

## CHAPTER III.2. IND-SCHEMES AND INF-SCHEMES

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### INTRODUCTION

0.1. **Inf-schemes.** As was explained in the Introduction to Part III, inf-schemes are our primary object of interest. In this Chapter we will finally define what they are.

By definition, an inf-scheme is a laft prestack  $\mathcal{X}$  such that:

- (a)  $\mathcal{X}$  admits deformation theory;

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(b)  $\text{red}\mathcal{X}$  is a (reduced) scheme.

It is quite remarkable that so general a definition produces a very reasonable object. Let us list some of the properties enjoyed by inf-schemes:

(i) Inf-schemes are well-adapted to the category  $\text{IndCoh}$ , i.e., the latter will extend to a functor of the  $(\infty, 2)$ -category of correspondences on inf-schemes. This will be realized in Chapter 3 of Part III.

(ii) Inf-schemes provide a unified language to talk about  $\mathcal{O}$ -modules and  $D$ -modules; in particular, one can talk about *relative*  $D$ -modules along the fibers of a morphism between schemes. This will be realized in Chapter 4 of Part III.

(iii) Inf-schemes are an adequate framework for *formal moduli problems* and the correspondence between group-objects and their Lie algebras. This will be realized in Chapters 2 and 3 of Part IV.

0.1.1. In this Chapter we will only initiate the study of inf-schemes. The main outcome of this Chapter is the following structural result that comes in two parts, Corollary 4.4.6:

Let  $\mathcal{X}$  be an inf-scheme such that  $\text{red}\mathcal{X} = X_0$  is a (reduced) affine scheme. Inside the category  $(\text{Sch}^{\text{aff}})_{/\mathcal{X}}$  one can single out a subcategory of those

$$S \xrightarrow{x} \mathcal{X},$$

for which  $S \in \text{Sch}_{\text{aft}}^{\text{aff}}$  and the map

$$\text{red}_x : \text{red}S \rightarrow \text{red}\mathcal{X} = X_0$$

is an isomorphism. The first assertion of Corollary 4.4.6 is that this subcategory is cofinal.

This means that the datum of  $\mathcal{X}$ , viewed as a functor from the category (opposite to that) of all affine schemes, is completely determined (i.e., is the left Kan extension) from its restriction to the category of pairs  $(S, x_0)$ , where  $S \in \text{Sch}_{\text{aft}}^{\text{aff}}$  and  $x_0$  is an isomorphism  $\text{red}S \rightarrow X_0$ .

In other words, in order to ‘know’  $\mathcal{X}$  we only need to know what the functor  $\mathcal{X}$  gives in nilpotent thickenings of  $X_0$ .

0.1.2. The second part of Corollary 4.4.6 provides a converse to the above restriction process.

Namely, let  $\mathcal{X}_{\text{nil-isom}}$  be an arbitrary functor from the category of affine eventually coconnective schemes almost of finite type, whose reduced subscheme is identified with a given (reduced) affine scheme  $X_0$  of finite type. I.e.,

$$\mathcal{X}_{\text{nil-isom}} : (\text{Sch}_{\text{aft}}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} \{X_0\})^{\text{op}} \rightarrow \text{Spc}.$$

We impose the condition that the value of  $\mathcal{X}_{\text{nil-isom}}$  on  $X_0$  itself be  $* \in \text{Spc}$ . Let  $\mathcal{X}$  be the left Kan extension of  $\mathcal{X}_{\text{nil-isom}}$  under the forgetful functor

$$\text{Sch}_{\text{aft}}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} \{X_0\} \rightarrow \text{Sch}^{\text{aff}}.$$

I.e., the value of  $\mathcal{X}$  on an (eventually coconnective) affine scheme  $S$  is the category of

$$S \rightarrow Z \rightarrow \mathcal{X}_{\text{nil-isom}}, \quad Z \in \text{Sch}_{\text{aft}}^{\text{aff}}, \quad \text{red}Z = X_0.$$

Thus, we can view  $\mathcal{X}$  as an object of  $\text{PreStk}_{\text{laff}}$ , and  $\text{red}\mathcal{X} = X_0$ . We would like  $\mathcal{X}$  to be an inf-scheme, but we cannot expect that because there is no reason that for an arbitrary  $\mathcal{X}_{\text{nil-isom}}$ , the prestack  $\mathcal{X}$  will admit deformation theory.

However, there is an obvious necessary condition on  $\mathcal{X}_{\text{nil-isom}}$  for  $\mathcal{X}$  to have a chance to admit deformation theory. Namely, recall that one of the conditions in admitting deformation theory is that it should take pushout diagrams of the form

$$(0.1) \quad \begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S'_1 & \longrightarrow & S'_2. \end{array}$$

where  $S_1 \rightarrow S'_1$  has a structure of square-zero extension, to pullback diagrams in  $\text{Spc}$ . Thus, a necessary condition on  $\mathcal{X}_{\text{nil-isom}}$  in order for  $\mathcal{X}$  to admit deformation theory is that  $\mathcal{X}_{\text{nil-isom}}$  have the same property with respect to the above push-outs when

$$S_1, S_2 \in {}^{<\infty}\text{Sch}_{\text{aft}}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} \{X_0\}.$$

Now, the second part of Corollary 4.4.6 says that the above condition is also sufficient.

0.1.3. To summarize, Corollary 4.4.6 says that the operation of restriction under

$${}^{<\infty}\text{Sch}_{\text{aft}}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} \{X_0\} \rightarrow {}^{<\infty}\text{Sch}^{\text{aff}}$$

defines a fully faithful functor from the category of inf-schemes  $\mathcal{X}$ , whose underlying reduced scheme is identified with  $X_0$  and the full subcategory of the category of functors

$$\mathcal{X}_{\text{nil-isom}} : ({}^{<\infty}\text{Sch}_{\text{aft}}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} \{X_0\})^{\text{op}} \rightarrow \text{Spc}, \quad \mathcal{X}_{\text{nil-isom}}(X_0) = *,$$

that take push-out (0.1) squares to pullback squares.

0.2. **Ind-schemes.** Prior to introducing inf-schemes, in Sects. 1 and 2, we study another type of algebro-geometric objects that often comes up in practice: ind-schemes.

0.2.1. Ind-schemes can be defined in any of the following three equivalent ways (but the equivalence is not altogether trivial):

A prestack  $\mathcal{X}$  is said to be an ind-scheme if it is convergent and:

Definition (a):  $\mathcal{X}$  can be written as a *filtered* colimit (in  $\text{PreStk}$ ) of quasi-compact schemes, where the transition maps are closed embeddings.

Definition (a'): Same as (a) with  $\mathcal{X}$  replaced by  $\leq^n \mathcal{X}$  for any  $n$ .

Definition (b): The subcategory of  $(\text{Sch}_{\text{qc}})_{/\mathcal{X}}$  consisting of *closed embeddings* is cofinal and filtered.

Definition (b'): Same as (a) with  $\mathcal{X}$  replaced by  $\leq^n \mathcal{X}$  for any  $n$ .

Definition (c):  $\mathcal{X}$  admits a connective deformation theory and  ${}^{\text{cl}}\mathcal{X}$  is a classical indscheme.

0.2.2. Beyond the equivalence of the above definitions, here is the summary of main results pertaining to ind-schemes:

- (i) Ind-schemes satisfy flat descent;
- (ii) If an ind-scheme  $\mathcal{X}$  is laft as a prestack, then the subcategory of  $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$  consisting of closed embeddings  $Z \rightarrow \mathcal{X}$  with  $S \in \mathrm{Sch}_{\mathrm{aft}}$  is cofinal and filtered.
- (iii) Let  $\mathcal{X}$  be a laft prestack that admits coconnective deformation theory, and such that  ${}^{\mathrm{red}}\mathcal{X}$  is a (reduced) indscheme. Then  $\mathcal{X}$  is an ind-scheme if and only if the following conditions hold:
  - (a) For any reduced affine scheme  $S$  and  $x : S \rightarrow \mathcal{X}$ , the object  $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$  can be written as a projective system with surjective transition maps;
  - (b) Either of the following equivalent conditions holds:
    - The map  ${}^{\mathrm{red}}\mathcal{X} \rightarrow {}^{\mathrm{cl}}\mathcal{X}$  is a monomorphism; or
    - For any reduced affine scheme  $S$  and  $x : S \rightarrow {}^{\mathrm{red}}\mathcal{X}$ , the object

$$T_x^*({}^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

belongs to  $\mathrm{Pro}(\mathrm{QCoh}(S)^-)^{\leq 0}$ .

0.2.3. *Formal schemes.* Normally, in the literature by a *formal scheme* one means an ind-scheme  $\mathcal{X}$  such that  ${}^{\mathrm{red}}\mathcal{X}$  is a (reduced) scheme. Since there many other usages of the word ‘formal’, in this book, we call these objects *nil-schematic ind-schemes*<sup>1</sup>.

We show that if  $\mathcal{X}$  is nil-schematic ind-scheme with  ${}^{\mathrm{red}}\mathcal{X} = X_0$ , then the subcategory of  $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$  consisting of *nilpotent embeddings* embeddings  $S \rightarrow \mathcal{X}$  is cofinal and filtered.

Moreover, we show that if  $\mathcal{X}$  is laft as a prestack, the subcategory of  $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$  consisting of *nilpotent embeddings* embeddings  $Z \rightarrow \mathcal{X}$  with  $Z \in \mathrm{Sch}_{\mathrm{aft}}$  is cofinal and filtered.

