^n}(\mathcal{G})) \right) \right),$$

and, due to the compactness of the object  $(\overset{\circ}{q}_n)_* (\omega_{\overset{\circ}{X}^{k,n}})$ , further as

$$\mathcal{H}om \left( \mathcal{G}, (\overset{\circ}{q}_n)_* (\omega_{\overset{\circ}{X}^{k,n}}) \right),$$

where  $\mathcal{H}om(-, -) \in \Lambda\text{-mod}$  denotes the complex of maps between given two objects.

Due to the assumption on  $\mathcal{G}$ , it remains to show that  $(\overset{\circ}{q}_n)_* (\omega_{\overset{\circ}{X}^{k,n}})$  is concentrated in perverse cohomological degrees  $\geq -n$ . The latter is immediate from the fact that  $\overset{\circ}{q}_n$  is quasi-finite, while  $\dim(X^n) = n$ .

**9.3. Proof of Theorem 8.3.6.** In this subsection we will be working in the context of constructible sheaves.

9.3.1. Let

$$\mathrm{ins}'^{\leq n} : \mathrm{Ran}'^{\leq n} \rightarrow \mathrm{Ran}' \quad \text{and} \quad \mathbf{j}_{\mathrm{Ran}^{\leq n}} : \mathrm{Ran}'^{\leq n} \rightarrow \mathrm{Ran}^{\leq n}$$

denote the corresponding morphisms. We have a tautological isomorphism

$$(\mathrm{ins}^{\leq n})_! \circ (\mathbf{j}_{\mathrm{Ran}^{\leq n}})_! \simeq (\mathbf{j}_{\mathrm{Ran}})_! \circ (\mathrm{ins}'^{\leq n})_!.$$

In addition, it is easy to see that the natural transformation

$$(\mathrm{ins}^{\leq n})_! \circ (\mathbf{j}_{\mathrm{Ran}^{\leq n}})_* \rightarrow (\mathbf{j}_{\mathrm{Ran}})_* \circ (\mathrm{ins}'^{\leq n})_!$$

is also an isomorphism.

As in Lemma 8.3.4, for  $\mathcal{G}' \in \mathrm{Shv}^!(\mathrm{Ran}'^{\leq n})$ , we have a canonical isomorphism

$$\mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathbf{j}_{\mathrm{Ran}^{\leq n}})_!(\mathcal{G}') \simeq (\mathbf{j}_{\mathrm{Ran}^{\leq n}})_* \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n}}(\mathcal{G}').$$

9.3.2. By Proposition 8.4.4, applied to the curve  $X'$ , as in Sect. 8.4.10, in order to prove Theorem 8.3.6, it suffices to establish the following variant of Proposition 8.4.9(i):

**Proposition 9.3.3.** *Let  $\mathcal{F} \in \mathrm{Shv}^!(\mathrm{Ran}')$  be such that there exists an integer  $m \geq 0$  such that the object  $(\mathrm{ins}'_j)^!(\mathcal{F})|_{\overset{\circ}{X}'_j}$  is concentrated in perverse cohomological degrees  $\leq -|j| - 2$  whenever  $|j| > m$ . Then the map*

$$\mathrm{C}_c^*(\mathrm{Ran}, (\mathbf{j}_{\mathrm{Ran}})_* \circ \mathbb{D}_{\mathrm{Ran}'}(\mathcal{F}')) \rightarrow \mathrm{C}_c^*(\mathrm{Ran}^{\leq n}, (\mathbf{j}_{\mathrm{Ran}^{\leq n}})_* \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n}} \circ (\mathrm{ins}'^{\leq n})^!(\mathcal{F}'))$$

induces an isomorphism in cohomological degrees  $\leq 0$  for  $n \geq m$ .

This, in turn, reduces to the following variant of Lemma 9.2.2:

**Lemma 9.3.4.** *Under the assumptions of Proposition 9.3.3, for  $n > m$  and any  $k$  the map*

$$\begin{aligned} \mathrm{C}_c^*(\mathrm{Ran}^{\leq k}, (\mathbf{j}_{\mathrm{Ran}^{\leq k}})_* \circ (\mathrm{ins}'^{\leq k})^! \circ (\mathrm{ins}'^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n}} \circ (\mathrm{ins}'^{\leq n})^!(\mathcal{F}')) &\rightarrow \\ \rightarrow \mathrm{C}_c^*(\mathrm{Ran}^{\leq k}, (\mathbf{j}_{\mathrm{Ran}^{\leq k}})_* \circ (\mathrm{ins}'^{\leq k})^! \circ (\mathrm{ins}'^{\leq n-1})_! \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n-1}} \circ (\mathrm{ins}'^{\leq n-1})^!(\mathcal{F}')) &\end{aligned}$$

induces an isomorphism in cohomological degrees  $\leq 0$ .

9.3.5. Note that for any  $m$ , the diagram

$$\begin{array}{ccc} \mathrm{Shv}(X^m/\Sigma_m) & \xleftarrow{(\mathbf{j}_{X^m/\Sigma_m})^*} & \mathrm{Shv}(X'^m/\Sigma_m) \\ (\mathrm{ins}'_m)_! \downarrow & & \downarrow (\mathrm{ins}_m)_! \\ \mathrm{Shv}(\mathrm{Ran}) & \xleftarrow{(\mathbf{j}_{\mathrm{Ran}})_*} & \mathrm{Shv}(\mathrm{Ran}') \end{array}$$

commutes.

As in Sect. 9.2.4, this implies that in order to prove Lemma 9.3.4, it suffices to show that the object

$$\begin{aligned} \mathrm{Fib}\left(\mathrm{C}^*\left(\overset{\circ}{X}'^k, (j'_k)^! \circ (\mathrm{ins}'_k)^! \circ (\mathrm{ins}'^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n}} \circ (\mathrm{ins}'^{\leq n})^!(\mathcal{F}')\right) \rightarrow \right. \\ \left. \mathrm{C}^*\left(\overset{\circ}{X}'^k, (j'_k)^! \circ (\mathrm{ins}'_k)^! \circ (\mathrm{ins}'^{\leq n-1})_! \circ \mathbb{D}_{\mathrm{Ran}'^{\leq n-1}} \circ (\mathrm{ins}'^{\leq n-1})^!(\mathcal{F}')\right)\right) \end{aligned}$$

is concentrated in cohomological degrees  $> 1$ .

However, this was proved in Sects. 9.2.5-9.2.7.

**9.4. Proof of Proposition 8.4.9(ii).** The proof was essentially given in the process of proof of Proposition 8.4.9(i):

9.4.1. The functor  $\text{ins}_k^!$  commutes with limits, so we need to show that the map

$$\lim_{n'} \text{ins}_k^! \circ (\text{ins}^{\leq n'})_! \circ \mathbb{D}_{\text{Ran} \leq n'} \circ (\text{ins}^{\leq n'})^!(\mathcal{F}) \rightarrow \text{ins}_k^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran} \leq n} \circ (\text{ins}^{\leq n})^!(\mathcal{F})$$

induces an isomorphism in degrees  $\leq 0$  for  $n \geq m$ .

For that, it suffices to show that for  $n' \geq n \geq m$ , the map

$$\text{ins}_k^! \circ (\text{ins}^{\leq n'})_! \circ \mathbb{D}_{\text{Ran} \leq n'} \circ (\text{ins}^{\leq n'})^!(\mathcal{F}) \rightarrow \text{ins}_k^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran} \leq n} \circ (\text{ins}^{\leq n})^!(\mathcal{F})$$

induces an isomorphism in cohomological degrees  $\leq 0$ .

The latter is equivalent to showing that

$$(9.7) \quad \text{ins}_k^! \left( \text{Fib} \left( (\text{ins}^{\leq n})_! \circ \mathbb{D}_{\text{Ran} \leq n} \circ (\text{ins}^{\leq n'})^!(\mathcal{F}) \rightarrow \text{ins}_i^{\leq n-1} \circ \mathbb{D}_{\text{Ran} \leq n-1} \circ (\text{ins}^{\leq n-1})^!(\mathcal{F}) \right) \right)$$

is concentrated in cohomological degrees  $> 1$  (i.e.,  $\geq 2$ ) for  $n > m$ .

9.4.2. By Lemmas 9.1.3 and 7.4.7, the object (9.7), identifies with

$$\left( \text{ins}_k^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n} \circ (j_n)_* \circ j_n^! \circ \text{ins}_n^!(\mathcal{F}) \right)^{\Sigma_n}.$$

So, it is enough to show that the object

$$\text{ins}_k^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n} \circ (j_n)_* \circ j_n^! \circ \text{ins}_n^!(\mathcal{F})$$

is concentrated in perverse cohomological degrees  $\geq 2$ .

The object  $(j_n)_* \circ j_n^! \circ \text{ins}_n^!(\mathcal{F})$  is concentrated in perverse cohomological degrees  $\leq -n - 2$ . We will show that for any  $\mathcal{G} \in \text{Shv}(X^n)$  which lives in such degrees, the object

$$\text{ins}_k^! \circ (\text{ins}_n)_! \circ \mathbb{D}_{X^n}(\mathcal{G})$$

lives in perverse degrees  $\geq 2$ .

9.4.3. By Corollary 8.1.5, we have

$$\text{ins}_k^! \circ (\text{ins}^{\leq n})_! \circ \mathbb{D}_{X^n}(\mathcal{G}) \simeq (q_k)_! \circ q_n^! \circ \mathbb{D}_{X^n}(\mathcal{G}).$$

Since the morphisms  $q_n$  and  $q_k$  are both finite, the functor  $(q_k)_! \circ q_n^!$  is right t-exact.

Hence, it suffices to show that the object  $\mathbb{D}_{X^n}(\mathcal{G})$  lives in perverse degrees  $\geq 2$ . However, this follows from the fact that  $\mathcal{G}$  lives in perverse degrees  $\leq -n - 2$ , while  $\dim(X^n) = n$ .<sup>18</sup>

**9.5. Proof of Theorem 8.2.6.** In this subsection we will be referring to the perverse t-structure on  $\text{Shv}(Z)$  for  $Z \in \text{Sch}$  (which the standard t-structure if we work in the context of D-modules).

9.5.1. We need to show that for every  $m_1, m_2$  and every cohomological degree  $k \geq 0$ , the map in (7.5)

$$(\text{ins}_{m_1} \times \text{ins}_{m_2})^! \left( (\mathbb{D}_{\text{Ran}}(\mathcal{F}_1) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{F}_2)) \right) \rightarrow (\text{ins}_{m_1} \times \text{ins}_{m_2})^! \circ \mathbb{D}_{\text{Ran} \times \text{Ran}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

induces an isomorphism in cohomological degrees  $\leq k$ . With no restriction of generality, we can assume that  $k = 0$ .

<sup>18</sup>Note that when working in the context of constructible sheaves, the object  $\mathbb{D}_{X^n}(\mathcal{G})$  actually lives in degrees  $\geq 2 + n$ .

9.5.2. First, we claim that the object

$$\mathrm{ins}_{m_i}^! \circ \mathbb{D}_{\mathrm{Ran}}(\mathcal{F}_i) \in \mathrm{Shv}(X^{m_i})$$

(for  $i = 1, 2$ ) is bounded below.

To prove this, by Proposition 8.4.9(ii), it suffices to show that the object

$$\mathrm{ins}_{m_i}^! \circ (\mathrm{ins}^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_i)$$

is bounded below for some  $n$  sufficiently large.

By Corollary 8.4.7, we have:

$$(\mathrm{ins}^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_i) \simeq \lim_{\mathcal{J} \in \mathrm{Fin}^{s \leq n}} (\mathrm{ins}_{\mathcal{J}})_! \circ \mathbb{D}_{X^{\mathcal{J}}} \circ (\mathrm{ins}_{\mathcal{J}})^!(\mathcal{F}_i).$$

Since the above limit is finite, it suffices to show that for each  $\mathcal{J}$ , the object

$$\mathrm{ins}_{m_i}^! \circ (\mathrm{ins}_{\mathcal{J}})_! \circ \mathbb{D}_{X^{\mathcal{J}}} \circ (\mathrm{ins}_{\mathcal{J}})^!(\mathcal{F}_i)$$

(for  $i = 1, 2$ ) is bounded below.

By assumption,  $(\mathrm{ins}_{\mathcal{J}})^!(\mathcal{F}_i)$  is bounded above. Hence,  $\mathbb{D}_{X^{\mathcal{J}}} \circ (\mathrm{ins}_{\mathcal{J}})^!(\mathcal{F}_i)$  is bounded below. Finally, the functor  $\mathrm{ins}_{m_i}^! \circ (\mathrm{ins}^{\leq n})_!$  has a finite cohomological dimension by Corollary 8.1.5.

9.5.3. Combining the facts that:

- (a) The ring of coefficients  $\Lambda$  has a finite cohomological dimension;
- (b)  $\mathrm{ins}_{m_i}^! \circ \mathbb{D}_{\mathrm{Ran}}(\mathcal{F}_i)$  are bounded below;
- (c) Proposition 8.4.9(ii),

we obtain that there exists an integer  $n \gg 0$ , such that the map

$$\begin{aligned} & (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \left( (\mathbb{D}_{\mathrm{Ran}}(\mathcal{F}_1) \boxtimes \mathbb{D}_{\mathrm{Ran}}(\mathcal{F}_2)) \right) \rightarrow \\ & \rightarrow (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \left( \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_2) \right) \right) \end{aligned}$$

induces an isomorphism in degrees  $\leq 0$ .

By an analog of Proposition 8.4.9(ii) for  $\mathrm{Ran} \times \mathrm{Ran}$ , we can choose  $n$  large enough so that the map

$$\begin{aligned} & (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \circ \mathbb{D}_{\mathrm{Ran} \times \mathrm{Ran}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \rightarrow \\ & (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}^{\leq n} \times \mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \end{aligned}$$

induces an isomorphism in degrees  $\leq 0$ .

Hence, it suffices to show that under the compactness assumption of the theorem, the map

$$(9.8) \quad (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \left( \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_2) \right) \right) \rightarrow (\mathrm{ins}_{m_1} \times \mathrm{ins}_{m_2})^! \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}^{\leq n} \times \mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

induces an isomorphism in cohomological degrees  $\leq 0$ . We will show that the map (9.8) is an isomorphism (in all degrees). In fact, we will show that the map

$$\begin{aligned} & (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \left( \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{\mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n})^!(\mathcal{F}_2) \right) \right) \rightarrow \\ & (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})_! \circ \mathbb{D}_{\mathrm{Ran}^{\leq n} \times \mathrm{Ran}^{\leq n}} \circ (\mathrm{ins}^{\leq n} \times \mathrm{ins}^{\leq n})^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \end{aligned}$$

is an isomorphism.



9.5.4. By Corollary 8.4.7, we have:

$$\begin{aligned} (\text{ins}^{\leq n} \times \text{ins}^{\leq n})! \left( \left( \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})! (\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{\text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n})! (\mathcal{F}_2) \right) \right) &\simeq \\ &\simeq \lim_{J_1, J_2 \in \text{Fin}^s, \leq n} (\text{ins}_{J_1} \times \text{ins}_{J_2})! \circ \left( \left( \mathbb{D}_{X^{J_1}} \circ \text{ins}_{J_1}^! (\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{X^{J_2}} \circ \text{ins}_{J_2}^! (\mathcal{F}_2) \right) \right), \end{aligned}$$

and

$$\begin{aligned} (\text{ins}^{\leq n} \times \text{ins}^{\leq n})! \circ \mathbb{D}_{\text{Ran}^{\leq n} \times \text{Ran}^{\leq n}} \circ (\text{ins}^{\leq n} \times \text{ins}^{\leq n})! (\mathcal{F}_1 \boxtimes \mathcal{F}_2) &\simeq \\ &\simeq \lim_{J_1, J_2 \in \text{Fin}^s, \leq n} (\text{ins}_{J_1} \times \text{ins}_{J_2})! \circ \mathbb{D}_{X^{J_1} \times X^{J_2}} \circ (\text{ins}_{J_1} \times \text{ins}_{J_2})! (\mathcal{F}_1 \boxtimes \mathcal{F}_2). \end{aligned}$$

Hence, it is sufficient to show that for every  $J_1, J_2$ , the map

$$\left( \mathbb{D}_{X^{J_1}} \circ \text{ins}_{J_1}^! (\mathcal{F}_1) \right) \boxtimes \left( \mathbb{D}_{X^{J_2}} \circ \text{ins}_{J_2}^! (\mathcal{F}_2) \right) \rightarrow \mathbb{D}_{X^{J_1} \times X^{J_2}} \circ (\text{ins}_{J_1} \times \text{ins}_{J_2})! (\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

is an isomorphism.

By assumption,  $\text{ins}_{J_1}^! (\mathcal{F}_1)$  and  $\text{ins}_{J_2}^! (\mathcal{F}_2)$  are bounded above with compact cohomologies. Hence, the required isomorphism follows from Lemma 7.3.2.

### Part III: Duality via sheaves on the augmented Ran space

#### 10. PAIRINGS ON THE AUGMENTED VS. USUAL RAN SPACE

In this section we will perform a manipulation crucial for our derivation of the cohomological product formula (0.5) from non-abelian Poincaré duality (0.8); this will be done in Sect. 15. It is for this manipulation that we need the unital augmented version of the Ran space, developed in Part I.

Say we start with two objects  $\mathcal{A}, \mathcal{B} \in \mathrm{Shv}^1(\mathrm{Ran})$  and we want to construct an isomorphism

$$(10.1) \quad C_c^*(\mathrm{Ran}, \mathcal{B}) \rightarrow C_c^*(\mathrm{Ran}, \mathcal{A})^\vee.$$

A natural first attempt would be to identify  $\mathcal{B}$  with the Verdier dual of  $\mathcal{A}$ . However, the object  $\mathcal{A}$  that we have in mind will be such that it will grossly violate the assumption of Theorem 8.2.4. So, it is unreasonable to expect that the (compactly supported) cohomology of the Verdier dual of  $\mathcal{A}$  has anything to do with the dual of the (compactly supported) cohomology of  $\mathcal{A}$ . In fact, in our case the Verdier dual of  $\mathcal{A}$  will be zero.

Instead, we will find that  $\mathcal{A}$  is naturally of the form

$$\mathcal{A} \simeq \mathrm{OblvUnit} \circ \mathrm{OblvAug}(\mathcal{A}_{\mathrm{untl}, \mathrm{aug}})$$

for  $\mathcal{A}_{\mathrm{untl}, \mathrm{aug}} \in \mathrm{Shv}^1(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})$ , and similarly for  $\mathcal{B}$ . Moreover, the objects  $\mathcal{A}_{\mathrm{untl}, \mathrm{aug}}, \mathcal{B}_{\mathrm{untl}, \mathrm{aug}} \in \mathrm{Shv}^1(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})$  satisfy the assumption of Theorem 5.4.3. Set

$$\mathcal{A}_{\mathrm{red}} := \mathrm{TakeOut}(\mathcal{A}_{\mathrm{untl}, \mathrm{aug}}) \text{ and } \mathcal{B}_{\mathrm{red}} := \mathrm{TakeOut}(\mathcal{B}_{\mathrm{untl}, \mathrm{aug}}).$$

By Corollary 5.4.5, we have the isomorphisms

$$C_c^*(\mathrm{Ran}, \mathcal{A}_{\mathrm{red}}) \simeq C_c^*(\mathrm{Ran}, \mathcal{A}) \text{ and } C_c^*(\mathrm{Ran}, \mathcal{B}_{\mathrm{red}}) \simeq C_c^*(\mathrm{Ran}, \mathcal{B}).$$

The reason for replacing  $\mathcal{A}$  by  $\mathcal{A}_{\mathrm{red}}$  is that the latter object does satisfy the assumption of Theorem 8.2.4. So, to establish the desired isomorphism (10.1), it now suffices to construct an isomorphism

$$\mathbb{D}_{\mathrm{Ran}}(\mathcal{A}_{\mathrm{red}}) \simeq \mathcal{B}_{\mathrm{red}}.$$

To do so we will first need to construct a pairing

$$(10.2) \quad \mathcal{A}_{\mathrm{red}} \boxtimes \mathcal{B}_{\mathrm{red}} \rightarrow (\mathrm{diag}_{\mathrm{Ran}})!(\omega_{\mathrm{Ran}}).$$

What we do in the present section is explain what kind of data on  $\mathcal{A}$  and  $\mathcal{B}$  is needed to construct a pairing (10.2). This will be addressed by Theorem 10.1.3.

#### 10.1. The notion of pairing for augmented sheaves.

10.1.1. Let

$$(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{\mathrm{sub}, \mathrm{disj}}$$

denote the following lax prestack: for  $S \in \mathrm{Sch}$ , the category  $(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{\mathrm{sub}, \mathrm{disj}}(S)$  is a full subcategory of  $(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})(S)$  that corresponds to quadruples

$$(K_1 \subseteq I_1), (K_2 \subseteq I_2)$$

for which  $K_1$  and  $I_2$  have disjoint images and  $K_2$  and  $I_1$  have disjoint images (see Sect. 5.4.1 what this means).

Let  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$  be two objects of  $\mathrm{Shv}^1(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})$ . By a *pairing* between them we shall mean a map in  $\mathrm{Shv}^1((\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{\mathrm{sub}, \mathrm{disj}})$

$$(10.3) \quad \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{\mathrm{sub}, \mathrm{disj}}} \rightarrow \omega_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{\mathrm{sub}, \mathrm{disj}}}$$

10.1.2. Our goal of this section is to prove the following:

**Theorem-Construction 10.1.3.**

(i) For  $\mathcal{F}, \mathcal{G} \in \text{Shv}^!(\text{Ran})$ , a datum of a pairing  $\mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}})$  gives rise to a pairing

$$\text{AddUnit}_{\text{aug}}(\mathcal{F}) \boxtimes \text{AddUnit}_{\text{aug}}(\mathcal{G})|_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}} \rightarrow \omega_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}}$$

(ii) For  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ , the datum of a pairing (10.3) gives rise to a pairing

$$(10.4) \quad \text{TakeOut}(\tilde{\mathcal{F}}) \boxtimes \text{TakeOut}(\tilde{\mathcal{G}}) \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}}).$$

(iii) The constructions in (i) and (ii) are compatible under adjunction maps

$$\mathcal{F} \rightarrow \text{TakeOut} \circ \text{AddUnit}_{\text{aug}}(\mathcal{F}), \quad \mathcal{G} \rightarrow \text{TakeOut} \circ \text{AddUnit}_{\text{aug}}(\mathcal{G})$$

and

$$\text{AddUnit}_{\text{aug}} \circ \text{TakeOut}(\tilde{\mathcal{F}}) \rightarrow \tilde{\mathcal{F}}, \quad \text{AddUnit}_{\text{aug}} \circ \text{TakeOut}(\tilde{\mathcal{G}}) \rightarrow \tilde{\mathcal{G}}.$$

(iv) Given a pairing (10.3) and the corresponding pairing (10.4), the diagram

$$\begin{array}{ccc} \text{OblvUnit} \circ \text{OblvAug}(\tilde{\mathcal{F}}) \boxtimes \text{OblvUnit} \circ \text{OblvAug}(\tilde{\mathcal{G}}) & \longrightarrow & \omega_{\text{Ran} \times \text{Ran}} \\ \uparrow (5.5) \boxtimes (5.5) & & \uparrow \\ \text{TakeOut}(\tilde{\mathcal{F}}) \boxtimes \text{TakeOut}(\tilde{\mathcal{G}}) & \longrightarrow & (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}}) \end{array}$$

commutes.

Combining this theorem with Theorem 5.4.3, we obtain:

**Corollary 10.1.4.** *There exists a canonical bijection between pairings  $\mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}})$  and pairings (10.3) for  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F})$  and  $\tilde{\mathcal{G}} := \text{AddUnit}_{\text{aug}}(\mathcal{G})$ .*

From point (iv) of the theorem we obtain:

**Corollary 10.1.5.** *Given a pairing  $\mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}})$ , for the corresponding pairing*

$$\tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}} \rightarrow \omega_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}}$$

with

$$\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F}), \quad \tilde{\mathcal{G}} := \text{AddUnit}_{\text{aug}}(\mathcal{G}),$$

the diagram

$$\begin{array}{ccccc} C_c^*(\text{Ran}, \mathcal{F}') \otimes C_c^*(\text{Ran}, \mathcal{G}') & \longrightarrow & C_c^*(\text{Ran} \times \text{Ran}, \omega_{\text{Ran} \times \text{Ran}}) & \longrightarrow & \Lambda \\ \uparrow (4.3) \boxtimes (4.3) & & & & \uparrow \text{id} \\ C_c^*(\text{Ran}, \mathcal{F}) \otimes C_c^*(\text{Ran}, \mathcal{G}) & \longrightarrow & C_c^*(\text{Ran}, \omega_{\text{Ran}}) & \longrightarrow & \Lambda \end{array}$$

commutes, where

$$\mathcal{F}' := \text{OblvUnit} \circ \text{OblvAug}(\tilde{\mathcal{F}}), \quad \mathcal{G}' := \text{OblvUnit} \circ \text{OblvAug}(\tilde{\mathcal{G}}).$$

*Remark 10.1.6.* Note that if  $X$  is connected, then in Corollary 10.1.5 the left vertical arrow is an isomorphism due to Corollary 5.4.5 and the two horizontal arrows on the right are also isomorphisms, due to Theorem 4.1.8.

**10.2. Proof of Theorem 10.1.3(i).**

10.2.1. Recall the notations from Sect. 5.2. Recall that the functor  $\text{AddUnit}_{\text{aug}}$  is defined as

$$\text{coFib} \left( (\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^! \rightarrow \pi^! \circ \text{AddUnit} \right).$$

We shall first construct a map

$$(10.5) \quad \pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G}) \rightarrow \omega_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}}},$$

and show that its restriction to  $(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{sub},\text{disj}}$  is equipped with:

- (a) a null-homotopy when precomposed with

$$(\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G}) \rightarrow \pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G});$$

- (b) a null-homotopy when precomposed with

$$\pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes (\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{G}) \rightarrow \pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G});$$

- (c) a datum of compatibility of the two null-homotopies when precomposed with

$$(\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{F}) \boxtimes (\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{G}) \rightarrow \pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G}).$$

10.2.2. Recall that

$$\text{AddUnit} = \psi_! \circ \xi^!,$$

where the morphisms  $\psi$  and  $\xi$  were introduced in Sect. 4.3.1.

The map (10.5) is the pullback by means of  $\pi$  of a map

$$(10.6) \quad \text{AddUnit}(\mathcal{F}) \boxtimes \text{AddUnit}(\mathcal{G}) \rightarrow \omega_{\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}}}.$$

In its turn, the map (10.6) comes by the  $(\psi \times \psi)_!, (\psi \times \psi)^!$  adjunction from a map

$$(10.7) \quad (\xi \times \xi)^!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow \omega_{\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow}}.$$

Finally, the map (10.7) is the pullback by means of  $\xi \times \xi$  of the map

$$(10.8) \quad \mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})! (\omega_{\text{Ran}}) \rightarrow \omega_{\text{Ran} \times \text{Ran}}.$$

10.2.3. Let us now calculate the composition

$$\begin{aligned} (\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G}) &\rightarrow \pi^! \circ \text{AddUnit}(\mathcal{F}) \boxtimes \pi^! \circ \text{AddUnit}(\mathcal{G}) \rightarrow \\ &\rightarrow \omega_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}}}. \end{aligned}$$

It comes by pullback by means of  $\text{id}_{\text{Ran}_{\text{untl},\text{aug}}} \times \pi$  of a map

$$(\psi_{\text{aug}})! \circ (\xi_{\text{aug}})^!(\mathcal{F}) \boxtimes \psi_! \circ \xi^!(\mathcal{G}) \rightarrow \omega_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}}},$$

which in turn comes by means of the  $(\psi_{\text{aug}} \times \psi)_!, (\psi_{\text{aug}} \times \psi)^!$  adjunction from a map

$$(\xi_{\text{aug}})^!(\mathcal{F}) \boxtimes \xi^!(\mathcal{G}) \rightarrow \omega_{\text{Ran}_{\text{aug}}^{\rightarrow} \times \text{Ran}^{\rightarrow}},$$

where the latter is the pullback of (10.8) by means of the map  $\xi_{\text{aug}} \times \xi$ .

