INTRODUCTION TO PART III: INF-SCHEMES

1. Why inf-schemes?

1.1. The primary new geometric object considered in this book is the notion of *inf-scheme*. Let us start with the definition: an inf-scheme is a prestack $\mathcal{X}$ such that:

(a) $\mathcal{X}$ is laft (locally almost of finite type, see [Chapter I.2, Sect. 1.7] for what this means);

(b) $\mathcal{X}$ admits deformation theory (i.e., has reasonable infinitesimal properties, see Chapter 1 of this part or Sect. 2 of this Introduction);

(c) The underlying reduced prestack ${}^{\mathrm{red}}\mathcal{X}$ is a (reduced) scheme.

Let us explain what are the favorable properties enjoyed by inf-schemes and how one is led to this definition.

1.2. Our initial goal was to have a geometric framework in which we could talk simultaneously about $\mathrm{QCoh}(X)$ and $\mathrm{Dmod}(X)$ (where $X \in \mathrm{Sch}_{\mathrm{aft}}$) equipped with the pair of adjoint functors of forgetting the D-module structure to that of an $\mathcal{O}$-module, and inducing an $\mathcal{O}$-module to a D-module.

However, as was explained in [GaRo2], if we replace $\mathrm{QCoh}(X)$ by $\mathrm{IndCoh}(X)$, the resulting adjoint pair has much better properties. So what we really want to consider is the functors

$$\mathbf{ind}_X : \mathrm{IndCoh}(X) \leftrightarrows \mathrm{Dmod}(X) : \mathbf{oblv}_X, \tag{1.1}$$

and their compatibility with the direct and inverse functors on $\mathrm{IndCoh}(-)$ and $\mathrm{Dmod}(-)$ for maps between schemes.

According to [GaRo2], the category $\mathrm{Dmod}(X)$ is *defined* as $\mathrm{IndCoh}(X_{\mathrm{dR}})$, where X_{dR} is the de Rham prestack of X (i.e., $\mathrm{Maps}(S, X_{\mathrm{dR}}) = \mathrm{Maps}({}^{\mathrm{red}}S, X)$).

So it is natural to set up the sought-for theory as IndCoh of a certain class of prestacks that contains schemes and de Rham prestacks of schemes. We would like the adjoint pair (1.1) to be given by the push-forward/pullback adjunct

$$(p_{\mathrm{dR},X})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \leftrightarrows \mathrm{IndCoh}(X_{\mathrm{dR}}) : (p_{\mathrm{dR},X})^!,$$

where $p_{\mathrm{dR},X}$ denotes the tautological map $X \to X_{\mathrm{dR}}$.

1.3. Let us try to be minimalistic and consider only prestacks $\mathcal{X}$ such that ${}^{\mathrm{red}}\mathcal{X}$ is a (reduced) scheme. Let us denote the sought-for class of prestacks by $\mathcal{C}$, and let us list some constructions that we would like to be possible within $\mathcal{C}$.

(i) For a map $f : \mathcal{X}_1 \to \mathcal{X}_2$ between objects of $\mathcal{C}$ we should have a well-defined push-forward functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2),$$

that satisfies base change with respect to the !-pullback. I.e., IndCoh restricted to $\mathcal{C}$ should extend to a functor out of the category of correspondences on $\mathcal{C}$.

Date: January 9, 2016.

(ii) Since we want to talk about base change, $\mathcal{C}$ should contain fiber products. Now, for a scheme X, the fiber product $X \underset{X_{\mathrm{dR}}}{\times} X$ is the formal completion $X^\wedge$ of X in $X \times X$. Hence, it is natural to ask that $\mathcal{C}$ contain *formal schemes*, i.e., *ind-schemes*, whose underlying reduced prestacks are (reduced) schemes.