0.2.4. Let us emphasize the difference between inf-schemes and nil-schematic ind-schemes.

Namely, in Corollary 4.3.3 (from which we deduce the first direction in Corollary 4.4.6 mentioned above) we show that if  $\mathcal{X}$  is an *inf-scheme*, the map

$$\mathrm{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil-isom\ to\ } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism. Moreover, we will show (see Proposition 4.3.6 that the category  $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil-isom\ to\ } \mathcal{X}}$  is *sifted*. In particular,  $\mathcal{X}$  can be written as a colimit

$$\mathrm{colim}_{\alpha \in A} Z_\alpha \rightarrow \mathcal{X},$$

for some sifted index category  $A$ , where  $Z_\alpha \in \mathrm{Sch}_{\mathrm{aft}}$  and the transition maps in this family are *nil-isomorphisms*.

However, if  $\mathcal{X}$  is a laft nil-schematic ind-scheme, the map

$$\mathrm{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nilp-emb\ into\ } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism, and the category  $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nilp-emb\ into\ } \mathcal{X}}$  is filtered. In particular,  $\mathcal{X}$  can be written as a colimit

$$\mathrm{colim}_{\alpha \in A} Z_\alpha \rightarrow \mathcal{X},$$

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<sup>1</sup>This is in line with our policy that the adjective ‘nil-?’ for a morphism  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  means that the corresponding morphism

$${}^{\mathrm{red}}f : {}^{\mathrm{red}}\mathcal{X}_1 \rightarrow {}^{\mathrm{red}}\mathcal{X}_2$$

has the property ?.

where  $Z_\alpha \in \text{Sch}_{\text{aft}}$  and the transition maps in this family are *nilpotent embeddings*, and where the index category  $A$  is filtered.

So, there are two points of difference: one is the ‘filtered’ vs. ‘sifted’ condition on the index category  $A$ , and the other is ‘nilpotent embeddings’ vs ‘nil-isomorphisms’.

In addition, given  $x : Z \rightarrow \mathcal{X}$  with  $Z$  not necessarily affine, if  $\mathcal{X}$  is a laft nil-schematic ind-scheme, the category of factorizations of  $x$  as

$$Z \rightarrow Z' \xrightarrow{x'} \mathcal{X}, \quad x' \text{ is a nilpotent embedding}$$

is contractible. If, however,  $Z$  is an inf-scheme, the category of factorizations of  $x$  as

$$Z \rightarrow Z' \xrightarrow{x'} \mathcal{X}, \quad x' \text{ is a nil-isomorphism}$$

need not be contractible (it may be empty).

### 0.3. Other definitions and results.

0.3.1. In Sect. 3 we define a notion slightly more general than inf-scheme, namely, that of *ind-inf-scheme*. Namely, a laft prestack  $\mathcal{X}$  is said to be an ind-inf-scheme if

- (a)  $\mathcal{X}$  admits deformation theory;
- (b)  ${}^{\text{red}}\mathcal{X}$  is a (reduced) ind-scheme.

Thus, the class of ind-inf-schemes contains both inf-schemes and ind-schemes. We give some infinitesimal criteria that allow to determine when an ind-inf-scheme is an ind-scheme.

0.3.2. In turns that the good behavior of  $\text{IndCoh}$  on inf-schemes extends at no cost to ind-inf-schemes; this will be done in Chapter 3 of Part III of the book.

We show that ind-inf-schemes satisfy Nisnevich (and with a little more work, also étale) descent.

0.3.3. Finally, we establish an extension of Corollaries 4.4.6 and Corollary 4.3.3 mentioned above, to the case of ind-inf-schemes; this is done in Sect. 4.

## 1. IND-SCHEMES

Ind-schemes are an approximation to our main object of interest (the latter being ind-inf-schemes). In this section we will mainly review various facts established in [GaRo1].

1.1. **The notion of ind-scheme.** Ind-schemes are defined in a very simple way: prestacks that can be presented as *filtered* colimits of schemes under closed embeddings. It is not altogether tautological that this is ‘the right notion’, but we will see that it is in the course of this section.

1.1.1. Let  $\mathcal{X}$  be an object of  $\text{PreStk}$ .

**Definition 1.1.2.** *We shall say that  $\mathcal{X}$  is an ind-scheme if:*

- $\mathcal{X}$  is convergent;
- As an object of  $\text{PreStk}$ , we can write  $\mathcal{X}$  as a filtered colimit

$$(1.1) \quad \text{colim}_{\alpha} X_{\alpha},$$

where  $X_{\alpha} \in \text{Sch}_{\text{qc}}$  and the maps  $X_{\alpha_1} \rightarrow X_{\alpha_2}$  are closed embeddings.

1.1.3. We let

$$\text{indSch} \subset \text{PreStk}$$

denote the full subcategory spanned by ind-schemes. We also denote

$$\text{indSch}_{\text{laft}} := \text{indSch} \cap \text{PreStk}_{\text{laft}}.$$

It is clear that the above subcategories are closed under finite limits taken in  $\text{PreStk}$ .

1.1.4. In addition:

**Definition 1.1.5.** *Let  $\mathcal{X}$  be an object of  $\leq^n \text{PreStk}$  (resp.,  $\text{clPreStk}$ ,  $\text{redPreStk}$ ). We shall say that  $\mathcal{X}$  is an  $n$ -coconnective (resp., classical, reduced) ind-scheme if as an object of  $\leq^n \text{PreStk}$  (resp.,  $\text{clPreStk}$ ,  $\text{redPreStk}$ ), we can write  $\mathcal{X}$  as a filtered colimit*

$$(1.2) \quad \text{colim}_{\alpha} X_{\alpha},$$

where  $X_{\alpha} \in \leq^n \text{Sch}_{\text{qc}}$  (resp.,  $X_{\alpha} \in \text{clSch}_{\text{qc}}$ ,  $X_{\alpha} \in \text{redSch}_{\text{qc}}$ ) and the maps  $X_{\alpha_1} \rightarrow X_{\alpha_2}$  are closed embeddings.

We let

$$\begin{aligned} \leq^n \text{indSch} &\subset \leq^n \text{PreStk}, & \leq^n \text{indSch}_{\text{laft}} &\subset \leq^n \text{PreStk}_{\text{laft}}, \\ \text{clindSch} &\subset \text{clPreStk}, & \text{clindSch}_{\text{laft}} &\subset \text{clPreStk}_{\text{laft}} \end{aligned}$$

and

$$\text{redindSch} \subset \text{redPreStk}, \quad \text{redindSch}_{\text{laft}} \subset \text{redPreStk}_{\text{laft}},$$

denote the corresponding subcategories.

These subcategories are closed under finite limits in  $\leq^n \text{PreStk}$  (resp.,  $\text{clPreStk}$ ,  $\text{redPreStk}$ ).

**1.2. Descent for ind-schemes.** In this subsection we show that ind-schemes satisfy flat descent.

1.2.1. We are going to prove:

**Proposition 1.2.2.** *Let  $\mathcal{X} \in \text{indSch}$ . Then  $\mathcal{X}$  satisfies flat descent.*

For the proof of the proposition we will need the following assertion of independent interest:

**Lemma 1.2.3.** *Let  $\mathcal{X}$  be an ind-scheme. Then for  $S \in \leq^n \text{Sch}_{\text{qc}}$ , the  $\infty$ -groupoid  $\text{Maps}(S, \mathcal{X})$  is  $n$ -truncated.*

*Proof.* First we take  $n = 0$ . In this case the assertion follows from the fact that filtered colimits of discrete objects of  $\text{Spc}$  are discrete. For general  $n$ , the assertion follows from [Chapter III.1, Lemma 6.3.2] and Proposition 1.3.2 below (which is proved independently).  $\square$

1.2.4. *Proof of Proposition 1.2.2, Step 1.* Let  $\mathcal{X}$  be written as  $\text{colim}_{\alpha \in A} X_{\alpha}$ , where  $X_{\alpha} \in \text{Sch}_{\text{qc}}$ , and the category  $A$  is filtered.

Let  $\overset{\circ}{S} \rightarrow S$  be a faithfully flat map in  $\text{Sch}^{\text{aff}}$ , and let  $\overset{\circ}{S^{\bullet}}/S$  be its Čech nerve. We need to show that the map

$$(1.3) \quad \text{Maps}(S, \mathcal{X}) \rightarrow \text{Tot}(\text{Maps}(\overset{\circ}{S^{\bullet}}/S, \mathcal{X}))$$

is an isomorphism.

Let us first assume that  $S$  is  $n$ -coconnective for some  $n$ . Since  $\overset{\circ}{S} \rightarrow S$  is flat, then  $\overset{\circ}{S}$  is also  $n$ -coconnective, and so are the terms of  $\overset{\circ}{S^{\bullet}}/S$ .

We have a commutative diagram in  $\mathrm{Spc}$ :

$$\begin{array}{ccc}
\mathrm{colim}_{\alpha} \mathrm{Maps}(S, X_{\alpha}) & \xrightarrow{\sim} & \mathrm{Maps}(S, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{\alpha} \mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) & \longrightarrow & \mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{\alpha} \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) & \longrightarrow & \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \\
\downarrow & & \downarrow \mathrm{id} \\
\mathrm{Tot}^{\leq n} \left( \mathrm{colim}_{\alpha} \mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}) \right) & \xrightarrow{\sim} & \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})),
\end{array}$$

where  $\mathrm{Tot}^{\leq n}$  denote the limit over the  $n$ -skeleton.

We note that the map

$$\mathrm{colim}_{\alpha} \mathrm{Maps}(S, X_{\alpha}) \rightarrow \mathrm{colim}_{\alpha} \mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}))$$

is an isomorphism since maps of schemes satisfy flat descent.

Now, Lemma 1.2.3, implies that

$$\mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \rightarrow \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X}))$$

and

$$\mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) \rightarrow \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})).$$

are isomorphisms.

Furthermore, the map

$$\mathrm{colim}_{\alpha} \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) \rightarrow \mathrm{Tot}^{\leq n} \left( \mathrm{colim}_{\alpha} \mathrm{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}) \right)$$

is an isomorphism, since finite limits commute with filtered colimits.