10.2.4. The open sub-prestack

$$(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{sub},\text{disj}} \subset \text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}}$$

is contained in the preimage by means of  $\text{id}_{\text{Ran}_{\text{untl},\text{aug}}} \times \pi$  of the open sub-prestack

$$(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \subset \text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}},$$

defined by the condition that  $K_1$  and  $I_2$  have disjoint images.

The required null-homotopy is provided by the fact that the fiber product

$$(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}}} (\text{Ran}_{\text{aug}}^{\rightarrow} \times \text{Ran}^{\rightarrow}) \times_{\text{Ran} \times \text{Ran}} \text{Ran}$$

is empty: indeed, it classifies the data of

$$(K_1 \subseteq I_1 \supseteq J_1, J_2 \subseteq I_2),$$

where  $K_1 \cap J_1 \neq \emptyset$ ,  $J_1 = J_2$ , while  $K_1$  and  $I_2$  have disjoint images.

10.2.5. The null-homotopy in point (b) is constructed similarly, and the compatibility of the two null-homotopies in (c) follows from the construction.

### 10.3. Proof of Theorem 10.1.3(ii).

10.3.1. *Preparations-I.* Recall that the map  $\text{diag}_{\text{Ran}}$  is finitary pseudo-proper. Hence, the complement of its scheme-theoretic image, denoted

$$(\text{Ran} \times \text{Ran})_{\neq},$$

is a well-defined open sub-prestack of  $\text{Ran} \times \text{Ran}$  (see Lemma 7.4.11(b)).

Let  $(\text{Ran} \times \text{Ran})_{\subseteq}$  (reps.,  $(\text{Ran} \times \text{Ran})_{\supseteq}$ ) be the prestack that assigns to  $S \in \text{Sch}$  the sub-groupoid of  $\text{Maps}(S, \text{Ran} \times \text{Ran})$  consisting of those  $I_1, I_2$  for which  $I_1 \subseteq I_2$  (resp.,  $I_1 \supseteq I_2$ ).

**Lemma 10.3.2.** *The map  $(\text{Ran} \times \text{Ran})_{\subseteq} \rightarrow \text{Ran} \times \text{Ran}$  is finitary pseudo-proper.*

*Proof.* For a pair of finite sets  $J_1$  and  $J_2$ , the fiber product

$$(X^{J_1} \times X^{J_2})_{\text{Ran} \times \text{Ran}} \times_{\text{Ran} \times \text{Ran}} (\text{Ran} \times \text{Ran})_{\subseteq}$$

equals the colimit of  $X^{\mathcal{K}}$ , taken over the category, whose objects are

$$J_1 \rightarrow \mathcal{K} \leftarrow J_2,$$

and morphisms are the surjections of the  $\mathcal{K}$ 's. □

Similarly, the map  $(\text{Ran} \times \text{Ran})_{\supseteq} \rightarrow \text{Ran} \times \text{Ran}$  is finitary pseudo-proper. Let

$$(\text{Ran} \times \text{Ran})_{\not\supseteq} \subset \text{Ran} \times \text{Ran} \supset (\text{Ran} \times \text{Ran})_{\not\subseteq}$$

be the complementary open sub-prestacks.

We have

$$(\text{Ran} \times \text{Ran})_{\subseteq} \times_{\text{Ran} \times \text{Ran}} (\text{Ran} \times \text{Ran})_{\supseteq} = \text{Ran},$$

where  $\text{Ran}$  maps to  $\text{Ran} \times \text{Ran}$  diagonally. Hence the map

$$(\text{Ran} \times \text{Ran})_{\not\subseteq} \cup (\text{Ran} \times \text{Ran})_{\not\supseteq} \rightarrow (\text{Ran} \times \text{Ran})_{\neq}$$

is a Zariski cover. Set

$$(\text{Ran} \times \text{Ran})_{\not\subseteq, \not\supseteq} := (\text{Ran} \times \text{Ran})_{\not\subseteq} \cap (\text{Ran} \times \text{Ran})_{\not\supseteq} \subset \text{Ran} \times \text{Ran}.$$

10.3.3. *Plan of the proof.* Let us be given a pairing

$$(10.9) \quad \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}} \rightarrow \omega_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}},$$

and we wish to construct a pairing

$$(10.10) \quad \text{TakeOut}(\tilde{\mathcal{F}}) \boxtimes \text{TakeOut}(\tilde{\mathcal{G}}) \rightarrow (\text{diag}_{\text{Ran}})_!(\omega_{\text{Ran}}).$$

Recall that

$$\begin{aligned} \text{TakeOut}(\tilde{\mathcal{F}}) &= \text{Fib} \left( \phi^! \circ \iota^!(\tilde{\mathcal{F}}) \rightarrow (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}) \right), \\ \text{TakeOut}(\tilde{\mathcal{G}}) &= \text{Fib} \left( \phi^! \circ \iota^!(\tilde{\mathcal{G}}) \rightarrow (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{G}}) \right). \end{aligned}$$

Thus, by Sect. 10.3.1, in order to construct (10.10), it would be sufficient to supply the following:

- (a) a map

$$(10.11) \quad \phi^! \circ \iota^!(\tilde{\mathcal{F}}) \boxtimes \phi^! \circ \iota^!(\tilde{\mathcal{G}}) \rightarrow \omega_{\text{Ran} \times \text{Ran}};$$

- (b) a map

$$(10.12) \quad (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}) \boxtimes \phi^! \circ \iota^!(\tilde{\mathcal{G}}) \rightarrow \omega_{(\text{Ran} \times \text{Ran})_{\not\subseteq}},$$

- (b') a datum of factoring of the restriction of the map (10.11) to  $(\text{Ran} \times \text{Ran})_{\not\subseteq}$  via the map (10.12).

- (c) a map
- $$(10.13) \quad \phi^! \circ \iota^!(\tilde{\mathcal{F}}) \boxtimes (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{G}}) \rightarrow \omega_{(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}};$$
- (c') a datum of factoring of the restriction of the map (10.11) to  $(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}$  via the map (10.13).
  - (d) a map
- $$(10.14) \quad (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}) \boxtimes (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{G}}) \rightarrow \omega_{(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}}};$$
- (d') data of factoring of the restriction of the maps (10.13) and (10.12) to  $(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}}$  via the map (10.14).
  - (d'') data of compatibility of the factorizations in (b') and (c') with that in (d').

The map (10.11) is obtained from the map (10.9) by pullback by means of the map  $\iota \circ \phi$ ; we note that the image of this map lands in  $(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}$ .

The construction of the maps (10.13), (10.12) and (10.14) is based on the material explained in the next subsection.

10.3.4. *Preparations-II.* Consider now the following lax prestacks:

$$(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}, (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}, (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}} :$$

For  $S \in \text{Sch}$ , the category  $(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}(S)$  is that of triples

$$(K_1 \subseteq J_1, J_2),$$

where  $K_1, J_1, J_2$  are finite non-empty subsets of  $\text{Maps}(S, X)$ , and where we require that  $K_1$  and  $J_2$  have disjoint images. As morphisms we allow inclusions of the  $K_1$ 's and isomorphisms of  $J_1$ 's and  $J_2$ 's.

Let  $\kappa_{\subseteq}$  denote the forgetful map

$$(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}.$$

The category  $(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}(S)$  is that of triples

$$(J_1, K_2 \subseteq J_2),$$

where  $K_2, J_1, J_2$  are finite non-empty subsets of  $\text{Maps}(S, X)$ , and where we require that  $K_2$  and  $J_1$  have disjoint images. As morphisms we allow inclusions of the  $K_2$ 's and isomorphisms of  $J_1$ 's and  $J_2$ 's.

Let  $\kappa_{\supseteq}$  denote the forgetful map

$$(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}.$$

The category  $(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}}(S)$  is that of triples

$$(K_1 \subseteq J_1, K_2 \subseteq J_2),$$

where  $K_1, K_2, J_1, J_2$  are finite non-empty subsets of  $\text{Maps}(S, X)$ , and where we require that  $K_2$  and  $J_1$  have disjoint images and  $K_1$  and  $J_2$  have disjoint images. As morphisms we allow inclusions of the  $K_1$ 's and  $K_2$ 's and isomorphisms of  $J_1$ 's and  $J_2$ 's.

Let  $\kappa_{\subseteq, \supseteq}$  denote the forgetful map

$$(\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}}.$$

We will need the following assertion, which will be proved in Sect. 10.4

**Proposition 10.3.5.** *The maps*

$$\begin{aligned} \kappa_{\subseteq} &: (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}, \\ \kappa_{\supseteq} &: (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}}, \\ \kappa_{\subseteq, \supseteq} &: (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}} \rightarrow (\text{Ran} \times \text{Ran})_{\underline{\mathbb{Z}}, \underline{\mathbb{Z}}} \end{aligned}$$

are universally homologically contractible<sup>19</sup>.

<sup>19</sup>See Sect. 2.5 for what this means.

10.3.6. *Constructions of the maps.* In order to construct the map (10.13), by Proposition 10.3.5 and Lemma 2.5.5, given  $S \in \text{Sch}$  and an  $S$ -point  $(J_1, J_2)$  of  $(\text{Ran} \times \text{Ran})_{\mathcal{Z}}$ , we need to construct a map

$$\left( (\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}) \boxtimes \phi^! \circ \iota^!(\tilde{\mathcal{G}}) \right)_{S, (J_1, J_2)} \rightarrow \lim_{\emptyset \neq K_1 \subseteq J_1, K_1 \text{ is disjoint from } J_2} \omega_S.$$

We will now use the description of the object

$$(\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}),$$

given by Proposition 5.3.2. Namely, we have

$$(\xi_{\text{aug}}^!)^R \circ \psi_{\text{aug}}^!(\tilde{\mathcal{F}}) \simeq \lim_{\emptyset \neq K_1 \subseteq J_1} \tilde{\mathcal{F}}_{S, K_1 \subseteq J_1}.$$

Thus, we need to construct a compatible family of maps, for every  $K_1 \subseteq J_1$  such that  $K_1$  and  $J_2$  have disjoint images, of maps

$$(10.15) \quad \tilde{\mathcal{F}}_{S, K_1 \subseteq J_1} \otimes \tilde{\mathcal{G}}_{S, \emptyset \subset J_2} \rightarrow \omega_S.$$

Note, however, that the quadruple

$$(K_1 \subseteq J_1), (\emptyset \subset J_2)$$

is an  $S$ -point of  $(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}$ , and the map (10.15) results from (10.9).

The maps in (c) and (d) are constructed similarly, and the compatibilities in (b'), (c') and (d') follow from the construction.

This completes the construction for Theorem 10.1.3(ii).

10.3.7. The compatibilities stated in Theorem 10.1.3(iii) and Theorem 10.1.3(iv) follows by unwinding the constructions.

10.4. **Proof of Proposition 10.3.5.** We will prove the assertion concerning  $\kappa_{\subseteq}$ , as the other two cases are similar.

10.4.1. It suffices to show that for any pair of finite sets  $(J_1, J_2)$ , the induced map

$$(X^{J_1} \times X^{J_2})_{\text{Ran} \times \text{Ran}} \times (\text{Ran} \times \text{Ran})_{\mathcal{Z}}^{\sim} \rightarrow (X^{J_1} \times X^{J_2})_{\text{Ran} \times \text{Ran}} \times (\text{Ran} \times \text{Ran})_{\mathcal{Z}}$$

is universally homologically contractible.

For a *non-empty* subset  $\mathcal{K}_1 \subset J_1$ , let

$$(X^{J_1} \times X^{J_2})_{\mathcal{K}_1 \cap J_2 = \emptyset} \subset (X^{J_1} \times X^{J_2})$$

be the open subscheme, corresponding to the condition that for every  $k_1 \in \mathcal{K}_1$  and  $i_2 \in J_2$  the corresponding maps  $S \rightrightarrows X$  have non-intersecting images. We have:

$$(X^{J_1} \times X^{J_2})_{\mathcal{K}'_1 \cap J_2 = \emptyset} \cap (X^{J_1} \times X^{J_2})_{\mathcal{K}''_1 \cap J_2 = \emptyset} = (X^{J_1} \times X^{J_2})_{(\mathcal{K}'_1 \cup \mathcal{K}''_1) \cap J_2 = \emptyset},$$

and

$$\bigcup_{\emptyset \neq \mathcal{K}_1} (X^{J_1} \times X^{J_2})_{\mathcal{K}_1 \cap J_2 = \emptyset} = (X^{J_1} \times X^{J_2})_{\text{Ran} \times \text{Ran}} \times (\text{Ran} \times \text{Ran})_{\mathcal{Z}}.$$

Hence, it suffices to show that for any  $\mathcal{K}_1$ , the map

$$(10.16) \quad (X^{J_1} \times X^{J_2})_{\mathcal{K}_1 \cap J_2 = \emptyset} \times (\text{Ran} \times \text{Ran})_{\mathcal{Z}}^{\sim} \rightarrow (X^{J_1} \times X^{J_2})_{\mathcal{K}_1 \cap J_2 = \emptyset}$$

is universally homologically contractible.

10.4.2. We claim that the map (10.16) is value-wise contractible (see Lemma 2.5.3). Namely, we claim that for a given  $S$ -point of  $(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\mathcal{K}_1 \cap \mathcal{J}_2 = \emptyset}$ , the category of its lifts to an  $S$ -point of

$$(10.17) \quad (X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\mathcal{K}_1 \cap \mathcal{J}_2 = \emptyset} \times_{(\text{Ran} \times \text{Ran})_{\mathcal{Z}}} (\text{Ran} \times \text{Ran})_{\mathcal{Z}}^{\sim}$$

is contractible.

For a given  $S$ -point of  $(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\mathcal{K}_1 \cap \mathcal{J}_2 = \emptyset}$  let  $I_1, I_2 \subset \text{Maps}(S, X)$  be the images of the maps

$$\mathcal{J}_1 \rightarrow \text{Maps}(S, X) \text{ and } \mathcal{J}_2 \rightarrow \text{Maps}(S, X),$$

respectively. Let  $K_1 \subset I_1$  be the image of  $\mathcal{K}_1 \subseteq \mathcal{J}_1$ .

The category of lifts of our given  $S$ -point of  $(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\mathcal{K}_1 \cap \mathcal{J}_2 = \emptyset}$  to an  $S$ -point of (10.17) is that of finite non-empty subsets  $K'_1 \subseteq I_1$  such that  $K'_1$  and  $I_2$  have disjoint images.

Note that left cofinal in this category is the full subcategory that consists of those  $K'_1$  that contain  $K_1$ . Indeed, the left adjoint to the embedding is given by  $K'_1 \mapsto K'_1 \cup K_1$ .

Hence, it is sufficient to show that the above subcategory is contractible. But it contains an initial object, namely, one with  $K'_1 = K_1$ .

## 11. AN EXPLICIT EXPRESSION FOR THE VERDIER DUAL

The contents of this section will be needed for one of the two proofs of the *pointwise duality* statement, Theorem 16.4.7. This section may be skipped on the first pass.

11.1. **The formula.** In this subsection we will be working in the context of constructible sheaves.

11.1.1. Let  $\mathcal{F}$  be an object of  $\text{Shv}^l(\text{Ran})$ , and consider the object  $\mathcal{G} := \mathbb{D}_{\text{Ran}}(\mathcal{F}) \in \text{Shv}^l(\text{Ran})$ .

Fix a point  $x \in X$  and consider the corresponding map  $\text{pt} \rightarrow \text{Ran}$ , which we symbolically denote by  $\{x\}$ . Consider the object

$$\mathcal{G}_{\{x\}} \in \Lambda\text{-mod.}$$

The goal of this subsection is to give an explicit description of  $\mathcal{G}_{\{x\}}$  in terms of  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F})$ .

11.1.2. Consider the following lax prestack, denoted  $(\text{Ran}_{\text{untl, aug}})_{x \notin}$ . For  $S \in \text{Sch}$ , the category  $(\text{Ran}_{\text{untl, aug}})_{x \notin}(S)$  is the full subcategory of  $\text{Ran}_{\text{untl, aug}}(S)$ , consisting of pairs  $K \subseteq J \neq \emptyset$  for which  $K$  has a disjoint image from  $\{x\}$ . I.e., for every  $k \in K$ , the image of the corresponding map  $S \rightarrow X$  avoids  $x \in X$ .

Set

$$\mathcal{F}_x^\dagger := C_c^* \left( (\text{Ran}_{\text{untl, aug}})_{x \notin}, \tilde{\mathcal{F}}|_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \right).$$

The goal of this section is to prove the following assertion:

**Theorem-Construction 11.1.3.** *There exists a canonical isomorphism*

$$(\mathcal{F}_x^\dagger)^\vee \simeq \mathcal{G}_{\{x\}}.$$

11.2. **Plan of the proof.**

11.2.1. Let  $\text{Ran}_{\neq \{x\}}$  be the open sub-prestack of  $\text{Ran}$ , which is the complement of the image of the (finitary, pseudo-proper) map  $\text{pt} \rightarrow \text{Ran}$ , corresponding to the point  $x$ .

The following results from the definitions:

**Lemma 11.2.2.** *We have a canonical isomorphism*

$$\mathcal{G}_{\{x\}} \simeq (\text{coFib}(C_c^*(\text{Ran}_{\neq \{x\}}, \mathcal{F}) \rightarrow C_c^*(\text{Ran}, \mathcal{F})))^\vee.$$



11.2.3. Recall now that  $\tilde{\mathcal{F}}$  was constructed as

$$\text{coFib} \left( (\psi_{\text{aug}})_! \circ (\xi_{\text{aug}})^!(\mathcal{F}) \rightarrow \pi^! \circ \psi_! \circ \xi^!(\mathcal{F}) \right).$$

We are going to construct a commutative diagram

$$\begin{array}{ccc} C_c^*(\text{Ran}_{\neq\{x\}}, \mathcal{F}) & \longrightarrow & C_c^* \left( (\text{Ran}_{\text{untl, aug}})_{x \notin} (\psi_{\text{aug}})_! \circ (\xi_{\text{aug}})^!(\mathcal{F})|_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \right) \\ \downarrow & & \downarrow \\ C_c^*(\text{Ran}, \mathcal{F}) & \longrightarrow & C_c^* \left( (\text{Ran}_{\text{untl, aug}})_{x \notin}, \pi^! \circ \psi_! \circ \xi^!(\mathcal{F})|_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \right), \end{array}$$

with the horizontal arrows being isomorphisms. This will prove Theorem 11.1.3.

11.2.4. To construct the isomorphism

$$C_c^*(\text{Ran}, \mathcal{F}) \simeq C_c^* \left( (\text{Ran}_{\text{untl, aug}})_{x \notin}, \pi^! \circ \psi_! \circ \xi^!(\mathcal{F})|_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \right)$$

we consider the Cartesian diagram

$$\begin{array}{ccccc} (\text{Ran}_{\text{untl, aug}})_{x \notin} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} & \longrightarrow & \text{Ran}^{\rightarrow} & \xrightarrow{\xi} & \text{Ran} \\ \downarrow & & \downarrow \psi & & \\ (\text{Ran}_{\text{untl, aug}})_{x \notin} & \xrightarrow{(K \subseteq I) \mapsto I} & \text{Ran}_{\text{untl}} & & \end{array}$$

As in the proof of Proposition 4.3.6, it suffices to show that the map

$$(\text{Ran}_{\text{untl, aug}})_{x \notin} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} \rightarrow \text{Ran}$$

has contractible fibers over any  $S$ -point of  $\text{Ran}$ .

For such a point  $J \subset \text{Maps}(S, X)$ , the category of its lifts to an  $S$ -point of the lax prestack  $(\text{Ran}_{\text{untl, aug}})_{x \notin} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow}$  is that of

$$(K \subseteq I \supseteq J), \quad K \text{ is disjoint from } x.$$

However, this subcategory has an initial object, namely, one with  $K = \emptyset$  and  $I = J$ .

11.3. **Construction.** To construct the isomorphism

$$(11.1) \quad C_c^*(\text{Ran}_{\neq\{x\}}, \mathcal{F}) \simeq C_c^* \left( (\text{Ran}_{\text{untl, aug}})_{x \notin}, (\psi_{\text{aug}})_! \circ (\xi_{\text{aug}})^!(\mathcal{F})|_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \right)$$

we proceed as follows.

11.3.1. Consider the lax prestack, denoted  $\text{Ran}_{\neq\{x\}}^{\sim}$ , whose category of  $S$ -points is that of pairs  $K \subseteq J$  such that  $K \neq \emptyset$  and  $K$  is disjoint from  $x$ . As morphisms we allow inclusions of the  $K$ 's and isomorphism of the  $J$ 's.

We have the evident forgetful map  $\kappa_x : \text{Ran}_{\neq\{x\}}^{\sim} \rightarrow \text{Ran}_{\neq\{x\}}$ . We will need the following assertion:

**Proposition 11.3.2.** *The map  $\kappa_x$  is universally homologically contractible.*

*Proof.* Recall that the map

$$\kappa_{\subseteq} : (\text{Ran} \times \text{Ran})_{\subseteq}^{\sim} \rightarrow (\text{Ran} \times \text{Ran})_{\subseteq}$$

is universally homologically contractible by Proposition 10.3.5.

Now, the map  $\kappa_x$  is obtained from  $\kappa_{\subseteq}$  as a base change by means of

$$\text{Ran}_{\neq\{x\}} \rightarrow (\text{Ran} \times \text{Ran})_{\subseteq}, \quad J \mapsto (J, \{x\}).$$

□

11.3.3. Note now that there is a canonically defined map

$$\mathrm{Ran}_{\neq\{x\}}^{\sim} \xrightarrow{\mu} \mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin} \quad (K \subseteq J) \mapsto ((K \subseteq J \supseteq J), (K \subseteq J))$$

that makes the diagram

$$\begin{array}{ccccc} \mathrm{Ran}_{\neq\{x\}}^{\sim} & \xrightarrow{\mu} & \mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin} & \longrightarrow & (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin} \\ \kappa_x \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ran}_{\neq\{x\}} & & \mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} & \xrightarrow{\psi_{\mathrm{aug}}} & \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \\ \downarrow & & \downarrow \xi_{\mathrm{aug}} & & \\ \mathrm{Ran} & \xrightarrow{\mathrm{id}} & \mathrm{Ran} & & \end{array}$$

commutative.

By Proposition 11.3.2, in order to construct (11.1), it suffices to construct an isomorphism

$$(11.2) \quad \mathrm{C}_c^* \left( \mathrm{Ran}_{\neq\{x\}}^{\sim}, \mu^! \left( \xi_{\mathrm{aug}}^! (\mathcal{F}) |_{\mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin}} \right) \right) \simeq \\ \simeq \mathrm{C}_c^* \left( (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin}, (\psi_{\mathrm{aug}})^! \circ (\xi_{\mathrm{aug}})^! (\mathcal{F}) |_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin}} \right).$$

11.3.4. Note that we can identify the right-hand side in (11.2) with

$$\mathrm{C}_c^* \left( \mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin}, \xi_{\mathrm{aug}}^! (\mathcal{F}) |_{\mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin}} \right).$$

Now, the required isomorphism follows from the next assertion (see Lemma 3.3.4):

**Lemma 11.3.5.** *The map  $\mu$  is value-wise homotopy-type equivalence.*

*Proof.* We need to show that for any  $S \in \mathrm{Sch}$ , the functor

$$\mathrm{Ran}_{\neq\{x\}}^{\sim}(S) \rightarrow \left( \mathrm{Ran}_{\mathrm{aug}}^{\rightarrow} \times_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} (\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{x \notin} \right)(S)$$

induces an equivalence of homotopy types. We will show that this happens after restriction to the fiber over any given point  $J \in \mathrm{Maps}(S, \mathrm{Ran})$ .

The fiber of the left-hand side is the category of all  $\emptyset \neq K \subseteq J$  so that  $K$  is disjoint from  $x$ . The fiber of the right-hand side is the category of

$$K' \subseteq I \supseteq J$$

with  $J \cap K' \neq \emptyset$  and where  $K'$  is disjoint from  $x$ . The functor in question is

$$(K \subseteq J) \mapsto (K \subseteq J \supseteq J).$$

Now, this functor admits a right adjoint, namely

$$(K' \subseteq I \supseteq J) \mapsto (K' \cap J \subseteq J).$$

□

## Part IV: Factorization

### 12. VERDIER DUALITY AND FACTORIZATION

As was mentioned in the preamble to Sect. 10, at some point in the proof of the cohomological product formula, we will need to show that a certain map in  $\text{Shv}^!(\text{Ran})$

$$\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$$

is an isomorphism. A priori, to check that some map in  $\text{Shv}^!(\text{Ran})$  is an isomorphism, we need to check that its pullback to  $X^{\mathcal{J}}$  for every finite non-empty set  $\mathcal{J}$  is an isomorphism.

What we will do in this section is show that there are certain pieces of structure on  $\mathcal{A}_{\text{red}}$  and  $\mathcal{B}_{\text{red}}$  that allow to consider only the case of  $\mathcal{J}$  being a one-element set. The relevant pieces of structure are that of *cocommutative factorization coalgebra* and *commutative factorization algebra*, respectively.

The material from this section will be used in Sect. 16 for the proof of the local duality statement, Theorem 15.3.3. The prerequisites for the present section are Sects. 1 and 8.2 (specifically, Theorem 8.2.6).

**12.1. Commutative and cocommutative factorization coalgebras.** Following [BD], [FG] or [Lu2], one can introduce the notion of *factorization sheaf* on the Ran space. The reason this notion is relevant for us is that if  $\mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a homomorphism of factorization sheaves, which induces an isomorphism when evaluated on  $X^{\{1\}}$ , then it is an isomorphism.

The general notion of factorization sheaf is indispensable for many purposes, but introducing it requires a longish detour of homotopy-theoretic nature. For our purposes it will be sufficient to deal with *commutative* or *cocommutative* factorization algebras, and those can be introduced directly, using the structure on Ran of commutative semi-group. This will be done in the present subsection.

12.1.1. Note that the prestack Ran carries a canonically defined commutative semi-group structure, symbolically denoted by

$$\text{Ran} \times \text{Ran} \xrightarrow{\text{union}} \text{Ran}.$$

Thus, pullback endows the category  $\text{Shv}^!(\text{Ran})$  with a symmetric comonoidal structure. Therefore, we can talk about commutative algebras in  $\text{Shv}^!(\text{Ran})$ . These are objects  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , equipped with a commutative binary operation

$$\mathcal{F} \boxtimes \mathcal{F} \rightarrow \text{union}^!(\mathcal{F}),$$

which satisfies a homotopy-coherent system of compatibilities.

