INTRODUCTION TO PART A: $(\infty,2)$-CATEGORIES

1. Why do we need them?

This part plays a service role for Part V, in which we develop the formalism of categories of correspondences.

1.1. As was explained before, an adequate framework to encode the information carried by the assignment

$$S \in \mathrm{Sch}_{\mathrm{aft}} \rightsquigarrow \mathrm{IndCoh}(S) \in \mathrm{DGCat}_{\mathrm{cont}}$$

is in terms of the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all,all}} \to (\mathrm{DGCat}_{\mathrm{cont}})^{2\text{-}\mathrm{Cat}}. \tag{1.1}$$

Now, the construction of the above functor is such that even if one is ultimately interested only in the 1-categorical data, i.e., the corresponding functor of $(\infty,1)$-categories

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{DGCat}_{\mathrm{cont}},$$

in order to produce it, one needs to construct (1.1).

So, $(\infty,2)$-categories are necessary in order to get IndCoh off the ground.

1.2. Now, one possible approach would be to believe that there exists a reasonable notion of $(\infty,2)$-category (and companion notions of functor, natural transformation, etc.) and not worry about the details. For example, just imagine that a $(\infty,2)$-category is a $(\infty,1)$-category enriched over the monoidal $(\infty,1)$-category 1-Cat. (The actual definition is indeed along these lines.)

The problem with that is that we need more than just the existence of these notions. We will actually need to perform some pretty non-trivial operations with them. Let us explain what these operations are.

1.3. First off, let us be given an $(\infty,2)$-category $\mathbb{S}$, equipped with a class of 1-morphisms $\mathbf{C}$ (closed under compositions and containing all isomorphisms). To this data we need to be able to associate a *bi-simplicial* space, denoted $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S},\mathbf{C})$.

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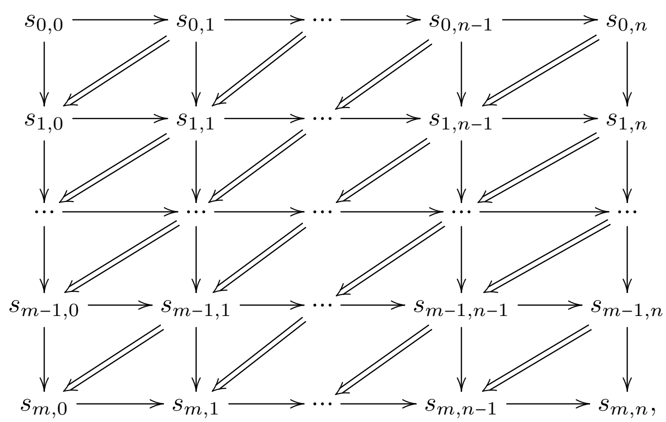
The corresponding space of (m,n)-simplices is that of diagrams



where the horizontal arrows are arbitrary 1-morphisms in $\mathbb{S}$, and the vertical arrows are 1-morphisms that belong to $\mathbf{C}$, and each square represents a (not necessarily invertible) 2-morphism.

Moreover, we need the assignment

$$(\mathbb{S}, \mathbf{C}) \rightsquigarrow \mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S}, \mathbf{C}),$$

viewed as a functor from the category $2\text{-Cat}^{\mathrm{Pair}}$ of pairs $(\mathbb{S}, \mathbf{C})$ to the category $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ of bi-simplicial spaces, to be *fully faithful* with essential image given by some explicit conditions (see [Chapter A.1, Theorem 5.2.3] for the latter).

1.4. Secondly, for a pair of $(\infty, 2)$-categories $\mathbb{S}$ and $\mathbb{T}$, in addition to the $(\infty, 2)$-category $\mathrm{Funct}(\mathbb{S}, \mathbb{T})$ of functors $\mathbb{S} \to \mathbb{T}$, we need to be able to form its two enlargements, denoted

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}} \text{ and } \mathrm{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}},$$

respectively, that have the same class of objects, but where we allow as 1-morphisms right-lax (resp., left-lax) natural transformations (see [Chapter A.1, Sect. 3.2.7] for the definition).

1.5. While the previous two properties of the sought-for notion of $(\infty, 2)$-category can still be taken for granted, the next one cannot. We will need to be able to perform the following manipulation:

For a 1-morphism in an $(\infty, 2)$-category, it makes sense to ask whether this 1-morphism admits a left or right adjoint (these are notions that take place in the underlying *ordinary* 2-category). Given a functor $\mathbb{S} \to \mathbb{T}$, we shall say that it is right (resp., left) *adjointable* if for every 1-morphism in $\mathbb{S}$, its image in $\mathbb{T}$ admits a right (resp., left) adjoint.