This implies that the map (1.3) is an isomorphism as well.

1.2.5. *Proof of Proposition 1.2.2, Step 2.* For an integer  $n$ , we consider the  $n$ -coconnective truncation  $\leq^n S$  of  $S$ . Note that since  $\mathring{S} \rightarrow S$  is flat, the map  $\leq^n \mathring{S} \rightarrow \leq^n S$  is flat, and the simplicial  $n$ -coconnective DG scheme  $\leq^n(\mathring{S}^{\bullet}/S)$  is the Čech nerve of  $\leq^n \mathring{S} \rightarrow \leq^n S$ .

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Maps}(S, \mathcal{X}) & \longrightarrow & \mathrm{Tot}(\mathrm{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \\
\downarrow & & \downarrow \\
\lim_{n \in \mathbb{N}^{\mathrm{op}}} \mathrm{Maps}(\leq^n S, \mathcal{X}) & \longrightarrow & \lim_{n \in \mathbb{N}^{\mathrm{op}}} \mathrm{Tot}(\mathrm{Maps}(\leq^n(\mathring{S}^{\bullet}/S), \mathcal{X}))
\end{array}$$

In this diagram the vertical arrows are isomorphisms, since  $\mathcal{X}$  is convergent. The bottom horizontal arrow is an isomorphism by Step 1. Hence, the top horizontal arrow is an isomorphism as well, as desired.  $\square$

1.2.6. As a corollary we obtain:

**Corollary 1.2.7.** *Let  $\mathcal{X} \in \text{indSch}$  be written as in (1.1). Then for  $Z \in \text{Sch}_{\text{qc}}$ , the map*

$$\text{colim}_{\alpha} \text{Maps}(Z, X_{\alpha}) \rightarrow \text{Maps}(Z, \mathcal{X})$$

*is an isomorphism.*

*Proof.* Follows from the Zariski descent property of  $\mathcal{X}$  and the fact that finite limits commute with filtered colimits.  $\square$

**1.3. Deformation theory of ind-schemes.** In this subsection we show that ind-schemes admit (connective) deformation theory, and that, moreover, they can essentially be characterized by this property.

1.3.1. We observe:

**Proposition 1.3.2.** *Let  $\mathcal{X}$  be an ind-scheme. Then  $\mathcal{X}$  admits a connective deformation theory.*

*Proof.* Follows from the fact that the formation of finite limits (involved in the definition of admitting connective deformation theory, see [Chapter III.1, Lemma 3.1.8]) commutes with filtered colimits.  $\square$

1.3.3. Note that [Chapter III.1, Lemma 2.5.5] gives an explicit expression to the value of pro-cotangent spaces of an ind-scheme:

**Lemma 1.3.4.** *For  $\mathcal{X} \in \text{indSch}$  written as in (1.1), and  $(Z, x) \in (\text{Sch}_{\text{qc}})_{/x}$  we have:*

$$T_x^*(\mathcal{X}) \simeq \lim_{(\alpha, x_{\alpha}) \in (A_{x/})^{\text{op}}} T_{x_{\alpha}}^*(X_{\alpha}),$$

where  $A_{x/}$  is the category of factorizations of  $x$  as

$$Z \rightarrow X_{\alpha} \rightarrow \mathcal{X}, \quad \alpha \in A.$$

In the above lemma, the limit is taken in  $\text{Pro}(\text{QCoh}(Z)^{-})$ , or equivalently  $\text{Pro}(\text{QCoh}(Z)^{\leq 0})$ , as  $A_{x/}$  is filtered.

1.3.5. We note the following feature of the pro-cotangent spaces of an ind-scheme.

**Definition 1.3.6.** *Let  $Z$  be a scheme and  $\mathcal{T}$  an object of  $\text{Pro}(\text{QCoh}(Z)^{\heartsuit})$ . We shall say that  $\mathcal{T}$  can be given by a surjective system if  $\mathcal{T}$  can be written as a limit*

$$\lim_{\alpha \in A^{\text{op}}} \mathcal{F}_{\alpha},$$

where  $A$  is a filtered category,  $\mathcal{F}_{\alpha} \in \text{QCoh}(Z)^{\heartsuit}$  and for  $\alpha_1 \rightarrow \alpha_2$ , the corresponding map  $\mathcal{F}_{\alpha_2} \rightarrow \mathcal{F}_{\alpha_1}$  is surjective.

We have:

**Lemma 1.3.7.** *An object  $\mathcal{T} \in \text{Pro}(\text{QCoh}(Z)^{\heartsuit})$  is given by a surjective system if and only if in the category*

$$((\text{QCoh}(Z)^{\heartsuit})_{\mathcal{T}})^{\text{op}}$$

*the full subcategory, spanned by surjections  $\mathcal{T} \rightarrow \mathcal{F}$ , is cofinal.*

For future reference, we note that

**Lemma 1.3.8.** *If  $i : \tilde{Z} \hookrightarrow Z$  is a nilpotent embedding, and  $\mathcal{T} \in \text{Pro}(\text{QCoh}(Z)^{\heartsuit})$  is such that  $\tilde{\mathcal{T}} := H^0((\text{Pro}(i^*))(\mathcal{T})) \in \text{Pro}(\text{QCoh}(\tilde{Z})^{\heartsuit})$  is given by a surjective system, then  $\mathcal{T}$  is also given by a surjective system.*

1.3.9. From Lemma 1.3.4 we obtain:

**Lemma 1.3.10.** *Let  $\mathcal{X}$  be an ind-scheme and  $x : Z \rightarrow \mathcal{X}$  a point, where  $Z \in \text{Sch}_{\text{qc}}$ . Then  $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(Z)^\heartsuit)$  is given by a surjective system.*

1.3.11. The next assertion provides a partial converse to Proposition 1.3.2:

**Theorem 1.3.12.** *Let  $\mathcal{X}$  be an object of  $\text{PreStk}$  that admits connective deformation theory and that for any  $(S, x : S \rightarrow \mathcal{X}) \in \text{clSch}_{/\mathcal{X}}^{\text{aff}}$ , the object  $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(S)^\heartsuit)$  is given by a surjective system. Assume that there exists a map  $f : \mathcal{X}_0 \rightarrow \mathcal{X}$  such that:*

- $\mathcal{X}_0$  is a classical ind-scheme;
- The map  $f : \mathcal{X}_0 \rightarrow \mathcal{X}$  is a monomorphism when evaluated on classical schemes;
- The map  $f$  is a pseudo-nilpotent embedding<sup>2</sup>.

*Then  $\mathcal{X}$  is an ind-scheme.*

The proof will be given in Sect. 2.

**Corollary 1.3.13.** *Let  $\mathcal{X}$  be an object of  $\text{PreStk}$  that admits connective deformation theory, and such that  $\text{cl}\mathcal{X}$  is a classical ind-scheme. Then  $\mathcal{X}$  is an ind-scheme.*

**1.4. Indschemes and truncations.** In this subsection we compare our present definition of ind-schemes with that of [GaRo1].

1.4.1. We have:

**Proposition 1.4.2.** *Let  $\mathcal{X}$  be an object of  ${}^{\text{conv}}\text{PreStk}$ . Then  $\mathcal{X} \in \text{indSch}$  if and only if for every  $n$ , we have  $\leq^n \mathcal{X} \in \leq^n \text{indSch}$ .*

*Proof.* The ‘only if’ part is evident. Conversely, let  $\mathcal{X}$  be convergent and such that for every  $n$ , we have  $\leq^n \mathcal{X} \in \leq^n \text{indSch}$ . By repeating the argument of Proposition 1.3.2, it follows from [Chapter III.1, Sect. 6.1.3] that  $\mathcal{X}$  admits a connective deformation theory.

Hence, such  $\mathcal{X}$  satisfies the conditions of Corollary 1.3.13, namely, we take  $\mathcal{X}_0 = \text{cl}\mathcal{X}$ . □

1.4.3. From Proposition 1.4.2 we obtain:

**Corollary 1.4.4.** *Let  $\alpha \mapsto X_\alpha$  be a filtered diagram of objects of  $\text{Sch}_{\text{qc}}$  with the maps being closed embeddings. Set  $\mathcal{X}' := \text{colim}_\alpha X_\alpha$ . Then  ${}^{\text{conv}}(\mathcal{X}')$  is an ind-scheme.*

Finally, from Corollary 1.4.4, we deduce:

**Corollary 1.4.5.** *Let  $\mathcal{X}_n$  (resp.,  $\mathcal{X}_{\text{cl}}$ ,  $\mathcal{X}_{\text{red}}$ ) be an  $n$ -coconnective (resp., classical, reduced) ind-scheme. Set*

$$\mathcal{X}' := \text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}(\mathcal{X}_?),$$

*where  $? = \leq n$  (resp.,  $? = \text{cl}$ ,  $? = \text{red}$ ). Then  ${}^{\text{conv}}(\mathcal{X}')$  is an ind-scheme.*

**1.5. Closed embeddings into an ind-scheme.** In this subsection we show that an ind-scheme  $\mathcal{X}$  can be recovered from the category of schemes equipped with a closed embedding into  $\mathcal{X}$ .

---

<sup>2</sup>See [Chapter III.1, Definition 8.1.2(d’)] for what this means.

1.5.1. We recall (see [Chapter III.1, Definition 8.1.2]) that a map of prestacks  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  is a closed embedding if for  $S_2 \in \text{Sch}_{/\mathcal{X}_2}^{\text{aff}}$ , the map

$$\text{cl}(S_2 \times_{\mathcal{X}_2} \mathcal{X}_1) \rightarrow \text{cl} S_2$$

is a closed embedding of classical affine schemes (in particular, the left hand side is a classical affine scheme).