(iii) Having included in $\mathcal{C}$ all formal schemes, one's appetite grows a little more. Let $\mathcal{G}$ be a formal groupoid over a scheme X that belongs to $\mathcal{C}$. One would like to be able to form the quotient of X by $\mathcal{G}$, which is still a prestack in our class $\mathcal{C}$. For example, the quotient of X of $X^\wedge$ should give us back X_{dR}.

(iv) Finally, we would like to have a description of *formal group-objects* in $\mathcal{C}$ over a scheme X in terms of their Lie algebras. As will be explained in [Chapter III.1], the latter, being tangent spaces at the unit section, are objects of $\mathrm{IndCoh}(X)$. So by a 'Lie algebra' we should understand a Lie algebra in the symmetric monoidal category $\mathrm{IndCoh}(X)$ with respect to the $\overset{!}{\otimes}$-tensor product.

1.4. As we will eventually see in Part IV of the book, properties (iii) and (iv) will force us to include into our class $\mathcal{C}$ all inf-schemes $\mathcal{X}$, defined as above.

However, one can consider it a strike of luck that as general a definition as one given in Sect. 1.1 above produces a workable notion, i.e., IndCoh on inf-schemes has the properties mentioned above.

2. Deformation theory

The definition of inf-schemes involves the notion of *admitting deformation theory*. In Chapter 1 of this part of the book we make a review of deformation theory.

2.1. A prestack $\mathcal{X}$ is said to admit deformation theory if:

(i) $\mathcal{X}$ is convergent, i.e., for $S \in \mathrm{Sch}$, the map

$$\mathrm{Maps}(S, \mathcal{X}) \to \lim_n \mathrm{Maps}({}^{\leq n}S, \mathcal{X})$$

is an isomorphism. (In other words, the values of $\mathcal{X}$ on all affine schemes are completely determined by its values on eventually coconnective affine schemes.)

(ii) For a push-out diagram

$$\begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2', \end{array}$$

of affine schemes, where the map $S_1 \to S_1'$ (and, hence, also $S_2 \to S_2'$) is a *nilpotent embedding*, the corresponding diagram

$$\begin{array}{ccc} \mathrm{Maps}(S_1, \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2, \mathcal{X}) \\ \uparrow & & \uparrow \\ \mathrm{Maps}(S_1', \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2', \mathcal{X}), \end{array}$$

is a pull-back diagram.

In condition (ii), we remind that a map of affine schemes $S \to S'$ is said to be nilpotent embedding if the corresponding map of classical schemes ${}^{\mathrm{cl}}S \to {}^{\mathrm{cl}}S'$ is a closed embedding with a nilpotent ideal of definition.

We also remind that if

$$S_1 = \mathrm{Spec}(A_1),\ S_2 = \mathrm{Spec}(A_2),\ S_1' = \mathrm{Spec}(A_1'),\ S_2' = \mathrm{Spec}(A_2'),$$

then to be a push-out diagram means that the map

$$A_2' \to A_1' \underset{A_1}{\times} A_2$$

should be an isomorphism in $\mathrm{Vect}^{\leq 0}$.

2.2. The above way of formulating what it means to admit deformation theory may at first appear mysterious (why these push-outs, and who has ever seen push-outs in algebraic geometry anyway?). And indeed, the more common definition, and the one we give in [Chapter III.1, Sect. 7] is different (but, of course, equivalent). The advantage of the definition given above is that it is concise.

It turns out that the infinitesimal behavior of a prestack that admits deformation theory is goverened by its *pro-cotangent complex*, where the latter is a functorial assignment for any

$$S \xrightarrow{x} \mathcal{X} \quad \in \quad \mathrm{Sch}_{/\mathcal{X}}$$

of an object $T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S^-))$. The precise meaning of the words 'governed by' is explained in the Introduction to [Chapter III.1].

Here we just say informally that, say when $\mathcal{X}$ is laft, the knowledge of the values of $\mathcal{X}$ on *reduced affine schemes* and some *linear* data (expressible in terms of the pro-cotangent complex of $\mathcal{X}$) allows to recover the values of $\mathcal{X}$ on all schemes.

