INTRODUCTION TO PART V: CATEGORIES OF CORRESPONDENCES

1. Why correspondences?

This part introduces one of the two main innovations in this book–the $(\infty, 2)$-category of correspondences as a way to encode bi-variant functors and the six functor formalism. This idea was suggested to us by J. Lurie.

1.1. Let us start with a category $\mathbf{C}$ (with finite limits), equipped with two classes of morphisms *vert* and *horiz* (both closed under composition). The category of correspondences is designed to perform the following function.

Suppose we want to encode a *bi-variant* functor Φ from $\mathbf{C}$ to some target $(\infty, 1)$-category $\mathbb{S}$. I.e., to $\mathbf{c} \in \mathbf{C}$ we assign $\Phi(\mathbf{c}) \in \mathbb{S}$, and to a 1-morphism $\mathbf{c}_1 \overset{\gamma}{\to} \mathbf{c}_2$ in $\mathbf{C}$, we assign a 1-morphism

$$\Phi(\gamma) : \Phi(\mathbf{c}_1) \to \Phi(\mathbf{c}_2) \text{ if } \gamma \in \mathit{vert}$$

and a 1-morphism

$$\Phi^!(\gamma) : \Phi(\mathbf{c}_2) \to \Phi(\mathbf{c}_1) \text{ if } \gamma \in \mathit{horiz},$$

equipped with the following pieces of structure:

(1) Compatibility of both $\Phi(-)$ and $\Phi^!(-)$ with compositions of 1-morphisms in $\mathbf{C}$;

(2) For a Cartesian square

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"'] & \mathbf{c}_{0,0} \arrow[d, "\beta_0"] \\ \mathbf{c}_{1,1} \arrow[r, "\alpha_1"] & \mathbf{c}_{1,0} \end{tikzcd}$$

with vertical arrows in *vert* and horizontal arrows in *horiz*, we are supposed to be *given* an identification (called *base change isomorhism*)

$$\Phi(\beta_1) \circ \Phi^!(\alpha_0) \simeq \Phi^!(\alpha_1) \circ \Phi(\beta_0).$$

The above pieces of data must satisfy a *homotopy-coherent system of compatibilities.* The partial list consists of the following:

- The data making Φ into a functor $\mathbf{C}_{vert} \to \mathbb{S}$, and the data making $\Phi^!$ into a functor $(\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}$.
- The compatibility of base-change isomorphisms with compositions;

However, the above is really only the beginning of an infinite tail of compatibilities, as it always happens in higher category theory.

So, if we want a workable theory, we need to find a convenient way to package this information, preferably in terms of one of the existing packages, such as the notion of functor between two given $(\infty, 1)$-categories.

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The category $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ allows us to do just that. Namely, we show ([Chapter V.1, Theorem 2.1.3]) that the datum of a functor as above is equivalent to the datum of a functor

$$\mathrm{Corr}(\mathbf{C})_{vert;horiz} \to \mathbb{S}.$$

1.2. The idea of $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ is very simple. Its objects are the same as objects of $\mathbf{C}$. But its 1-morphisms are diagrams

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ \mathbf{c}_1 \end{tikzcd} \tag{1.1}$$

with the vertical arrow in *vert* and the horizontal arrow in *horiz*.

The composition of the 1-morphism (1.1) with a 1-morphism

$$\begin{tikzcd} \mathbf{c}_{1,2} \arrow[r] \arrow[d] & \mathbf{c}_1 \\ \mathbf{c}_2 \end{tikzcd}$$

is the 1-morphism

$$\begin{tikzcd} \mathbf{c}_{1,2} \underset{\mathbf{c}_1}{\times} \mathbf{c}_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ \mathbf{c}_2. \end{tikzcd}$$

What may be a little less obvious is to how to give the definition of $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ in the ∞-context (and without appealing to a particular model of $(\infty,1)$-categories, i.e., we do not want to talk about simplicial sets).