It is easy to see that if  $\mathcal{X}_1 \rightarrow \mathcal{X}_2 \rightarrow \mathcal{X}_3$  are such that  $\mathcal{X}_1 \rightarrow \mathcal{X}_3$  and  $\mathcal{X}_2 \rightarrow \mathcal{X}_3$  are closed embeddings, then so is  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ .

In what follows, for  $\mathcal{X} \in \text{PreStk}$  we let

$$\text{PreStk}_{\text{closed in } \mathcal{X}} \subset \text{PreStk}_{/\mathcal{X}}$$

denote the full subcategory, consisting of those  $(\mathcal{X}', f : \mathcal{X}' \rightarrow \mathcal{X})$ , for which  $f$  is a closed embedding. We will use a similar notation for any category that maps to  $\text{PreStk}$ , e.g.,

$$\text{Sch}_{\text{closed in } \mathcal{X}}.$$

1.5.2. Let  $\mathcal{X}$  be an ind-scheme, written as in (1.1). It is clear that for  $Z \in \text{Sch}_{\text{qc}}$ , a map  $Z \rightarrow \mathcal{X}$  is a closed embedding if and only if for some/any  $\alpha$ , for which the above map factors through a map  $Z \rightarrow X_\alpha$ , the latter is a closed embedding.

1.5.3. We claim:

**Proposition 1.5.4.** *An object  $\mathcal{X} \in {}^{\text{conv}}\text{PreStk}$  is an ind-scheme if and only if the following two conditions are satisfied:*

- *The functor  $\text{Sch}_{\text{closed in } \mathcal{X}} \rightarrow (\text{Sch}_{\text{qc}})_{/\mathcal{X}}$  is cofinal.*
- *The category  $\text{Sch}_{\text{closed in } \mathcal{X}}$  is filtered.*

*Proof.* Clearly, the conditions of the proposition are sufficient for  $\mathcal{X}$  to be an ind-scheme.

Assume now that  $\mathcal{X}$  is an ind-scheme. The fact that

$$\text{Sch}_{\text{closed in } \mathcal{X}} \rightarrow (\text{Sch}_{\text{qc}})_{/\mathcal{X}}$$

is cofinal follows from Corollary 1.2.7.

To prove that the category  $\text{Sch}_{\text{closed in } \mathcal{X}}$  is filtered, it is enough to show that it contains finite colimits. Let

$$(1.4) \quad i \mapsto Z_i, \quad i \in I$$

be a finite diagram in  $\text{Sch}_{\text{closed in } \mathcal{X}}$ . Write  $\mathcal{X}$  as in (1.1). Then by Corollary 1.2.7, and since the category of indices  $A$  is filtered, there exists an index  $\alpha \in A$  such that (1.4) factors through a diagram in  $\text{Sch}_{\text{closed in } X_\alpha}$ .

By [Chapter II.2, Proposition 1.1.3], the resulting diagram admits a colimit, denote it

$$Z_\alpha \in \text{Sch}_{\text{closed in } X_\alpha}.$$

Furthermore, by [Chapter II.2, Lemma 1.1.5], for an arrow  $\alpha \rightarrow \alpha'$  in  $A$ , the resulting map  $Z_\alpha \rightarrow Z_{\alpha'}$  is an isomorphism. Since  $A$  is filtered, this implies that  $Z_\alpha$  maps isomorphically to the sought-for colimit in  $\text{Sch}_{\text{closed in } \mathcal{X}}$ . □

**1.6. Topological conditions.** In this subsection we introduce several classes of maps between prestacks, imposing the condition that they behave as ‘relative ind-schemes’.

1.6.1. We give the following definitions:

**Definition 1.6.2.** *We shall say that a reduced ind-scheme  $\mathcal{X}$  is ind-affine if it can be written as in (1.2) with  $X_\alpha \in \text{redSch}_{\text{qc}}$  being affine.*

It is easy to see that  $\mathcal{X}$  is ind-affine if and only if for any closed embedding  $X \rightarrow \mathcal{X}$  with  $X \in \text{redSch}_{\text{qc}}$ , the scheme  $X$  is affine.

**Definition 1.6.3.** *We shall say that an ind-scheme (resp.,  $n$ -coconnective ind-scheme)  $\mathcal{X}$  is ind-affine if  $\text{red}\mathcal{X}$  has this property.*

Again, it is easy to see that  $\mathcal{X}$  is ind-affine if and only if for any closed embedding  $X \rightarrow \mathcal{X}$  with  $X \in \text{Sch}_{\text{qc}}$  (resp.,  $\leq^n \text{Sch}_{\text{qc}}$ ), the scheme  $X$  is affine<sup>3</sup>.

It is easy to see that  $\mathcal{X}$  is ind-affine if for any/some presentation (1.1) (resp., (1.2)), the schemes  $X_\alpha$  are affine.

1.6.4. We give the following definition:

**Definition 1.6.5.**

(a) *We shall say that a morphism  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  of prestacks (resp.,  $n$ -coconnective prestacks, classical prestacks, reduced prestacks) is ind-schematic if its base change by an affine scheme (resp.,  $n$ -connective affine scheme, classical scheme, reduced scheme) yields an ind-scheme (resp.,  $n$ -coconnective, classical, reduced ind-scheme).*

(b) *We shall say that a morphism  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  of prestacks (resp.,  $n$ -coconnective prestacks, classical prestacks, reduced prestacks) is ind-affine if its base change by an affine scheme (resp.,  $n$ -connective affine scheme, classical scheme, reduced scheme) yields an ind-affine ind-scheme (resp.,  $n$ -coconnective, classical, reduced ind-affine ind-scheme).*

1.6.6. We also give the following definitions:

**Definition 1.6.7.**

(a) *We shall say that a map from a classical prestack  $\mathcal{X}$  to a classical affine scheme  $S$  is an ind-closed embedding if  $\mathcal{X}$  is a classical ind-scheme and for any closed embedding  $X \rightarrow \mathcal{X}$ , where  $X \in \text{clSch}_{\text{qc}}$ , the composed map  $X \rightarrow S$  is a closed embedding.*

(b) *We shall say that a map of classical prestacks  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  is an ind-closed embedding if its base change by a classical affine scheme yields a map which is an ind-closed embedding.*

(c) *We shall say that a map of prestacks  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  is an ind-closed embedding if the corresponding map  $\text{cl}\mathcal{X}_1 \rightarrow \text{cl}\mathcal{X}_2$  is.*

*Remark 1.6.8.* Note that ‘closed embedding’ is stronger than ‘ind-closed embedding’. E.g.,

$$\text{Spf}(k[[t]]) \rightarrow \text{Spec}(k[t])$$

is an ind-closed embedding, but not a closed embedding. And similarly, for

$$\bigsqcup_I \text{pt} \rightarrow \mathbb{A}^1,$$

where  $I$  is an arbitrary infinite set of distinct  $k$ -points in  $\mathbb{A}^1$ .

---

<sup>3</sup>Here we use the fact that if a classical scheme  $X$  is such that  $\text{red}X$  is affine, then  $X$  itself is affine.

1.6.9. Let  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  be a map of (classical) ind-schemes. It is easy to see that it is an ind-closed embedding (resp., ind-affine) if and only if the following is satisfied:

If

$$\mathcal{X}_1 := \operatorname{colim}_{\alpha \in A} X_{1,\alpha} \text{ and } \mathcal{X}_2 := \operatorname{colim}_{\beta \in A} X_{2,\beta},$$

then for every index  $\alpha$ , and every/some index  $\beta$  for which  $X_{1,\alpha} \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_2$  factors as

$$X_{1,\alpha} \rightarrow X_{2,\beta} \rightarrow \mathcal{X}_2,$$

the map  $X_{1,\alpha} \rightarrow X_{2,\beta}$  is a closed embedding (resp., an affine morphism between schemes).

In addition,  $f$  is an ind-closed embedding if and only if for every closed embedding  $X_1 \rightarrow \mathcal{X}_1$ , the composition  $X_1 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_2$  is a closed embedding.

1.6.10. For future reference we also give the following definitions:

**Definition 1.6.11.**

(a) We shall say that a map from a reduced prestack  $\mathcal{X}$  to a reduced affine scheme  $S$  is ind-proper (resp., ind-finite) if  $\mathcal{X}$  is a reduced ind-scheme and for any closed embedding  $X \rightarrow \mathcal{X}$ , where  $X \in \operatorname{redSch}_{\text{qc}}$ , the composite map  $X \rightarrow S$  is proper (resp., finite).

(b) We shall say that a map of reduced prestacks  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  is (ind)-proper (resp., (ind)-finite) if its base change by a reduced affine scheme yields a map which is (ind)-proper (resp., (ind)-finite).

(c) We shall say that a map of prestacks  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  is (ind)-proper (resp., (ind)-finite) if the corresponding map  $\operatorname{red}\mathcal{X}_1 \rightarrow \operatorname{red}\mathcal{X}_2$  is (ind)-proper (resp., (ind)-finite).

1.6.12. Let  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  be a map of reduced ind-schemes. It is easy to see that it is ind-proper (resp., ind-finite) if and only if the following is satisfied:

If

$$\mathcal{X}_1 := \operatorname{colim}_{\alpha \in A} X_{1,\alpha} \text{ and } \mathcal{X}_2 := \operatorname{colim}_{\beta \in A} X_{2,\beta},$$

then for every index  $\alpha$ , and every/some index  $\beta$  for which  $X_{1,\alpha} \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_2$  factors as

$$X_{1,\alpha} \rightarrow X_{2,\beta} \rightarrow \mathcal{X}_2,$$

the map  $X_{1,\alpha} \rightarrow X_{2,\beta}$  is proper (resp., finite).

**1.7. Indschemes and the finite type condition.** In this subsection we show that ind-schemes are nicely compatible with the ‘locally almost of finite type’ condition.

