

## INTRODUCTION TO PART I: PRELIMINARIES

### WHY DO WE NEED THESE PRELIMINARIES?

**0.1.** None of the contents of Part I is original mathematics.

Chapter I.1 is a review of higher category theory and higher algebra, mostly following [Lu1] and [Lu4].

Chapter I.2 is a review of the basic definitions of derived algebraic geometry (derived schemes, Artin stack and general prestacks), mostly following [TV1, TV2].

Chapter I.3 is a review of the basics of quasi-coherent sheaves (there are no deep theorems there, so one can say that it is mostly folklore).

**0.2.** We wish to emphasize that by no means do these chapters supply a self-contained exposition of elements of the theory required for the rest of the book. Our goal is rather to give the reader a concise account of the most basic pieces of structure, in order to enable him/her to start reading the subsequent chapters.

Our hope is that once he/she gets started, he/she will gradually acquire the ability to look up or reconstruct the necessary bits of foundational material.

### 1. $\infty$ -CATEGORIES AND HIGHER ALGEBRA

**1.1.** Let us accept the inevitable: when we talk about algebraic geometry, we need to talk in the language of categories.

For one thing, geometric objects (such as schemes and their generalizations) form a category. But even more importantly, the flora to be found on these geometric objects (sheaves of various sorts) is a category. And there is no way to develop the theory of sheaves without using categories.

Since the introduction of the categorical language to the study of algebraic geometry by Grothendieck in the 1950's, and up until the late 2000's, the methods of *usual* (=ordinary) category theory sufficed for most purposes. People used either the *abelian* category of quasi-coherent sheaves or its derived category, which is a *triangulated category*.

However, there are some instances where triangulated categories are not enough. Perhaps the main example of this is the failure of gluing: one cannot glue the derived category of quasi-coherent sheaves on a scheme from just knowing it on an open cover.

Now, the problem of inadequacy of triangulated categories becomes even more acute in the context of *derived algebraic geometry* (DAG). So, having accepted the inevitability of categories for usual algebraic geometry, we now have no choice but accept the inevitability of  $\infty$ -categories if we want to work in DAG. This is further reinforced by the fact that the geometric objects themselves (derived schemes or, more generally, prestacks) now form an  $\infty$ -category.

**1.2.** In [Chapter I.1, Sects. 1 and 2] we give a concise review of the basics of  $\infty$ -categories.

We mostly focus on the syntax: how to use the language of  $\infty$ -categories. In other words, the reader does not have to be familiar with a particular model for  $\infty$ -categories, be it topological categories, simplicial categories, or the model that finally won the day—Joyal’s quasi-categories, put into action by Lurie in [Lu1].

We introduce the key notions of Cartesian/coCartesian fibration, Yoneda, limit/colimit, cofinality, left/right Kan extension, adjunction for functors.

**1.3.** In [Chapter I.1, Sects. 3 and 4] we give the first taste of *higher algebra*. We introduce the notions of *monoidal  $\infty$ -category* and of associative algebra inside a monoidal  $\infty$ -category. We also introduce the corresponding commutative notions.

We also introduce the corresponding notions of module (that is, a module category for a given monoidal  $\infty$ -category, and the notion of module for an associative algebra).

We then proceed to the discussion of duality. We discuss the notion of left/right dualizability of an object in a monoidal  $\infty$ -category, and the related notion of dualizability of left/right module over an algebra.

**1.4.** In [Chapter I.1, Sects. 5, 6, 7] we discuss the notion of *stable  $\infty$ -category*.

Stable  $\infty$ -categories are the higher categorical replacement of triangulated categories, i.e., this is where we really do algebra.

An operation that will play a key role in the book is that of *Lurie tensor product* of (co-complete) stable  $\infty$ -categories, that gives the totality of the latter, denoted,  $1\text{-Cat}_{\text{cont}}^{\text{St,coimpl}}$  a structure of symmetric monoidal  $\infty$ -category.

**1.5.** In [Chapter I.1, Sects. 8 and 9] we supply a framework for “really doing algebra”:

We talk about (symmetric) monoidal stable  $\infty$ -categories, i.e., associative (resp., commutative) algebra objects in the symmetric monoidal category  $1\text{-Cat}_{\text{cont}}^{\text{St,coimpl}}$ .

**1.6.** Finally, in [Chapter I.1, Sect. 10] we introduce the notion of *DG category*, i.e., a DG category that is equipped with a linear structure over a given ground field  $k$ .

## 2. BASICS OF DERIVED ALGEBRAIC GEOMETRY

In [Chapter I.2] we start our discussion of derived algebraic geometry proper, i.e., we introduce the  $\infty$ -category of the corresponding geometric objects.

**2.1.** We start with the category of (derived)<sup>1</sup> affine schemes over  $k$ , denoted  $\text{Sch}^{\text{aff}}$ , which is, by definition, the category opposite to that of *connective commutative DG algebras over  $k$* .

In [Chapter I.2, Sect. 1] we introduce the most general class of geometric objects: prestacks. The  $\infty$ -category of the latter is simply that of functors

$$(\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \text{Spc},$$

where  $\text{Spc}$  is the  $\infty$ -category of spaces (a.k.a.  $\infty$ -groupoids). I.e., a prestack is just something that has a Grothendieck functor of points.