The definition of $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ is the subject of [Chapter V.1, Sect. 1]. In fact, the construction is not difficult and quite natural: it is formulated in terms of the interpretation of $(\infty,1)$-categories as *complete Segal spaces*.

1.3. At this point let us comment on the relationship between our approach and (our interpretation of) [LZ1, LZ2].

Consider the following bi-simplicial space $\mathrm{Grid}_{\bullet,\bullet}(\mathbf{C})$: its space $\mathrm{Grid}_{m,n}(\mathbf{C})$ of $[m]\times[n]$-simplicies are $m\times n$-grids of objects of $\mathbf{C}$, in which every square is Cartesian, all vertical arrows are in *vert* and all horizontal arrow are in *horiz*.

Then [Chapter V.1, Theorem 2.1.3] says that the datum of a functor

$$\mathrm{Corr}(\mathbf{C})_{vert;horiz} \to \mathbb{S}$$

is equivalent to that of a map of bi-simplicial spaces

$$\mathrm{Grid}_{\bullet,\bullet}(\mathbf{C}) \to \mathrm{Maps}([\bullet]\times[\bullet],\mathbb{S}).$$

The authors of [LZ1, LZ2] construct their datum in terms of the latter map of bi-simplicial spaces.

2. The six functor formalism

Let us now explain how the $(\infty,2)$-category of correspondences encodes the six functor formalism.

2.1. The setup. The general setup for the six functor formalism is the following. Suppose that we have a category $\mathbf{C}$ of 'geometric objects', e.g., the category of topological spaces, schemes, prestacks, etc. To each object $X \in \mathbf{C}$, we associate a category

$$X \rightsquigarrow \mathrm{Sh}(X) \in \mathrm{DGCat}_{\mathrm{cont}},$$

of 'sheaves on X', e.g., $\mathrm{IndCoh}(X)$ or $\mathrm{Dmod}(X)$. This assignment comes with the following additional data, in particular making it natural in $X \in \mathbf{C}$:

(1) (functoriality) For every map $f : X \to Y \in \mathbf{C}$, there are two pairs of adjoint functors

$$f_! : \mathrm{Sh}(X) \rightleftarrows \mathrm{Sh}(Y) : f^!, \text{ and } f^* : \mathrm{Sh}(Y) \rightleftarrows \mathrm{Sh}(X) : f_*$$

which are natural in f, i.e. each of them is given by a functor $\mathbf{C} \to \mathrm{DGCat}_{\mathrm{cont}}$ (or $\mathbf{C}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$). There are four of the six functors in the six functor formalism.

Note the data of an adjoint pair is uniquely determined by one of the functors. In the case of IndCoh and D-modules, we only have the right adjoint functors $f^!$ and f_* exist in general. For this reason, we will describe the formalism in terms of these functors without explicit reference to their adjoints.

(2) (proper adjunction) Given $f : X \to Y \in \mathbf{C}$, there is a natural transformation

$$f_! \to f_*,$$

which is natural in f and is an isomorphism when f is proper.

Equivalently, there is a natural transformation

$$\mathrm{id} \to f^! \circ f_*,$$

which is the unit of an adjunction when f is proper.

(3) (open adjunction) If $f : X \to Y \in \mathbf{C}$ is an open immersion, there is a natural isomorphism

$$f^! \circ f_* \simeq \mathrm{id},$$

which is the counit of an adjunction. In particular, in this case, we have an isomorphism

$$f^! \simeq f^*.$$

(4) (proper base change) For a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in $\mathbf{C}$, there is a natural base change isomorphism

$$f'_* \circ g'^! \simeq g^! \circ f_*.$$

In the case that f is proper (resp. open), this isomorphism is given by the natural transformation arising from proper (resp. open) adjunction above.

(5) (duality) For each $X \in \mathbf{C}$, the DG category $\mathrm{Sh}(X)$ is self-dual; see [Chapter II.2, Sect. 4.6] for an explanation of how this recovers the usual Verdier or Serre duality on sheaves. Moreover, for each morphism $f : X \to Y \in \mathbf{C}$, the functors $f^!$ and $f_!$ are dual to f_* and f^*, respectively.