1.7.1. We have the following assertion:

**Proposition 1.7.2.** *Let  $\mathcal{X}$  be an object of  $\operatorname{indSch}_{\text{laft}}$ . Then the category  $(\operatorname{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$  is filtered and the functor*

$$(\operatorname{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}} \rightarrow \operatorname{Sch}_{\text{closed in } \mathcal{X}}$$

*is cofinal.*

The proof will be given in Sect. 2.7.

1.7.3. From Proposition 1.7.2 we obtain:

**Corollary 1.7.4.** *An object  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$  is an ind-scheme if and only if the following two conditions are satisfied:*

- *The functor  $(\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}} \rightarrow (\text{Sch}_{\text{qc}})_{/\mathcal{X}}$  is cofinal.*
- *The category  $(\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$  is filtered.*

*Proof.* It is clear that the conditions of the proposition are sufficient for  $\mathcal{X}$  to be an ind-scheme. The converse implication follows by combining Propositions 1.5.4 and 1.7.2, and the fact that a category cofinal in a filtered category is filtered.  $\square$

Now, from Corollary 1.7.4 we obtain:

**Corollary 1.7.5.** *Let  $\mathcal{X}$  be an object of  $\text{indSch}_{\text{laft}}$ .*

(a) *As an object of  $\text{PreStk}$ , it can be written as a filtered colimit*

$$(1.5) \quad \mathcal{X} \simeq \underset{\alpha}{\text{colim}} X_{\alpha},$$

where  $X_{\alpha} \in \text{Sch}_{\text{aft}}$  and the maps  $X_{\alpha_1} \rightarrow X_{\alpha_2}$  are closed embeddings.

(a') *The map*

$$\underset{Z \in (\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}}}{\text{colim}} Z \rightarrow \mathcal{X},$$

where the colimit is taken in  $\text{PreStk}$ , is an isomorphism.

(b) *The functors*

$$(\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}} \rightarrow (\text{Sch}_{\text{aft}})_{/\mathcal{X}}$$

and

$$(\text{Sch}_{\text{aft}})_{/\mathcal{X}} \rightarrow (\text{Sch}_{\text{qc}})_{/\mathcal{X}}$$

are cofinal.

1.7.6. The following theorem is a variant of Theorem 1.3.12 in the locally of finite type case.

**Theorem 1.7.7.** *Let  $\mathcal{X}$  be an object of  $\text{PreStk}_{\text{laft}}$ , which admits a connective deformation theory, and such that:*

- *For any  $(S, x : S \rightarrow \mathcal{X}) \in (\text{redSch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$ , the object  $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(S)^{\heartsuit})$  is given by a surjective system.*
- *$\text{red}\mathcal{X}$  is a reduced ind-scheme.*

*Then the following conditions are equivalent:*

- (a)  $\mathcal{X}$  is an ind-scheme;
- (b)  $\text{cl}\mathcal{X}$  is a classical ind-scheme;
- (c) The map  $\text{LKE}_{(\text{redSch}^{\text{aff}})_{\text{op}} \hookrightarrow (\text{clSch}^{\text{aff}})_{\text{op}}}(\text{red}\mathcal{X}) \rightarrow \text{cl}\mathcal{X}$ , is a monomorphism.
- (d) For any  $S \in \text{redSch}^{\text{aff}}$  and a map  $x : S \rightarrow \mathcal{X}$ , the object

$$T_x^*(\text{red}\mathcal{X}/\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^{-})$$

belongs to  $\text{Pro}(\text{QCoh}(S)^{\leq -1})$ . (Here by a slight abuse of notation, we denote by  $\text{red}\mathcal{X}$  the ind-scheme obtained by the procedure of Corollary 1.4.5.)

The proof will be given in Sect. 2.6.

**1.8. Nil-schematic ind-schemes.** In this subsection we study ind-schemes, whose underlying reduced ind-scheme is a scheme<sup>4</sup>.

1.8.1. We give the following definition:

**Definition 1.8.2.** *We shall say that an ind-scheme  $\mathcal{X}$  is nil-schematic if the reduced ind-scheme  $\text{red}\mathcal{X}$  is a scheme.*

1.8.3. For  $\mathcal{X} \in \text{PreStk}$  let

$$\text{PreStk}_{\text{nilp-emb into } \mathcal{X}} \subset \text{PreStk}/\mathcal{X}$$

be the full subcategory spanned by objects  $f : \mathcal{X}' \rightarrow \mathcal{X}$  for which  $f$  is a nilpotent embedding.

We will use a similar notation for full subcategories of  $\text{PreStk}$ , e.g.,

$$\text{Sch}_{\text{nilp-emb to } \mathcal{X}} \subset \text{Sch}/\mathcal{X},$$

etc.

1.8.4. In this subsection we will prove the following:

**Proposition 1.8.5.** *Let  $\mathcal{X}$  be a nil-schematic ind-scheme locally almost of finite type. Then the category  $(\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}}$  is filtered, and the functor*

$$(\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}} \rightarrow (\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$$

*is cofinal.*

As a formal consequence we obtain:

**Corollary 1.8.6.** *Let  $\mathcal{X}$  be a nil-schematic ind-scheme locally almost of finite type.*

(a) *As an object of  $\text{PreStk}$ , it can be written as a filtered colimit*

$$(1.6) \quad \mathcal{X} \simeq \text{colim}_{\alpha} X_{\alpha},$$

*where  $X_{\alpha} \in \text{Sch}_{\text{aft}}$  and the maps  $X_{\alpha_1} \rightarrow X_{\alpha_2}$  are nilpotent embeddings.*

(a') *The map*

$$\text{colim}_{Z \in (\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

*where the colimit is taken in  $\text{PreStk}$ , is an isomorphism.*

(a'') *The category  $(\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}}$  is filtered.*

(b) *The functor*

$$(\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}} \rightarrow (\text{Sch}_{\text{qc}})_{/\mathcal{X}}$$

*is cofinal.*

---

<sup>4</sup>Elsewhere in the literature, such ind-schemes are called ‘formal schemes’. We do not use this terminology to avoid clashing with other usages of the word ‘formal’.

1.8.7. *Proof of Proposition 1.8.5.* Since the category  $(\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}}$  is filtered, it suffices to show that for any object

$$(Z \rightarrow \mathcal{X}) \in (\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}},$$

there exists a factorization

$$Z \rightarrow W \rightarrow \mathcal{X},$$

where  $(W \rightarrow \mathcal{X}) \in (\text{Sch}_{\text{aft}})_{\text{nilp-emb into } \mathcal{X}}$ .

Note that since  $Z$  is almost of finite type, the map  $\text{red}Z \rightarrow Z$  is a nilpotent embedding. The sought-for scheme  $W$  is constructed as

$$Z \sqcup_{\text{red}Z} \text{red}\mathcal{X},$$

using [Chapter III.1, Corollary 7.2.3]. □

## 2. PROOFS OF RESULTS CONCERNING IND-SCHEMES

2.1. **Proof of Theorem 1.3.12, Plan.** Let  $\mathcal{X}$  and  $\mathcal{X}_0$  be as in Theorem 1.3.12.

2.1.1. We consider the following full subcategory of  $(\text{Sch}_{\text{qc}})_{/\mathcal{X}}$ , to be denoted  $A$ . Its objects are those

$$Z \rightarrow \mathcal{X},$$

for which there *exists* a commutative diagram

$$\begin{array}{ccc} Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathcal{X}, \end{array}$$

where  $Z_0 \in {}^{\text{cl}}\text{Sch}_{\text{qc}}$ , the map  $Z_0 \rightarrow Z$  is a nilpotent embedding, and  $Z_0 \rightarrow \mathcal{X}_0$  is a closed embedding.

Note that this condition implies that the map  $Z \rightarrow \mathcal{X}$  is nil-closed. In particular, any map  $Z_1 \rightarrow Z_2$  in  $A$  is nil-closed, and hence affine.

2.1.2. Let  $B$  be a full subcategory of  $A$ , consisting of those

$$x : Z \rightarrow \mathcal{X}$$

that satisfy the following condition:

The map  $(dx)^* : T_x^*(\mathcal{X}) \rightarrow T^*(Z)$  induces a surjection  $H^0(T_x^*(\mathcal{X})) \twoheadrightarrow H^0(T^*(Z))$ .

2.1.3. We will prove Theorem 1.3.12 by establishing the following facts:

- (1) Any map  $Z_1 \rightarrow Z_2$  in  $B$  is a closed embedding;
- (2) The category  $B$  is filtered;
- (3) The map  $\text{colim}_{(Z \rightarrow \mathcal{X}) \in B} Z \rightarrow \mathcal{X}$  is an isomorphism.

2.2. **Step 1: proof that the maps are closed embeddings.** We will prove a slightly stronger assertion: any map  $Z_1 \rightarrow Z_2$ , where  $Z_1 \in B$  and  $Z_2 \in A$ , is a closed embedding.

2.2.1. Let  $Z_1 \rightarrow Z_2$  be a map in  $A$ . Consider the corresponding nilpotent embeddings  $Z_{0,i} \hookrightarrow Z_i$ ,  $Z_{0,i} \in \text{clSch}_{\text{qc}}$ ,  $i = 1, 2$ .

Let  $Z_0$  be the intersection of the closed subschemes  $Z_{0,1}$  and  ${}^{\text{cl}}Z_1 \times_{{}^{\text{cl}}Z_2} Z_{0,2}$  in  ${}^{\text{cl}}Z_1$ .

The map  $Z_0 \rightarrow Z_{0,1}$  is a nilpotent embedding. The map  $Z_0 \rightarrow Z_{0,2}$  is a closed embedding, because the composition  $Z_0 \rightarrow Z_{0,2} \rightarrow X_0$  is.