12.1.2. We have:

**Lemma 12.1.3.** *For any  $n$ , the product map*

$$\text{Ran}^{\leq n} \rightarrow \text{Ran}$$

*is finitary pseudo-proper.*

*Proof.* For simplicity, let us consider the case of  $n = 2$ . The proof is parallel to Sect. 8.1.2. For a finite set  $\mathcal{J}$  let us calculate the fiber product

$$X^{\mathcal{J}} \times_{\text{Ran}} (\text{Ran} \times \text{Ran}).$$

It equals the colimit

$$\text{colim}_{\mathcal{J} \rightarrow \mathcal{K} \simeq \mathcal{K}_1 \cup \mathcal{K}_2} X^{\mathcal{K}_1} \times X^{\mathcal{K}_2},$$

where the category of indices has as objects the diagrams

$$\mathcal{J} \rightarrow \mathcal{K} \simeq \mathcal{K}_1 \cup \mathcal{K}_2, \quad \mathcal{K}_1, \mathcal{K}_2 \neq \emptyset,$$

and as morphisms commutative diagrams

$$\begin{array}{ccccc} \mathcal{J} & \longrightarrow & \mathcal{K}' & \longleftarrow & \mathcal{K}'_1 \sqcup \mathcal{K}'_2 \\ \text{id} \downarrow & & \downarrow & & \downarrow \\ \mathcal{J} & \longrightarrow & \mathcal{K}'' & \longleftarrow & \mathcal{K}''_1 \sqcup \mathcal{K}''_2 \end{array}$$

with surjective vertical maps. □

12.1.4. In particular, by Corollary 1.5.4, the functor  $\mathbf{union}^!$  admits a left adjoint, denoted  $\mathbf{union}_!$ , and thus the symmetric comonoidal structure on  $\text{Shv}^!(\text{Ran})$ , gives rise by passing to left adjoints, to a symmetric *monoidal* structure. We shall refer to it as the *convolution* symmetric monoidal structure.

Tautologically, commutative algebras in  $\text{Shv}^!(\text{Ran})$  as defined in Sect. 12.1.1 are the same as commutative algebras with respect to the convolution symmetric monoidal structure.

12.1.5. Let

$$(\text{Ran} \times \text{Ran})_{\text{disj}} \subset \text{Ran} \times \text{Ran}$$

be the open subfunctor corresponding to the condition that  $I_1, I_2 \subset \text{Maps}(S, X)$  have disjoint images.

Let  $\mathcal{F}$  be a commutative algebra in  $\text{Shv}^!(\text{Ran})$  (with respect to the convolution symmetric monoidal structure). We shall say that  $\mathcal{F}$  is a *commutative factorization algebra* if the following condition holds: the map

$$\mathcal{F} \boxtimes \mathcal{F} \rightarrow \mathbf{union}^!(\mathcal{F}),$$

has the property that the induced map

$$\mathcal{F} \boxtimes \mathcal{F}|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathbf{union}^!(\mathcal{F})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}}$$

is an isomorphism.

The following (nearly evident) observation will be of crucial importance:

**Lemma 12.1.6.** *Let  $\mathcal{F}_1 \rightarrow \mathcal{F}_2$  be a homomorphism of commutative factorization algebras. Suppose that the induced map  $(\mathcal{F}_1)_X \rightarrow (\mathcal{F}_2)_X$  is an isomorphism in  $\text{Shv}(X)$ . Then the initial map is an isomorphism.*

Here for  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  we denote by  $\mathcal{F}_X \in \text{Shv}(X)$  the pullback of  $\mathcal{F}$  under the map

$$\text{ins}_{\{1\}} : X = X^{\{1\}} \rightarrow \text{Ran}.$$

12.1.7. By a *cocommutative coalgebra* in  $\text{Shv}^!(\text{Ran})$  we shall mean a cocommutative coalgebra object in  $\text{Shv}^!(\text{Ran})$  with respect to the convolution symmetric monoidal structure.

I.e., this is an object  $\mathcal{G} \in \text{Shv}^!(\text{Ran})$  equipped with a cocommutative cobinary map

$$(12.1) \quad \mathcal{G} \rightarrow \mathbf{union}_!(\mathcal{G} \boxtimes \mathcal{G}),$$

which satisfies a homotopy-coherent system of compatibilities.

12.1.8. Let us now note the following:

**Lemma 12.1.9.**

- (a) *The restriction of the map  $\mathbf{union}$  to  $(\text{Ran} \times \text{Ran})_{\text{disj}}$  is étale.*
- (b) *The diagonal map*

$$(\text{Ran} \times \text{Ran})_{\text{disj}} \rightarrow (\text{Ran} \times \text{Ran})_{\text{disj}} \times_{\text{Ran}} (\text{Ran} \times \text{Ran})$$

*is open and closed (i.e., is the inclusion of a union of connected components).*

*Remark 12.1.10.* Let  $f : Z \rightarrow Y$  be a (separated) map between schemes, and let  $\overset{\circ}{Z} \subset Z$  be an open subscheme, such that  $f|_{\overset{\circ}{Z}}$  is étale. In this case, the diagonal map

$$\overset{\circ}{Z} \rightarrow Z \times_Y \overset{\circ}{Z}$$

is open and closed.

Point (b) of Lemma 12.1.9 states that the above phenomenon takes place for the map

$$\mathbf{union} : (\text{Ran} \times \text{Ran}) \rightarrow \text{Ran}$$

and  $(\text{Ran} \times \text{Ran})_{\text{disj}} \subset (\text{Ran} \times \text{Ran})$ , but it requires a proof since we are dealing not with schemes but prestacks.

*Proof of Lemma 12.1.9.* To prove point (a), it suffices to show that the map

$$X^{\mathcal{J}} \times_{\text{Ran}} (\text{Ran} \times \text{Ran})_{\text{disj}} \rightarrow X^{\mathcal{J}}$$

is étale for any finite set  $\mathcal{J}$ .

However, it is easy to see that the above fiber product identifies with

$$\bigsqcup_{\mathcal{J}=\mathcal{J}_1 \sqcup \mathcal{J}_2} (X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\text{disj}},$$

where

$$(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\text{disj}} \subset X^{\mathcal{J}_1} \times X^{\mathcal{J}_2}$$

is the open subset corresponding to the condition that for any  $i_1 \in \mathcal{J}_1$  and  $i_2 \in \mathcal{J}_2$ , the corresponding maps  $S \rightrightarrows X$  have non-intersecting images.

For point (b) it suffices to show that for any pair of finite sets  $\mathcal{J}_1 \times \mathcal{J}_2$ , the map

$$(12.2) \quad (X^{\mathcal{J}_1} \times X^{\mathcal{J}_2}) \times_{\text{Ran} \times \text{Ran}} (\text{Ran} \times \text{Ran})_{\text{disj}} \rightarrow (X^{\mathcal{J}_1} \times X^{\mathcal{J}_2}) \times_{\text{Ran} \times \text{Ran}} (\text{Ran} \times \text{Ran}) \times_{\text{Ran}} (\text{Ran} \times \text{Ran})_{\text{disj}}$$

is open and closed.

The left-hand side in (12.2) identifies with  $(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\text{disj}}$ . The right-hand side is the disjoint union of the schemes  $(X^{\mathcal{J}_1} \times X^{\mathcal{J}_2})_{\text{disj}}$  over the set of isomorphisms

$$\mathcal{J}_1 \sqcup \mathcal{J}_2 \simeq \mathcal{J}_1 \sqcup \mathcal{J}_2,$$

where  $\mathcal{J}_1$  and  $\mathcal{J}_2$  are finite non-empty sets.

The map in (12.2) is the inclusion of the connected component with  $\mathcal{J}_1 = \mathcal{J}_1$  and  $\mathcal{J}_2 = \mathcal{J}_2$ . □

12.1.11. From point (b) of Lemma 12.1.9, by base change (i.e., using the fact that the map **union** is pseudo-proper) we obtain:

**Corollary 12.1.12.** *For  $\mathcal{G}' \in \text{Shv}^1(\text{Ran} \times \text{Ran})$  there exists a canonically defined map*

$$\mathbf{union}^! \circ \mathbf{union}_!(\mathcal{G}')|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathcal{G}'|_{(\text{Ran} \times \text{Ran})_{\text{disj}}}.$$

12.1.13. Given a *cocommutative coalgebra*  $\mathcal{G}$  in  $\text{Shv}^1(\text{Ran})$ , we shall say that it is a *cocommutative factorization coalgebra* if the composed map

$$(12.3) \quad \mathbf{union}^!(\mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathbf{union}^! \circ \mathbf{union}_!(\mathcal{G} \boxtimes \mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathcal{G} \boxtimes \mathcal{G}|_{(\text{Ran} \times \text{Ran})_{\text{disj}}},$$

where the first arrow is induced by (12.1), and the second by Corollary 12.1.12, is an isomorphism.

12.2. **Interaction with duality.** As we shall see shortly, if  $\mathcal{G}$  is a cocommutative coalgebra in  $\text{Shv}^1(\text{Ran})$ , then its Verdier dual  $\mathbb{D}_{\text{Ran}}(\mathcal{G})$  has a natural structure of commutative algebra in  $\text{Shv}^1(\text{Ran})$ .

Suppose now that  $\mathcal{G}$  was actually a cocommutative factorization coalgebra. Will it be true that  $\mathbb{D}_{\text{Ran}}(\mathcal{G})$  is a commutative factorization algebra? The answer turns out to be “yes”, provided that  $\mathcal{G}$  satisfies a certain connectivity hypothesis; it is here that Theorem 8.2.6 will play a role.

12.2.1. Let  $\mathcal{G}$  be a cocommutative coalgebra in  $\mathrm{Shv}^!(\mathrm{Ran})$ . We claim that

$$\mathcal{F} := \mathbb{D}_{\mathrm{Ran}}(\mathcal{G})$$

acquires a canonically defined structure of commutative algebra in  $\mathrm{Shv}^!(\mathrm{Ran})$ .

Indeed, starting from the coproduct map

$$\mathcal{G} \rightarrow \mathbf{union}_!(\mathcal{G} \boxtimes \mathcal{G}),$$

and applying the functor  $\mathbb{D}_{\mathrm{Ran}}$ , we obtain a map

$$(12.4) \quad \mathbb{D}_{\mathrm{Ran}} \circ \mathbf{union}_!(\mathcal{G} \boxtimes \mathcal{G}) \rightarrow \mathbb{D}_{\mathrm{Ran}}(\mathcal{G}).$$

Precomposing with (7.3), we obtain a map

$$\mathbf{union}_!(\mathbb{D}_{\mathrm{Ran}} \times_{\mathrm{Ran}}(\mathcal{G} \boxtimes \mathcal{G})) \rightarrow \mathbb{D}_{\mathrm{Ran}}(\mathcal{G}).$$

Finally, precomposing with (7.5), we obtain the desired product map

$$(12.5) \quad \mathbf{union}_!(\mathbb{D}_{\mathrm{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\mathrm{Ran}}(\mathcal{G})) \rightarrow \mathbb{D}_{\mathrm{Ran}}(\mathcal{G}).$$

The higher compatibilities for (12.5) are constructed similarly.

12.2.2. We now claim:

**Proposition 12.2.3.** *Let  $\mathcal{G} \in \mathrm{Shv}^!(\mathrm{Ran})$  be a cocommutative coalgebra, and let  $\mathcal{F} := \mathbb{D}_{\mathrm{Ran}}(\mathcal{G})$  be its Verdier dual commutative algebra. Assume that  $\mathcal{G}$  is a cocommutative factorization algebra and:*

- *We are working in the context of constructible sheaves;*
- *The ring of coefficients  $\Lambda$  is of finite cohomological dimension;*
- *$X$  is a curve;*
- *The object  $\mathcal{G}_X \in \mathrm{Shv}^!(X)$  lives in (perverse) cohomological degrees  $\leq -2$  and all of its cohomologies are compact.*

*Then  $\mathcal{F}$  is a commutative factorization algebra.*

The rest of this subsection is devoted to the proof of Proposition 12.2.3.

12.2.4. For a finite set  $J$  consider the corresponding object

$$\mathcal{G}_{X^J} := \mathrm{ins}_J^!(\mathcal{G}) \in \mathrm{Shv}(X^J).$$

The factorization hypothesis on  $\mathcal{G}$  implies that the restriction of  $\mathcal{G}_{X^J}$  to  $\overset{\circ}{X}^J \subset X^J$  is isomorphic to the restriction of  $(\mathcal{G}_X)^{\boxtimes J}$ .

Hence, the assumption that  $\mathcal{G}_X$  lives in cohomological degrees  $\leq -2$  implies that  $\mathcal{G}_{X^J}|_{\overset{\circ}{X}^J}$  lives in degrees  $\leq -2i$ . In particular, it satisfies the assumption of Theorem 8.2.4.

Furthermore, we obtain that all of the cohomologies of  $\mathcal{G}_{X^J}|_{\overset{\circ}{X}^J}$  are compact. Since we are working in the constructible category and since  $X^J$  is stratified by  $\overset{\circ}{X}^J$  for  $J \leftarrow J$ , we obtain that  $\mathcal{G}_{X^J}$  itself is bounded above and all of its cohomologies are compact.

Hence, by Theorem 8.2.6, the map

$$(12.6) \quad \mathbb{D}_{\mathrm{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\mathrm{Ran}}(\mathcal{G}) \rightarrow \mathbb{D}_{\mathrm{Ran}} \times_{\mathrm{Ran}}(\mathcal{G} \boxtimes \mathcal{G})$$

is an isomorphism.

12.2.5. We claim that there exists a commutative diagram

$$(12.7) \quad \begin{array}{ccc} (\mathbb{D}_{\text{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{G}))|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} & \longrightarrow & \mathbb{D}_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \left( (\mathcal{G} \boxtimes \mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right) \\ \downarrow & & \downarrow \\ (\mathbf{union}^!(\mathbb{D}_{\text{Ran}}(\mathcal{G})))|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} & \longrightarrow & \mathbb{D}_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \left( \mathbf{union}^!(\mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right). \end{array}$$

The top horizontal arrow is the composition

$$(12.8) \quad (\mathbb{D}_{\text{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{G}))|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow (\mathbb{D}_{\text{Ran} \times \text{Ran}}(\mathcal{G} \times \mathcal{G}))_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathbb{D}_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \left( (\mathcal{G} \boxtimes \mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right),$$

where the first arrow is (7.5), and the second arrow comes from the fact that  $(\text{Ran} \times \text{Ran})_{\text{disj}} \hookrightarrow \text{Ran} \times \text{Ran}$  is an open embedding (see Sect. 7.3.3).

The bottom arrow comes from the fact that the map  $\mathbf{union}|_{(\text{Ran} \times \text{Ran})_{\text{disj}}}$  is étale (again, by Sect. 7.3.3).

The left vertical arrow is the restriction to  $(\text{Ran} \times \text{Ran})_{\text{disj}}$  of the map

$$\mathbb{D}_{\text{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{G}) \rightarrow \mathbf{union}^!(\mathbb{D}_{\text{Ran}}(\mathcal{G})),$$

obtained by the  $(\mathbf{union}!, \mathbf{union}^!)$  adjunction from the map

$$\mathbf{union}^!(\mathbb{D}_{\text{Ran}}(\mathcal{G}) \boxtimes \mathbb{D}_{\text{Ran}}(\mathcal{G})) \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{G})$$

of (12.5).

The right vertical arrow is obtained by applying the (contravariant) functor  $\mathbb{D}_{(\text{Ran} \times \text{Ran})_{\text{disj}}}$  to the map

$$\mathbf{union}^!(\mathcal{G})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathcal{G} \boxtimes \mathcal{G}|_{(\text{Ran} \times \text{Ran})_{\text{disj}}}$$

of (12.3).

The fact that the above diagram is commutative follows by unwinding the definitions.

12.2.6. We need to prove that the left vertical arrow in (12.7) is an isomorphism. We will show that all other arrows in this diagram are isomorphisms.

The right vertical arrow is an isomorphism due to the assumption that  $\mathcal{G}$  is a cocommutative factorization algebra.

In the top horizontal arrow, which is given by (12.8), the first map is an isomorphism because (12.6) is an isomorphism.

Thus, it remains to show that the second map in (12.8) and the bottom horizontal arrow in (12.7) are isomorphisms. However, this follows from Lemma 7.5.8.

### 13. FACTORIZATION FOR AUGMENTED SHEAVES

In this section we will introduce several more ingredients required for the proof of the local duality statement, Theorem 15.3.3, to be used in Sect. 16.

Let us recall the set-up of the preamble of Sect. 10. We start with  $\mathcal{A}_{\text{untl, aug}}, \mathcal{B}_{\text{untl, aug}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ , and we want to establish an isomorphism  $\mathcal{B}_{\text{red}} \simeq \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$ , where

$$\mathcal{A}_{\text{red}} := \text{TakeOut}(\mathcal{A}_{\text{untl, aug}}) \text{ and } \mathcal{B}_{\text{red}} := \text{TakeOut}(\mathcal{B}_{\text{untl, aug}}).$$

As was explained in Sect. 10, the map in one direction

$$(13.1) \quad \mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$$

comes from a pairing between  $\mathcal{A}_{\text{untl, aug}}$  and  $\mathcal{B}_{\text{untl, aug}}$ . What we do in this section is explain what additional pieces of structure on  $\mathcal{A}_{\text{untl, aug}}, \mathcal{B}_{\text{untl, aug}}$  and a pairing between them are needed to make (13.1) into a homomorphism of commutative factorization algebras.

The prerequisite for this section are: all of Part I and Sects. 10 and 12.

**13.1. (Co)commutative and (co)algebras in unital augmented sheaves.** In this subsection we will explain what kind of structure on  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F}) \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  corresponds to a structure on  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  of (co)commutative (co)algebra with respect to the convolution symmetric monoidal structure.

13.1.1. We consider the category  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  as equipped with a symmetric monoidal structure given by the *pointwise* tensor product

$$\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \mapsto \tilde{\mathcal{F}} \otimes^! \tilde{\mathcal{G}}.$$

Thus, we can speak about commutative algebras and cocommutative coalgebras in  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ . We will now connect these notions to the notions of commutative algebra and cocommutative coalgebra in  $\text{Shv}^!(\text{Ran})$  with respect to the convolution product, introduced in Sect. 12.1. This relies on the following assertion:

**Theorem-Construction 13.1.2.** *The functor*

$$\text{AddUnit}_{\text{aug}} : \text{Shv}^!(\text{Ran}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$$

*has a natural symmetric monoidal structure.*

Combining with Theorem 5.4.3, we obtain:

**Corollary 13.1.3.** *For an object  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ , to specify on it a structure on it of (co)commutative (co)algebra is equivalent to a specifying a structure of (co)commutative (co)algebra on  $\text{AddUnit}_{\text{aug}}(\mathcal{F})$ .*

13.1.4. *Proof of Theorem 13.1.2, Step 1.* We will construct the data of compatibility of binary operations:

$$(13.2) \quad \text{AddUnit}_{\text{aug}}(\mathcal{F}) \otimes^! \text{AddUnit}_{\text{aug}}(\mathcal{G}) \simeq \text{AddUnit}_{\text{aug}}(\mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G})).$$

The datum for higher compatibility is constructed similarly.

Recall that

$$\text{AddUnit}_{\text{aug}} = \text{coFib} \left( (\psi_{\text{aug}})_! \circ \xi_{\text{aug}}^! \rightarrow \pi^! \circ \psi_! \circ \xi^! \right).$$

We will construct a map

$$(13.3) \quad \text{AddUnit}_{\text{aug}}(\mathcal{F}) \otimes^! \text{AddUnit}_{\text{aug}}(\mathcal{G}) \rightarrow \text{AddUnit}_{\text{aug}}(\mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G}))$$

by constructing the following maps:

- (a) A map (in fact, an isomorphism)

$$\text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\pi \times \pi)^! \circ (\psi \times \psi)_! \circ (\xi \times \xi)^!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow \pi^! \circ \psi_! \circ \xi^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G});$$

- (b) A map

$$\text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\pi \times \text{id}_{\text{Ran}_{\text{untl, aug}}})^! \circ (\psi \times \psi_{\text{aug}})_! \circ (\xi \times \xi_{\text{aug}})^!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow (\psi_{\text{aug}})_! \circ \xi_{\text{aug}}^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G});$$

- (b') A homotopy between the composition

$$\begin{aligned} \text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\pi \times \text{id}_{\text{Ran}_{\text{untl, aug}}})^! \circ (\psi \times \psi_{\text{aug}})_! \circ (\xi \times \xi_{\text{aug}})^!(\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \\ &\rightarrow (\psi_{\text{aug}})_! \circ \xi_{\text{aug}}^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow \pi^! \circ \psi_! \circ \xi^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G}) \end{aligned}$$

and

$$\begin{aligned} \text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\pi \times \text{id}_{\text{Ran}_{\text{untl, aug}}})^! \circ (\psi \times \psi_{\text{aug}})_! \circ (\xi \times \xi_{\text{aug}})^!(\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \\ \rightarrow \text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\pi \times \pi)^! \circ (\psi \times \psi)_! \circ (\xi \times \xi)^!(\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \pi^! \circ \psi_! \circ \xi^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G}); \end{aligned}$$

- (c) A map

$$\text{diag}_{\text{Ran}_{\text{untl, aug}}}^! \circ (\text{id}_{\text{Ran}_{\text{untl, aug}}} \times \pi)^! \circ (\psi_{\text{aug}} \times \psi)_! \circ (\xi_{\text{aug}} \times \xi)^!(\mathcal{F} \boxtimes \mathcal{G}) \rightarrow (\psi_{\text{aug}})_! \circ \xi_{\text{aug}}^! \circ \mathbf{union}^!(\mathcal{F} \boxtimes \mathcal{G});$$



- (c') A homotopy between the composition

$$\begin{aligned} \text{diag}_{\text{Ran}_{\text{untl},\text{aug}}}^! \circ (\text{id}_{\text{Ran}_{\text{untl},\text{aug}}} \times \pi)^! \circ (\psi_{\text{aug}} \times \psi)^! \circ (\xi_{\text{aug}} \times \xi)^! (\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \\ &\rightarrow (\psi_{\text{aug}})^! \circ \xi_{\text{aug}}^! \circ \mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}) \rightarrow \pi^! \circ \psi^! \circ \xi^! \circ \mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}) \end{aligned}$$

and

$$\begin{aligned} \text{diag}_{\text{Ran}_{\text{untl},\text{aug}}}^! \circ (\text{id}_{\text{Ran}_{\text{untl},\text{aug}}} \times \pi)^! \circ (\psi_{\text{aug}} \times \psi)^! \circ (\xi_{\text{aug}} \times \xi)^! (\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \\ \rightarrow \text{diag}_{\text{Ran}_{\text{untl},\text{aug}}}^! \circ (\pi \times \pi)^! \circ (\psi \times \psi)^! \circ (\xi \times \xi)^! (\mathcal{F} \boxtimes \mathcal{G}) &\rightarrow \pi^! \circ \psi^! \circ \xi^! \circ \mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}); \end{aligned}$$

- (d) A homotopy between the resulting two maps

$$\text{diag}_{\text{Ran}_{\text{untl},\text{aug}}}^! \circ (\pi \times \pi)^! \circ (\psi_{\text{aug}} \times \psi_{\text{aug}})^! \circ (\xi_{\text{aug}} \times \xi_{\text{aug}})^! (\mathcal{F} \boxtimes \mathcal{G}) \rightrightarrows \pi^! \circ \psi^! \circ \xi^! \circ \mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}).$$

13.1.5. *Proof of Theorem 13.1.2, Step 2.* The map in (a) is obtained as pullback by means of  $\pi$  from a map

$$(13.4) \quad \text{diag}_{\text{Ran}_{\text{untl}}}^! \circ (\psi \times \psi)^! \circ (\xi \times \xi)^! (\mathcal{F} \boxtimes \mathcal{G}) \rightarrow \psi^! \circ \xi^! \circ \mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}).$$

The latter map is obtained by base change from the isomorphism of lax prestacks

$$\text{Ran}_{\text{untl}} \times_{\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}}} (\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow}) \simeq \text{Ran}^{\rightarrow} \times_{\text{Ran}} (\text{Ran} \times \text{Ran}).$$

The map in (b) is obtained from a map of lax prestacks

$$\text{Ran}_{\text{untl},\text{aug}} \times_{\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl},\text{aug}}} (\text{Ran}^{\rightarrow} \times \text{Ran}_{\text{aug}}^{\rightarrow}) \rightarrow \text{Ran}_{\text{aug}}^{\rightarrow} \times_{\text{Ran}} (\text{Ran} \times \text{Ran}),$$

and the map in (c) is obtained from a map of lax prestacks

$$\text{Ran}_{\text{untl},\text{aug}} \times_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl}}} (\text{Ran}_{\text{aug}}^{\rightarrow} \times \text{Ran}^{\rightarrow}) \rightarrow \text{Ran}_{\text{aug}}^{\rightarrow} \times_{\text{Ran}} (\text{Ran} \times \text{Ran}).$$

The data of homotopies in (b'), (c') and (d') follows from the construction.

13.1.6. *Proof of Theorem 13.1.2, Step 3.* We will now show that the map (13.3) is an isomorphism. First, it is easy to see that the left-hand side satisfies the conditions (\*) and (\*\*) in Theorem 5.4.3.

Note now that the functor  $\iota^! : \text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}}) \rightarrow \text{Shv}^!(\text{Ran}_{\text{untl}})$  is conservative on the full subcategory consisting of objects satisfying conditions (\*) and (\*\*) in Theorem 5.4.3. Hence, it remains to show that the induced map

$$\left( \iota^! \circ \text{AddUnit}_{\text{aug}}(\mathcal{F}) \right)^! \otimes \left( \iota^! \circ \text{AddUnit}_{\text{aug}}(\mathcal{G}) \right)^! \rightarrow \iota^! \circ \text{AddUnit}_{\text{aug}}(\mathbf{union}_! (\mathcal{F} \boxtimes \mathcal{G}))$$

is an isomorphism.

However, by Lemma 5.2.8, the latter map identifies with the map (13.4), and thus is an isomorphism.

**13.2. Relation to factorization.** In the previous subsection we proved that for  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F})$ , a structure or (co)commutative (co)algebra on it with respect to the pointwise tensor product is equivalent to the structure on  $\mathcal{F}$  of (co)commutative (co)algebra with respect to the convolution product on  $\text{Shv}^!(\text{Ran})$ .

In this subsection we will address the following question: what property of the (co)commutative (co)algebra  $\mathcal{F}$  guarantees that that  $\mathcal{F}$  is a (co)commutative factorization algebra.

13.2.1. Let  $(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{compl},\text{disj}}$  be the following lax prestack.

For  $S \in \text{Sch}$ , the category  $(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{compl},\text{disj}}(S)$  is the full subcategory of  $(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})(S)$ , consisting of quadruples  $(K_1 \subseteq I_1), (K_2 \subseteq I_2)$ , satisfying the following condition: for any  $i_1 \in I_1 - K_1$  and  $i_2 \in I_2$  the corresponding two maps  $S \rightrightarrows X$  have non-intersecting images, and any  $i_2 \in I_2 - K_2$  and  $i_1 \in I_1$  the corresponding two maps  $S \rightrightarrows X$  have non-intersecting images.