2.3. Going back to inf-schemes, requiring the condition that they admit deformation theory makes them reasonable objects: by condition (c) in Sect. 1.1, the values of an inf-scheme $\mathcal{X}$ on a reduced scheme are given by a scheme $X = {}^{\mathrm{red}}\mathcal{X}$, and when we want evaluate $\mathcal{X}$ on an arbitrary scheme S, the fibers of the map

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\mathrm{red}}S, \mathcal{X}) \simeq \mathrm{Maps}({}^{\mathrm{red}}S, X)$$

are controlled by linear data.

3. Inf-schemes

In Chapter 2 we introduce inf-schemes and study their basic properties. The main results of this chapter are Theorems 4.1.3 and 4.2.5. Here we will informally explain what these theorems say.

3.1. Let $\mathcal{X}$ be an inf-scheme such that ${}^{\mathrm{red}}\mathcal{X} = X \in {}^{\mathrm{red}}\mathrm{Sch}$. Consider the full subcategory

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to }\mathcal{X}} \subset (\mathrm{Sch})_{/\mathcal{X}}$$

consisting of those maps $Z \to \mathcal{X}$ that are *nil-isomorphisms*, i.e., induce an isomorphism

$${}^{\mathrm{red}}Z \to {}^{\mathrm{red}}\mathcal{X} = X.$$

The assertion of [Chapter III.2, Theorem 4.1.3] (in the guise of [Chapter III.2, Corollary 4.3.3]) is that the resulting map

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to }\mathcal{X}}}{\mathrm{colim}} Z \to \mathcal{X}$$

is an isomorphism of prestacks.

The latter means, by definition, that for an *affine* scheme S, the map

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to }\mathcal{X}}}{\mathrm{colim}} \mathrm{Maps}(S, Z) \to \mathrm{Maps}(S, \mathcal{X}) \tag{3.1}$$

is an isomorphism.

Equivalently, for an *affine* scheme S and a map $S \to \mathcal{X}$, the category $\text{Factor}(x, \text{aft}, \text{nil-isom})$ of its factorizations as

$$S \to Z \to \mathcal{X}$$

with $Z \in \text{Sch}_{\text{aft}}$ and the map $Z \to \mathcal{X}$ being a nil-isomorphism, is contractible.

3.2. Let us emphasize, however, that it is *not* true that the map (3.1) is an isomorphism if S is non-affine. Equivalently, it is *not* true that the category $\text{Factor}(x, \text{aft}, \text{nil-isom})$ is contractible if S is non-affine.

We remark, however, that in Sects. 1 and 2 we study a more restricted class of objects, commonly called *formal schemes* (but we choose to call *nil-schematic ind-schemes*), for which the map (3.1) is an isomorphism (and the category $\text{Factor}(x, \text{aft}, \text{nil-isom})$ is contractible).

In fact, we consider the full subcategory

$$(\text{Sch}_{\text{aft}})_{\text{nilp-embed into } \mathcal{X}} \subset (\text{Sch}_{\text{aft}})_{\text{nil-isom to } \mathcal{X}},$$

consisting of those $Z \to \mathcal{X}$ that are *nilpotent embeddings*.

We show that if $\mathcal{X}$ is a nil-schematic ind-scheme, then the category $(\text{Sch}_{\text{aft}})_{\text{nil-embedinto } \mathcal{X}}$ is *filtered* and the map

$$\underset{Z \in (\text{Sch}_{\text{aft}})_{\text{nilp-embed into } \mathcal{X}}}{\text{colim}} \text{Maps}(S, Z) \to \text{Maps}(S, \mathcal{X})$$

is an isomorphism for *any* (i.e., not necessarily affine) $S \in \text{Sch}$.