2.2.2. The assertion of Step 1 follows now from the following general statement:

Let

$$\begin{array}{ccc} & Z_0 & \\ g_1 \swarrow & & \searrow g_2 \\ Z_1 & \xrightarrow{f} & Z_2, \end{array}$$

be a diagram in  $\text{Sch}_{\text{qc}}$ , where  $g_2$  is a closed embedding, and  $g_1$  a nilpotent embedding. We have:

**Proposition 2.2.3.** *The following conditions are equivalent:*

- (a)  $f$  is a closed embedding;
- (b)  $f$  is a monomorphism when evaluated on classical affine schemes;
- (c) The map  $T_{g_2}^*(Z_2) \rightarrow T_{g_1}^*(Z_1)$ , induced by  $(df)^*$ , gives rise to a surjection

$$H^0(T_{g_2}^*(Z_2)) \rightarrow H^0(T_{g_1}^*(Z_1)).$$

The rest of this subsection is devoted to the proof of Proposition 2.2.3.

2.2.4. Clearly, (a) implies (b) and (b) implies (c). Let us show that (c) implies (a). Clearly, the statement reduces to one about classical schemes. So, we can assume that  $Z_0, Z_1$  and  $Z_2$  are classical.

Let  $Z_0$  be given in  $Z_1$  by an ideal that vanishes to the power  $n$ . We will argue by induction on  $n$ , starting with  $n = 2$ .

2.2.5. For  $n = 2$ , the map  $Z_0 \rightarrow Z_1$  is a square-zero extension, say by  $\mathcal{J}_1 \in \text{QCoh}(Z_0)^\heartsuit$ . Replacing  $Z_2$  by the classical 1st infinitesimal neighborhood of  $Z_0$ , we can assume that  $Z_0 \rightarrow Z_2$  is also a square-zero extension, say by  $\mathcal{J}_2 \in \text{QCoh}(Z_0)^\heartsuit$ . We have:

$$H^{-1}(T^*(Z_0/Z_i)) \simeq \mathcal{J}_i.$$

We have a map of exact sequences

$$\begin{array}{ccccccc} H^{-1}(T^*(Z_0)) & \longrightarrow & H^{-1}(T^*(Z_0/Z_2)) & \longrightarrow & H^0(T_{g_2}^*(Z_2)) & \longrightarrow & H^0(T^*(Z_0)) \\ \text{id} \downarrow & & \downarrow & & \downarrow & & \text{id} \downarrow \\ H^{-1}(T^*(Z_0)) & \longrightarrow & H^{-1}(T^*(Z_0/Z_1)) & \longrightarrow & H^0(T_{g_1}^*(Z_1)) & \longrightarrow & H^0(T^*(Z_0)), \end{array}$$

hence assumption (c) implies that  $\mathcal{J}_2 \rightarrow \mathcal{J}_1$  is surjective, as required.

2.2.6. To carry out the induction step, let  $Z_{1/2}$  be a closed subscheme of  $Z_1$  such that

$$Z_0 \subset Z_{1/2} \subset Z_1,$$

and such that the ideal of  $Z_0$  in  $Z_{1/2}$  and the ideal of  $Z_{1/2}$  in  $Z_1$  vanish to a smaller power.

Note that the assumption of (c) holds for the map  $Z_{1/2} \rightarrow Z_2$ . Hence, by induction hypothesis applied to

$$\begin{array}{ccc} & Z_0 & \\ & \swarrow & \searrow \\ Z_{1/2} & \longrightarrow & Z_2, \end{array}$$

the map  $Z_{1/2} \rightarrow Z_2$  is a closed embedding.

We now apply the induction hypothesis to

$$\begin{array}{ccc} & Z_{1/2} & \\ & \swarrow & \searrow \\ Z_1 & \longrightarrow & Z_2, \end{array}$$

and deduce that  $Z_1 \rightarrow Z_2$  is a closed embedding. □

**2.3. Step 2: construction of a left adjoint.** In order to proceed with the proof of Theorem 1.3.12 we will now show that the inclusion

$$B \hookrightarrow A$$

admits a left adjoint.

2.3.1. Thus, given an object  $Z \rightarrow \mathcal{X}$  of  $A$ , we need to show that the category  $\mathbf{D}(Z)$  of factorizations

$$Z \rightarrow Z' \rightarrow \mathcal{X},$$

where  $(Z' \rightarrow \mathcal{X}) \in B$ , admits an initial object.

2.3.2. We first reduce to the case of classical schemes. Indeed, let

$${}^{\text{cl}}Z \rightarrow Z'_{\text{cl}} \rightarrow \mathcal{X}$$

be the initial object in the category  $\mathbf{D}({}^{\text{cl}}Z)$  (in this case  $Z'_{\text{cl}}$  is automatically classical). Then the object

$$Z \sqcup_{{}^{\text{cl}}Z} Z'_{\text{cl}}$$

is initial in  $\mathbf{D}(Z)$ .

2.3.3. From now, until the end of Step 2, all schemes will be classical, and we shall sometimes omit “cl” from the notation.

By assumption, there exists a diagram

$$\begin{array}{ccc} Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z & \xrightarrow{x} & \mathcal{X}, \end{array}$$

where  $Z_0 \rightarrow Z$  is a nilpotent embedding, and  $Z_0 \rightarrow \mathcal{X}_0$  is a closed embedding (and in particular, a monomorphism when evaluated on classical schemes).

Note that for any  $(Z \rightarrow Z' \rightarrow \mathcal{X}) \in \mathbf{D}(Z)$ , the composed map  $Z_0 \rightarrow Z'$  is a closed embedding (e.g., by Step 1).

Let  $Z_0 \hookrightarrow Z$  be given by the ideal that vanishes to the power  $n$ . We will argue by induction on  $n$ , starting with  $n = 2$ .

2.3.4. Thus, we first assume that  $Z_0 \hookrightarrow Z$  is a square-zero extension. Let

$$\mathbf{D}(Z)_{\text{SqZ}} \subset \mathbf{D}(Z)$$

be the full subcategory spanned by those objects, for which  $Z_0 \rightarrow Z'$  is a square-zero extension. (Note that since we are working with classical schemes, being a square-zero extension is a property and not an extra structure.)

Note that the embedding  $\mathbf{D}(Z)_{\text{SqZ}} \hookrightarrow \mathbf{D}(Z)$  admits a right adjoint, which sends  $Z'$  to the classical 1st infinitesimal neighborhood of  $Z_0$  in  $Z'$ . Hence, it is enough to show that  $\mathbf{D}(Z)_{\text{SqZ}}$  admits an initial object.

2.3.5. Denote  $x_0 = x|_{Z_0}$ . The data of a square-zero extension  $Z_0 \hookrightarrow Z$  and a map  $x$ , extending  $x_0$  is given by the data of  $\mathcal{J} \in \text{QCoh}(Z_0)^\heartsuit$  and a map

$$(2.1) \quad \text{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1] \rightarrow \mathcal{J}.$$

By assumption,  $x_0$  is a monomorphism. This implies that

$$(dx_0)^* : T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0)$$

induces a surjection

$$(2.2) \quad H^0(T_{x_0}^*(\mathcal{X})) \twoheadrightarrow H^0(T^*(Z_0)).$$

Hence,

$$T^*(Z_0/\mathcal{X})[-1] = \text{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1] \in \text{Pro}(\text{QCoh}(Z_0)^{\geq 0}).$$

Now, the assumption on  $T^*(\mathcal{X})$  implies that

$$H^0(\text{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1])$$

is also given by a surjective family.

Hence, the map (2.1) canonically factors as

$$\text{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1] \rightarrow \mathcal{J}' \rightarrow \mathcal{J},$$

where

$$H^0(T^*(Z_0/\mathcal{X})[-1]) = H^0(\text{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1]) \rightarrow \mathcal{J}'$$

is surjective.

Let  $Z'$  be the square-zero extension of  $Z_0$  that corresponds to

$$\mathrm{coFib}(T_{x_0}^*(\mathcal{X}) \rightarrow T^*(Z_0))[-1] \rightarrow \mathcal{J}'.$$

It is easy to see that  $Z'$  is the initial object in  $\mathbf{D}(Z)_{\mathrm{sqz}}$ .

2.3.6. We are now ready to carry out the induction step. Choose a classical subscheme  $Z_{1/2}$

$$Z_0 \hookrightarrow Z_{1/2} \hookrightarrow Z,$$

such that

$$Z_{1/2} \hookrightarrow Z$$

is a square-zero extension, and the ideal of  $Z_0$  in  $Z_{1/2}$  vanishes to a smaller power.

By the induction hypothesis, the category  $\mathbf{D}(Z_{1/2})$  admits an initial object, denote it by  $Z'_{1/2}$ . Denote

$$\tilde{Z} := Z \sqcup_{Z_{1/2}} Z'_{1/2},$$

and let

$$\tilde{x} : \tilde{Z} \rightarrow \mathcal{X}$$

denote the resulting map.

Note that

$$Z'_{1/2} \hookrightarrow \tilde{Z}$$

is a square-zero extension. Hence, by Sects. 2.3.4 and 2.3.5, the category  $\mathbf{D}(\tilde{Z})$  admits an initial object. Indeed, the proof only used the fact that (2.2) was surjective, which is satisfied for the map  $Z'_{1/2} \rightarrow \mathcal{X}$  by construction.

Let

$$\tilde{Z} \rightarrow Z' \rightarrow \mathcal{X}$$

an initial object of  $\mathbf{D}(\tilde{Z})$ . It is easy to see that the resulting object

$$Z \rightarrow Z' \rightarrow \mathcal{X}$$

is the initial one in  $\mathbf{D}(Z)$ .