Denote also

$$(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} := (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}}) \times_{\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}}} (\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{compl},\text{disj}}.$$

Note that we have a Cartesian diagram

$$\begin{array}{ccc} (\text{Ran} \times \text{Ran})_{\text{disj}} & \longrightarrow & (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \\ \downarrow & & \downarrow \\ \text{Ran} \times \text{Ran} & \xrightarrow{\iota \times \iota} & \text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}}. \end{array}$$

13.2.2. Let  $\tilde{\mathcal{F}}$  be a commutative algebra in  $\text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}})$  (with respect to the pointwise tensor product, i.e., the operation  $\overset{!}{\otimes}$ ). We shall say that  $\tilde{\mathcal{F}}$  is a *commutative factorization algebra*, if the following condition holds:

For any  $S \in \text{Sch}$  and an object  $(I_1, I_2) \in (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}(S)$ , the composed map

$$(13.5) \quad \tilde{\mathcal{F}}_{\emptyset \subset I_1} \overset{!}{\otimes} \tilde{\mathcal{F}}_{\emptyset \subset I_2} \rightarrow \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2} \overset{!}{\otimes} \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2} = (\tilde{\mathcal{F}} \overset{!}{\otimes} \tilde{\mathcal{F}})_{\emptyset \subset I_1 \cup I_2} \rightarrow \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2}$$

and the maps

$$\tilde{\mathcal{F}}_{\emptyset \subset I_1} \rightarrow \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2} \quad \text{and} \quad \tilde{\mathcal{F}}_{\emptyset \subset I_2} \rightarrow \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2}$$

induce an isomorphism

$$(13.6) \quad \tilde{\mathcal{F}}_{\emptyset \subset I_1} \oplus (\tilde{\mathcal{F}}_{\emptyset \subset I_1} \overset{!}{\otimes} \tilde{\mathcal{F}}_{\emptyset \subset I_2}) \oplus \tilde{\mathcal{F}}_{\emptyset \subset I_2} \rightarrow \tilde{\mathcal{F}}_{\emptyset \subset I_1 \cup I_2}.$$

Note that the condition of being a *commutative factorization algebra* only depends on the restriction of  $\tilde{\mathcal{F}}$  to  $\text{Ran}_{\text{untl}}$  under the map  $\iota : \text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl},\text{aug}}$ .

*Remark 13.2.3.* Suppose in addition that  $\tilde{\mathcal{F}}$  satisfies conditions (\*) and (\*\*) in Theorem 5.4.3.

In this case, it is easy to see that for any  $S$  and an  $S$ -point  $(K_1 \subseteq I_1), (K_2 \subseteq I_2)$  of  $(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{compl},\text{disj}}$ , the composed map

$$\tilde{\mathcal{F}}_{K_1 \subseteq I_1} \overset{!}{\otimes} \tilde{\mathcal{F}}_{K_2 \subseteq I_2} \rightarrow \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \overset{!}{\otimes} \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} = (\tilde{\mathcal{F}} \overset{!}{\otimes} \tilde{\mathcal{F}})_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2}$$

and the maps

$$\tilde{\mathcal{F}}_{K_1 \subseteq I_1} \rightarrow \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \quad \text{and} \quad \tilde{\mathcal{F}}_{K_2 \subseteq I_2} \rightarrow \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2}$$

induce an isomorphism

$$\tilde{\mathcal{F}}_{K_1 \subseteq I_1} \oplus (\tilde{\mathcal{F}}_{K_1 \subseteq I_1} \overset{!}{\otimes} \tilde{\mathcal{F}}_{K_2 \subseteq I_2}) \oplus \tilde{\mathcal{F}}_{K_2 \subseteq I_2} \rightarrow \tilde{\mathcal{F}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2}.$$

13.2.4. Let  $\tilde{\mathcal{G}}$  be a cocommutative coalgebra in  $\text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}})$  (with respect to the *pointwise* tensor product). We shall say that  $\tilde{\mathcal{G}}$  is a *cocommutative factorization coalgebra*, if the following conditions hold:

- (1)  $\tilde{\mathcal{G}}$  satisfies conditions (\*) and (\*\*) in Theorem 5.4.3;
- (2) For any  $S \in \text{Sch}$  and an  $S$ -point  $(K_1 \subseteq I_1), (K_2 \subseteq I_2)$  of

$$(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{compl},\text{disj}},$$

the composed map

$$\begin{aligned} \tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} &\rightarrow (\tilde{\mathcal{G}} \overset{!}{\otimes} \tilde{\mathcal{G}})_{K_1 \cup K_2 \subseteq I_1 \cup I_2} = \tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \overset{!}{\otimes} \tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \\ &\rightarrow \tilde{\mathcal{G}}_{K_1 \cup I_2 \subseteq I_1 \cup I_2} \overset{!}{\otimes} \tilde{\mathcal{G}}_{I_1 \cup K_2 \subseteq I_1 \cup I_2}, \end{aligned}$$

together with the maps

$$\tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \tilde{\mathcal{G}}_{K_1 \cup I_2 \subseteq I_1 \cup I_2} \quad \text{and} \quad \tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \tilde{\mathcal{G}}_{I_1 \cup K_2 \subseteq I_1 \cup I_2}$$

induce an isomorphism

$$(13.7) \quad \tilde{\mathcal{G}}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \tilde{\mathcal{G}}_{K_1 \cup I_2 \subseteq I_1 \cup I_2} \oplus (\tilde{\mathcal{G}}_{K_1 \cup I_2 \subseteq I_1 \cup I_2} \overset{!}{\otimes} \tilde{\mathcal{G}}_{I_1 \cup K_2 \subseteq I_1 \cup I_2}) \oplus \tilde{\mathcal{G}}_{I_1 \cup K_2 \subseteq I_1 \cup I_2}.$$

13.2.5. We are going to prove the following result:

**Proposition 13.2.6.**

(a) Let  $\mathcal{F} \in \text{Shv}^1(\text{Ran})$  be a commutative algebra, and let  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F})$  be the corresponding commutative algebra in  $\text{Shv}^1(\text{Ran}_{\text{untl}, \text{aug}})$ . Then  $\mathcal{F}$  is a commutative factorization algebra (in the sense of Sect. 12.1.5) if and only if  $\tilde{\mathcal{F}}$  is a commutative factorization algebra (in the sense of Sect. 13.2.2).

(b) Let  $\mathcal{G} \in \text{Shv}^1(\text{Ran})$  be a cocommutative coalgebra, and let  $\tilde{\mathcal{G}} := \text{AddUnit}_{\text{aug}}(\mathcal{G})$  be the corresponding cocommutative coalgebra in  $\text{Shv}^1(\text{Ran}_{\text{untl}, \text{aug}})$ . Then  $\mathcal{G}$  is a cocommutative factorization coalgebra (in the sense of Sect. 12.1.13) if and only if  $\tilde{\mathcal{G}}$  is a cocommutative factorization coalgebra (in the sense of Sect. 13.2.4).

The rest of this subsection is devoted to the proof of Proposition 13.2.6.

13.2.7. *Proof of Proposition 13.2.6(a).* Let  $\mathcal{F}$  be a commutative algebra object in  $\text{Shv}^1(\mathcal{F})$ .

For  $S \in \text{Sch}$  and an object  $(I_1, I_2) \in (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}(S)$ , the right-hand side in (13.6) is obtained as the value on  $S$  and  $(I_1, I_2)$  of the object of  $\text{Shv}^1((\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}})$ , given by pull-push along the Cartesian diagram

$$\begin{array}{ccc} (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} & \longrightarrow & \text{Ran}^{\rightarrow} \xrightarrow{\xi} \text{Ran} \\ \downarrow & & \downarrow \psi \\ (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} & \xrightarrow{\mathbf{union}_{\text{untl}}} & \text{Ran}_{\text{untl}}, \end{array}$$

starting from  $\mathcal{F} \in \text{Shv}^1(\text{Ran})$ . In the above diagram,  $\mathbf{union}_{\text{untl}}$  denotes the naturally defined morphism

$$\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl}}, \quad I_1, I_2 \mapsto I_1 \cup I_2.$$

The three direct summands in the left-hand side of (13.6) are given, respectively, by evaluating at the same  $S$  and  $(I_1, I_2)$  of  $(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}$  the following objects of  $(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}$ :

(1) Pull-push along the Cartesian diagram

$$\begin{array}{ccc} (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} & \longrightarrow & \text{Ran}^{\rightarrow} \xrightarrow{\xi} \text{Ran} \\ \downarrow & & \downarrow \psi \\ (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} & \xrightarrow{\text{pr}_1} & \text{Ran}_{\text{untl}}, \end{array}$$

starting from  $\mathcal{F} \in \text{Shv}^1(\text{Ran})$ .

(2) Pull-push along the diagram

$$(13.8) \quad \begin{array}{ccc} (\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{\text{disj}} & \xrightarrow{(\xi \times \xi)_{\text{disj}}} & (\text{Ran} \times \text{Ran})_{\text{disj}} \\ (\psi \times \psi)_{\text{disj}} \downarrow & & \\ (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}, & & \end{array}$$

starting from  $(\mathcal{F} \boxtimes \mathcal{F})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}}$ , where

$$(\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{\text{disj}} := (\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})} \times (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}.$$

(3) Pull-push along the Caretsian diagram

$$\begin{array}{ccc} (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} & \longrightarrow & \text{Ran}^{\rightarrow} \xrightarrow{\xi} \text{Ran} \\ \downarrow & & \downarrow \psi \\ (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} & \xrightarrow{\text{pr}_2} & \text{Ran}_{\text{untl}}, \end{array}$$

starting from  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$ .

Now, we note that  $(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow}$  is canonically isomorphic to the disjoint union of

$$\begin{aligned} & (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{pr}_1, \text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow}, \\ & (\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{\text{disj}}, \text{ and} \\ & (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{pr}_2, \text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow}. \end{aligned}$$

Now, the map in (13.6) is obtained from this identification, where for the middle direct summand we note that the composition

$$(\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{\text{disj}} \hookrightarrow (\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}^{\rightarrow} \rightarrow \text{Ran}^{\rightarrow} \xrightarrow{\xi} \text{Ran}$$

identifies with

$$(\text{Ran}^{\rightarrow} \times \text{Ran}^{\rightarrow})_{\text{disj}} \rightarrow (\text{Ran} \times \text{Ran})_{\text{disj}} \xrightarrow{\text{union}} \text{Ran}$$

and the map (13.5) is induced by the map

$$(\mathcal{F} \boxtimes \mathcal{F})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \rightarrow \mathbf{union}^!(\mathcal{F})|_{(\text{Ran} \times \text{Ran})_{\text{disj}}},$$

given by the (commutative) algebra structure.

Hence, if  $\mathcal{F}$  is a commutative factorization algebra, then the map (13.6) is an isomorphism.

The converse implication follows from the fact that the functor

$$((\psi \times \psi)_{\text{disj}})^! \circ ((\xi \times \xi)_{\text{disj}})^! : \text{Shv}^!((\text{Ran} \times \text{Ran})_{\text{disj}}) \rightarrow \text{Shv}^!((\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}})$$

is conservative, which in turn follows from Proposition 4.3.6 and Lemma 5.2.9.

13.2.8. *Proof of Proposition 13.2.6(b).* It is easy to see that condition (1) in Sect. 13.2.4 implies that condition (2) holds if and only if it holds for  $K_1 = K_2 = \emptyset$ . In this case, (13.7) comes from a map of objects in  $\text{Shv}^!((\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}})$ .

As in the proof of Proposition 13.2.6(a), the left-hand side in (13.7) is the direct sum

$$(13.9) \quad \begin{aligned} & \left( \text{pr}_1^! \circ \text{AddUnit}(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \oplus \\ & \oplus \left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ \mathbf{union}^!(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \oplus \\ & \left( \oplus \text{pr}_2^! \circ \text{AddUnit}(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}}. \end{aligned}$$

The right-hand side in (13.7) is the direct sum

$$(13.10) \quad \begin{aligned} & \left( \text{shift}_1^! \circ \text{AddUnit}_{\text{aug}}(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \oplus \\ & \oplus \left( \mathbf{union}_{\text{aug}}^! \circ (\text{AddUnit}_{\text{aug}} \boxtimes \text{AddUnit}_{\text{aug}})^!(\mathcal{G} \boxtimes \mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \oplus \\ & \oplus \left( \text{shift}_2^! \circ \text{AddUnit}_{\text{aug}}(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}}, \end{aligned}$$

where

$$\mathbf{union}_{\text{aug}} : \text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}}$$

is the map  $(I_1, I_2) \mapsto (I_2 \subseteq I_1 \sqcup I_2, I_1 \subseteq I_1 \sqcup I_2)$ , and  $\text{shift}_1$  and  $\text{shift}_2$  are the maps

$$\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl},\text{aug}},$$

given by

$$(I_1, I_2) \mapsto (I_2 \subseteq I_1 \cup I_2) \text{ and } (I_1, I_2) \mapsto (I_1 \subseteq I_1 \cup I_2),$$

respectively.

Note, however, that condition (1) in Sect. 13.2.4 on  $\tilde{\mathcal{G}} = \text{AddUnit}_{\text{aug}}(\mathcal{G})$  implies that the map (13.7) maps the first (resp., third) direct summand in (13.9) isomorphically onto the first (reps., third) direct summand in (13.10).

In addition, condition (1) on  $\tilde{\mathcal{G}}$  implies that the middle direct summand in (13.10) receives an isomorphism from

$$\left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ (\mathcal{G} \boxtimes \mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}},$$

and in terms of this identification, the map in (13.7) comes from a map

$$(13.11) \quad \left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ \mathbf{union}^!(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \rightarrow \\ \rightarrow \left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ (\mathcal{G} \boxtimes \mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}},$$

defined as follows:

Consider the diagram (13.8) and note that

$$\left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ \mathbf{union}^!(\mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \simeq \\ \simeq ((\psi \times \psi)_{\text{disj}})^! \circ ((\xi \times \xi)_{\text{disj}})^! \left( \mathbf{union}^!(\mathcal{G}) |_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right)$$

and

$$\left( (\text{AddUnit} \boxtimes \text{AddUnit})^! \circ (\mathcal{G} \boxtimes \mathcal{G}) \right) |_{(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}}} \simeq \\ \simeq ((\psi \times \psi)_{\text{disj}})^! \circ ((\xi \times \xi)_{\text{disj}})^! \left( \mathcal{G} \boxtimes \mathcal{G} |_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right).$$

Now, the map in (13.11) identifies with the map

$$((\psi \times \psi)_{\text{disj}})^! \circ ((\xi \times \xi)_{\text{disj}})^! \left( \mathbf{union}^!(\mathcal{G}) |_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right) \rightarrow ((\psi \times \psi)_{\text{disj}})^! \circ ((\xi \times \xi)_{\text{disj}})^! \left( \mathcal{G} \boxtimes \mathcal{G} |_{(\text{Ran} \times \text{Ran})_{\text{disj}}} \right),$$

induced by (12.3).

Thus, if (12.3) is an isomorphism (i.e., if  $\mathcal{G}$  is a cocommutative factorization coalgebra), the map (13.11) is an isomorphism and hence the map (13.7) is an isomorphism.

The converse implication follows as in point (a).

**13.3. Pairings and augmentation.** Finally, in this subsection we will address the following issue. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a commutative algebra and a cocommutative coalgebra in  $\text{Shv}^!(\text{Ran})$ , respectively. Let us be given a pairing between  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{B}}$  as mere objects of  $\text{Shv}^!(\text{Ran}_{\text{untl},\text{aug}})$ . We shall explain what structure on this pairing guarantees that the induced map  $\mathcal{B} \rightarrow \mathbb{D}_{\text{Ran}}\mathcal{A}$  is a homomorphism of commutative algebras.

13.3.1. Let  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{G}}$  be two objects in  $\mathrm{Shv}^!(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})$ , and let us be given a pairing

$$(13.12) \quad \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \rightarrow \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}}.$$

Suppose now that  $\tilde{\mathcal{F}}$  is endowed with a structure of commutative algebra and  $\tilde{\mathcal{G}}$  is endowed with a structure of cocommutative coalgebra (with respect to the pointwise symmetric monoidal structure). In this case, there is a naturally defined notion of *structure of compatibility* on (13.12) with the above pieces of structure on  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{G}}$ .

The initial data in a structure of compatibility is that of a homotopy between the maps

$$\begin{aligned} (\tilde{\mathcal{F}} \overset{!}{\otimes} \tilde{\mathcal{F}}) \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} &\rightarrow \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \rightarrow \\ &\rightarrow \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \end{aligned}$$

and

$$\begin{aligned} (\tilde{\mathcal{F}} \overset{!}{\otimes} \tilde{\mathcal{F}}) \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} &\rightarrow (\tilde{\mathcal{F}} \overset{!}{\otimes} \tilde{\mathcal{F}}) \boxtimes (\tilde{\mathcal{G}} \overset{!}{\otimes} \tilde{\mathcal{G}})|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \simeq \\ &\simeq \left( \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \right) \overset{!}{\otimes} \left( \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \right) \rightarrow \\ &\rightarrow \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \overset{!}{\otimes} \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}} \simeq \\ &\simeq \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}}. \end{aligned}$$

The higher compatibilities amount to a compatible family of homotopies for each surjection of finite sets  $J_1 \twoheadrightarrow J_2$

$$\tilde{\mathcal{F}}^{\overset{!}{\otimes} J_1} \boxtimes \tilde{\mathcal{G}}^{\overset{!}{\otimes} J_2} \rightrightarrows \omega_{(\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}} \times \mathrm{Ran}_{\mathrm{untl},\mathrm{aug}})_{\mathrm{compl},\mathrm{disj}}}.$$

13.3.2. Let now  $\mathcal{F}$  and  $\mathcal{G}$  be two objects of  $\mathrm{Shv}^!(\mathrm{Ran})$ , and let

$$(13.13) \quad \mathcal{F} \boxtimes \mathcal{G} \rightarrow (\mathrm{diag}_{\mathrm{Ran}})_!(\omega_{\mathrm{Ran}})$$

be a datum of pairing.

Let now  $\mathcal{F}$  be endowed with a structure of commutative algebra, and let  $\mathcal{G}$  be endowed with a structure of cocommutative coalgebra (both with respect to the convolution symmetric monoidal structure). In this case, there is a naturally defined notion of *structure of compatibility* on (13.13) with the above pieces of structure on  $\mathcal{F}$  and  $\mathcal{G}$ .

The initial data in a structure of compatibility is that of a homotopy between the maps

$$\mathbf{unior}_!(\mathcal{F} \boxtimes \mathcal{F}) \boxtimes \mathcal{G} \rightarrow \mathcal{F} \boxtimes \mathcal{G} \rightarrow (\mathrm{diag}_{\mathrm{Ran}})_!(\omega_{\mathrm{Ran}})$$

and

$$\begin{aligned} \mathbf{unior}_!(\mathcal{F} \boxtimes \mathcal{F}) \boxtimes \mathcal{G} &\rightarrow \mathbf{unior}_!(\mathcal{F} \boxtimes \mathcal{F}) \boxtimes \mathbf{unior}_!(\mathcal{G} \boxtimes \mathcal{G}) \rightarrow \\ &\rightarrow (\mathrm{diag}_{\mathrm{Ran}})_! \circ \mathbf{unior}_!(\omega_{\mathrm{Ran}} \boxtimes \omega_{\mathrm{Ran}}) \rightarrow (\mathrm{diag}_{\mathrm{Ran}})_!(\omega_{\mathrm{Ran}}). \end{aligned}$$

The higher compatibilities amount to a compatible family of homotopies for each surjection of finite sets  $J_1 \twoheadrightarrow J_2$  as in Sect. 13.3.1.

The following assertion results from the definition:

**Lemma 13.3.3.** *For  $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^!(\mathrm{Ran})$  and a pairing (13.13), a structure of compatibility with a given commutative algebra structure on  $\mathcal{F}$  and a given cocommutative coalgebra structure on  $\mathcal{G}$  is equivalent to a structure on the resulting map  $\mathcal{F} \rightarrow \mathbb{D}_{\mathrm{Ran}}(\mathcal{G})$  of homomorphism of commutative algebras in  $\mathrm{Shv}^!(\mathrm{Ran})$ .*

13.3.4. Finally, we have the following assertion, which results by unwinding the constructions:

**Lemma 13.3.5.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be a commutative algebra and a cocommutative coalgebra in  $\text{Shv}^!(\text{Ran})$ , respectively. Let us be given a pairing*

$$(13.14) \quad \mathcal{F} \boxtimes \mathcal{G} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}})$$

and consider the corresponding pairing

$$(13.15) \quad \tilde{\mathcal{F}} \boxtimes \tilde{\mathcal{G}}|_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{compl, disj}}} \rightarrow \omega_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{compl, disj}}},$$

where  $\tilde{\mathcal{F}} := \text{AddUnit}_{\text{aug}}(\mathcal{F})$ ,  $\tilde{\mathcal{G}} := \text{AddUnit}_{\text{aug}}(\mathcal{G})$ .

Then a structure of compatibility on (13.14) with the commutative algebra structure on  $\mathcal{F}$  and the cocommutative coalgebra structure on  $\mathcal{G}$  is equivalent to a structure of compatibility on (13.15) with the commutative algebra structure on  $\tilde{\mathcal{F}}$  and the cocommutative coalgebra structure on  $\tilde{\mathcal{G}}$ .

### Part V: The cohomological product formula

In this part we will be working in the context of constructible sheaves, and we will assume that the ring of coefficients  $\Lambda$  has a finite cohomological dimension.

After the preparations in Parts 0-IV, this part we will constitute the core of the paper—the derivation of the cohomological product formula.

#### 14. REDUCTION TO A GLOBAL DUALITY STATEMENT

In order to understand the contents of this section one only needs to know what  $\text{Ran}$  is and how to makes sense of sheaves on prestacks. In other words, the prerequisites for the present section are Sects. 1 and 4.1.

In this section we take  $X$  to be a smooth, connected and complete curve, and let  $G$  be a smooth fiber-wise connected group-scheme over  $X$ , which is simply connected at the generic point of  $X$ .

We are going to state the *cohomological product formula* for the cohomology of  $\text{Bun}_G$ . It says that a naturally defined map

$$C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C_{\text{red}}^*(\text{Bun}_G)$$

is an isomorphism, where  $\mathcal{B} \in \text{Shv}^!(\text{Ran})$  is obtained by considering the cohomology of the classifying space of  $G$ .

We will reduce the proof of the cohomological product formula to a combination of two statements. The first statement will be the *non-abelian Poincaré duality* that says that a naturally defined map

$$C_c^*(\text{Ran}, \mathcal{A}) \rightarrow C_{\text{red}}^*(\text{Bun}_G)$$

is an isomorphism, where  $\mathcal{A} \in \text{Shv}^!(\text{Ran})$  is obtained by considering the homology of the affine Grassmannian of  $G$ . The second statement will be the *global duality* statement, which says that the (compactly supported) cohomologies of  $\mathcal{A}$  and  $\mathcal{B}$  are related by duality.

**14.1. The cohomological product formula.** Let  $\text{Bun}_G$  denote the moduli stack of  $G$ -bundles on  $X$ . We are interested in the (reduced) cohomology of  $\text{Bun}_G$

$$C_{\text{red}}^*(\text{Bun}_G) := \left( C_*^{\text{red}}(\text{Bun}_G) \right)^\vee.$$

In this subsection we will state the cohomological product formula for  $C_{\text{red}}^*(\text{Bun}_G)$ .

14.1.1. First, we are going to define an object, denoted,  $\mathcal{B} \in \text{Shv}^!(\text{Ran})$  that encodes the reduced cohomology of  $BG$ .

For  $S \in \text{Sch}$  and an  $S$ -point  $I$  of  $\text{Ran}$ , let  $D_I \subset S \times X$  be the corresponding Cartier divisor. We let  $BG_I$  denote the Artin stack over  $S$  that classifies  $G$ -bundles over  $D_I$ . Let  $f_I : BG_I \rightarrow S$  denote the resulting forgetful map.

Define

$$\mathcal{B}_{S,I} := \mathbb{D}_S \left( \text{Fib} \left( (f_I)_! \circ (f_I)^!(\Lambda_S) \rightarrow \Lambda_S \right) \right).$$

In the above formula  $\Lambda_S$  is the *constant sheaf* on  $S$ , i.e.,  $\Lambda_S = \mathbb{D}_S(\omega_S)$ , where  $\mathbb{D}_S$  denotes Verdier duality on  $S$ .

For example, for a  $k$ -point  $\{x_1, \dots, x_n\}$  of  $\text{Ran}$ , the  $!$ -fiber of  $\mathcal{B}$  at this point equals

$$\text{Fib} \left( \bigotimes_{i=1, \dots, n} C^*(BG_{x_i}) \rightarrow \Lambda \right).$$

The assignment  $(S, I) \mapsto \mathcal{B}_{S,I}$  forms an object of  $\text{Shv}^!(\text{Ran})$ , which is the sought-for  $\mathcal{B}$ .



*Remark 14.1.2.* Note that there is a minor subtlety involved in the assertion that the assignment  $(S, I) \mapsto \mathcal{B}_{S, I}$  is compatible under pullbacks  $S' \rightarrow S$ :

For a given point  $I \in \text{Maps}(S, \text{Ran})$  and the resulting point  $I' \in \text{Maps}(S', \text{Ran})$ , we have a *map*

$$S' \times_S BG_I \rightarrow BG_{I'},$$

but this map is *not necessarily* an isomorphism. However, this map has contractible fibers, and hence induces an isomorphism  $\mathcal{B}_{S', I'} \rightarrow \mathcal{B}_{S, I}|_{S'}$ . See [Main Text, Proposition 5.4.3] for more details.

14.1.3. For every  $S \in \text{Sch}$  and  $I \in \text{Ran}(S)$  we have a map of stacks over  $S$

$$\text{ev}_{S, I} : S \times \text{Bun}_G \rightarrow BG_I,$$

given by restriction. From here we obtain a map in  $\text{Shv}(S)$

$$\Lambda_S \otimes C_*(\text{Bun}_G) \rightarrow (f_I)_! \circ (f_I)^!(\Lambda_S)$$

and hence

$$\Lambda_S \otimes C_*^{\text{red}}(\text{Bun}_G) \rightarrow \text{Fib}((f_I)_! \circ (f_I)^!(\Lambda_S) \rightarrow \Lambda_S).$$

Applying  $\mathbb{D}_S$  we obtain a map

$$(14.1) \quad \mathcal{B}_{S, I} \rightarrow \mathbb{D}_S \left( \Lambda_S \otimes C_*^{\text{red}}(\text{Bun}_G) \right).$$

Now, for any  $V \in \Lambda\text{-mod}$ , the map

$$\omega_S \otimes V^\vee \rightarrow \mathbb{D}_S(\Lambda_S \otimes V)$$

is an isomorphism.

Hence, from (14.1) we obtain a map

$$(14.2) \quad \text{ev}_{S, I} : \mathcal{B}_{S, I} \rightarrow \omega_S \otimes C_{\text{red}}^*(\text{Bun}_G).$$

14.1.4. The maps (14.2) combine into a map

$$\mathcal{B} \rightarrow \omega_{\text{Ran}} \otimes C_{\text{red}}^*(\text{Bun}_G).$$

Finally, applying the functor  $C_c^*(\text{Ran}, -)$ , we obtain a map

$$(14.3) \quad C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C_c^*(\text{Ran}, \omega_{\text{Ran}} \otimes C_{\text{red}}^*(\text{Bun}_G)) \simeq C_c^*(\text{Ran}, \omega_{\text{Ran}}) \otimes C_{\text{red}}^*(\text{Bun}_G) \simeq C_{\text{red}}^*(\text{Bun}_G),$$

where the last isomorphism is Theorem 4.1.8.

14.1.5. The cohomological product formula says:

**Theorem 14.1.6.** *The map (14.3)*

$$C_c^*(\text{Ran}, \mathcal{B}) \rightarrow C_{\text{red}}^*(\text{Bun}_G)$$

*is an isomorphism.*

Our goal in the rest of Part V is to prove Theorem 14.1.6. We emphasize that the map (14.3) makes sense for any  $G$ . However, the assertion of Theorem 14.1.6 only holds if the generic fiber of  $G$  is simply connected.<sup>20</sup>

*Remark 14.1.7.* As was noted in Sect. 0.2.2, one can informally think of  $C_c^*(\text{Ran}, \mathcal{B})$  as an ‘‘Euler product’’

$$(14.4) \quad \bigotimes_x C^*(BG_x).$$

This is why we call the assertion of Theorem 14.1.6 the ‘‘cohomological product formula’’: it gives an expression for the cohomology of  $\text{Bun}_G$  as the Euler product of the cohomologies of  $BG_x$ .