Equivalently, for a given $S \overset{x}{\to} \mathcal{X}$, the corresponding category $\text{Factor}(x, \text{aft}, \text{nilp-embed})$ is contractible.

3.3. We will now explain the content of the second main result of this Chapter, namely, [Chapter III.2, Theorem 4.2.5] (in its guise as [Chapter III.2, Corollary 4.4.6]).

Let $\mathcal{X}$ be an inf-scheme, such that $X := {}^{\text{red}}\mathcal{X}$ is affine. It follows from Theorem 4.1.3 that $\mathcal{X}$, when viewed as a functor

$$(\text{Sch}^{\text{aff}})^{\text{op}} \to \text{Spc}$$

is completely determined[1] by its restriction to the category

$$\text{Sch}^{\text{aff}} \underset{{}^{\text{red}}\text{Sch}^{\text{aff}}}{\times} \{X\}.$$

In words, the above category is that of affine schemes, whose reduced subscheme is of finite type and is equipped with a map to X.

Now, [Chapter III.2, Theorem 4.2.5] is a converse to the above assertion. Namely, it says that any functor

$$\left(\text{Sch}^{\text{aff}} \underset{{}^{\text{red}}\text{Sch}^{\text{aff}}}{\times} \{X\}\right)^{\text{op}} \to \text{Spc},$$

that satisfies deformation theory-like conditions, gives rise to an inf-scheme $\mathcal{X}$ with ${}^{\text{red}}\mathcal{X} \simeq X$.

4. Ind-coherent sheaves on inf-schemes

Chapter 3 of this part is a central one for this book. In it we study the category IndCoh on inf-schemes.

[1]Technically, by 'completely determined' we mean 'is the left Kan extension from'.

4.1. What makes this theory manageable is [Chapter III.2, Theorem 4.1.3] mentioned above. Namely, when we write

$$\mathcal{X} \simeq \underset{\alpha \in A}{\operatorname{colim}} Z_\alpha,$$

where $Z_\alpha \in \operatorname{Sch}_{\mathrm{aft}}$ and the transition maps $f_{\alpha,\beta} : Z_\alpha \to Z_\beta$ are nil-isomorphisms, we have:

$$\operatorname{IndCoh}(\mathcal{X}) \simeq \lim_{\alpha \in A^{\mathrm{op}}} Z_\alpha,$$

where the limit is formed with respect to the functors of !-pullback

$$f_{\alpha,\beta} \rightsquigarrow f^!_{\alpha,\beta} : \operatorname{IndCoh}(Z_\beta) \to \operatorname{IndCoh}(Z_\alpha).$$

The above presentation of $\operatorname{IndCoh}(\mathcal{X})$ as a limit tells what the objects and morphisms are in this category. However, since the functors $f^!_{\alpha,\beta}$ admit left adjoints, by [Chapter I.1, Proposition 2.5.7] we also have:

$$\operatorname{IndCoh}(\mathcal{X}) \simeq \underset{\alpha \in A}{\operatorname{colim}} \operatorname{IndCoh}(Z_\alpha), \tag{4.1}$$

where the colimit is formed with respect to the push-forward functors

$$f_{\alpha,\beta} \rightsquigarrow (f_{\alpha,\beta})^{\operatorname{IndCoh}_*} : \operatorname{IndCoh}(Z_\alpha) \to \operatorname{IndCoh}(Z_\beta).$$

The latter presentation tells us what it takes to construct a functor *out of* $\operatorname{IndCoh}(\mathcal{X})$. Namely, such a functor amounts to a compatible family of functors out of $\operatorname{IndCoh}(X_\alpha)$.

4.2. The main construction in Chapter 3 is that of the direct image functor. Namely, let $f : \mathcal{X}^1 \to \mathcal{X}^2$ be a map between inf-schemes. We want to construct the functor

$$f^{\operatorname{IndCoh}}_* : \operatorname{IndCoh}(\mathcal{X}^1) \to \operatorname{IndCoh}(\mathcal{X}^2). \tag{4.2}$$

For example, when $\mathcal{X}^i = X^i_{\mathrm{dR}}$ where $X^i \in \operatorname{Sch}_{\mathrm{ft}}$, the resulting functor will be the de Rham (D-module) direct image.