### 2.4. Step 3: proof of filteredness.

2.4.1. We consider two auxiliary categories. We let  $B'$  be the category

$$(\mathrm{clSch}_{\mathrm{qc}})_{\mathrm{closed\ in\ }x_0}.$$

We let  $B''$  be the category of commutative diagrams

$$\begin{array}{ccc} Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathcal{X}, \end{array}$$

where  $(Z \rightarrow \mathcal{X}) \in B$ ,  $(Z_0 \rightarrow \mathcal{X}_0) \in B'$ , and  $Z_0 \rightarrow Z$  is a nilpotent embedding.

We have the naturally defined forgetful functors:

$$B \leftarrow B'' \rightarrow B'.$$

2.4.2. Note that the category  $B'$  is filtered by the assumption on  $\mathcal{X}_0$ . We will now show that the category  $B''$  is filtered as well.

We will use the following general assertion:

**Lemma 2.4.3.** *Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a co-Cartesian fibration. Assume that  $\mathbf{D}$  is filtered and that the fibers of  $F$  are also filtered. Then  $\mathbf{C}$  is filtered.*

We claim that the above lemma is applicable to the above functor

$$B'' \rightarrow B'.$$

This would imply that  $B''$  is filtered.

2.4.4. Let us show that  $B'' \rightarrow B'$  is a co-Cartesian fibration. Given a diagram

$$\begin{array}{ccc} Z_0^1 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z^1 & \longrightarrow & \mathcal{X}, \end{array}$$

and a map  $Z_0^1 \rightarrow Z_0^2$  we construct the sought-for object

$$\begin{array}{ccc} Z_0^2 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z^2 & \longrightarrow & \mathcal{X} \end{array}$$

as follows. First, we set

$$\tilde{Z}_2 := Z^1 \sqcup_{Z_0^1} Z_0^2,$$

which is equipped with a canonical map to  $\mathcal{X}$ .

We have  $(\tilde{Z}_2 \rightarrow \mathcal{X}) \in A$  and the required object  $(Z_2 \rightarrow \mathcal{X}) \in B$  is obtained by applying the left adjoint  $A \rightarrow B$  constructed in Step 2.

2.4.5. Let us now show that the fiber of  $B''$  over a given object  $(Z_0 \rightarrow \mathcal{X}_0) \in B'$  is filtered. We claim that the fiber in question admits coproducts and push-outs.

For products, given two objects  $Z^1$  and  $Z^2$ , we take

$$\tilde{Z} := Z^1 \sqcup_{Z_0} Z^2,$$

which is equipped with a canonical map to  $\mathcal{X}$ .

We have  $(\tilde{Z} \rightarrow \mathcal{X}) \in A$  and the sought-for coproduct in  $B$  is obtained by applying the left adjoint  $A \rightarrow B$ .

The proof for push-outs is similar (using the fact that all maps in  $B$  are closed embeddings).

2.4.6. Thus, we have shown that  $B''$  is filtered. To prove that  $B$  is filtered, we will use the following general statement:

**Lemma 2.4.7.** *Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between  $(\infty, 1)$ -categories. Assume that  $F$  is cofinal and  $\mathbf{C}$  is filtered. Then  $\mathbf{D}$  is filtered.*

We claim that the above lemma is applicable to the functor  $B'' \rightarrow B$ . This would imply that  $B$  is filtered.

We have the following general statement:

**Lemma 2.4.8.** *Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a Cartesian fibration. Then  $F$  is cofinal if and only if it has contractible fibers.*

Hence, it is enough to show that  $B'' \rightarrow B$  is a Cartesian fibration and that it has contractible fibers.

The fact that  $B'' \rightarrow B$  is a Cartesian fibration is obvious via the formation of fiber products (again, using the fact that any map in  $B$  is a closed embedding).

The fact that the fibers of  $B'' \rightarrow B$  are contractible is proved in Sect. 2.5.4 below.

**2.5. Step 4: proof of the isomorphism.** We will now show that the map

$$\operatorname{colim}_{(Z \rightarrow \mathcal{X}) \in B} Z \rightarrow \mathcal{X}$$

is an isomorphism, thereby proving Theorem 1.3.12.

2.5.1. We need to show that for  $S \in \operatorname{Sch}^{\text{aff}}$  and a map  $S \rightarrow \mathcal{X}$ , the category  $\mathbf{C}$  of factorizations

$$S \rightarrow Z \rightarrow \mathcal{X},$$

with  $(Z \rightarrow \mathcal{X}) \in B$  is contractible.

We introduce several auxiliary categories.

2.5.2. We let  $\mathbf{C}'$  be the category of diagrams

$$(2.3) \quad \begin{array}{ccccc} & & Z_0 & \longrightarrow & \mathcal{X}_0 \\ & & \downarrow & & \downarrow \\ S & \longrightarrow & Z & \longrightarrow & \mathcal{X}, \end{array}$$

where  $(Z \rightarrow \mathcal{X}) \in B$ ,  $Z_0 \in {}^{\text{cl}}\operatorname{Sch}_{\text{qc}}$ , the map  $Z_0 \rightarrow Z$  is a nilpotent embedding, and  $Z_0 \rightarrow \mathcal{X}_0$  is a closed embedding.

We let  $\mathbf{C}''$  be the category of diagrams

$$\begin{array}{ccccc} S_0 & \longrightarrow & Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & Z & \longrightarrow & \mathcal{X}, \end{array}$$

where  $Z \rightarrow \mathcal{X}$ ,  $Z_0 \rightarrow Z$ ,  $Z_0 \rightarrow \mathcal{X}_0$  are as above,  $S_0 \in {}^{\text{cl}}\operatorname{Sch}_{\text{qc}}$ , and  $S_0 \rightarrow S$  is a nilpotent embedding.

We let  $\mathbf{C}'''$  denote the category of diagrams

$$(2.4) \quad \begin{array}{ccccc} S_0 & \longrightarrow & Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & & & \downarrow \\ S & \longrightarrow & & \longrightarrow & \mathcal{X} \end{array}$$

where  $S_0 \rightarrow S$  and  $Z_0 \rightarrow \mathcal{X}_0$  are as above.

We let  $\mathbf{C}''''$  denote the category of diagrams

$$(2.5) \quad \begin{array}{ccc} S_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{X}, \end{array}$$

where  $S_0 \rightarrow S$  is as above.

2.5.3. We have the forgetful functors

$$\mathbf{C} \leftarrow \mathbf{C}' \leftarrow \mathbf{C}'' \rightarrow \mathbf{C}''' \rightarrow \mathbf{C}''''.$$

We claim that all of the above functors are homotopy equivalences and that  $\mathbf{C}''''$  is contractible. This will imply that  $\mathbf{C}$  is contractible.

2.5.4. The functor  $\mathbf{C}' \rightarrow \mathbf{C}$  is a Cartesian fibration (via the formation of fiber products). Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of  $\mathbf{C}' \rightarrow \mathbf{C}$  over an object  $(S \rightarrow Z \rightarrow \mathcal{X}) \in \mathbf{C}$  is the category of ways to complete

$$\begin{array}{ccc} & & \mathcal{X}_0 \\ & & \downarrow \\ Z & \longrightarrow & \mathcal{X} \end{array}$$

to a commutative diagram

$$\begin{array}{ccc} Z_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathcal{X}, \end{array}$$

where  $Z_0 \rightarrow Z$  is a nilpotent embedding, and  $Z_0 \rightarrow \mathcal{X}_0$  is a closed embedding.

The assumption that  $(Z \rightarrow \mathcal{X})$  belongs to  $A$  means that the above category is non-empty. To prove that this category is contractible, it is sufficient to show that it contains products. These are given by intersecting the corresponding closed subschemes inside  ${}^{\text{cl}}Z$  (here we use the fact that  $\mathcal{X}_0 \rightarrow \mathcal{X}$  is a monomorphism of classical prestacks).

2.5.5. The functor  $\mathbf{C}'' \rightarrow \mathbf{C}'$  is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of  $\mathbf{C}'' \rightarrow \mathbf{C}'$  over an object (2.3) is the category of fillings of

$$\begin{array}{ccc} & & Z_0 \\ & & \downarrow \\ S & \longrightarrow & Z \end{array}$$

to a commutative diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & Z, \end{array}$$

where  $S_0 \rightarrow S$  is a nilpotent embedding. This category is contractible, because it contains the final object, namely,  $S_0 := S \times_Z Z_0$ .

2.5.6. Consider the functor  $\mathbf{C}'' \rightarrow \mathbf{C}'''$ . We claim that it is a co-Cartesian fibration. Indeed, given a map from a diagram

$$\begin{array}{ccccc} S_{0,1} & \longrightarrow & Z_{0,1} & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & & & \downarrow \\ S & \longrightarrow & & \longrightarrow & \mathcal{X} \end{array}$$

to a diagram

$$\begin{array}{ccccc} S_{0,2} & \longrightarrow & Z_{0,2} & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & & & \downarrow \\ S & \longrightarrow & & \longrightarrow & \mathcal{X}, \end{array}$$

and a diagram

$$\begin{array}{ccccc} S_{0,1} & \longrightarrow & Z_{0,1} & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & Z_1 & \longrightarrow & \mathcal{X}, \end{array}$$

we construct the corresponding diagram

$$\begin{array}{ccccc} S_{0,2} & \longrightarrow & Z_{0,2} & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & Z_2 & \longrightarrow & \mathcal{X} \end{array}$$

as follows.

Set  $\tilde{Z}_2 := Z_1 \sqcup_{Z_{0,1}} Z_{0,2}$ . We have  $(Z_{0,2} \rightarrow \tilde{Z}_2) \in A$ . The sought-for object  $(Z_{0,2} \rightarrow Z_2) \in B$  is obtained from  $Z_{0,2} \rightarrow \tilde{Z}_2$  by applying the left adjoint functor to  $B \hookrightarrow A$  from Step 2.