<sup>20</sup>For example, if the generic fiber of  $G$  is semi-simple but not simply connected, then  $\text{Bun}_G$  is disconnected, and the isomorphism in (14.3) cannot hold because the left-hand side is insensitive to the isogeny class of  $G$ .

**14.2. The key input: non-abelian Poincaré duality.** The assertion of Theorem 14.1.6 is a local-to-global result. We will deduce it from another local-to-global result of a *dual nature*, namely, the non-abelian Poincaré duality. The precise meaning in which the two results are dual to each other will constitute the bulk of the proof of Theorem 14.1.6.

14.2.1. In [Main Text, Lemma 7.1.1 and Proposition A.3.11] it was shown that we can (and from now on will) assume that  $G$  has the following properties:

- $G$  is semi-simple and simply connected over a non-empty open subset  $X' \subset X$ ;
- The fibers of  $G$  over points in  $X - X'$  are cohomologically contractible.<sup>21</sup>

In other words, if  $G_1$  and  $G_2$  are two group-schemes over  $X$  (both smooth with connected fibers) that are isomorphic at the generic point of  $X$ , then Theorem 14.1.6 holds for  $G_1$  if and only if it does for  $G_2$ . And for any given  $G_1$  we can find a  $G_2$  satisfying the above two conditions.

14.2.2. Let  $\text{Ran}' \subset \text{Ran}$  be the Ran space of  $X'$ . We will now introduce an object of  $\text{Shv}(\text{Ran}')$ , denoted  $\mathcal{A}'$ . It will encode the reduced homology of the affine Grassmannians built out of  $G|_{X'}$ .

For  $S \in \text{Sch}$  and an  $S$ -point  $I$  of  $\text{Ran}$ , let  $S \times X - \text{Graph}_I \subset S \times X$  be the open subset equal to the complement of the union of the graphs of the maps  $S \rightarrow X$ ,  $i \in I$ .

Let  $\text{Gr}_{\text{Ran}'}$  denote the following prestack over  $\text{Ran}'$ : for  $S \in \text{Sch}$  an  $S$ -point of  $\text{Gr}_{\text{Ran}'}$  is a datum of  $I \subset \text{Maps}(S, X')$ , a  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$ , and a trivialization of  $\mathcal{P}_G|_{S \times X - \text{Graph}_I}$ .

Let  $g : \text{Gr}_{\text{Ran}'} \rightarrow \text{Ran}'$  denote the natural projection. It is well-known that this map is pseudo-proper (because  $G$  was assumed reductive over  $X'$ ).

Set

$$\mathcal{A}' := \text{Fib}(g_!(\omega_{\text{Gr}_{\text{Ran}'}}) \rightarrow \omega_{\text{Ran}'}) \in \text{Shv}(\text{Ran}').$$

For example, for a point  $\{x_1, \dots, x_n\}$  of  $\text{Ran}'$ , the  $!$ -fiber of  $\mathcal{A}'$  at this point is

$$\text{Fib} \left( \bigotimes_{i=1, \dots, n} C_*(\text{Gr}_{x_i}) \rightarrow \Lambda \right).$$

Note that we have a tautological isomorphism

$$(14.5) \quad C_c^*(\text{Ran}', \mathcal{A}') \simeq \text{Fib} \left( C_c^*(\text{Gr}_{\text{Ran}'}, \omega_{\text{Gr}_{\text{Ran}'}}) \rightarrow C_c^*(\text{Ran}', \omega_{\text{Ran}'}) \right) \simeq \text{Fib} \left( C_c^*(\text{Gr}_{\text{Ran}'}, \omega_{\text{Gr}_{\text{Ran}'}}) \rightarrow \Lambda \right) = C_*^{\text{red}}(\text{Gr}_{\text{Ran}'}).$$

14.2.3. We have a canonically defined map

$$\text{Gr}_{\text{Ran}'} \rightarrow \text{Bun}_G.$$

According to [Main Text, Theorem 3.2.13] and the assumption on  $G$  in Sect. 14.2.1, we have:

**Theorem 14.2.4.** *The map  $\text{Gr}_{\text{Ran}'} \rightarrow \text{Bun}_G$  is universally homologically contractible, and in particular induces an isomorphism  $C_*^{\text{red}}(\text{Gr}_{\text{Ran}'}) \rightarrow C_*^{\text{red}}(\text{Bun}_G)$ .*

Combining with (14.5), from Theorem 14.2.4 we obtain an isomorphism

$$(14.6) \quad C_c^*(\text{Ran}', \mathcal{A}') \simeq C_*^{\text{red}}(\text{Bun}_G).$$

**14.3. Reduction to a duality statement on chiral homology.** In this subsection we will explain how the assertion of Theorem 14.1.6 is related to that of Theorem 14.2.4.

<sup>21</sup>In fact, we can arrange that each of these fibers of  $G$  is isomorphic to a vector group.

14.3.1. We claim that there is a canonically defined pairing

$$(14.7) \quad C_c^*(\text{Ran}', \mathcal{A}') \otimes C_c^*(\text{Ran}, \mathcal{B}) \rightarrow \Lambda.$$

In fact, we claim that there is a canonically defined map in  $\text{Shv}^1(\text{Ran}' \times \text{Ran})$

$$(14.8) \quad \mathcal{A}' \boxtimes \mathcal{B} \rightarrow \omega_{\text{Ran}' \times \text{Ran}},$$

from which the map (14.7) is obtained by composing with the trace map

$$C_c^*(\text{Ran}' \times \text{Ran}, \omega_{\text{Ran}' \times \text{Ran}}) \rightarrow \Lambda.$$

14.3.2. The map (14.8) is constructed as follows.

For  $S' \in \text{Sch}$  and  $I' \subset \text{Maps}(S', \text{Ran}')$  denote  $\text{Gr}_{I'}$  denote the prestack  $S' \times_{\text{Ran}'} \text{Gr}_{\text{Ran}'}$ ; let  $g_{I'}$  denote the resulting map  $\text{Gr}_{I'} \rightarrow S'$ .

Let  $S \in \text{Sch}$  and  $I \subset \text{Maps}(S, \text{Ran})$ . We need to construct a map

$$\text{Fib}\left((g_{I'})_! \circ (g_{I'})^!(\omega_{S'}) \rightarrow \omega_{S'}\right) \boxtimes \mathbb{D}_S\left(\text{Fib}\left((f_I)_! \circ (f_I)^!(\Lambda_S) \rightarrow \Lambda_S\right)\right) \rightarrow \omega_{S' \times S},$$

which is equivalent to constructing a map

$$\text{Fib}\left((g_{I'})_! \circ (g_{I'})^!(\omega_{S'}) \rightarrow \omega_{S'}\right) \boxtimes \Lambda_S \rightarrow \omega_{S'} \boxtimes \text{Fib}\left((f_I)_! \circ (f_I)^!(\Lambda_S) \rightarrow \Lambda_S\right),$$

or equivalently

$$(14.9) \quad (g_{I'} \times \text{id})_! \circ (g_{I'} \times \text{id})^!(\omega_{S'} \boxtimes \Lambda_S) \rightarrow (\text{id} \times f_I)_! \circ (\text{id} \times f_I)^!(\omega_{S'} \boxtimes \Lambda_S).$$

Note now that we have a canonically defined map of prestacks over  $S' \times S$

$$\text{Gr}_{I'} \times S \rightarrow \text{Bun}_G \times S' \times S \rightarrow S' \times BG_I,$$

which induces the sought-for map in (14.9).

14.3.3. By construction, the diagram

$$\begin{array}{ccc} C_c^*(\text{Ran}', \mathcal{A}') \otimes C_c^*(\text{Ran}, \mathcal{B}) & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ C_*^{\text{red}}(\text{Bun}_G) \otimes C_{\text{red}}^*(\text{Bun}_G) & \longrightarrow & \Lambda \end{array}$$

commutes.

Hence, Theorem 14.1.6 is equivalent to the following assertion:

**Theorem 14.3.4.** *The map (14.7) defines an isomorphism*

$$C_c^*(\text{Ran}, \mathcal{B}) \rightarrow (C_c^*(\text{Ran}', \mathcal{A}'))^\vee.$$

We refer to Theorem 14.3.4 as the “global duality” statement.

## 15. THE LOCAL DUALITY STATEMENT

In order to understand the contents of this section, one needs to know the contents of Part I (the unital augmented version of the Ran space), Part II (only Sects. 7.1, 7.2, 8.2 and 8.3) and Part III (only Sect. 10).

In this section we will reduce the assertion of Theorem 14.3.4 to a *local duality* statement, namely, Theorem 15.3.3. This theorem says that a certain map in  $\text{Shv}^1(\text{Ran}')$  is an isomorphism.

More precisely, Theorem 15.3.3 says that certain two objects of  $\text{Shv}^1(\text{Ran}')$  are related by Verdier duality. These objects are obtained from  $\mathcal{B}$  and  $\mathcal{A}'$  by the procedure of taking the units out, indicated in the preamble to Sect. 10.

15.1. **The unital augmented and reduced versions of  $\mathcal{B}$ .** First, we claim that the object  $\mathcal{B}$  is obtained as

$$\text{OblvUnit} \circ \text{OblvAug}(\mathcal{B}_{\text{untl, aug}})$$

for a canonically defined object  $\mathcal{B}_{\text{untl, aug}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$ , which is in turn of the form

$$\text{AddUnit}_{\text{aug}}(\mathcal{B}_{\text{red}})$$

for a canonically defined object  $\mathcal{B}_{\text{red}} \in \text{Shv}^!(\text{Ran})$ .

To understand the meaning of the above formulas, the reader should be familiar with the contents of Sect. 5.

15.1.1. To specify  $\mathcal{B}_{\text{untl, aug}}$  we need to construct a compatible family of assignments

$$S \in \text{Sch}, K \subseteq I \subset \text{Maps}(S, X) \rightsquigarrow (\mathcal{B}_{\text{untl, aug}})_{S, K \subseteq I} \in \text{Shv}(S).$$

We construct  $(\mathcal{B}_{\text{untl, aug}})_{S, K \subseteq I}$  in the same way as  $(\mathcal{B}_{\text{untl, aug}})_{S, I}$ , with the difference that we replace the stack  $BG_I$  by  $BG_{K \subseteq I}$ , that classifies  $G$ -bundles on  $D_I$ , equipped with a trivialization on  $D_K$ .

For example, for  $S = \text{pt}$  and  $I$  given by an  $I$ -tuple of *distinct*  $k$ -points of  $X$ , the  $!$ -fiber of  $\mathcal{B}$  at  $K \subseteq I$  is

$$\text{Fib} \left( \bigotimes_{i \in I-K} C^*(BG_{x_i}) \rightarrow \Lambda \right).$$

By construction,  $(\mathcal{B}_{\text{untl, aug}})_{S, \emptyset \subseteq I} = \mathcal{B}_{S, I}$ , so

$$\mathcal{B} \simeq \text{OblvUnit} \circ \text{OblvAug}(\mathcal{B}_{\text{untl, aug}}),$$

as desired.

15.1.2. We claim that  $\mathcal{B}_{\text{untl, aug}}$  satisfies conditions  $(*)$  and  $(**)$  in Theorem 5.4.3. Indeed, this follows from the corresponding property of the assignment

$$(S, I) \mapsto (\mathcal{B}_{\text{red}})_{S, K \subseteq I}.$$

Namely, for  $L \subset \text{Maps}(S, X)$  such that  $I$  and  $L$  have disjoint images, the restriction map

$$BG_{K \cup L \subseteq I \cup L} \rightarrow BG_{K \subseteq I}$$

is an isomorphism.

15.1.3. Applying Theorem 5.4.3, we obtain that there exists a canonically defined object

$$\mathcal{B}_{\text{red}} \in \text{Shv}^!(\text{Ran}),$$

so that

$$\mathcal{B}_{\text{untl, aug}} = \text{AddUnit}_{\text{aug}}(\mathcal{B}_{\text{red}}).$$

For example, the  $!$ -fiber of  $\mathcal{B}_{\text{red}}$  at a point  $\{x_1, \dots, x_n\} \in \text{Ran}$  is

$$\bigotimes_{i=1, \dots, n} C_{\text{red}}^*(BG_{x_i}).$$

So the effect of replacing  $\mathcal{B}$  by  $\mathcal{B}_{\text{red}}$  consists at the level of  $!$ -fibers of replacing the *augmentation ideal in the tensor product* of  $C^*(BG_{x_i})$  by the *tensor product of the augmentation ideals* in each  $C^*(BG_{x_i})$ , i.e.,

$$\text{Fib} \left( \bigotimes_{i=1, \dots, n} C^*(BG_{x_i}) \rightarrow \Lambda \right) \rightsquigarrow \bigotimes_{i=1, \dots, n} \text{Fib}(C^*(BG_{x_i}) \rightarrow \Lambda).$$

Denote

$$\mathcal{B}'_{\text{red}} := \mathcal{B}_{\text{red}}|_{\text{Ran}'} \text{ and } \mathcal{B}' := \mathcal{B}|_{\text{Ran}'}$$

15.2. **The unital augmented and reduced versions of  $\mathcal{A}$ .** Next, we claim that the object  $\mathcal{A}'$  is obtained as

$$\text{OblvUnit} \circ \text{OblvAug}(\mathcal{A}'_{\text{untl, aug}})$$

for a canonically defined object  $\mathcal{A}'_{\text{untl, aug}} \in \text{Shv}^!(\text{Ran}'_{\text{untl, aug}})$ , which is in turn of the form

$$\text{AddUnit}_{\text{aug}}(\mathcal{A}'_{\text{red}})$$

for a canonically defined object  $\mathcal{A}'_{\text{red}} \in \text{Shv}^!(\text{Ran}')$ .

15.2.1. The object  $\mathcal{A}'_{\text{untl, aug}}$  is constructed in a manner parallel to  $\mathcal{A}'$ , where instead of  $\text{Gr}_{\text{Ran}'} \rightarrow \text{Ran}'$ , we use the lax prestack

$$(15.1) \quad g_{\text{untl, aug}} : \text{Gr}_{\text{Ran}'_{\text{untl, aug}}} \rightarrow \text{Ran}'_{\text{untl, aug}},$$

constructed as follows.

For  $S \in \text{Sch}$ , an  $S$ -point of  $\text{Gr}_{\text{Ran}'_{\text{untl, aug}}}$  is the category whose objects are:

- $K \subseteq I \subset \text{Maps}(S, X')$ ;
- a  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$ ;
- a trivialization  $\gamma$  of  $\mathcal{P}_G|_{S \times X - \text{Graph}_I}$ .

Given two such objects

$$(K^1 \subseteq I^1, \mathcal{P}_G^1, \gamma^1) \text{ and } (K^2 \subseteq I^2, \mathcal{P}_G^2, \gamma^2),$$

a morphism between them is an inclusion  $K_1 \subseteq K_2$  and  $I_1 \subseteq I_2$ , and an isomorphism

$$\mathcal{P}_G^1|_{S \times X - \text{Graph}_{K_2}} \simeq \mathcal{P}_G^2|_{S \times X - \text{Graph}_{K_2}},$$

which is compatible with the trivializations of  $\mathcal{P}_G^1|_{S \times X - \text{Graph}_{I_2}}$  and  $\mathcal{P}_G^2|_{S \times X - \text{Graph}_{I_2}}$ , given by  $\gamma^1|_{S \times X - \text{Graph}_{I_2}}$  and  $\gamma^2$ , respectively.

It is easy to see that the map (15.1) is pseudo-proper, i.e., satisfies the assumption of Corollary 1.5.4, so that the functor  $(g_{\text{untl, aug}})!$  is well-behaved.

For example, for  $S = \text{pt}$  and  $I$  given by an  $I$ -tuple of *distinct*  $k$ -points of  $X'$ , the  $!$ -fiber of  $\mathcal{A}'_{\text{untl, aug}}$  at  $K \subseteq I$  is

$$\text{Fib} \left( \bigotimes_{i \in I - K} C_*(\text{Gr}_{x_i}) \rightarrow \Lambda \right).$$

15.2.2. By construction,

$$\text{Ran}'_{\text{untl, aug}} \times \text{Gr}_{\text{Ran}'_{\text{untl, aug}}} \simeq \text{Gr}_{\text{Ran}'},$$

so

$$\mathcal{A}' \simeq \text{OblvUnit} \circ \text{OblvAug}(\mathcal{A}'_{\text{untl, aug}}),$$

as desired.

*Notation:* For  $S \in \text{Sch}$  and an  $S$ -point  $K \subseteq I$  of  $\text{Ran}_{\text{untl, aug}}$ , we let  $\text{Gr}_{K \subseteq I}$  denote the prestack

$$S \times_{\text{Ran}'_{\text{untl, aug}}} \text{Gr}_{\text{Ran}'_{\text{untl, aug}}}.$$

*Remark 15.2.3.* Note that in defining  $\text{Gr}_{\text{Ran}'_{\text{untl, aug}}}$ , we can avoid referring to the complete curve  $X$ . Indeed, instead of  $\mathcal{P}_G$  being a  $G$ -bundle on  $X$  and  $\gamma$  its trivialization over  $S \times X - \text{Graph}_I$ , we can let  $\mathcal{P}_G$  be a  $G$ -bundle on  $X'$  and  $\gamma'$  its trivialization over  $S \times X' - I$ . And when defining morphisms, we take isomorphisms

$$\mathcal{P}_G^1|_{S \times X' - \text{Graph}_{K_2}} \simeq \mathcal{P}_G^2|_{S \times X' - \text{Graph}_{K_2}}.$$

15.2.4. We claim that  $\mathcal{A}'_{\text{untl, aug}}$  satisfies conditions  $(*)$  and  $(**)$  in Theorem 5.4.3. Indeed, this follows from the corresponding property of the lax prestack

$$\text{Gr}_{\text{Ran}'_{\text{untl, aug}}} \rightarrow \text{Ran}'_{\text{untl, aug}}.$$

Namely, for  $I, K, L \subset \text{Maps}(S, X)$  such that  $I$  and  $L$  have disjoint images, the restriction map

$$\text{Gr}_{K \subseteq I} \rightarrow \text{Gr}_{K \cup L \subseteq I \cup L}$$

is an isomorphism.

Hence, from Theorem 5.4.3 we obtain that there exists a canonically defined object

$$\mathcal{A}'_{\text{red}} \in \text{Shv}^!(\text{Ran}'),$$

so that

$$\mathcal{A}'_{\text{untl, aug}} = \text{AddUnit}_{\text{aug}}(\mathcal{A}'_{\text{red}}).$$

For example, the  $!$ -fiber of  $\mathcal{A}'_{\text{red}}$  at a point  $\{x_1, \dots, x_n\} \in \text{Ran}'$  is

$$\bigotimes_{i=1, \dots, n} C_*^{\text{red}}(\text{Gr}_{x_i}).$$

So the effect of replacing  $\mathcal{A}$  by  $\mathcal{A}'_{\text{red}}$  consists at the level of  $!$ -fibers of replacing

$$\text{Fib} \left( \bigotimes_{i=1, \dots, n} C_*(\text{Gr}_{x_i}) \rightarrow \Lambda \right) \rightsquigarrow \bigotimes_{i=1, \dots, n} \text{Fib}(C_*(\text{Gr}_{x_i}) \rightarrow \Lambda).$$

15.2.5. Recall now (see Sect. 8.3) that we denote by  $\mathbf{j}_{\text{Ran}}$  the open embedding  $\text{Ran}' \rightarrow \text{Ran}$ . Recall also that the restriction functor  $\mathbf{j}^!$  admits both a left and a right adjoints, denoted  $(\mathbf{j}_{\text{Ran}})!$  and  $(\mathbf{j}_{\text{Ran}})_*$ , respectively.

Set

$$\mathcal{A}'_{\text{red}} := (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}}).$$

Denote also

$$\mathcal{A}_{\text{untl, aug}} := \text{AddUnit}_{\text{aug}}(\mathcal{A}'_{\text{red}}) \text{ and } \mathcal{A} := \text{OblvUnit} \circ \text{OblvAug}(\mathcal{A}_{\text{untl, aug}}).$$

15.3. **The unital augmented and reduced versions of the pairing.** Recall the notion of pairing between two sheaves on  $\text{Ran}_{\text{untl, aug}}$ , see Sect. 10.1.

We will show that the pairing (14.8) extends to a canonically defined map

$$(15.2) \quad \mathcal{A}'_{\text{untl, aug}} \boxtimes \mathcal{B}_{\text{untl, aug}} \rightarrow \omega_{(\text{Ran}'_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}},$$

where

$$(\text{Ran}'_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}} := (\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}} \cap (\text{Ran}'_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}}).$$

We will use the map (15.2) and Theorem 10.1.3 to define a pairing between  $\mathcal{A}'_{\text{red}}$  and  $\mathcal{B}'_{\text{red}}$ .

15.3.1. To construct the map (15.2), given  $S_1 \in \text{Sch}$  with  $K_1 \subseteq I_1 \subset \text{Maps}(S_1, X')$  and  $S_2 \in \text{Sch}$  with  $K_2 \subseteq I_2 \subset \text{Maps}(S_2, X)$  so that the resulting object of  $(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})(S_1 \times S_2)$  belongs to

$$(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}(S_1 \times S_2) \subset (\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})(S_1 \times S_2),$$

we need to construct a map

$$(\mathcal{A}'_{\text{untl, aug}})_{S_1, K_1 \subseteq I_1} \boxtimes (\mathcal{B}_{\text{untl, aug}})_{S_2, K_2 \subseteq I_2} \rightarrow \omega_{S_1 \times S_2}.$$

This map is constructed as in Sect. 14.3.2 from the map of lax prestacks

$$\text{Gr}_{K_1 \subseteq I_1} \times S_2 \rightarrow S_1 \times \text{BG}_{K_2 \subseteq I_2}.$$

15.3.2. From (15.2) we in particular obtain a pairing

$$(15.3) \quad \mathcal{A}'_{\text{untl, aug}} \boxtimes \mathcal{B}'_{\text{untl, aug}} \rightarrow \omega_{(\text{Ran}'_{\text{untl, aug}} \times \text{Ran}'_{\text{untl, aug}})_{\text{sub, disj}}}$$

Applying Corollary 10.1.4, from (15.3) we obtain a pairing

$$(15.4) \quad \mathcal{A}'_{\text{red}} \boxtimes \mathcal{B}'_{\text{red}} \rightarrow (\text{diag}_{\text{Ran}'})!(\omega_{\text{Ran}'}).$$

Hence, we obtain a map

$$(15.5) \quad \mathcal{B}'_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}'}(\mathcal{A}'_{\text{red}}).$$

We claim:

**Theorem 15.3.3.** *The map in (15.5) is an isomorphism.*

We refer to Theorem 15.3.3 as the “local duality” statement.

*Remark 15.3.4.* The assertion of Theorem 15.3.3, as well as its proof, are valid for any  $G$  that is *reductive* over  $X'$ . I.e., we do not need the “semi-simple simply connected” hypothesis for the validity of Theorem 15.3.3. It is for the deduction

$$\text{Local duality} \Rightarrow \text{Global duality}$$

that such a hypothesis is needed: it will be used in showing that  $\mathcal{A}'_{\text{red}}$  verifies the conditions of Theorem 8.2.4.

**15.4. Local duality implies global duality.** In this subsection we will show how Theorem 15.3.3 implies Theorem 14.3.4. Here we will use the material from Part II in a crucial way, specifically Theorem 8.3.6.

15.4.1. The map (15.4) gives rise to a map

$$(15.6) \quad \mathcal{A}'_{\text{red}} \boxtimes \mathcal{B}_{\text{red}} \rightarrow (\text{id}_{\text{Ran}} \times \mathbf{j}_{\text{Ran}})_* \circ (\text{diag}_{\text{Ran}'})!(\omega_{\text{Ran}'}) \simeq (\text{Graph}_{\mathbf{j}_{\text{Ran}}})!(\omega_{\text{Ran}'}),$$

and by applying  $C_c^*(\text{Ran}' \times \text{Ran}, -)$  to a map

$$(15.7) \quad C_c^*(\text{Ran}', \mathcal{A}'_{\text{red}}) \otimes C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) \rightarrow \Lambda.$$

We claim:

**Lemma 15.4.2.** *The diagram*

$$\begin{array}{ccc} C_c^*(\text{Ran}', \mathcal{A}') \otimes C_c^*(\text{Ran}, \mathcal{B}) & \longrightarrow & \Lambda \\ (4.3) \otimes (4.3) \uparrow & & \uparrow \\ C_c^*(\text{Ran}', \mathcal{A}'_{\text{red}}) \otimes C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) & \longrightarrow & \Lambda \end{array}$$

*commutes.*

*Proof.* We will show that the diagram

$$(15.8) \quad \begin{array}{ccc} \mathcal{A}' \boxtimes \mathcal{B} & \longrightarrow & \omega_{\text{Ran}' \times \text{Ran}} \\ (4.3) \otimes (4.3) \uparrow & & \uparrow \\ \mathcal{A}'_{\text{red}} \boxtimes \mathcal{B}_{\text{red}} & \xrightarrow{(15.6)} & (\text{Graph}_{\mathbf{j}_{\text{Ran}}})!(\omega_{\text{Ran}'}) \end{array}$$

commutes. This would imply the assertion of the lemma.

Consider the map

$$\mathbf{j}_{\text{Ran}_{\text{untl, aug}}} : \text{Ran}'_{\text{untl, aug}} \rightarrow \text{Ran}_{\text{untl, aug}},$$

and consider the object

$$(\mathcal{A}_{\text{untl, aug}})_{\text{bad}} := (\mathbf{j}_{\text{Ran}_{\text{untl, aug}}})!(\mathcal{A}'_{\text{untl, aug}}) \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}}).$$

We have

$$(\mathbf{j}_{\text{Ran}_{\text{untl, aug}}})^!((\mathcal{A}_{\text{untl, aug}})_{\text{bad}}) \simeq \mathcal{A}'_{\text{untl, aug}}.$$

(We remark that the object  $(\mathcal{A}_{\text{untl, aug}})_{\text{bad}}$  does not have a clear geometric meaning; for example it does not satisfy conditions  $(*)$  and  $(**)$  of Theorem 5.4.3; in particular, it is *not* isomorphic to  $\mathcal{A}_{\text{untl, aug}}$ .)

By adjunction, the map (15.2) gives rise to a pairing

$$(\mathcal{A}_{\text{untl, aug}})_{\text{bad}} \boxtimes \mathcal{B}_{\text{untl, aug}} \rightarrow \omega_{(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{sub, disj}}}.$$

Applying Theorem 10.1.3(ii and iv), we obtain a pairing

$$\text{TakeOut}((\mathcal{A}_{\text{untl, aug}})_{\text{bad}}) \boxtimes \mathcal{B}_{\text{red}} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}}),$$

so that the diagram

$$\begin{array}{ccc} \text{OblvUnit} \circ \text{OblvAug}((\mathcal{A}_{\text{untl, aug}})_{\text{bad}}) \boxtimes \mathcal{B}_{\text{untl, aug}} & \longrightarrow & \omega_{\text{Ran}} \times \text{Ran} \\ \uparrow (4.3) \otimes (4.3) & & \uparrow \\ \text{TakeOut}((\mathcal{A}_{\text{untl, aug}})_{\text{bad}}) \boxtimes \mathcal{B}_{\text{red}} & \longrightarrow & (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}}) \end{array}$$

commutes.