All other geometric objects that we will consider (schemes, Artin stacks, etc.) will be prestacks. That is, for example, a scheme (resp., Artin stack) will be a prestack with certain properties (as opposed to additional pieces of structure).

<sup>1</sup>Henceforth the adjective ‘derived’ will be dropped, because everything will be derived.

In later Chapters of the book we will be interested in yet another class of prestacks–indschemes.

**2.2.** In [Chapter I.2, Sect. 2] we will introduce the descent condition with respect to the Zariski, étale or faithfully flat topology. We call prestacks that satisfy the descent condition *stacks*.

We study how the descent condition interacts with the basic properties that a prestack can possess (such as being locally of finite type).

**2.3.** In [Chapter I.2, Sect. 3] we introduce what is, arguably, the main object of study in derived algebraic geometry: (derived) schemes.

According to what was said above, we *do not* introduce schemes as locally ringed spaces. Rather, we define schemes as prestacks that admit an open covering by affine schemes.

**2.4.** In [Chapter I.2, Sect. 4] we introduce the hierarchy of  $k$ -Artin stacks,  $k \geq 0$ . We should say that we call a  $k$ -Artin stack for a particular  $k$  may diverge from elsewhere in the literature (for example, for us, a 0-Artin stack is a stack that is a (possibly infinite) disjoint of affine schemes). However, the union over all  $k$  produces the same class of objects. The advantage of our particular system of definitions is that it makes inductive proofs of various properties of  $k$ -Artin stacks very simple.

We should also point out that from the point of view of our hierarchy of  $k$ -Artin stacks, schemes are a red herring. They are more general than 0-Artin stacks, but are a tiny particular case of 1-Artin stacks.

### 3. QUASI-COHERENT SHEAVES

In [Chapter I.3] we introduce what is perhaps the main object of study of (derived) algebraic geometry: quasi-coherent sheaves.

**3.1.** In [Chapter I.3, Sect. 1] we start with the functor

$$\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}^* : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}, \quad S = \mathrm{Spec}(A) \rightsquigarrow A\text{-mod}, \quad (S' \xrightarrow{f} S) \rightsquigarrow f^*.$$

We apply the procedure of *right Kan extension* along the (Yoneda) embedding  $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$  and thus obtain a functor

$$\mathrm{QCoh}_{\mathrm{PreStk}}^* : (\mathrm{PreStk})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

Thus, for any prestack  $\mathcal{Y}$  we have a well-defined DG category  $\mathrm{QCoh}(\mathcal{Y})$  and for a morphism  $f : \mathcal{Y}' \rightarrow \mathcal{Y}$  we have a pullback functor  $f^* : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{QCoh}(\mathcal{Y}')$ .

**3.2.** Note, in particular, that if  $Z$  is a scheme, we obtain a category  $\mathrm{QCoh}(Z)$ . This definition of  $\mathrm{QCoh}$  of a scheme is equivalent to any other (correct) definition. However, we note that we *do not* approach it via first considering *all sheaves of  $\mathcal{O}$ -modules in Zariski topology*, and then passing to a subcategory. Instead, we directly glue  $\mathrm{QCoh}(Z)$  from affines.

A similar feature of our definition is also present in the case of Artin stacks.

**3.3.** In [Chapter I.3, Sect. 2] we study the functor of *direct image* for quasi-coherent sheaves

$$f_* : \mathrm{QCoh}(\mathcal{Y}') \rightarrow \mathrm{QCoh}(\mathcal{Y})$$

for a morphism  $f : \mathcal{Y}' \rightarrow \mathcal{Y}$ . By definition,  $f_*$  is the right adjoint of  $f^*$ , and exists for abstract reasons (the Adjoint Functor Theorem).

For a general morphism  $f$ , the functor  $f_*$  is very badly behaved. For example, it fails to satisfy the base change formula. However, by imposing some additional assumptions on  $f$  one can ensure that it is reasonable. One such assumption is that  $f$  should be schematic quasi-compact.

**3.4.** In [Chapter I.3, Sect. 3] we study the natural *right lax* symmetric monoidal structure on the functor  $\mathrm{QCoh}_{\mathrm{PreStk}}^*$ . Concretely, this structure amounts to (a compatible family of) functors

$$\mathrm{QCoh}(\mathcal{Y}_1) \boxtimes \mathrm{QCoh}(\mathcal{Y}_2) \rightarrow \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2), \quad \mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}.$$

We study the question of when the above functor is an equivalence.

The symmetric monoidal structure on  $\mathrm{QCoh}_{\mathrm{PreStk}}^*$  induces a symmetric monoidal structure on the category  $\mathrm{QCoh}(\mathcal{Y})$  for an individual prestack  $\mathcal{Y} \in \mathrm{PreStk}$ .

We study how various properties of a prestack  $\mathcal{Y}$  reflect in properties of  $\mathrm{QCoh}(\mathcal{Y})$  (compact generation, dualizability, rigidity).