Let

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T})^R_{\text{right-lax}} \subset \mathrm{Funct}(\mathbb{S}, \mathbb{T})_{\text{right-lax}}$$

and

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T})^L_{\text{left-lax}} \subset \mathrm{Funct}(\mathbb{S}, \mathbb{T})_{\text{left-lax}}$$

be the full $(\infty, 2)$-subcategories that correspond to functors that are left (resp., right) adjointable.

What we need is to have a canonical equivalence

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T})^R_{\text{right-lax}} \simeq \mathrm{Funct}(\mathbb{S}^{1\&2\text{-op}}, \mathbb{T})^L_{\text{left-lax}}, \tag{1.2}$$

given by *passage to adjoint 1-morphisms*.

The construction of this equivalence will be the subject of [Chapter A.3].

1.6. That said, it is a sensible strategy to get the idea of how we approach $(\infty, 2)$-categories by reading the rest of this introduction, and skipping the bulk of Part A on the first pass.

2. Setting up the theory of $(\infty, 2)$-categories

2.1. In [Chapter A.1] we define what we mean by $(\infty, 2)$-categories.

The idea is to mimic the approach to $(\infty, 1)$-categories via *complete Segal spaces*. And this is what one obtains if one wants to express the idea is that an $(\infty, 2)$-category is just an $(\infty, 1)$-category, *enriched over* 1-Cat: we upgrade the *spaces* of morphisms to $(\infty, 1)$*-categories*.

2.2. So, for us the datum of an $(\infty, 2)$-category $\mathbb{S}$ is that of a *simplicial* $(\infty, 1)$*-category* that we denote $\mathrm{Seq}_\bullet(\mathbb{S})$ (here "Seq" stands for sequences).

Namely, the $(\infty, 1)$-category $\mathrm{Seq}_0(\mathbb{S})$ is actually a *space*, formed by objects of $\mathbb{S}$. I.e., it is the same as one of the underlying $(\infty, 1)$-category $\mathbb{S}^{1\text{-Cat}}$, obtained by discarding non-invertible 1-morphisms in $\mathbb{S}$.

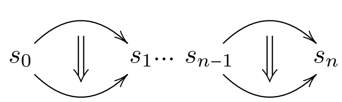
The $(\infty, 1)$-category $\mathrm{Seq}_1(\mathbb{S})$ has as objects 1-morphisms in $\mathbb{S}$. Again, these are the same as objects of $\mathrm{Seq}_1(\mathbb{S}^{1\text{-Cat}})$.

However, whereas the latter is a *space* (i.e.., we only allow homotopies between 1-morphisms in $\mathbb{S}^{1\text{-Cat}}$), in the case of $\mathrm{Seq}_1(\mathbb{S})$, we have non-invertible morphisms. Namely, morphisms between two objects $s_0 \overset{\alpha}{\to} s_1$ and $s_0 \overset{\beta}{\to} s_1$ are 2-morphisms $\alpha \Rightarrow \beta$ in $\mathbb{S}$.

The higher $\mathrm{Seq}_n(\mathbb{S})$ have as objects sequences

$$s_0 \to s_1 \to \ldots \to s_{n-1} \to s_n$$

of objects of $\mathbb{S}$, and as morphisms sequences of 2-morphisms



2.3. Formally, we define the $(\infty, 1)$-category of $(\infty, 2)$-categories 2-Cat to be a full subcategory in $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, given by explicit conditions that are analogous to the condition on an object of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ to be a complete Segal space.

In [Chapter A.1, Sect. 2.4] we introduce the main example of $(\infty, 2)$-category: this is the $(\infty, 2)$-category of $(\infty, 1)$-categories, denoted **1-Cat**.

The $(\infty, 2)$-category **1-Cat** plays the same role vis-à-vis 2-Cat as the $(\infty, 1)$-category Spc vis-à-vis 1-Cat. In particular, it is the recipient of the 2-categorical Yoneda functor, discussed below.

2.4. The above definition $(\infty,2)$-categories is amenable to introducing the notion of *right-lax* functor $\mathbb{S} \dashrightarrow \mathbb{T}$ between two $(\infty,2)$-categories $\mathbb{S}$ to $\mathbb{T}$. The idea of right-lax functors is that they do not strictly preserve compositions of 1-morphisms, but only do so up to (not necessarily invertible) 2-morphisms.

By definition, right-lax functors $\mathbb{S} \dashrightarrow \mathbb{T}$ are functors (subject to a certain non-degeneracy conditions) between the coCartesian fibrations

$$\mathbb{S}^{\oint} \to \mathbf{\Delta}^{\mathrm{op}} \text{ and } \mathbb{T}^{\oint} \to \mathbf{\Delta}^{\mathrm{op}}$$

corresponding to

$$\mathrm{Seq}_\bullet(\mathbb{S}) : \mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat} \text{ and } \mathrm{Seq}_\bullet(\mathbb{T}) : \mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat},$$

respectively, see [Chapter A.1, Sect. 3.1].