Restricting to  $\text{Ran}' \times \text{Ran}$ , we obtain a commutative diagram that identifies with (15.8).  $\square$

15.4.3. Note that the left vertical arrow in the diagram in Lemma 15.4.2 is an isomorphism by Corollary 4.5.2. Hence, in order to prove Theorem 14.3.4, it suffices to show that the map

$$(15.9) \quad C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) \rightarrow (C_c^*(\text{Ran}', \mathcal{A}'_{\text{red}}))^\vee,$$

defined by the pairing (15.7), is an isomorphism.

15.4.4. Recall that by Lemma 8.3.4, we have a canonical isomorphism

$$(\mathbf{j}_{\text{Ran}})_* \circ \mathbb{D}_{\text{Ran}'}(\mathcal{A}'_{\text{red}}) \simeq \mathbb{D}_{\text{Ran}} \circ (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}}).$$

We have a commutative diagram

$$\begin{array}{ccc} C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) & \xrightarrow{(15.9)} & (C_c^*(\text{Ran}', \mathcal{A}'_{\text{red}}))^\vee \\ \downarrow & & \uparrow \sim \\ C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_*(\mathcal{B}'_{\text{red}})) & & (C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}})))^\vee \\ (15.5) \downarrow & & \uparrow (7.4) \\ C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_* \circ \mathbb{D}_{\text{Ran}'}(\mathcal{A}'_{\text{red}})) & \xrightarrow{\sim} & C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}} \circ (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}})). \end{array}$$

Hence, in order to prove that (15.9) is an isomorphism (and thus finish the proof of Theorem 14.1.6), we need to show that the maps

$$C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_*(\mathcal{B}'_{\text{red}})) \rightarrow C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_* \circ \mathbb{D}_{\text{Ran}'}(\mathcal{A}'_{\text{red}}))$$

and

$$C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) \rightarrow C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_*(\mathcal{B}'_{\text{red}}))$$

and

$$C_c^*(\text{Ran}, \mathbb{D}_{\text{Ran}} \circ (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}})) \rightarrow (C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})!(\mathcal{A}'_{\text{red}})))^\vee,$$

appearing in the above commutative diagram, are isomorphisms.

Now, Theorem 15.3.3 implies that the first of these three maps is an isomorphism. The fact that the second map is an isomorphism follows from the next assertion (proved right below):

**Proposition 15.4.5.** *For  $G$  satisfying the assumption of Sect. 14.2.1, the map*

$$\mathcal{B}_{\text{red}} \rightarrow (\mathbf{j}_{\text{Ran}})_*(\mathcal{B}'_{\text{red}})$$

*is an isomorphism.*



Finally, the fact that the third of the above maps is an isomorphism follows from Theorem 8.3.6 using the next assertion (also proved right below):

**Proposition 15.4.6.** *The object  $\mathcal{A}'_{\text{red}} \in \text{Shv}(\text{Ran}')$  satisfies the cohomological estimate of Theorem 8.2.4<sup>22</sup> over the curve  $X'$ .*

*Proof of Proposition 15.4.5.* As we shall see in Sect. 16.2.1, the object  $\mathcal{B}_{\text{red}} \in \text{Shv}^!(\text{Ran})$  has a structure of commutative factorization algebra. We have the following general assertion:

**Lemma 15.4.7.** *Let  $\mathcal{F} \in \text{Shv}^!(\text{Ran})$  have a structure of commutative factorization (co)algebra. Then the map  $\mathcal{F} \rightarrow (\mathbf{j}_{\text{Ran}})_*(\mathcal{F}|_{\text{Ran}'})$  is an isomorphism if and only if the map  $\mathcal{F}_X \rightarrow \mathbf{j}_*(\mathcal{F}_X|_{X'})$  is.*

The proof of the lemma is immediate from the description of the functor  $(\mathbf{j}_{\text{Ran}})_*$  given by (8.4).

Hence, in order to prove the proposition, it suffices to show that for any  $x \in X - X'$ , we have  $(\mathcal{B}_{\text{red}})_{\{x\}} = 0$ . Note that for a singleton set we have

$$(\mathcal{B}_{\text{red}})_{\{x\}} \simeq (\text{OblvUnit} \circ \text{OblvAug}(\mathcal{B}_{\text{untl, aug}}))_{\{x\}} = \mathcal{B}_{\{x\}},$$

and the latter is by definition  $C_{\text{red}}^*(BG_x)$ .

Now, the assumption that  $G_x$  is contractible implies that  $BG_x$  is universally homologically contractible. Hence,  $C_{\text{red}}^*(BG_x) = 0$  as required.  $\square$

*Proof of Proposition 15.4.6.* We will show that the restriction of  $\mathcal{A}'_{\text{red}}$  to  $\overset{\circ}{X}^n$  lives in (perverse) cohomological degrees  $\leq -3n$ .

As we shall see in Sect. 16.1.1, the object  $\mathcal{A}'_{\text{red}} \in \text{Shv}^!(\text{Ran}')$  has a structure of cocommutative factorization algebra.<sup>23</sup> Hence, in order to prove the required cohomological estimate, it suffices to do so for  $n = 1$ . I.e., we are considering the affine Grassmannian

$$\text{Gr}_{X'} \xrightarrow{g_{X'}} X',$$

and we need to show that

$$\text{Fib}((g_{X'})_!(\omega_{\text{Gr}_{X'}}) \rightarrow \omega_{X'})$$

lives in (perverse) cohomological degrees  $\leq -3$ .

By passing to the étale cover of  $X'$ , we can assume that  $G|_{X'}$  is a constant group-scheme. Hence, it is enough to show that for some/any point  $x \in X$ , we have that  $C_{*}^{\text{red}}(\text{Gr}_x)$  lives in cohomological degrees  $\leq -2$ .

However, the latter is a well-known property of the affine Grassmannian of *semi-simple simply connected* groups. (Note that if  $G$  is reductive but not semi-simple simply connected, then  $\text{Gr}_x$  is *disconnected* and the estimate of Proposition 15.4.6 does not hold.)  $\square$

## 16. REDUCTION TO A POINTWISE DUALITY STATEMENT

Our goal of the rest of Part V is to prove the local duality statement, Theorem 15.3.3. The material in this section will use Part IV of the paper.

In this section we shall take  $X$  to be a smooth (but not necessarily complete) curve. We will take  $G$  to be a smooth group-scheme over  $X$ , whose fibers are semi-simple and simply connected. We will reduce Theorem 15.3.3 to Theorem 16.4.7 that says that the map

$$\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$$

<sup>22</sup>We repeat that this estimate says that every integer  $k \geq 0$  there exists an integer  $n_k \geq 0$ , such that the object  $\text{ins}_{\mathcal{J}}^1(\mathcal{A}'_{\text{red}})|_{\overset{\circ}{X}^n, \mathcal{J}}$  is concentrated in *perverse* cohomological degrees  $\leq -k - |\mathcal{J}|$  whenever  $|\mathcal{J}| > n_k$ .

<sup>23</sup>We say “factorization algebra” because we have not introduced the general notion of *factorization sheaf*. For our purposes here it is the “factorization” and not the “cocommutative” part that is important.

induces an isomorphism

$$(\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

for *some* curve  $X$  and *some* point  $x \in X$ .

**16.1. The structure on  $\mathcal{A}$  of cocommutative factorization coalgebra.** We wish to show that the map  $\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$ , given by (15.5) is an isomorphism. As was explained in the preamble to Sect. 12.1, a convenient tool for this would be to first endow  $\mathcal{A}_{\text{red}}$  (resp.,  $\mathcal{B}_{\text{red}}$ ) with a structure of cocommutative (resp., commutative) factorization coalgebra (resp., algebra).

In this subsection we will define the relevant structure on  $\mathcal{A}_{\text{red}}$ , by deducing it from the corresponding structure on  $\mathcal{A}_{\text{untl, aug}}$ .

16.1.1. Let us note that the diagonal map

$$\text{Gr}_{\text{Ran}_{\text{untl, aug}}} \rightarrow \text{Gr}_{\text{Ran}_{\text{untl, aug}}} \times_{\text{Ran}_{\text{untl, aug}}} \text{Gr}_{\text{Ran}_{\text{untl, aug}}}$$

defines on  $\mathcal{A}_{\text{untl, aug}}$  a structure of cocommutative coalgebra on  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  (with respect to the pointwise symmetric monoidal structure).

By Corollary 13.1.3 we obtain that  $\mathcal{A}_{\text{red}}$  acquires a structure of cocommutative coalgebra in  $\text{Shv}^!(\text{Ran})$  (with respect to the convolution symmetric monoidal structure).

16.1.2. We have the following crucial observation:

**Proposition 16.1.3.** *The cocommutative coalgebra  $\mathcal{A}_{\text{untl, aug}} \in \text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  is a cocommutative factorization coalgebra (in the sense of Sect. 13.2.4).*

*Proof.* The assertion of the proposition follows from the corresponding property of the lax prestack  $\text{Gr}_{\text{Ran}_{\text{untl, aug}}}$ . Namely, we claim that for any  $S \in \text{Sch}$  and an  $S$ -point  $(K_1 \subseteq I_1), (K_2 \subseteq I_2)$  of

$$(\text{Ran}_{\text{untl, aug}} \times \text{Ran}_{\text{untl, aug}})_{\text{compl, disj}},$$

the map

$$\text{Gr}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \xrightarrow{\text{diag}} \text{Gr}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \times \text{Gr}_{K_1 \cup K_2 \subseteq I_1 \cup I_2} \rightarrow \text{Gr}_{K_1 \cup I_2 \subseteq I_1 \cup I_2} \times \text{Gr}_{I_1 \cup K_2 \subseteq I_1 \cup I_2}$$

is an isomorphism. □

By Proposition 13.2.6(b), from Proposition 16.1.3 we obtain:

**Corollary 16.1.4.** *The cocommutative coalgebra  $\mathcal{A}_{\text{red}} \in \text{Shv}^!(\text{Ran})$  admits a canonical structure of cocommutative factorization coalgebra (in the sense of Sect. 12.1.13).*

16.1.5. Consider the object  $\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}) \in \text{Shv}^!(\text{Ran})$ . By Sect. 12.2.1, we obtain that  $\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$  acquires a structure of commutative algebra in  $\text{Shv}^!(\text{Ran})$  (with respect to the convolution symmetric monoidal structure).

Applying Proposition 12.2.3, and using Proposition 15.4.6, we obtain:

**Corollary 16.1.6.** *The commutative algebra  $\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}})$  admits a canonical structure of commutative factorization algebra (in the sense of Sect. 12.1.5).*

**16.2. The structure on  $\mathcal{B}$  of commutative factorization algebra.** In this subsection we will define a structure of commutative factorization algebra on  $\mathcal{B}_{\text{red}}$ , by deducing it from the corresponding structure on  $\mathcal{B}_{\text{untl, aug}}$ .

16.2.1. Next, we note that the diagonal maps

$$BG_{K \subset I} \rightarrow BG_{K \subset I} \times BG_{K \subset I}$$

define on  $\mathcal{B}_{\text{untl, aug}}$  a structure of commutative algebra on  $\text{Shv}^!(\text{Ran}_{\text{untl, aug}})$  (with respect to the pointwise symmetric monoidal structure).

Hence, by Corollary 13.1.3 we obtain that  $\mathcal{B}_{\text{red}}$  acquires a structure of commutative algebra in  $\text{Shv}^!(\text{Ran})$  (with respect to the convolution product).

16.2.2. We note:

**Proposition 16.2.3.** *The commutative algebra  $\mathcal{B}_{\text{untl},\text{aug}}$  is a commutative factorization algebra (in the sense of Sect. 13.2.2).*

*Proof.* This follows from the corresponding property of the assignment

$$(S, K \subseteq I) \rightsquigarrow BG_{K \subset I}.$$

Namely, for  $S \in \text{Sch}$  and an  $S$ -point  $(I_1, I_2)$  of

$$(\text{Ran}_{\text{untl}} \times \text{Ran}_{\text{untl}})_{\text{disj}},$$

the map

$$BG_{x_{I_1 \cup I_2}} \xrightarrow{\text{diag}} BG_{x_{I_1 \cup I_2}} \times BG_{x_{I_1 \cup I_2}} \rightarrow BG_{x_{I_1}} \times BG_{x_{I_2}}$$

is an isomorphism, and the fact that  $BG_{x_I}$  are quasi-compact Artin stacks over  $S$ , so that the Künneth formula holds (see [Main Text, Proposition A.5.19]).  $\square$

By Proposition 13.2.6(a), from Proposition 16.2.3, we obtain:

**Corollary 16.2.4.** *The commutative algebra  $\mathcal{B}_{\text{red}} \in \text{Shv}^!(\text{Ran})$  has a canonical structure of commutative factorization algebra (in the sense of Sect. 12.1.5).*

**16.3. Compatibility of the pairing with the algebra structure.** We have endowed both  $\mathbb{D}_{\text{Ran}}(\mathcal{A})$  and  $\mathcal{B}$  with a structure of commutative algebra in  $\text{Shv}^!(\text{Ran})$ . In this subsection we will show that the map  $\mathcal{B} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A})$  has a natural structure of homomorphism of commutative algebras.

16.3.1. Observe that the pairing

$$\mathcal{B}_{\text{untl},\text{aug}} \boxtimes \mathcal{A}_{\text{untl},\text{aug}} \rightarrow \omega_{(\text{Ran}_{\text{untl},\text{aug}} \times \text{Ran}_{\text{untl},\text{aug}})_{\text{sub},\text{disj}}}$$

of (15.3) has a structure of compatibility with the algebra and coalgebra structure on  $\mathcal{B}_{\text{untl},\text{aug}}$  and  $\mathcal{A}_{\text{untl},\text{aug}}$  (see Sect. 13.3.1 for what this means).

Hence, by Lemma 13.3.5, we obtain that the induced pairing

$$\mathcal{B}_{\text{red}} \boxtimes \mathcal{A}_{\text{red}} \rightarrow (\text{diag}_{\text{Ran}})!(\omega_{\text{Ran}})$$

of (15.4) has a structure of compatibility with the algebra and coalgebra structure on  $\mathcal{B}_{\text{red}}$  and  $\mathcal{A}_{\text{red}}$  (see Sect. 13.3.2) for what this means).

Therefore, by Lemma 13.3.3, the map

$$\mathcal{B}_{\text{red}} \rightarrow \mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}),$$

appearing in Theorem 15.3.3 has a structure of homomorphism of commutative algebras.

16.3.2. Combining with Corollaries 16.1.6 and 16.2.4, and Lemma 12.1.6 we obtain that in order to prove Theorem 15.3.3, it suffices to show that the map

$$(16.1) \quad (\mathcal{B}_{\text{red}})_X \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_X$$

is an isomorphism.

**16.4. Reduction to the constant group-scheme case.** In this subsection we will exploit some properties of the assertion of Theorem 15.3.3 that embody its locality property.

We will show that for the validity of the isomorphism (16.1) is insensitive to changes of the curve or forms of  $G$ .

16.4.1. Let  $f : X_1 \rightarrow X_2$  be an étale map between (not necessarily complete) curves. Let  $f_{\text{Ran}}$  denote the resulting map  $\text{Ran}_1 \rightarrow \text{Ran}_2$ . The map  $f_{\text{Ran}}$  is not étale, but we shall now single out a locus of  $\text{Ran}_1$  over which it is well-behaved.

Let  $(X_1 \times X_1)_{\text{rel.disj}} \subset (X_1 \times X_1)$  be the open subset obtained by removing the closed subset

$$X_1 \times_{X_2} X_1 - \text{diag}_{X_1},$$

(where  $\text{diag}_{X_1}$  is a connected component in  $X_1 \times_{X_2} X_1$  due to the assumption that  $X_1$  be étale over  $X_2$ ).

For a finite set  $\mathcal{J}$ , let  $(X_1)_{\text{rel.disj}}^{\mathcal{J}} \subset (X_1)^{\mathcal{J}}$  be the open subset

$$\bigcap_{i_1 \neq i_2} (X_1)^{\mathcal{J} - \{i_1, i_2\}} \times (X_1 \times X_1)_{\text{rel.disj}}.$$

For example  $k$ -points of  $(X_1)_{\text{rel.disj}}^{\mathcal{J}}$  are maps

$$\mathcal{J} \rightarrow X_1(k)$$

with the following property: if for a pair of indices  $i_1$  and  $i_2$  we have  $f(x_{i_1}) = f(x_{i_2})$  then  $x_{i_1} = x_{i_2}$ .

Set

$$(\text{Ran}_1)_{\text{rel.disj}} := \text{colim}_{\mathcal{J} \in (\text{Fin}^s)^{\text{op}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}}.$$

We claim:

**Proposition 16.4.2.**

- (a) *The forgetful map  $(\text{Ran}_1)_{\text{rel.disj}} \rightarrow \text{Ran}_1$  is an open embedding.*
- (b) *The composed map*

$$(\text{Ran}_1)_{\text{rel.disj}} \rightarrow \text{Ran}_1 \xrightarrow{f_{\text{Ran}}} \text{Ran}_2$$

*is étale, and is surjective if  $f$  is.*

The proof of Proposition 16.4.2 relies on the following observation:

**Lemma 16.4.3.** *For a surjection of finite sets  $\mathcal{J} \twoheadrightarrow \mathcal{K}$ , we have*

$$X_1^{\mathcal{K}} \times_{X_1^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} = (X_1)_{\text{rel.disj}}^{\mathcal{K}},$$

and

$$X_2^{\mathcal{K}} \times_{X_2^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} = (X_1)_{\text{rel.disj}}^{\mathcal{K}},$$

as open subsets of  $X_1^{\mathcal{K}}$

*Proof of Proposition 16.4.2.* To prove point (a) we need to show that for a finite non-empty set  $\mathcal{J}$ , the fiber product

$$X_1^{\mathcal{J}} \times_{\text{Ran}_1} (\text{Ran}_1)_{\text{rel.disj}}$$

is an open subscheme of  $X_1^{\mathcal{J}}$ . In fact, we will show that the above fiber product identifies with  $(X_1)_{\text{rel.disj}}^{\mathcal{J}}$ .

We have:

$$\begin{aligned} X_1^{\mathcal{J}} \times_{\text{Ran}_1} (\text{Ran}_1)_{\text{rel.disj}} &\simeq \text{colim}_{\mathcal{J}} X_1^{\mathcal{J}} \times_{\text{Ran}_1} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \text{colim}_{\mathcal{J}} (X_1^{\mathcal{J}} \times_{\text{Ran}_1} X_1^{\mathcal{J}}) \times_{X_1^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \\ &\simeq \text{colim}_{\mathcal{J}} \text{colim}_{\mathcal{J} \twoheadrightarrow \mathcal{K} \leftarrow \mathcal{J}} X_1^{\mathcal{K}} \times_{X_1^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \text{colim}_{\mathcal{J}} \text{colim}_{\mathcal{J} \twoheadrightarrow \mathcal{K} \leftarrow \mathcal{J}} (X_1)_{\text{rel.disj}}^{\mathcal{K}} \simeq \text{colim}_{\mathcal{K} \leftarrow \mathcal{J}} (X_1)_{\text{rel.disj}}^{\mathcal{K}} \simeq (X_1)_{\text{rel.disj}}^{\mathcal{J}}. \end{aligned}$$

To prove point (b), we need to show that for a finite non-empty set  $\mathcal{J}$ , the fiber product

$$X_2^{\mathcal{J}} \times_{\text{Ran}_2} (\text{Ran}_1)_{\text{rel.disj}}$$

is étale over  $X_2^{\mathcal{J}}$ . We will show that the above fiber product identifies with  $(X_1)_{\text{rel.disj}}^{\mathcal{J}}$ .

We have:

$$\begin{aligned} X_2^{\mathcal{J}} \times_{\text{Ran}_2} (\text{Ran}_1)_{\text{rel.disj}} &\simeq \text{colim}_{\mathcal{J}} X_2^{\mathcal{J}} \times_{\text{Ran}_2} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \text{colim}_{\mathcal{J}} (X_2^{\mathcal{J}} \times_{\text{Ran}_2} X_2^{\mathcal{J}}) \times_{X_2^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \\ &\simeq \text{colim}_{\mathcal{J}} \text{colim}_{\mathcal{J} \rightarrow \mathcal{K} \leftarrow \mathcal{I}} X_2^{\mathcal{K}} \times_{X_2^{\mathcal{J}}} (X_1)_{\text{rel.disj}}^{\mathcal{J}} \simeq \text{colim}_{\mathcal{J}} \text{colim}_{\mathcal{J} \rightarrow \mathcal{K} \leftarrow \mathcal{I}} (X_1)_{\text{rel.disj}}^{\mathcal{K}} \simeq (X_1)_{\text{rel.disj}}^{\mathcal{J}}. \end{aligned}$$

□

16.4.4. Let  $G_2$  be a group-scheme on  $X_2$ , and let  $G_1$  be its pullback to  $X_1$ .

According to Remark 15.2.3, we have well-defined objects  $\mathcal{A}_{\text{red},i} \in \text{Shv}(\text{Ran}_i)$ ,  $i = 1, 2$ .

It follows from the definitions that we have a canonical isomorphism

$$(f_{\text{Ran}})^!(\mathcal{A}_{\text{red},2})|_{(\text{Ran}_1)_{\text{rel.disj}}} \simeq \mathcal{A}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}}.$$

Note now that we also have well-defined objects  $\mathcal{B}_{\text{red},i} \in \text{Shv}(\text{Ran}_i)$ ,  $i = 1, 2$ , and a canonical isomorphism

$$(f_{\text{Ran}})^!(\mathcal{B}_{\text{red},2})|_{(\text{Ran}_1)_{\text{rel.disj}}} \simeq \mathcal{B}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}}.$$

By Lemma 7.5.8 and Proposition 16.4.2, we have canonical isomorphisms

$$\left( (f_{\text{Ran}})^! \circ \mathbb{D}_{\text{Ran}_2}(\mathcal{A}_{\text{red},2}) \right) |_{(\text{Ran}_1)_{\text{rel.disj}}} \simeq \mathbb{D}_{(\text{Ran}_1)_{\text{rel.disj}}} \left( (f_{\text{Ran}})^!(\mathcal{A}_{\text{red},2})|_{(\text{Ran}_1)_{\text{rel.disj}}} \right)$$

and

$$\mathbb{D}_{\text{Ran}_1}(\mathcal{A}_{\text{red},1})|_{(\text{Ran}_1)_{\text{rel.disj}}} \simeq \mathbb{D}_{(\text{Ran}_1)_{\text{rel.disj}}}(\mathcal{A}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}}).$$

Furthermore, by unwinding the constructions, we obtain that the diagram

$$\begin{array}{ccc} \mathcal{B}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}} & \longrightarrow & \mathbb{D}_{\text{Ran}_1}(\mathcal{A}_{\text{red},1})|_{(\text{Ran}_1)_{\text{rel.disj}}} \\ \sim \downarrow & & \downarrow \sim \\ (f_{\text{Ran}})^!(\mathcal{B}_{\text{red},2})|_{(\text{Ran}_1)_{\text{rel.disj}}} & & \mathbb{D}_{(\text{Ran}_1)_{\text{rel.disj}}}(\mathcal{A}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}}) \\ \downarrow & & \downarrow \sim \\ \left( (f_{\text{Ran}})^! \circ \mathbb{D}_{\text{Ran}_2}(\mathcal{A}_{\text{red},2}) \right) |_{(\text{Ran}_1)_{\text{rel.disj}}} & \xrightarrow{\sim} & \mathbb{D}_{(\text{Ran}_1)_{\text{rel.disj}}} \left( (f_{\text{Ran}})^!(\mathcal{A}_{\text{red},2})|_{(\text{Ran}_1)_{\text{rel.disj}}} \right) \end{array}$$

commutes.

16.4.5. Thus, we obtain that if  $f : X_1 \rightarrow X_2$  is étale and surjective, then so is the map

$$(\text{Ran}_1)_{\text{rel.disj}} \rightarrow \text{Ran}_2,$$

and hence the assertion of Theorem 15.3.3 for  $G_1$  implies that for  $G_2$ .

Conversely, if the assertion of Theorem 15.3.3 holds for  $G_2$ , then the map

$$\mathcal{B}_{\text{red},1}|_{(\text{Ran}_1)_{\text{rel.disj}}} \rightarrow \mathbb{D}_{\text{Ran}_1}(\mathcal{A}_{\text{red},1})|_{(\text{Ran}_1)_{\text{rel.disj}}}$$

is an isomorphism. I.e., the assertion of Theorem 15.3.3 holds for  $G_1$  over  $(\text{Ran}_1)_{\text{rel.disj}}$ .

However, since  $X_1 \subset (\text{Ran}_1)_{\text{rel.disj}}$  and using Sect. 16.3.2, we obtain that the assertion of Theorem 15.3.3 for  $G_2$  implies the assertion of Theorem 15.3.3 for  $G_1$ .

16.4.6. For a reductive group scheme  $G$  on  $X$ , let  $X_1 \rightarrow X$  be an étale cover such that  $G_1 := G|_{X_1}$  is a constant group-scheme. Thus, we obtain that it is enough to prove Theorem 15.3.3 in the case of constant group schemes.

Now, if  $G$  is a constant group scheme on  $X$ , let  $X' \rightarrow X$  be an open cover such that  $X'$  admits an étale map to  $\mathbb{A}^1$ . Hence, we obtain that it is enough to prove Theorem 15.3.3 (or, equivalently, that (16.1) is an isomorphism) in the case when  $G$  is a constant group scheme and  $X = \mathbb{A}^1$ .

Note that for  $X = \mathbb{A}^1$  and the constant group-scheme, both sides in (16.1) are *constant*. This is due to the  $\mathbb{A}^1$ -equivariance structure on all the objects involved. Hence, the map (16.1) is an isomorphism in this case if and only if the map

$$(16.2) \quad (\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

is an isomorphism for some/any  $x \in \mathbb{A}^1$ .

By the same logic as in Sect. 16.4.5, the assertion that (16.2) if and only if it is an isomorphism for some/any curve  $X$  and some  $x \in X$ .

Hence, we obtain that Theorem 15.3.3 follows from the next assertion:

**Theorem 16.4.7.** *The map*

$$(\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

*of (16.2) is an isomorphism for some/any  $X$  and some/any  $x \in X$  and the constant group-scheme  $G$ .*

We shall refer to Theorem 16.4.7 as the *pointwise duality* statement.

## 17. FIRST PROOF OF THE POINTWISE DUALITY STATEMENT

The goal of this section is to prove Theorem 16.4.7, and thereby finish the proof of Theorem 14.1.6. A prerequisite for the present section is Sect. 11 and the notion for a map between lax prestacks to be *universally homologically contractible* from Sect. 3.4.

We let  $X$  be a smooth curve and  $G$  a constant reductive group scheme over  $X$ .

We will essentially reproduce the proof of [Main Text, Theorem 7.2.10] in a simplified context of the constant group-scheme.

### 17.1. Local non-abelian Poincaré duality.

17.1.1. Recall the lax prestack  $(\text{Ran}_{\text{untl, aug}})_{x \notin}$ , see Sect. 11.1.2. Denote

$$\text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} := (\text{Ran}_{\text{untl, aug}})_{x \notin} \times_{\text{Ran}_{\text{untl, aug}}} \text{Gr}_{\text{Ran}_{\text{untl, aug}}}.$$

Explicitly, the above lax prestack attaches to a test scheme  $S$  the following category. Its objects are triples:

- $K \subseteq I \subset \text{Maps}(S, X)$ , where the images of the maps  $S \rightarrow X$  corresponding to elements of  $K$  are disjoint from  $x$ ;
- a  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$ ;
- a trivialization  $\gamma$  of  $\mathcal{P}_G|_{S \times X - \text{Graph}_I}$ .