(6) (tensor structure) For each $X, Y \in \mathbf{C}$ we have a functor

$$\boxtimes : \mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y),$$

natural in X and Y.

Moreover, $*$-pullback along the diagonal $X \to X \times X$ defines a *closed* symmetric monoidal structure on $\mathrm{Sh}(X)$ for every $X \in \mathbf{C}$, i.e., each $\mathrm{Sh}(X)$ comes with a tensor product $\otimes$ and an inner hom $\underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}$ functor – these are the remaining two of the six functors. Furthermore, for every map $f : X \to Y \in \mathbf{C}$, the functor

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$$

is equipped with a symmetric monoidal structure.

In the case of IndCoh and D-modules, the functor $\boxtimes$ above is an isomorphism in the case of (ind-inf-)schemes X and Y. However, we only have the !-pullback functor and so we can only define the dual !-tensor structure $\overset{!}{\otimes}$ on $\mathrm{Sh}(X)$ given by !-pullback along the diagonal. In this case, the functor

$$f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$$

is equipped with a symmetric monoidal structure with respect to the $\overset{!}{\otimes}$ tensor product.

(7) (projection formula) Let $f : X \to Y$ be a morphism in $\mathbf{C}$. Since f^* is a tensor functor by the above, we have that $\mathrm{Sh}(X)$ is a module category over the tensor category $\mathrm{Sh}(Y)$. We further require that the functor $f_!$ be equipped with the structure of a functor of module categories over $\mathrm{Sh}(Y)$. In particular, from this we obtain the familiar natural isomorphisms:

$$f_!(M \otimes f^*(N)) \simeq f_!(M) \otimes N,$$
$$\underline{\mathrm{Hom}}(f_!(M), N) \simeq f_*(\underline{\mathrm{Hom}}(M, f^!(N))), \text{ and}$$
$$f^!(\underline{\mathrm{Hom}}(M, N)) \simeq \underline{\mathrm{Hom}}(f^*(M), f^!(N)),$$

for $M \in \mathrm{Sh}(X)$ and $N \in \mathrm{Sh}(Y)$.

In the case of IndCoh and D-modules, where we have the dual tensor product $\overset{!}{\otimes}$, we require that the functor f_* be equipped with the structure of a functor of module categories over $\mathrm{Sh}(Y)$ with respect to the $\overset{!}{\otimes}$ tensor product. In particular, we obtain the projection formula

$$f_*(M \overset{!}{\otimes} f^!(N)) \simeq f_*(M) \overset{!}{\otimes} N,$$

dual to the one above.

2.2. As explained in Sect. 1, the data of functoriality and proper base change above is equivalent to the data of a functor of $(\infty, 1)$-categories

$$\mathrm{Sh} : \mathrm{Corr}(\mathbf{C})_{\mathrm{all};\mathrm{all}} \to \mathrm{DGCat}_{\mathrm{cont}};$$

namely, an object $X \in \mathrm{Corr}(\mathbf{C})_{\mathrm{all};\mathrm{all}}$ maps to $\mathrm{Sh}(X)$ and a morphism

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

maps to $g_* \circ f^! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

The idea is that we can enlarge $\mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$ to a symmetric monoidal $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$ so that *all* of the above data will be encoded by a symmetric monoidal functor of $(\infty,2)$-categories

$$\mathrm{Sh} : \mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\text{2-Cat}}_{\mathrm{cont}}. \tag{2.1}$$

Suppose that we are in the situation of Sect. 1.1, and in addition to *vert* and *horiz*, we are given a third class of 1-morphisms

$$adm \subset vert \cap horiz.$$

We define the $(\infty,2)$-category

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

so that its underlying $(\infty,1)$-category is the $(\infty,1)$-category $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ discussed above, but we now allow non-invertible 2-morphisms. Namely, a 2-morphism from the 1-morphism (1.1) to the 1-morphism

$$\begin{tikzcd} \mathbf{c}'_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ \mathbf{c}_1 & \end{tikzcd}$$

is a commutative diagram

$$\begin{tikzcd} \mathbf{c}_{0,1} \arrow[dr, "\gamma"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathbf{c}'_{0,1} \arrow[r] \arrow[d] & \mathbf{c}_0 \\ & \mathbf{c}_1 & \end{tikzcd}$$

with $\gamma \in adm$.