2.5. Having defined right-lax functors, we can now define the $(\infty,2)$-category

$$\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}.$$

Namely, for a test $(\infty,2)$-category $\mathbb{X}$, the space of maps

$$\mathbb{X} \to \mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}$$

is a certain full subspace in the space of right-lax functors $\mathbb{X} \times \mathbb{S} \dashrightarrow \mathbb{T}$, see [Chapter A.1, Sect. 3.2.7]. Namely, we take those right-lax functors that:

- For every $x \in \mathbb{X}$ the corresponding right-lax functor $\{x\} \times \mathbb{S} \to \mathbb{T}$ is strict;
- For every $s \in \mathbb{S}$ the corresponding right-lax functor $\mathbb{X} \times \{s\} \to \mathbb{T}$ is strict;
- For every $x_0 \overset{\alpha}{\to} x_1$ and $s_0 \overset{\beta}{\to} s_1$, the 2-morphism in $\mathbb{T}$, corresponding to the composition

$$(x_0,s_0) \overset{(\alpha,\mathrm{id})}{\longrightarrow} (x_1,s_0) \overset{(\mathrm{id},\beta)}{\longrightarrow} (x_1,s_1)$$

  is invertible.

2.6. Having defined the $(\infty,2)$-categories $\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}$, we can define the functor

$$\mathrm{Sq}^{\mathrm{Pair}} : 2\text{-Cat}^{\mathrm{Pair}} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}, \tag{2.1}$$

mentioned in Sect. 1.3.

Namely, given a pair $(\mathbb{S},\mathbf{C})$, we let the space of (m,n)-simplices in $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S},\mathbf{C})$ be the subspace of the space of functors

$$[m] \to \mathrm{Funct}([n],\mathbb{S})_{\text{right-lax}},$$

such that for every $i \in [n]$, the corresponding functor $[m] \to \mathbb{S}$ factors through $\mathbf{C}$.

2.7. We made the decision to leave some statements in [Chapter A.1] without proof. The majority of the these have to do with the notion of *Gray product*. The most important of them is the theorem that says that the functor (2.1) is fully faithful. The missing proofs will be supplied elsewhere.

3. The rest of Part A

3.1. We start [Chapter A.2] by upgrading the structure of $(\infty,1)$-category on the totality of $(\infty,2)$-categories to that of $(\infty,2)$-category. We denote the latter by $\mathbf{2}\text{-}\mathbf{Cat}$, so that

$$(\mathbf{2}\text{-}\mathbf{Cat})^{1\text{-Cat}} = 2\text{-Cat}.$$

3.2. In [Chapter A.2, Sect. 2], we introduce the notions of what it means for a functor $\mathbb{T} \to \mathbb{S}$ to be a 1-Cartesian and 2-Cartesian fibration. Both of these notion are obtained by imposing certain *conditions* (as opposed to additional pieces of structure).

The main result of [Chapter A.2] is the straightening/unstraightening theorem. It says that the $(\infty, 2)$-category of 2-Cartesian (resp., 1-Cartesian) fibrations over $\mathbb{S}$ (with 1-morphisms being functors preserving Cartesian arrows) is equivalent to $(\infty, 2)$-category of functors

$$\mathbb{S}^{1\text{-op}} \to \mathbf{2\text{-}Cat} \quad (\text{resp., } \mathbb{S}^{1\text{-op}} \to \mathbf{1\text{-}Cat}).$$

3.3. Having at our disposal the straightening theorem, starting from the $(\infty, 2)$-category

$$\text{Funct}([1], \mathbb{S})_{\text{right-lax}},$$

projecting to $\mathbb{S} \times \mathbb{S}$ (by evaluation on the two ends of $[1]$), we obtain the 2-categorical Yoneda functor

$$\mathbb{S} \hookrightarrow \text{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat})$$

that we prove to be a fully faithful embedding.

3.4. Having developed the basics of $(\infty, 2)$-categories, in [Chapter A.3] we finally address the construction of functors obtained by *passing to adjoints along 1-morphisms*, mentioned in Sect. 1.5.

The main construction of [Chapter A.3] is given in Sects. 2.2 and 2.3. Namely, given an $(\infty, 2)$-category $\mathbb{S}$, we explicitly describe another $(\infty, 2)$-category, denoted $\mathbb{S}^R$, equipped with a functor

$$\mathbb{S} \to \mathbb{S}^R,$$

which is *universal* with respect to the property of being *left adjointable*.

The construction of $\mathbb{S}^R$ is given in terms of the functor $\text{Sq}_{\bullet,\bullet}$, mentioned in Sect. 1.3, and its left adjoint, denoted $\mathfrak{L}^{Sq}$.

Having this explicit description of $\mathbb{S}^R$ allows to to establish the desired equivalence (1.2).