In Theorem 4.3.2 we show that there exists a functor (4.2) that is *uniquely* characterized by the requirement that whenever

$$\begin{array}{ccc} Z^1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z^2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

is a commutative diagram with $Z^i \in \operatorname{Sch}_{\mathrm{aft}}$ and the maps g^i nil-isomorphisms, then the diagram of functors

$$\begin{array}{ccc} \operatorname{IndCoh}(Z^1) & \xrightarrow{(g_1)^{\operatorname{IndCoh}}_*} & \operatorname{IndCoh}(\mathcal{X}_1) \\ {\scriptstyle (f')^{\operatorname{IndCoh}}_*}\downarrow & & \downarrow{\scriptstyle f^{\operatorname{IndCoh}}_*} \\ \operatorname{IndCoh}(Z^2) & \xrightarrow{(g_2)^{\operatorname{IndCoh}}_*} & \operatorname{IndCoh}(\mathcal{X}_2) \end{array}$$

commutes, where the functors $(g_1)^{\operatorname{IndCoh}}_*$ are the ones from the presentation (4.1).

4.3. Having constructed direct images, we show that they satisfy the proper base change property. Then, by applying the general machinery from [Chapter V.2, Sect. 1], we show that IndCoh, viewed as a functor out of the category of correspondences

$$\operatorname{IndCoh}_{(\operatorname{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{corr:all;all}}} : (\operatorname{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{corr:all;all}} \to \operatorname{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$$

uniquely extends to a functor

$$\operatorname{IndCoh}_{(\operatorname{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{corr:all;all}}} : (\operatorname{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{corr:all;all}} \to \operatorname{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}. \tag{4.3}$$

5. Crystals and D-modules

In Chapter 4 we apply the theory of IndCoh on inf-schemes to construct the theory of D-modules, viewed as a functor

$$\text{Dmod}_{(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{corr:all;all}}} : (\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{corr:all;all}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}}. \quad (5.1)$$

5.1. Namely, we stipulate that for $X \in \text{Sch}_{\text{aft}}$

$$\text{Dmod}(X) := \text{IndCoh}(X_{\text{dR}}). \quad (5.2)$$

Now, the operation $X \mapsto X_{\text{dR}}$ defines a functor

$$(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{corr:all;all}} \to (\text{indinfSch}_{\text{laft}})^{\text{ind-proper}}_{\text{corr:all;all}}.$$

Thus, composing this functor with (4.3), we obtain the desired functor (5.1).

5.2. The definition of D-modules as in (5.2) gives also a natural framework for the induction functor

$$\text{IndCoh}(X) \to \text{Dmod}(X), \quad (5.3)$$

left adjoint to the tautological forgetful functor.

Namely, the functor (5.3) is the functor of direct image with respect to the tautological morphism

$$X \to X_{\text{dR}}.$$

5.3. In Sect. 4 of this Chapter, we explain why the definition of D-modules (5.2) is the right thing to do.

Namely, we show that when X is a smooth affine scheme, $\text{IndCoh}(X_{\text{dR}})$ does indeed recover the category of modules over the ring Diff_X of differential operators on X.

Moreover, we show that for a map $f : X \to Y$ between smooth schemes, the functors

$$(f_{\text{dR}})^! : \text{IndCoh}(Y_{\text{dR}}) \to \text{IndCoh}(X_{\text{dR}}) \text{ and } (f_{\text{dR}})^{\text{IndCoh}}_* : \text{IndCoh}(X_{\text{dR}}) \to \text{IndCoh}(Y_{\text{dR}})$$

correspond to the usual functors of pullback and push-forward on the corresponding categories of D-modules.