Given two such objects

$$(K^1 \subseteq I^1, \mathcal{P}_G^1, \gamma^1) \text{ and } (K^2 \subseteq I^2, \mathcal{P}_G^2, \gamma^2),$$

a morphism between them is an inclusion  $K_1 \subseteq K_2$  and  $I_1 \subseteq I_2$ , and an isomorphism

$$\mathcal{P}_G^1|_{S \times X - \text{Graph}_{K_2}} \simeq \mathcal{P}_G^2|_{S \times X - \text{Graph}_{K_2}},$$

which is compatible with the trivializations of  $\mathcal{P}_G^1|_{S \times X - \text{Graph}_{I_2}}$  and  $\mathcal{P}_G^2|_{S \times X - \text{Graph}_{I_2}}$ , given by  $\gamma^1|_{S \times X - \text{Graph}_{I_2}}$  and  $\gamma^2$ , respectively.

17.1.2. Note that by taking the fiber at  $x \in X$ , we obtain a map

$$(17.1) \quad \text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \rightarrow BG_x.$$

We will prove:

**Theorem 17.1.3.** *The map (17.1) is a universally homologically contractible<sup>24</sup>. In particular, it induces an isomorphism on homology.*

One can view Theorem 17.1.3 as a local version of non-abelian Poincaré duality.

We will now show how Theorem 17.1.3 implies Theorem 16.4.7.

17.1.4. *Proof of Theorem 16.4.7.* Recall that in Theorem 11.1.3, we constructed a canonical isomorphism

$$((\mathcal{A}_{\text{red}}^\dagger)_x)^\vee \simeq (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}.$$

Consider the resulting pairing

$$(\mathcal{A}_{\text{red}}^\dagger)_x \otimes (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}} \rightarrow k.$$

Note that by construction

$$(\mathcal{A}_{\text{red}}^\dagger)_x \simeq C_*^{\text{red}}(\text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}}).$$

By unwinding the construction in Theorem 16.4.7, we obtain that the following diagram commutes

$$\begin{array}{ccc} (\mathcal{A}_{\text{red}}^\dagger)_x \otimes (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}} & \longrightarrow & k \\ \sim \uparrow & & \uparrow \\ C_*^{\text{red}}(\text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}}) \otimes (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}} & & C_*^{\text{red}}(BG_x) \otimes C_{\text{red}}^*(BG_x) \\ \uparrow & & \uparrow \sim \\ C_*^{\text{red}}(\text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}}) \otimes (\mathcal{B}_{\text{red}})_{\{x\}} & \longrightarrow & C_*^{\text{red}}(BG_x) \otimes (\mathcal{B}_{\text{red}})_{\{x\}}, \end{array}$$

where the lower left vertical arrow is induced by the map of Theorem 16.4.7, and the bottom horizontal arrow is induced by the map of (17.1).

Applying Theorem 11.1.3, we deduce that the map

$$(\mathcal{B}_{\text{red}})_{\{x\}} \rightarrow (\mathbb{D}_{\text{Ran}}(\mathcal{A}_{\text{red}}))_{\{x\}}$$

is an isomorphism, as required.  $\square$

17.1.5. Thus, our remaining goal is to prove Theorem 17.1.3. Note that the above derivation of Theorem 16.4.7 from Theorem 11.1.3 shows that these two theorems are logically equivalent. In particular, since Theorem 16.4.7 is a *local* statement (i.e., it makes sense for a not necessarily complete curve), we obtain that so is Theorem 11.1.3.

However, in order to prove Theorem 11.1.3, we will use global methods. In particular, for the proof we will assume that  $X$  is complete (which we could have from the start). The proof of Theorem 11.1.3 will amount to a combination of two results, namely, Theorems 17.2.4 and 17.2.6, both global in nature. It is reasonable to think that the two global phenomena used in the proof of Theorem 17.1.3 cancel each other out.

## 17.2. The prestack of germs of bundles.

<sup>24</sup>See Sect. 2.5 for what it means to be universally homologically contractible for a map from a lax prestack to a prestack.

17.2.1. Let  $\mathrm{Bun}_G^{\mathrm{around}_x}$  denote the following prestack. For  $S \in \mathrm{Sch}$ , the groupoid of  $S$ -points of  $\mathrm{Bun}_G^{\mathrm{around}_x}$  has as objects  $G$ -bundles  $\mathcal{P}_G$  on  $S \times X$ , and as morphisms isomorphisms between  $G$ -bundles defined on an open subset of  $S \times X$  of the form  $S \times X - \mathrm{Graph}_K$  for *some* finite set  $K \subset \mathrm{Maps}(S, X - x)$ .

*Remark 17.2.2.* The prestack  $\mathrm{Bun}_G^{\mathrm{around}_x}$  should be thought of as classifying germs of  $G$ -bundles on  $X$  defined in an (unspecified, but non-empty) neighborhood of  $x$ .

17.2.3. Restriction to  $x \in X$  defines a map

$$(17.2) \quad \mathrm{Bun}_G^{\mathrm{around}_x} \rightarrow BG_x.$$

In Sect. 17.3 we will prove:

**Theorem 17.2.4.** *The map (17.2) is a universally homologically contractible. In particular, it induces an isomorphism on homology.*

We note that Theorem 17.2.4 is essentially the same as [Main Text, Proposition 7.3.17].

17.2.5. Note now that we also have a tautologically defined forgetful map

$$(17.3) \quad \mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}} \rightarrow \mathrm{Bun}_G^{\mathrm{around}_x}.$$

In Sect. 17.4 we will prove:

**Theorem 17.2.6.** *The map (17.3) is universally homologically contractible. In particular, it induces an isomorphism on homology.*

We note that Theorem 17.2.6 is essentially the same as [Main Text, Proposition 7.3.16] (in the simplified context of a constant group-scheme).

17.2.7. Note that the map (17.1) is the composition of the maps (17.3) and (17.2). So, Theorem 17.1.3 follows from the combination of Theorems 17.2.4 and 17.2.6.

### 17.3. Proof of Theorem 17.2.4.

17.3.1. Since the map  $\mathrm{pt} \rightarrow BG_x$  is a smooth cover, it suffices to show that the prestack

$$\mathrm{Bun}_G^{\mathrm{around}_x} \times_{BG_x} \mathrm{pt}$$

is universally homologically contractible.

Let  $\mathbf{Maps}((X; x), (G; 1))^{\mathrm{around}_x}$  be the group-prestack whose  $S$ -points are maps from  $S \times X$  to  $G$ , defined on an open subset of  $S \times X$  of the form  $S \times X - \mathrm{Graph}_I$  for *some* finite set  $I \subset \mathrm{Maps}(S, X - x)$  and equal to the constant map with value  $1 \in G$  when restricted to  $S \times x \subset S \times X$ .

By [Main Text, Theorem 3.3.6] (applied in the case of a constant group-scheme), for any  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$  equipped with a trivialization along  $S \times x$ , there exists an étale cover of  $\tilde{S} \rightarrow S$  and  $K \subset \mathrm{Maps}(\tilde{S}, X - x)$ , such that  $\mathcal{P}_G|_{\tilde{S} \times X - \mathrm{Graph}_K}$  can be trivialized in a way compatible with the given trivialization on  $\tilde{S} \times x$ .

This implies that the prestack  $\mathrm{Bun}_G^{\mathrm{around}_x} \times_{BG_x} \mathrm{pt}$  identifies with the étale sheaffication of the prestack

$$B(\mathbf{Maps}((X; x), (G; 1))^{\mathrm{around}_x}).$$

Thus, it remains to show that the prestack  $B(\mathbf{Maps}((X; x), (G; 1))^{\mathrm{around}_x})$  is universally homologically contractible.

It is easy to see that if  $H$  is a group-prestack, which is universally homologically contractible (as a mere prestack), then so is  $BH$ .

Hence, we obtain that in order to prove Theorem 17.2.4, it suffices to show that the prestack  $\mathbf{Maps}((X; x), (G; 1))^{\mathrm{around}_x}$  is universally homologically contractible.



17.3.2. Let  $\mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x}$  denote the following lax prestack. For  $S \in \text{Sch}$ , the category of  $S$ -points of  $\mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x}$  has as objects pairs  $K \subset \text{Maps}(S, X - x)$  and a map

$$z : S \times X - \text{Graph}_K \rightarrow G,$$

which takes value  $1 \in G$  over  $S \times x \subset S \times X$ . Morphisms between  $(K^1, z^1)$  and  $(K^2, z^2)$  are inclusions  $K^1 \subseteq K^2$  such that

$$z^1|_{S \times X - \text{Graph}_{K^2}} = z^2.$$

We have the forgetful map

$$(17.4) \quad \mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x} \rightarrow \mathbf{Maps}((X; x), (G; 1))^{\text{around}_x}.$$

We claim:

**Lemma 17.3.3.** *The map (17.4) is universally homologically contractible.*

*Proof.* By Lemma 2.5.3, it is enough to show that for any  $S$ -point of  $\mathbf{Maps}((X; x), (G; 1))^{\text{around}_x}$ , the category of its lifts to an  $S$ -point of  $\mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x}$  is contractible.

Let an  $S$ -point of  $\mathbf{Maps}((X; x), (G; 1))^{\text{around}_x}$  be given by  $(K, z : S \times X - \text{Graph}_K \rightarrow Z)$ . The category of its lifts to an  $S$ -point of  $\mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x}$  is that of

$$(K', z' : S \times X - \text{Graph}_{K'} \rightarrow Z, z|_{S \times X - \text{Graph}_{K \cup K'}} = z'|_{S \times X - \text{Graph}_{K \cup K'}}).$$

Now, left cofinal in this category is the subcategory consisting of those  $K'$  for which  $K \subset K'$ . Finally, this subcategory is contractible since it has an initial object.  $\square$

17.3.4. Hence, by Lemma 17.3.3, it suffices to show that the lax prestack  $\mathbf{Maps}((X; x), (G; 1))_{\text{param}}^{\text{around}_x}$  is universally homologically contractible. However, this is the assertion of [Main Text, Theorem 3.3.2] (applied in the case of a constant group-scheme). (Note that the theorem from [Main Text] quoted above is a global result; this is what makes Theorem 17.2.4 a global assertion.)

#### 17.4. Proof of Theorem 17.2.6.

17.4.1. Let  $(\text{Bun}_G^{\text{around}_x})_{\text{param}}$  be the lax prestack defined as follows. For  $S \in \text{Sch}$ , the category of  $S$ -points of  $(\text{Bun}_G^{\text{around}_x})_{\text{param}}$  has as objects pairs  $K \subset \text{Maps}(S, X - x)$  and  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$ .

For two such objects  $(K^1, \mathcal{P}_G^1)$  and  $(K^2, \mathcal{P}_G^2)$ , a datum of a morphism between them is an inclusion  $K_1 \subseteq K_2$  and an isomorphism

$$\mathcal{P}_G^1|_{S \times X - \text{Graph}_{K_2}} \simeq \mathcal{P}_G^2|_{S \times X - \text{Graph}_{K_2}}.$$

The map (17.3) clearly factors as

$$(17.5) \quad \text{Gr}_{(\text{Ran}_{\text{untl, aug}})_{x \notin}} \rightarrow (\text{Bun}_G^{\text{around}_x})_{\text{param}} \rightarrow \text{Bun}_G^{\text{around}_x}.$$

We will prove that both arrows in (17.5) are universally homologically contractible.<sup>25</sup> We note, however, that the fact that the second arrow in (17.5) is universally homologically contractible is proved by repeating the proof of Lemma 17.3.3.

<sup>25</sup>Here we are using the notion of being universally homologically contractible for a map between lax prestacks, see Sect. 3.4 for what this means.

17.4.2. By Proposition 3.4.6, in order to prove that

$$(17.6) \quad \mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}} \rightarrow (\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}$$

is universally homologically contractible, it suffices to show that for any  $S \in \mathrm{Sch}$ , the functor

$$\mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}}(S) \rightarrow (\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}(S)$$

is a coCartesian fibration and that for any given object of  $(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}(S)$  the map

$$S \times_{(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}} \mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}} \rightarrow S$$

is universally homologically contractible.

17.4.3. The coCartesian property is established as follows. For a map

$$(K^1, \mathcal{P}_G^1) \rightarrow (K^2, \mathcal{P}_G^2)$$

in  $(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}(S)$  and an object  $\mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}}(S)$ , given by  $I^1 \supseteq K^1$  and a trivialization  $\gamma^1$  of  $\mathcal{P}_G^1$  over  $S \times X - \mathrm{Graph}_{I^1}$ , we produce a new object of  $(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}(S)$  by setting  $I^2 := I^1 \cup K^2$  with  $\gamma^2$ , given by restriction.

17.4.4. Let us now be given an object  $(K, \mathcal{P}_G)$  of  $(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}(S)$ . The fiber of (17.6) over it is the lax prestack over  $S$  that assigns to  $S' \rightarrow S$  the category of

$$((K' \subseteq I' \subset \mathrm{Maps}(S', X)), \gamma'),$$

where  $K'$  is the image of  $K$  in  $\mathrm{Maps}(S', X)$ , and where  $\gamma'$  is a trivialization of  $\mathcal{P}_G|_{S' \times X}$  over

$$S' \times X - \mathrm{Graph}_{I'}.$$

Consider now the lax prestack

$$S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}}.$$

By definition, it assigns to  $S' \rightarrow S$  the category of

$$(I' \subset \mathrm{Maps}(S', X), \gamma'),$$

where  $\gamma'$  is a trivialization of  $\mathcal{P}_G|_{S' \times X}$  over  $S' \times X - \mathrm{Graph}_{I'}$ .

We have a tautological functor

$$S \times_{(\mathrm{Bun}_G^{\mathrm{around}_x})_{\mathrm{param}}} \mathrm{Gr}_{(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})_{x \notin}} \rightarrow S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}},$$

which admits a left adjoint (take the union of  $I'$  with  $K'$ ).

Hence, by Lemma 3.3.4, it is enough to show that the map

$$S \times_{\mathrm{Bun}_G} \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \rightarrow S$$

is universally homologically contractible.

17.4.5. By [Main Text, Theorem 3.3.6] (applied in the case of a constant group-scheme), we can assume that the given  $G$ -bundle on  $S \times X$  admits a rational trivialization. In this case, the fiber product

$$S \times_{\text{Bun}_G} \text{GrRan}_{\text{untl}}$$

identifies with

$$S \times \mathbf{Maps}(X, G)_{\text{param}},$$

where  $\mathbf{Maps}(X, G)_{\text{param}}$  is the prestack that assigns to  $S' \rightarrow S$  the groupoid of

$$(I' \subset \text{Maps}(S', X), \gamma' : S' \times X - \text{Graph}_{I'} \rightarrow G).$$

Thus, it remains to show that the prestack  $\mathbf{Maps}(X, G)_{\text{param}}$  is universally homologically contractible. However, the latter is given by [Main Text, Theorem 3.3.2] (applied in the case of a constant group-scheme). (We note again that the theorem from [Main Text] quoted above is a global result, making Theorem 17.2.6 into a global assertion).

## 18. SECOND PROOF OF THE POINTWISE DUALITY STATEMENT

In this section we will give another proof of Theorem 16.4.7, also using global methods. One component of this proof relies on the notion of *universal homological left cofinality*, see Sect. 3.5.

In this section we let  $X$  be a proper curve with a marked point  $x_\infty \in X$ . We denote  $X' := X - x_\infty$ , and we let  $x_0$  be a marked point on  $X'$ . In practice, we will take  $X = \mathbb{P}^1$  with points  $\infty$  and  $0$ , respectively.

We let  $G$  be a constant reductive group-scheme on  $X$ .

**18.1. Local non-abelian Poincaré duality,  $\mathbb{P}^1$ -version.** In this subsection we will formulate a theorem, Theorem 18.1.3, which would be a semi-global  $\mathbb{P}^1$ -analog of Theorem 17.1.3, and deduce from it Theorem 16.4.7.

18.1.1. Let  $\mathbf{union}_{x_\infty, \text{untl}, \text{aug}}$  denote the map

$$\text{Ran}_{\text{untl}, \text{aug}} \rightarrow \text{Ran}_{\text{untl}, \text{aug}}, \quad (K \subseteq I) \mapsto ((K \cup x_\infty) \subseteq (I \cup x_\infty)).$$

Consider the map

$$\mathbf{union}_{x_\infty, \text{untl}, \text{aug}} \circ \iota \circ \phi : \text{Ran} \rightarrow \text{Ran}_{\text{untl}, \text{aug}}, \quad I \mapsto (\{x_\infty\} \subset (I \cup x_\infty)).$$

Consider the fiber product

$$\text{Ran} \times_{\text{Ran}_{\text{untl}, \text{aug}}} \text{GrRan}_{\text{untl}, \text{aug}}.$$

This is a prestack that assigns to a test scheme  $S$  the category, whose objects are triples:

- $I \subset \text{Maps}(S, X)$ ;
- a  $G$ -bundle  $\mathcal{P}_G$  on  $S \times X$ ;
- a trivialization  $\gamma$  of the restriction of  $\mathcal{P}_G$  to  $S \times X - (\text{Graph}_I \cup (S \times x_\infty))$ .

Morphisms between  $(I^1, \mathcal{P}_G^1, \gamma^1)$  and  $(I^2, \mathcal{P}_G^2, \gamma^2)$  are non-empty only when  $I^1 = I^2$ , and in the latter case consist of isomorphisms

$$\mathcal{P}_G^1|_{S \times (X - x_\infty)} \simeq \mathcal{P}_G^2|_{S \times (X - x_\infty)},$$

compatible with the data of  $\gamma^1$  and  $\gamma^2$ , respectively.

18.1.2. We have a naturally defined map

$$(18.1) \quad \mathrm{Ran}_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} \times \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl},\mathrm{aug}}} \rightarrow BG,$$

given by restricting the bundle to  $x_0 \in X'$ .

We will prove:

**Theorem 18.1.3.** *Let  $(X, x_\infty, x_0) = (\mathbb{P}^1, \infty, 0)$ . Then the map (18.1) is universally homologically contractible.*

We regard Theorem 18.1.3 as a semi-global version of Theorem 14.2.4.

We shall now show how Theorem 18.1.3 implies Theorem 16.4.7.

18.1.4. Consider the object

$$(\mathbf{j}_{\mathrm{Ran}})_*(\mathcal{A}'_{\mathrm{red}}) \in \mathrm{Shv}^1(\mathrm{Ran}(X)).$$

We have a tautologically defined map

$$(18.2) \quad (\mathbb{D}_{\mathrm{Ran}'}(\mathcal{A}'_{\mathrm{red}}))_{\{x_0\}} \rightarrow (\mathrm{C}_c^*(\mathrm{Ran}, (\mathbf{j}_{\mathrm{Ran}})_*(\mathcal{A}'_{\mathrm{red}})))^\vee.$$

We claim:

**Lemma 18.1.5.** *Assume that  $(X, x_\infty, x_0) = (\mathbb{P}^1, \infty, 0)$ . Then the map (18.2) is an isomorphism.*

*Proof.* We will prove the lemma for  $\mathcal{A}'_{\mathrm{red}}$  replaced by any  $\mathcal{F} \in \mathrm{Shv}^1(\mathrm{Ran}')$ , which is equivariant with respect to the action of  $\mathbb{G}_m$  by dilations.

By Corollary 8.2.2, we can assume that  $\mathcal{F} = (\mathrm{ins}'_{\mathcal{J}})_!(\mathcal{F}_{\mathcal{J}})$  for some finite set  $\mathcal{J}$  and  $\mathcal{F}_{\mathcal{J}} \in \mathrm{Shv}(X'^{\mathcal{J}})$  with  $\mathcal{F}_{\mathcal{J}}$  being  $\mathbb{G}_m$ -equivariant.

Then it suffices to show that the map

$$(18.3) \quad (\mathbb{D}_{X'^{\mathcal{J}}}(\mathcal{F}_{\mathcal{J}}))_{x_0^{\mathcal{J}}} \rightarrow (\mathrm{C}_c^*(X^{\mathcal{J}}, (\mathbf{j}_{X^{\mathcal{J}}})_*(\mathcal{F}_{\mathcal{J}})))$$

is an isomorphism, where  $x_0^{\mathcal{J}}$  denotes the corresponding point of  $X^{\mathcal{J}}$ .

Let  $(\mathcal{F}_{\mathcal{J}})_{x_0^{\mathcal{J}}}^*$  denote the \*-fiber of  $\mathcal{F}_{\mathcal{J}}$  at the point  $x_0^{\mathcal{J}} \in X'^{\mathcal{J}}$ . Then the map (18.3) is the dual of the canonical map

$$\mathrm{C}^*(X'^{\mathcal{J}}, \mathcal{F}_{\mathcal{J}}) \rightarrow (\mathcal{F}_{\mathcal{J}})_{x_0^{\mathcal{J}}}^*.$$

Now, the assertion follows from the contraction principle for  $\mathbb{G}_m$ -equivariant sheaves on  $(\mathbb{A}^1)^{\mathcal{J}}$ , see [DrGa, Theorem C.5.3]. □

18.1.6. Recall the map  $\mathbf{union}_{x_\infty, \mathrm{untl}, \mathrm{aug}}$  introduced above. It follows from Theorem 5.4.3 that we have a canonical isomorphism

$$\mathcal{A}'_{\mathrm{red}} \simeq \mathrm{TakeOut} \circ \mathbf{union}_{x_\infty, \mathrm{untl}, \mathrm{aug}}^!(\mathcal{A}_{\mathrm{untl}, \mathrm{aug}})|_{\mathrm{Ran}'}$$

From here we obtain a map

$$(18.4) \quad \mathrm{TakeOut} \circ \mathbf{union}_{x_\infty, \mathrm{untl}, \mathrm{aug}}^!(\mathcal{A}_{\mathrm{untl}, \mathrm{aug}}) \rightarrow (\mathbf{j}_{\mathrm{Ran}})_*(\mathcal{A}'_{\mathrm{red}}).$$

We claim:

**Lemma 18.1.7.** *The map (18.4) is an isomorphism.*

*Proof.* Follows easily from Theorem 5.4.3: it is true for  $\mathcal{A}_{\mathrm{red}}$  replaced by an arbitrary  $\mathcal{F} \in \mathrm{Shv}^1(\mathrm{Ran})$ ,  $\mathcal{A}'_{\mathrm{red}}$  by  $\mathcal{F}|_{\mathrm{Ran}'}$  and  $\mathcal{A}_{\mathrm{untl}, \mathrm{aug}}$  by  $\tilde{\mathcal{F}} := \mathrm{AddUnit}_{\mathrm{aug}}(\mathcal{F}) \in \mathrm{Shv}^1(\mathrm{Ran}_{\mathrm{untl}, \mathrm{aug}})$ . □

18.1.8. By construction, the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{B}'_{\text{red}})_{\{x_0\}} & \xrightarrow{(16.2)} & (\mathbb{D}_{\text{Ran}'(\mathcal{A}'_{\text{red}})})_{\{x_0\}} \\
= \downarrow & & \downarrow (18.2) \\
C_{\text{red}}^*(BG) & & (C_c^*(\text{Ran}, (\mathbf{j}_{\text{Ran}})_*(\mathcal{A}'_{\text{red}})))^\vee \\
(18.1) \downarrow & & \downarrow (18.4) \\
C_{\text{red}}^* \left( \text{Ran}_{\text{Ran}_{\text{untl, aug}}} \times \text{Gr}_{\text{Ran}_{\text{untl, aug}}} \right) & & C_c^*(\text{Ran}, \text{TakeOut} \circ \mathbf{union}_{x_\infty, \text{untl, aug}}^!(\mathcal{A}_{\text{untl, aug}}))^\vee \\
\sim \downarrow & & \downarrow (5.5) \\
& & C_c^*(\text{Ran}, \text{OblvUnit} \circ \text{OblvAug} \circ \mathbf{union}_{x_\infty, \text{untl, aug}}^!(\mathcal{A}_{\text{untl, aug}}))^\vee \\
& & \downarrow \sim \\
C_*^{\text{red}} \left( \text{Ran}_{\text{Ran}_{\text{untl, aug}}} \times \text{Gr}_{\text{Ran}_{\text{untl, aug}}} \right)^\vee & \xrightarrow{\sim} & (C_c^*(\text{Ran}, \phi^! \circ \iota^! \circ \mathbf{union}_{x_\infty}^!(\mathcal{A})))^\vee.
\end{array}$$

We need to show that the top horizontal arrow in this diagram is an isomorphism. For this we can take  $(X, x_\infty, x_0) = (\mathbb{P}^1, \infty, 0)$ . We claim that in this case all the other arrows in this diagram are isomorphisms.

Indeed, for  $(X, x_\infty, x_0) = (\mathbb{P}^1, \infty, 0)$ , Theorem 18.1.3 says that the second from the top left vertical arrow is an isomorphism, and Lemma 18.1.5 says that the top right vertical arrow is an isomorphism.

The other arrows are isomorphism for any  $(X, x_\infty, x_0)$ , which follows from Lemma 18.1.7 and Corollary 5.4.5.

This proves Theorem 16.4.7.

**18.2. The stack of bundles on the punctured curve.** In this subsection we shall reduce Theorem 18.1.3 to a combination of another two theorems. These theorems are parallel to Theorems 17.2.4 and 17.2.6, respectively.

18.2.1. Let  $\text{Bun}'_G$  be the prestack that assigns to  $S \in \text{Sch}$  the groupoid whose objects are  $G$ -bundles on  $S \times X$ , and whose morphisms are isomorphisms of  $G$ -bundles on  $S \times X'$ . (In other words, this is the subgroupoid of  $G$ -bundles on  $S \times X'$  whose objects are those  $G$ -bundles that can be extended to  $S \times X$ .)

Restriction to  $x_0 \in X'$  defines a map

$$(18.5) \quad \text{Bun}'_G \rightarrow BG.$$

We will prove:

**Theorem 18.2.2.** *Assume that  $(X, x_\infty, x_0) = (\mathbb{P}^1, \infty, 0)$ . Then the map (18.5) is universally homologically contractible. In particular, it induces an isomorphism on homology.*

18.2.3. Recall the lax prestack

$$\text{Ran}_{\text{Ran}_{\text{untl, aug}}} \times \text{Gr}_{\text{Ran}_{\text{untl, aug}}},$$

see Sect. 18.1.1 (in particular, its definition involved the distinguished point  $x_\infty$ ).

We have a naturally defined map

$$(18.6) \quad \text{Ran}_{\text{Ran}_{\text{untl, aug}}} \times \text{Gr}_{\text{Ran}_{\text{untl, aug}}} \rightarrow \text{Bun}'_G$$

We will prove:

**Theorem 18.2.4.** *The map (18.6) is universally homologically contractible. In particular, it induces an isomorphism on homology.*

Note that Theorem 18.2.4 is stated for any  $(X, x_\infty)$ , i.e., it is not linked to the  $\mathbb{P}^1$  situation.

18.2.5. Clearly, the composition of the maps (18.6) and (18.5) equals the map (18.1). Hence, Theorems 18.2.2 and 18.2.4 imply Theorem 18.1.3.

### 18.3. Proof of Theorem 18.2.2.

18.3.1. Let  $\text{Bun}_{G, \text{level}_{x_0}}$  be the moduli stack of  $G$ -bundles with structure of level 1 at  $x_0$  (i.e.,  $G$ -bundles equipped with a trivialization of the fiber at  $x_0$ ), and let  $\text{Bun}'_{G, \text{level}_{x_0}}$  be the corresponding variant, i.e.,

$$\text{Bun}'_{G, \text{level}_{x_0}} := \text{Bun}'_G \times_{BG} \text{pt}.$$

It suffices to show that  $\text{Bun}'_{G, \text{level}_{x_0}}$  is universally homologically contractible.