If $\mathbf{C}$ has a Cartesian symmetric monoidal structure that preserves each of the subcategories $adm, vert$ and $horiz$, then it induces a symmetric monoidal structure on the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$. In particular, in the situation of Sect. 2.1, we obtain that the $(\infty,2)$-category

$$\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$$

has a canonical symmetric monoidal structure such that the functor

$$\mathbf{C} \to \mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$$

given by "vertical morphisms" is symmetric monoidal with respect to the Cartesian symmetric monoidal structure on $\mathbf{C}$.

2.3. We will now explain how to recover all of the data in Sect. 2.1 from the data of the functor (2.1). We have already seen how functoriality and proper base change is encoded as a functor out of correspondences.

Proper adjunction.

Let $f : X \to Y \in \mathbf{C}$. In this case, we have that the functor $f_* \circ f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(Y)$ is the image under Sh of the morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Y & \end{tikzcd}$$

in $\mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$. Similarly, $f^! \circ f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$ is given by the image of the composite

$$\begin{tikzcd} X \underset{Y}{\times} X \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \\ X \arrow[r] \arrow[d, equal] & Y & \\ X & & \end{tikzcd}$$

If the diagonal morphism $X \to X \underset{Y}{\times} X$ is proper (as is the case with a separated morphism of schemes), we obtain the desired natural transformation

$$\mathrm{id} \to f^! \circ f_*.$$

Furthermore, if the map $f : X \to Y$ is proper, we also obtain a natural transformation $f_* \circ f^! \to \mathrm{id}$ and it is easy to see that the two natural transformations give the unit and counit of an adjunction.

2.3.1. *Open adjunction.* Similarly, if $f : X \to Y$ is an open embedding, we have that

$$X \underset{Y}{\times} X \simeq X$$

and therefore we obtain the desired isomorphism

$$f^! \circ f_* \simeq \mathrm{id}\,.$$

The assertion that this isomorphism gives a counit of an adjunction is an additional *condition.*

Duality.

A key feature of the symmetric monoidal category $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$ is that every object X is self-dual. In particular, it is easy to see that the morphisms

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Delta"'] & * \\ X \times X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & X \times X \\ * & \end{tikzcd}$$

give the unit and counit maps, respectively. Applying the symmetric monoidal functor Sh, we obtain that the DG category $\mathrm{Sh}(X)$ is self-dual.

Moreover, it is straightforward to check that given a map $f : X \to Y \in \mathbf{C}$, the morphisms

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \\ X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \\ Y & \end{tikzcd}$$

are dual in $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$. Hence the functors $f^!$ and f_* are dual to each other.

Tensor structure.

The symmetric monoidal structure on the functor Sh gives a natural isomorphism

$$\boxtimes : \mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \xrightarrow{\sim} \mathrm{Sh}(X \times Y);$$

in the case that Sh is only right-lax symmetric monoidal, we would only have a functor.

Moreover, by construction of the symmetric monoidal structure on $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$, we have that the functor

$$\mathbf{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$$

given by 'horizontal morphisms' is symmetric monoidal, where the symmetric monoidal structure on $\mathbf{C}^{\mathrm{op}}$ is given by coproduct. In particular, every object $X \in \mathbf{C}^{\mathrm{op}}$ has a canonical structure of a commutative algebra with multiplication given by the opposite of the diagonal map

$$(X \to X \times X)^{\mathrm{op}}$$

(see [Chapter I.1, Sect. 5.1.8]). Thus, $\mathrm{Sh}(X)$ carries a symmetric monoidal structure $\overset{!}{\otimes}$ given by !-restriction along the diagonal $\Delta : X \to X \times X$.

Projection formula.

Suppose that we have an object $Y \in \mathbf{C}$. By the above, we have that Y has a canonical structure of a commutative algebra object in $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$. Furthermore, if $f : X \to Y$ is a morphism in Y, we have that X has a canonical structure of a module over Y. It is straightforward to see that in this case the morphism

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \\ Y & \end{tikzcd}$$

has the structure of a morphism of Y-modules in $\mathrm{Corr}(\mathbf{C})^{\mathrm{proper}}_{\mathrm{all;all}}$. In particular, applying the symmetric monoidal functor Sh, we obtain that the functor

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

has the structure of a functor of $\mathrm{Sh}(Y)$-modules, as desired.

3. Constructing functors

Having constructed the categories $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$ and $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$, our next problem is how to construct functors

$$\mathrm{Corr}(\mathbf{C})_{vert;horiz} \to 1\text{-}\mathrm{Cat}\,.$$

In our main application, $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$, $\mathbb{S} = 1\text{-}\mathrm{Cat}$, and Φ is supposed to send a scheme X to the category $\mathrm{IndCoh}(X)$. We take *vert* and *horiz* to be all morphisms in $\mathrm{Sch}_{\mathrm{aft}}$.

3.1. It turns out, however, that in order to construct functors out of $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$, it is convenient (and necessary, if one wants to retain canonicity) to enlarge it to an $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

Suppose that $\mathbb{S}$ is an $(\infty,2)$-category. A functor

$$\Phi : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S}$$

encodes the following data (in addition to that of its restriction to $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$):

For a 1-morphism $(\mathbf{c} \overset{\gamma}{\to} \mathbf{c}') \in adm$, the 1-morphism

$$\Phi^!(\gamma) : \Phi(\mathbf{c}') \to \Phi(\mathbf{c})$$

in $\mathbb{S}$ identifies with the *right adjoint* of

$$\Phi(\gamma) : \Phi(\mathbf{c}) \to \Phi(\mathbf{c}').$$

We recall that the notion of adjoint morphisms makes sense in an arbitrary $(\infty,2)$-category.

3.2. The above 2-categorical enhancement plays a crucial role for the following reason.

Suppose that $horiz \subset vert$, and consider the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{horiz}_{vert;horiz}$. We have a tautological functor

$$\mathbf{C}_{vert} \simeq \mathrm{Corr}(\mathbf{C})_{vert;\mathrm{isom}} \to \mathrm{Corr}(\mathbf{C})_{vert;horiz} \to \mathrm{Corr}(\mathbf{C})^{horiz}_{vert;horiz}.$$

Then the basic result ([Chapter V.1, Theorem 3.2.2]) is that for any target $\mathbb{S}$, restriction along the above functor identifies the space of functors

$$\mathrm{Corr}(\mathbf{C})^{horiz}_{vert;horiz} \to \mathbb{S}$$

with the full subspace of functors

$$\mathbf{C}_{vert} \to \mathbb{S},$$

consisting of those functions for which for every $\alpha \in horiz$, the corresponding 1-morphism $\Phi(\alpha)$ in $\mathbb{S}$ admits a right adjoint, and the *Beck-Chevalley* conditions are satisfied (see [Chapter V.1, Sect. 3.1] for what this means).

The above theorem is the initial input for any functor out of any category of correspondences considered in this book.

3.3. For example, let us take $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ with $vert = \mathrm{all}$ and $horiz = \mathrm{proper}$. Then starting from IndCoh, viewed as a functor

$$\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{3.1}$$

(with respect to the operation of direct image), we canonically extend it to a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;proper}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

Similarly, taking $horiz = \mathrm{open}$, and inverting the direction of 2-morphisms, we canonically extend (3.1) to a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}} \to (\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}})^{2\text{-}\mathrm{op}}.$$

3.4. In [Chapter V.1] we prove two fundamental theorems that allow to (uniquely) extend functors defined on one category of correspondences to a larger one. Rather than giving the abstract formulation, we will consider the example of $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$.