Consider the simplicial object  $(\text{Bun}_{G, \text{level}_{x_0}} / \text{Bun}'_{G, \text{level}_{x_0}})^\bullet$  of  $\text{PreStk}$  equal to the Čech nerve of the map

$$\text{Bun}_{G, \text{level}_{x_0}} \rightarrow \text{Bun}'_{G, \text{level}_{x_0}}.$$

It suffices to show that the map

$$(18.7) \quad |C_*(\text{Bun}_{G, \text{level}_{x_0}} / \text{Bun}'_{G, \text{level}_{x_0}})^\bullet| \rightarrow \Lambda$$

is an isomorphism.

18.3.2. Consider the simplicial object  $(\text{Gr}_{x_\infty})^\bullet$ , of  $\text{PreStk}$  given by taking the powers of the indscheme  $\text{Gr}_{x_\infty}$ . Clearly,

$$|C_*((\text{Gr}_{x_\infty})^\bullet)| \rightarrow \Lambda$$

is an isomorphism.

Thus, to prove that (18.7), it suffices to construct a simplicial map

$$(18.8) \quad (\text{Gr}_{x_\infty})^\bullet \rightarrow (\text{Bun}_{G, \text{level}_{x_0}} / \text{Bun}'_{G, \text{level}_{x_0}})^\bullet,$$

so that it induces a term-wise isomorphism on homology.

18.3.3. The map (18.8) is constructed as follows: it sends an  $n$ -tuple of  $G$ -bundles  $(\mathcal{P}^1, \dots, \mathcal{P}^n)$  on  $S \times X$  each equipped with a trivialization  $\gamma^i$  over  $S \times X'$  to the same  $n$ -tuple, where the trivializations of each  $\mathcal{P}^i$  at  $S \times x_0$  is given by  $\gamma^i|_{S \times x_0}$ , and where the identifications

$$\mathcal{P}^i|_{S \times X'} \simeq \mathcal{P}^j|_{S \times X'}$$

are given by  $\gamma_j \circ \gamma_i^{-1}$ .

Now, each of the above maps

$$(\text{Gr}_{x_\infty})^n \rightarrow (\text{Bun}_{G, \text{level}_{x_0}} / \text{Bun}'_{G, \text{level}_{x_0}})^n,$$

is a fibration, locally trivial in the Zariski topology, with the typical fiber being

$$\mathbf{Maps}((X', x_0), (G, 1)).$$

Thus, it remains to show that the trace map

$$C_*(\mathbf{Maps}((X', x_0), (G, 1))) \rightarrow \Lambda$$

is an isomorphism.

18.3.4. We shall now use the fact that  $(X', x_0)$  equals  $(\mathbb{A}^1, 0)$ . In fact, we claim that for any affine scheme  $Z$  with a point  $z$ , the map

$$C_*(\mathbf{Maps}((\mathbb{A}^1, 0), (Z, z))) \rightarrow \Lambda$$

is an isomorphism.

Indeed, let us embed  $Z$  into an affine space  $\mathbb{A}^n$ , so that  $z = 0 \in \mathbb{A}^n$ . We can represent  $\mathbf{Maps}((\mathbb{A}^1, 0), (Z, z))$  as the colimit of affine schemes each equal to

$$\mathbf{Maps}((\mathbb{A}^1, 0), (Z, z)) \cap \mathbf{Maps}((\mathbb{A}^1, 0), (\mathbb{A}^n, 0))^{\leq d},$$

where  $\mathbf{Maps}((\mathbb{A}^1, 0), (\mathbb{A}^n, 0))^{\leq d}$  is the vector space of polynomial maps of degree  $\leq d$ .

Hence, it suffices to show that each  $\mathbf{Maps}((\mathbb{A}^1, 0), (Z, z)) \cap \mathbf{Maps}((\mathbb{A}^1, 0), (\mathbb{A}^n, 0))^{\leq d}$  has a trivial homology.

Now, the  $\mathbb{G}_m$ -action on  $\mathbb{A}^1$  defines an action of  $\mathbb{G}_m$  on  $\mathbf{Maps}((\mathbb{A}^1, 0), (\mathbb{A}^n, 0))$ , which preserves each  $\mathbf{Maps}((\mathbb{A}^1, 0), (Z, z)) \cap \mathbf{Maps}((\mathbb{A}^1, 0), (\mathbb{A}^n, 0))^{\leq d}$ , and contracts it to the constant map  $\mathbb{A}^1 \rightarrow 0 \in Z$ .

18.4. **Proof of Theorem 18.2.4.** We emphasize again that Theorem 18.2.4 is stated for any  $(X, x_\infty)$ , i.e., it is not linked to the  $(\mathbb{P}^1, \infty)$  situation.

18.4.1. Since the map

$$\text{Bun}_G \rightarrow \text{Bun}'_G$$

is surjective on isomorphism classes of  $S$ -points, it suffices to show that the base-changed map

$$\text{Bun}_G \times_{\text{Bun}'_G} (\text{Ran}_{\text{Bun}'_G} \times_{\text{Ran}_{\text{untl}, \text{aug}}} \text{Gr}_{\text{Ran}_{\text{untl}, \text{aug}}}) \rightarrow \text{Bun}_G$$

is universally homologically contractible.

Note that

$$\text{Bun}_G \times_{\text{Bun}'_G} (\text{Ran}_{\text{Bun}'_G} \times_{\text{Ran}_{\text{untl}, \text{aug}}} \text{Gr}_{\text{Ran}_{\text{untl}, \text{aug}}}) \simeq \text{Gr}_{\text{Ran}} \times_{\text{Ran}}$$

where  $\text{Ran} \rightarrow \text{Ran}$  is the map

$$\mathbf{union}_{x_\infty}, \quad I \mapsto I \cup x_\infty.$$

Thus, it remains to show that the composed map

$$(18.9) \quad \text{Gr}_{\text{Ran}} \times_{\text{Ran}} \text{Ran} \rightarrow \text{Gr}_{\text{Ran}} \rightarrow \text{Bun}_G$$

is universally homologically contractible.

We will deduce this from the fact that the usual uniformization map

$$(18.10) \quad \text{Gr}_{\text{Ran}} \rightarrow \text{Bun}_G$$

is universally homologically contractible (the latter due to Theorem 14.2.4).

18.4.2. Note that the map  $\text{Gr}_{\text{Ran}} \rightarrow \text{Bun}_G$  factors as

$$\text{Gr}_{\text{Ran}} \rightarrow \text{Gr}_{\text{Ran}_{\text{untl}}} \rightarrow \text{Bun}_G,$$

where

$$\text{Gr}_{\text{Ran}_{\text{untl}}} := \text{Ran}_{\text{untl}} \times_{\text{Ran}_{\text{untl}, \text{aug}}} \text{Gr}_{\text{Ran}_{\text{untl}, \text{aug}}}$$

for  $\text{Gr}_{\text{Ran}_{\text{untl}, \text{aug}}}$  introduced in Sect. 15.2. I.e., we have a commutative diagram

$$(18.11) \quad \begin{array}{ccccc} \text{Gr}_{\text{Ran}} \times_{\text{Ran}} \text{Ran} & \longrightarrow & \text{Gr}_{\text{Ran}} & \longrightarrow & \text{Bun}_G \\ \downarrow & & \downarrow & & \downarrow \text{id}_S \\ \text{Gr}_{\text{Ran}_{\text{untl}}} \times_{\text{Ran}_{\text{untl}}} \text{Ran}_{\text{untl}} & \longrightarrow & \text{Gr}_{\text{Ran}_{\text{untl}}} & \longrightarrow & \text{Bun}_G, \end{array}$$

where  $\text{Ran}_{\text{untl}} \rightarrow \text{Ran}_{\text{untl}}$  is the map

$$\mathbf{union}_{x_\infty, \text{untl}}, \quad I \mapsto I \cup x_\infty.$$

18.4.3. We claim that the maps

$$\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}}, \quad \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \times_{\mathrm{Ran}_{\mathrm{untl}}} \mathrm{Ran}_{\mathrm{untl}} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \quad \text{and} \quad \mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}} \mathrm{Ran} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \times_{\mathrm{Ran}_{\mathrm{untl}}} \mathrm{Ran}_{\mathrm{untl}},$$

appearing in the diagram (18.11), are universally homologically left cofinal.

Indeed, consider the Cartesian square

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{Ran}} & \longrightarrow & \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \\ \downarrow & & \downarrow \\ \mathrm{Ran} & \xrightarrow{\phi} & \mathrm{Ran}_{\mathrm{untl}}, \end{array}$$

and the assertion follows from the combination of the following facts:

- (i) Theorem 4.6.2 and Proposition 4.6.5;
- (ii) The map  $\mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \rightarrow \mathrm{Ran}_{\mathrm{untl}}$  is a value-wise coCartesian fibration;
- (iii) Proposition 3.5.9.

18.4.4. Since  $\mathrm{Bun}_G$  is a prestack (as opposed to a lax prestack), since the map

$$\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}} \mathrm{Ran} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \times_{\mathrm{Ran}_{\mathrm{untl}}} \mathrm{Ran}_{\mathrm{untl}},$$

is universally homologically left cofinal, by Proposition 3.5.11, it suffices to show that the composed map

$$\mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \times_{\mathrm{Ran}_{\mathrm{untl}}} \mathrm{Ran}_{\mathrm{untl}} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \rightarrow \mathrm{Bun}_G$$

is universally homologically contractible.

Since

$$\mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \times_{\mathrm{Ran}_{\mathrm{untl}}} \mathrm{Ran}_{\mathrm{untl}} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}}$$

is universally homologically left cofinal, it suffices to show that the map

$$\mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}} \rightarrow \mathrm{Bun}_G$$

is universally homologically contractible.

However, the latter follows from the fact that  $\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Gr}_{\mathrm{Ran}_{\mathrm{untl}}}$  is universally homologically left cofinal and the fact that the map  $\mathrm{Gr}_{\mathrm{Ran}} \rightarrow \mathrm{Bun}_G$  is universally homologically contractible.



## Part VI: The Atiyah-Bott formula and the numerical product formula

### 19. THE ATIYAH-BOTT FORMULA

In this section we let  $X$  be a smooth connected complete curve. Let  $G$  be a group-scheme over  $X$  satisfying the assumptions of Sect. 14.2.1. We will assume that our sheaf theory is such that the ring of coefficients  $\Lambda$  is a field of characteristic 0.

In this section we will apply Theorem 14.1.6 to deduce a much more explicit expression for the cohomology of  $\text{Bun}_G$ , the Atiyah-Bott formula.

The prerequisites for this section are Sects. 14.1 and 16.2 and [Main Text, Theorem 5.6.4].

#### 19.1. Statement of the Atiyah-Bott formula.

19.1.1. Let  $G_0$  be the split form of  $G$ . Consider the commutative algebra in  $\Lambda\text{-mod}$ :

$$C^*(BG_0) =: B_0.$$

Consider the corresponding  $\mathbb{Z}$ -graded (classical) commutative algebra  $H^*(B_0)$ . Let  $M_0$  denote the  $\mathbb{Z}$ -graded  $\Lambda$ -vector space  $\mathfrak{m}/\mathfrak{m}^2$ , where  $\mathfrak{m} \subset H^*(B_0)$  is the augmentation ideal of  $B_0$  (the set of elements of strictly positive degree).

We shall regard  $M_0$  as a semi-simple object of  $\Lambda\text{-mod}$  (turning the  $\mathbb{Z}$ -grading into a cohomological one).

19.1.2. The (graded) vector space  $M_0$  is the sum

$$\bigoplus_e \Lambda[-2e],$$

where the  $e$ 's are the exponents of  $G_0$ . Note that because  $G_0$  was assumed semi-simple, we have  $e \geq 2$ . When  $k = \overline{\mathbb{F}}_q$  (and our sheaf theory is that of  $\mathbb{Q}_\ell$ -adic sheaves),  $M_0$  carries a canonical action of the geometric Frobenius, where the action on the  $e$ -th direct summand is given by  $q^e$ .

It is well-known that  $H^*(B_0)$  is isomorphic to  $\text{Sym}_\Lambda(M_0)$ . When  $k = \overline{\mathbb{F}}_q$ , this isomorphism can be chosen compatible with an action of the Frobenius.

Lifting the generators we can therefore choose an isomorphism of commutative algebras in  $\Lambda\text{-mod}$

$$\text{Sym}_\Lambda(M_0) \simeq C^*(B_0).$$

For  $k = \overline{\mathbb{F}}_q$  the latter can also be chosen compatible with an action of the Frobenius.

19.1.3. The assignment  $G_0 \mapsto M_0$  is functorial with respect to automorphisms of  $G_0$ . Hence over the open curve  $X'$  (see Sect. 14.2.1 for the notation) we have a well-defined lisse sheaf  $M \in \text{Shv}^!(X')$ , whose  $!$ -fiber  $M_x$  at  $x \in X$  is identified with  $M_0$  for every choice of identification of  $G_x$  with  $G_0$ .

The Atiyah-Bott formula reads:

#### Theorem 19.1.4.

(a) *There exists a (non-canonical) isomorphism*

$$C^*(\text{Bun}_G) \simeq \text{Sym}_\Lambda(C^*(X', M)).$$

(b) *When  $k = \overline{\mathbb{F}}_q$  and  $X$  and  $G$  are defined over  $\mathbb{F}_q$ , the above isomorphism can be chosen compatible with the Frobenius-equivariance structure.*

#### 19.2. Proof of the Atiyah-Bott formula.

19.2.1. The starting point for the proof is the isomorphism

$$C_{\text{red}}^*(\text{Bun}_G) \simeq C_c^*(\text{Ran}, \mathcal{B}),$$

provided by Theorem 14.1.6. So, from now on our goal will be to construct a (non-canonical) isomorphism

$$(19.1) \quad C_c^*(\text{Ran}, \mathcal{B}) \simeq \text{Sym}_{\Lambda}^+(C^*(X', M)),$$

where  $\text{Sym}^+$  denotes the augmentation ideal of  $\text{Sym}$ .

19.2.2. Recall the object  $\mathcal{B}_{\text{red}} \in \text{Shv}^1(\text{Ran})$  (see Sect. 15.1.3), and recall also that by Corollary 5.4.5 we have a canonical isomorphism

$$(19.2) \quad C_c^*(\text{Ran}, \mathcal{B}) \simeq C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}).$$

Recall also that by Sect. 16.2.1,  $\mathcal{B}_{\text{red}}$  has a structure of *commutative algebra* in the symmetric monoidal category  $\text{Shv}^1(\text{Ran})$  (endowed with the convolution product). We claim:

**Proposition 19.2.3.** *There is a (non-canonical) isomorphism between  $\mathcal{B}_{\text{red}}$  and the (non-unital) free commutative algebra on the object  $(\text{ins}_X)_!(\mathbf{j}_*(M))$  in the symmetric monoidal category  $\text{Shv}^1(\text{Ran})$ .*

In the above proposition,  $\text{ins}_X$  denotes the map  $X \rightarrow \text{Ran}$  and  $\mathbf{j}$  is the open embedding  $X' \hookrightarrow X$ .

19.2.4. Before proving Proposition 19.2.3, let us show that it implies the existence of a non-canonical isomorphism

$$C_c^*(\text{Ran}, \mathcal{B}_{\text{red}}) \simeq \text{Sym}_{\Lambda}^+(C^*(X', M))$$

(while the latter implies (19.1) by (19.2)).

Indeed, the functor

$$\mathcal{F} \mapsto C_c^*(\text{Ran}, \mathcal{F}), \quad \text{Shv}^1(\text{Ran}) \rightarrow \Lambda\text{-mod}$$

has a natural symmetric monoidal structure. Hence,

$$C_c^*(\text{Ran}, \text{FreeCom}_{\text{Shv}^1(\text{Ran})}((\text{ins}_X)_!(\mathbf{j}_*(M)))) \simeq \text{FreeCom}_{\Lambda\text{-mod}}(C_c^*(\text{Ran}, (\text{ins}_X)_!(\mathbf{j}_*(M)))),$$

while

$$C_c^*(\text{Ran}, (\text{ins}_X)_!(\mathbf{j}_*(M))) \simeq C_c^*(X, \mathbf{j}_*(M)) \simeq C^*(X', M)$$

and

$$\text{FreeCom}_{\Lambda\text{-mod}}(-) \simeq \text{Sym}_{\Lambda}^+(-).$$

### 19.3. Proof of Proposition 19.2.3.

19.3.1. First, it is easy to see that for any  $\mathcal{F} \in \text{Shv}(X)$ , the commutative algebra

$$\text{FreeCom}_{\text{Shv}^1(\text{Ran})}((\text{ins}_X)_!(\mathcal{F}))$$

in  $\text{Shv}^1(\text{Ran})$  is actually a *commutative factorization algebra* (see [Main Text, Lemma 5.6.15]).

Second, by Corollary 16.2.4,  $\mathcal{B}_{\text{red}}$  is also a commutative factorization algebra.

Therefore, by [Main Text, Theorem 5.6.4], the existence of an isomorphism stated in the proposition is equivalent to the existence of an isomorphism in the category of commutative algebras in  $\text{Shv}(X)$  (with respect to the pointwise tensor product):

$$(\mathcal{B}_{\text{red}})_X \simeq \text{FreeCom}_{\text{Shv}(X)}(\mathbf{j}_*(M)).$$

19.3.2. Note that by the assumption on  $G$ , we have

$$(\mathcal{B}_{\text{red}})_X \simeq \mathbf{j}_*((\mathcal{B}'_{\text{red}})_{X'}).$$

Hence, it is enough to establish the existence of an isomorphism in the category of commutative algebras in  $\text{Shv}(X')$ :

$$(19.3) \quad (\mathcal{B}'_{\text{red}})_{X'} \simeq \text{FreeCom}_{\text{Shv}(X')}(M).$$

19.3.3. Let  $\tilde{X} \rightarrow X'$  be an étale Galois cover such that  $G|_{\tilde{X}}$  is the constant group-scheme with fiber  $G_0$ . Let  $\Gamma$  denote the Galois group of  $\tilde{X}$  over  $X'$ .

We have:

$$(19.4) \quad (\mathcal{B}'_{\text{red}})_{X'}|_{\tilde{X}} \simeq (B_0)_{\text{red}} \otimes \omega_{\tilde{X}} \text{ and } M|_{\tilde{X}} \simeq M_0 \otimes \omega_{\tilde{X}}.$$

Thus, the datum of an isomorphism (19.3) is equivalent to that of a  $\Gamma$ -equivariant isomorphism

$$(B_0)_{\text{red}} \otimes \omega_{\tilde{X}} \simeq \text{Sym}_{\Lambda}^+(M_0) \otimes \omega_{\tilde{X}},$$

where the  $\Gamma$ -equivariant structure on each side comes from the isomorphisms in (19.4).

19.3.4. The datum of a  $\Gamma$ -equivariant map of commutative algebras

$$(19.5) \quad \text{Sym}_{\Lambda}^+(M_0) \otimes \omega_{\tilde{X}} \rightarrow (B_0)_{\text{red}} \otimes \omega_{\tilde{X}}$$

is equivalent to that of a  $\Gamma$ -equivariant map in  $\text{Shv}(\tilde{X})$ :

$$(19.6) \quad M_0 \otimes \Lambda_{\tilde{X}} \rightarrow (B_0)_{\text{red}} \otimes \Lambda_{\tilde{X}}$$

We will find a map in (19.6) so that at the level of cohomology sheaves, it induces a map that fits into a commutative diagram

$$(19.7) \quad \begin{array}{ccc} H^*(M_0) \otimes \Lambda_{\tilde{X}} & \longrightarrow & H^*((B_0)_{\text{red}}) \otimes \Lambda_{\tilde{X}} \\ \sim \downarrow & & \downarrow \sim \\ M_0 \otimes \Lambda_{\tilde{X}} & \longleftarrow & \mathfrak{m} \otimes \Lambda_{\tilde{X}}, \end{array}$$

where we recall that  $M_0$  was defined as  $\mathfrak{m}/\mathfrak{m}^2$ , where  $\mathfrak{m}$  is the augmentation ideal in  $H^*(B_0)$ .

The map in (19.5), corresponding to a map in (19.6) with the property of making (19.7) commutative, is automatically an isomorphism.

19.3.5. Consider the space  $\mathcal{V}$  of *all*, i.e., not necessarily  $\Gamma$ -equivariant, (resp., in the situation (b) of Theorem 19.1.4, Frobenius-equivariant) maps in (19.6) that make the diagram (19.7) commute.

The space  $\mathcal{V}$  is non-empty since  $(B_0)_{\text{red}}$  is non-canonically (resp., in the situation (b) of Theorem 19.1.4, Frobenius-equivariantly) isomorphic to  $\text{Sym}_{\Lambda}^+(M_0)$ , see Sect. 19.1.2.

The  $\Gamma$ -equivariant structure on both sides of (19.6) defines an action of  $\Gamma$  on  $\mathcal{V}$ , and our task it show that  $\mathcal{V}$  contains a  $\Gamma$ -fixed point.

19.3.6. Note, however, that  $\mathcal{V}$  is naturally a torsor for a canonically defined object  $V \in \Lambda\text{-mod}^{\leq 0}$ . Furthermore,  $V$  is also endowed with an action of  $\Gamma$ , compatible with the action of  $V$  on  $\mathcal{V}$ .

Now, the required assertion follows from the fact that  $H^1(\Gamma, V) = 0$  (the latter because  $\Gamma$  is a finite group and  $\Lambda$  is a field of characteristic 0).

## 20. THE NUMERICAL PRODUCT FORMULA

The only prerequisite for this section is the statement of Theorem 19.1.4. We will give a derivation of the numerical product formula (0.3) from the cohomological product formula (0.5), slightly different from the one in [Main Text, Sect. 6].

20.1. **The setting.** In this section we take the ground field  $k$  to be  $\overline{\mathbb{F}}_q$ , but we assume that  $X$  and  $G$  come by extension of scalars from  $X^0$  and  $G^0$ , respectively, defined over  $\mathbb{F}_q$ . We take our sheaf theory to be that of  $\mathbb{Q}_\ell$ -adic sheaves. We fix an embedding

$$\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

20.1.1. On the one hand, we consider the cohomology

$$H^*(\mathrm{Bun}_G),$$

where each  $H^*(\mathrm{Bun}_G)$  is endowed with an (invertible) action of the geometric Frobenius, denoted  $\mathrm{Frob}$ .

On the other hand, for each closed point  $x$  of  $X^0$ , we consider the finite group  $G(k_x)$  (i.e., the group of rational points of the fiber  $G_x$  of  $G^0$  over the residue  $k_x$  field at  $x$ ), and the number

$$\frac{|k_x|^{\dim(G_x)}}{|G(k_x)|} \in \mathbb{R} \subset \mathbb{C}.$$

20.1.2. The numerical product formula says:

**Theorem 20.1.3.**

(a) *Each cohomology  $H^i(\mathrm{Bun}_G)$  is finite-dimensional and the sum of complex numbers*

$$\sum_i (-1)^i \cdot \tau \left( \mathrm{Tr}(\mathrm{Frob}^{-1}, H^i(\mathrm{Bun}_G)) \right)$$

*converges absolutely.*

(b) *The product (of real numbers)*

$$\prod_{x \in |X^0|} \frac{|k_x|^{\dim(G_x)}}{|G(k_x)|}$$

*converges absolutely.*

(c) *The expressions in (a) and (b) are equal to one another (as complex numbers).*

The goal of this section is to prove Theorem 20.1.3.

*Remark 20.1.4.* In the course of the proof we will show that  $\tau \left( \mathrm{Tr}(\mathrm{Frob}^{-1}, C_c^*(\mathrm{Ran}, \mathcal{B})) \right)$  equals

$$\prod_{x \in |X^0|} \tau \left( \mathrm{Tr}(\mathrm{Frob}_x^{-1}, H^*(BG_x)) \right).$$

As was noted in Sect. 0.2.2, this justifies why we want to think of  $C_c^*(\mathrm{Ran}, \mathcal{B})$  as the Euler product (14.4).

## 20.2. Proof of the numerical product formula.

20.2.1. First, it is not difficult to see that the validity of the assertion of Theorem 20.1.3 only depends on the generic fiber of  $G$ . Hence, we can (and will) assume that  $G$  satisfies the assumptions of Sect. 14.2.1, where the open subset  $X' \subset X$  is also defined over  $\mathbb{F}_q$ .

Under this assumption, we can apply Theorem 19.1.4 and rewrite  $H^*(\mathrm{Bun}_G)$  as the cohomology of  $\mathrm{Sym}_{\mathbb{Q}_\ell}(C^*(X', M))$ , which is the same as

$$\mathrm{Sym}_{\mathbb{Q}_\ell}(H^*(X', M)).$$

Note now that the finite-dimensionality assertion in point (a) of the theorem readily follows.

20.2.2. For every closed point  $x$  of  $X^0$ , the  $\tau$ -weights of  $\mathrm{Frob}_x^{-1}$  on the  $!$ -fiber  $M_x$  of  $M$  at  $x$  are  $\leq -4$  (see Sect. 19.1.2). Hence, by [Del], the  $\tau$ -weights of  $\mathrm{Frob}^{-1}$  on  $H^*(X', M)$  are  $\leq -2$ . I.e., if we view the pair  $(\mathrm{Frob}^{-1}, H^*(X', M))$  as a complex vector space equipped with an endomorphism via  $\tau$ , and if  $\lambda$  is an eigenvalue, then  $|\lambda| < 2$ .

This implies the convergence assertion in point (a) of the theorem.

Consider the formal series

$$(20.1) \quad \sum_{n \geq 0} \tau \left( \mathrm{Tr} \left( \mathrm{Frob}^{-1}, \mathrm{Sym}_{\mathbb{Q}_\ell}^n(H^*(X', M)) \right) \right) \cdot t^n.$$

We obtain that this series converges absolutely for  $|t| < 2$  and its value at  $t = 1$  equals the sum in point (a) of the theorem.

*Remark 20.2.3.* We obtain that the sum in point (a) of the theorem equals

$$\tau \left( \frac{1}{\det(1 - \text{Frob}^{-1}, \text{H}^*(X', M'))} \right).$$

20.2.4. Next, note that for a closed point  $x$  of  $X^0$  the quantity  $\frac{1}{|G(k_x)|}$  equals the number of  $k_x$ -points of the stack  $BG_x$ . Hence, applying the Grothendieck-Lefschetz formula for the stack  $BG_x$ , we obtain

$$\frac{|k_x|^{\dim(G_x)}}{|G(k_x)|} = \sum_i (-1)^i \cdot \tau \left( \text{Tr}(\text{Frob}_x^{-1}, \text{H}^i(BG_x)) \right).$$

We have an isomorphism

$$\text{H}^*(BG_x) \simeq \begin{cases} \text{Sym}_{\mathbb{Q}_\ell}(M_x) & \text{for } x \in X' \\ \mathbb{Q}_\ell & \text{for } x \notin X'. \end{cases}$$

Consider the formal series

$$(20.2) \quad \prod_{x \in |X^0|, x \in X'} \left( \sum_{n \geq 0} \tau \left( \text{Tr}(\text{Frob}_x^{-1}, \text{Sym}_{\mathbb{Q}_\ell}^n(M_x)) \right) \cdot t^{n \cdot \deg(x)} \right).$$

The Grothendieck-Lefschetz formula for  $X'$  implies that the series (20.1) equals the series (20.2). In particular, we obtain that the series (20.2) converges absolutely for  $|t| < 2$  and its value at  $t = 1$  equals the sum in point (a) of the theorem.

The absolute convergence of (20.2) at  $t = 1$  is equivalent to the statement of point (b) of the theorem, and the product in point (b) of the theorem equals the value of (20.2) at  $t = 1$ . This implies the equality in point (c) of the theorem.  $\square$

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