Together, these theorems allow to start with IndCoh, viewed as a functor as in (3.1), and extend it to a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

3.5. The first of these theorems, [Chapter V.1, Theorem 4.1.3], allows to treat the following situation:

Let us be given a functor

$$\Phi : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{closed}}_{\mathrm{all;all}} \to \mathbb{S},$$

and we want to extend to a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathbb{S}.$$

I.e., the initial functor was only defined on 2-morphisms given by closed embeddings, and we want to extend it to 2-morphisms given by proper maps.

The assertion of [Chapter V.1, Theorem 4.1.3] is that if such an extension exists, it is unique, and one can give explicit conditions for the existence.

The idea here is that for a separated map $f : S \to S'$, the diagram

$$\begin{tikzcd}
S \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S \underset{S'}{\times} S \arrow[r] \arrow[d] & S \arrow[d, "f"] \\
& S \arrow[r, "f"] & S'
\end{tikzcd}$$

provides a 2-morphism

$$\mathrm{id} \to \Phi^!(f) \circ \Phi(f),$$

which will be the unit of an adjunction if f is proper.

3.6. The second theorem, [Chapter V.1, Theorem 5.2.4], is designed to treat the following situation. Let us be given a functor

$$\Phi : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}} \to \mathbb{S} \tag{3.2}$$

(see Sect. 3.3 for the example that we have in mind) and we want to extend to a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathbb{S}. \tag{3.3}$$

Again, the claim is that if such an extension exists, it is unique, and one can give explicit conditions for the existence.

The idea here is the following: as a first step we restrict Φ to

$$\mathrm{Sch}_{\mathrm{aft}} \simeq \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;isom}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}},$$

and then extend it to $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;proper}}$, using [Chapter V.1, Theorem 4.1.3] (see Sect. 3.3 for the example we have in mind).

Thus, we now have $\Phi^!(f)$ defined separately for f open and proper. One now uses Nagata's theorem that any morphism can be factored into a composition of an open morphism, followed by a proper one.

The bulk of the proof consists of showing how the existence of such factorizations leads to the existence and uniqueness of the functor (3.3).

We emphasize that in this theorem the 2-categorical structure on the category of correspondences is essential. I.e., even if we are only interested in the functor

$$\Phi^! : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathbb{S},$$

we need to pass by 2-categories in order to obtain it from the initial functor (3.2).

4. Extension theorems

The Chapter V.2 contains two results (Theorems 1.1.9 and 6.1.5) that allow to (uniquely) extend a given functor

$$\Phi : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S}$$

to a functor

$$\Psi : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S},$$

along the functor $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}$, corresponding to a functor between $(\infty,1)$-categories $\mathbf{C} \to \mathbf{D}$.

4.1. Let us explain the typical situation that [Chapter V.2, Theorem 1.1.9] is applied to. We start with IndCoh, viewed as a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and we want to (canonically) extend it to a functor[1]

$$\mathrm{Corr}(\mathrm{infSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

We do not really know what is the general 2-categorical paradigm in which such an extension fits (it has features of both the left and right Kan extension).

Again, the 2-categorical structure on the category of correspondences here is essential.

4.2. Let us now explain what [Chapter V.2, Theorem 6.1.5] says. We start with the functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathbb{S}, \tag{4.1}$$

and we wish to (canonically) extend it to a functor

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}} \to \mathbb{S}. \tag{4.2}$$

Here the subscript sch stands for the class of schematic maps, and the superscript sch & proper stands for the class of schematic and proper maps.

The required extension is the 2-categorical right Kan extension. However, the particular properties of the functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}}$$

make this extension procedure very manageable.

Namely, it turns out that the restriction of (4.2) along

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}}$$

equals the right Kan extension along the functor of $(\infty,1)$-categories

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch;all}}$$

[1] Here we really have to work with the class of nil-closed morphisms rather than proper ones, because [Chapter V.2, Theorem 1.1.9] only applies in this situation. The further extension to proper morphisms is obtained by the procedure described in Sect. 3.5.

of the restriction of (4.1) along

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}.$$

I.e., the above 2-categorical right Kan extension is essentially 1-categorical.

Moreover, the further restriction of (4.2) along

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \simeq \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{isom;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch;all}}$$

equals the right Kan extension along

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}$$

of the further restriction of (4.1) along

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \simeq \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{isom;all}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}.$$

I.e., this extension procedure 'does the right thing' on objects and pullbacks.

A similar discussion applies when we replace $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$ by $\mathrm{Corr}(\mathrm{infSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}$ and $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}}$ by $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{infsch\ \&\ proper}}_{\mathrm{infsch;all}}$.

5. (Symmetric) monoidal structures

In [Chapter V.3] we study the symmetric monoidal structure that arises on the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$, induced by the *Cartesian* symmetric monoidal structure on $\mathbf{C}$. But in fact, our primary focus will be on the $(\infty,1)$-category $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$.

The essence of [Chapter V.3] is the following two observations. Assume for simplicity that $vert = horiz = \mathrm{all}$, and consider the $(\infty,1)$-category $\mathrm{Corr}(\mathbf{C}) = \mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$.

5.1. The first observation is the following. We note that the category $\mathrm{Corr}(\mathbf{C})$ carries a canonical anti-involution, given by swapping the roles of vertical and horizontal arrows.

We show that this involution canonically identifies with the dualization functor on $\mathrm{Corr}(\mathbf{C})$ for the symmetric monoidal structure on the latter.

As a corollary, we obtain that whenever

$$\Phi : \mathrm{Corr}(\mathbf{C}) \to \mathbf{O}$$

is a symmetric monoidal functor, where $\mathbf{O}$ is a target symmetric monoidal category, for every $\mathbf{c} \in \mathbf{C}$, the corresponding object $\Phi(\mathbf{c}) \in \mathbf{O}$ is canonically self-dual.

This fact is responsible for the Serre duality on IndCoh on schemes: apply the above observation to the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

5.2. The second observation has to do with the construction of *convolution categories*.

Let $\mathbf{c}^\bullet$ be a *Segal object* of $\mathbf{C}$. I.e., this is a simplicial object such that for every $n \geq 2$, the map

$$\mathbf{c}^1 \underset{\mathbf{c}^0}{\times} ... \underset{\mathbf{c}^0}{\times} \mathbf{c}^1,$$

given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

is an isomorphism[2].

We show that $\mathbf{c}^1$, regarded as as object of $\mathrm{Corr}(\mathbf{C})$, carries a canonical structure of associative algebra, (with respect to the symmetric monoidal structure on $\mathrm{Corr}(\mathbf{C})$). For example, the binary operation on $\mathbf{c}^1$ is given by the diagram

$$\begin{tikzcd} \mathbf{c}^2 \arrow[r] \arrow[d] & \mathbf{c}^1 \times \mathbf{c}^1 \\ \mathbf{c}^1, & \end{tikzcd}$$

in which the vertical map is given by the active map $[1] \to [2]$, and the horizontal map is given by the product of the two inert maps $[1] \to [2]$.

As a corollary, we obtain that whenever we are given a monoidal functor

$$\Phi : \mathrm{Corr}(\mathbf{C}) \to \mathbf{O},$$

where $\mathbf{O}$ is a monoidal category, the object $\Phi(\mathbf{c}^1) \in \mathbf{O}$ acquires a structure of associative algebra.

In particular, taking $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$, $\mathbf{O} = \mathrm{DGCat}_{\mathrm{cont}}$, and Φ to be the functor IndCoh, we obtain that for a Segal object $X^\bullet$ in the category of schemes, the category $\mathrm{IndCoh}(X^1)$ is endowed with a monoidal structure, given by *convolution*. I.e., it is given by pull-push along the diagram

$$\begin{tikzcd} X_1 \underset{X_0}{\times} X_1 \arrow[r] \arrow[d] & X^1 \times X^1 \\ X^1. & \end{tikzcd}$$

[2]An alternative terminology is *category object*.