2.5.7. Hence, in order to show that  $\mathbf{C}'' \rightarrow \mathbf{C}'''$  is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of  $\mathbf{C}'' \rightarrow \mathbf{C}'''$  over an object (2.4) is the category of factorizations of the map

$$S \sqcup_{S_0} Z_0 \rightarrow \mathcal{X}$$

as

$$S \sqcup_{S_0} Z_0 \rightarrow Z \rightarrow \mathcal{X},$$

where the composition

$$Z_0 \rightarrow S \sqcup_{S_0} Z_0 \rightarrow Z$$

is a nilpotent embedding, and  $(Z \rightarrow \mathcal{X}) \in B$ .

We claim that the above category of factorizations contains an initial object. Indeed, set  $\tilde{Z} := S \sqcup_{S_0} Z_0$ , where the formation of the push-out is well-behaved because the map  $S_0 \rightarrow Z_0$  is affine (recall that all our schemes were assumed separated).

We have  $(\tilde{Z} \rightarrow \mathcal{X}) \in A$ . Now, the sought-for initial object is obtained by applying to  $\tilde{Z}$  the left adjoint to  $B \hookrightarrow A$ .

2.5.8. The functor  $\mathbf{C}''' \rightarrow \mathbf{C}''''$  is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of  $\mathbf{C}''' \rightarrow \mathbf{C}''''$  over an object (2.5) is the category of factorizations of  $S_0 \rightarrow \mathcal{X}_0$  via a closed embedding

$$S_0 \rightarrow Z_0 \rightarrow \mathcal{X}.$$

The latter category is contractible by the assumption that  $\mathcal{X}_0$  is a classical ind-scheme.

2.5.9. Finally, we claim that  $\mathbf{C}''''$  is contractible. Indeed, it is non-empty by the assumption that  $\mathcal{X}_0 \rightarrow \mathcal{X}$  is a pseudo-nilpotent embedding. Furthermore, it contains finite products: these are obtained by intersecting the corresponding closed subschemes in  ${}^{\text{cl}}S$  as in Sect. 2.5.4.

□(Theorem 1.3.12)

## 2.6. Proof of Theorem 1.7.7.

2.6.1. The implication (a)  $\Rightarrow$  (b) is tautological.

Note also that at this point we also know that (b) implies (a): this follows from Corollary 1.3.13.

2.6.2. The implication (b)  $\Rightarrow$  (c) is easy: we need to show that for  $S \in {}^{\text{cl}}\text{Sch}^{\text{aff}}$ , the map

$$\text{Maps}(S, {}^{\text{red}}\mathcal{X}) \rightarrow \text{Maps}(S, {}^{\text{cl}}\mathcal{X})$$

is a monomorphism of groupoids.

Writing  ${}^{\text{cl}}\mathcal{X} = \text{colim}_{\alpha \in A} X_\alpha$  with  $X_\alpha \in {}^{\text{cl}}\text{Sch}_{\text{qc}}$  and  $A$  filtered, the above map becomes

$$\text{colim}_{\alpha \in A} \text{Maps}(S, {}^{\text{red}}X_\alpha) \rightarrow \text{Maps}(S, X_\alpha),$$

which is an injection (of sets), since  $A$  is filtered.

2.6.3. We now claim that (c) implies (a). Indeed, we apply Theorem 1.3.12 to  $\mathcal{X}_0 = {}^{\text{red}}\mathcal{X}$ . Thus, we only have to show that  ${}^{\text{red}}\mathcal{X} \rightarrow \mathcal{X}$  is a pseudo-nilpotent embedding. However, this follows from [Chapter III.1, Lemma 8.1.5].

2.6.4. The implication (c)  $\Rightarrow$  (d) is tautological from the definition of pro-cotangent spaces. Hence, it remains to show that (d) implies (c).

By [Chapter I.2, Lemma 1.6.8], the functor

$$\text{LKE}_{({}^{\text{cl}}\text{Sch}_{\text{ft}}^{\text{aff}})^{\text{op}} \hookrightarrow ({}^{\text{cl}}\text{Sch}^{\text{aff}})^{\text{op}}}$$

commutes with finite limits, and in particular, preserves monomorphisms. Hence, it is sufficient to show that for  $S \in {}^{\text{cl}}\text{Sch}_{\text{ft}}^{\text{aff}}$ , the map

$$\text{Maps}(S, {}^{\text{red}}\mathcal{X}) \rightarrow \text{Maps}(S, \mathcal{X})$$

is a monomorphism of groupoids.

The map

$$\text{Maps}(S_0, {}^{\text{red}}\mathcal{X}) \rightarrow \text{Maps}(S_0, \mathcal{X})$$

is a monomorphism (in fact, an isomorphism) for  $S_0 = {}^{\text{red}}S$ .

Since  $S$  is of finite type, there exists a finite sequence of square-zero extensions

$${}^{\text{red}}S = S_0 \hookrightarrow S_1 \hookrightarrow \dots \hookrightarrow S_n = S.$$

We will show by induction that the maps

$$\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X}) \rightarrow \mathrm{Maps}(S_i, \mathcal{X})$$

are monomorphisms.

2.6.5. The case  $i = 0$  has been considered above. To carry out the induction step, we need to show that for any  $x_i : S_i \rightarrow {}^{\mathrm{red}}\mathcal{X}$ , the map

$$\{x_i\}_{\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X})} \times \mathrm{Maps}(S_{i+1}, {}^{\mathrm{red}}\mathcal{X}) \rightarrow \{x_i\}_{\mathrm{Maps}(S_i, \mathcal{X})} \times \mathrm{Maps}(S_{i+1}, \mathcal{X})$$

is a monomorphism.

Let the square-zero extension  $S_i \hookrightarrow S_{i+1}$  be given by an object

$$\mathcal{J} \in \mathrm{QCoh}(S_i)_{T^*(S_i)[-1]}^{\heartsuit}.$$

Then the groupoid

$$\{x_i\}_{\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X})} \times \mathrm{Maps}(S_{i+1}, {}^{\mathrm{red}}\mathcal{X})$$

identifies with that of null-homotopies of the composition

$$T_{x_i}^*({}^{\mathrm{red}}\mathcal{X}) \rightarrow T^*(S_i) \rightarrow \mathcal{J}[1],$$

while the groupoid

$$\{x_i\}_{\mathrm{Maps}(S_i, \mathcal{X})} \times \mathrm{Maps}(S_{i+1}, \mathcal{X})$$

identifies with that of null-homotopies of the composition

$$T_{x_i}^*(\mathcal{X}) \rightarrow T^*(S_i) \rightarrow \mathcal{J}[1].$$

Hence, the required monomorphism property follows from the fact that

$$T_{x_i}^*({}^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S_i)^{\leq -1}),$$

which in turn follows from condition (d) and the fact that  $S_0 \rightarrow S_i$  is a nilpotent embedding.

□(Theorem 1.7.7)

## 2.7. Proof of Proposition 1.7.2.

2.7.1. *Step 0.* Since the category  $\mathrm{Sch}_{\mathrm{closed\ in\ } \mathcal{X}}$  is filtered, in order to prove the proposition, it is sufficient to show that every closed embedding  $Z \rightarrow \mathcal{X}$  can be factored as

$$Z \rightarrow \tilde{Z} \rightarrow \mathcal{X},$$

where  $\tilde{Z} \in \mathrm{Sch}_{\mathrm{aft}}$  and  $\tilde{Z} \rightarrow \mathcal{X}$  is a closed embedding.

Given a closed embedding  $Z \rightarrow \mathcal{X}$ , we will construct a *compatible* system of factorizations

$$\leq^n Z \rightarrow \tilde{Z}_n \rightarrow \mathcal{X}, \quad \tilde{Z}_n \in (\leq^n \mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed\ in\ } \mathcal{X}}, \quad \leq^{n-1} \tilde{Z}_n \simeq \tilde{Z}_{n-1}.$$

We shall proceed by induction on  $n$ , starting from  $n = 0$ .

2.7.2. *Step 1.* We claim that  ${}^{\text{cl}}Z$  is already of finite type. I.e., we claim that the functor

$$({}^{\text{cl}}\text{Sch}_{\text{aft}})_{\text{closed in } \mathcal{X}} \rightarrow ({}^{\text{cl}}\text{Sch}_{\text{qc}})_{\text{closed in } \mathcal{X}}$$

is an equivalence. Indeed, write

$${}^{\text{cl}}\mathcal{X} \simeq \text{colim}_{\alpha} X_{\alpha}, \quad X_{\alpha} \in {}^{\text{cl}}\text{Sch}_{\text{qc}}.$$

Since a closed classical subscheme of a classical scheme of finite type is itself of finite type, it suffices to show that all  $X_{\alpha}$  are of finite type.

Recall the following characterization of classical schemes of finite type:  $X \in {}^{\text{cl}}\text{Sch}_{\text{qc}}$  is of finite type if and only if for a classical commutative  $k$ -algebra  $R$  and a filtered family of *subalgebras*  $R_i$  with  $\bigcup_i R_i = R$ , the map

$$\text{colim}_i \text{Maps}(\text{Spec}(R_i), X) \rightarrow \text{Maps}(\text{Spec}(R), X)$$

is an equivalence.

For a  $R$  and  $R_i$  as above, and any index  $\alpha$ , the diagram

$$\begin{array}{ccc} \text{colim}_i \text{Maps}(\text{Spec}(R_i), X_{\alpha}) & \longrightarrow & \text{colim}_i \text{Maps}(\text{Spec}(R_i), \mathcal{X}) \\ \downarrow & & \downarrow \\ \text{Maps}(\text{Spec}(R), X_{\alpha}) & \longrightarrow & \text{Maps}(\text{Spec}(R), \mathcal{X}) \end{array}$$

is a pullback square. Hence, the fact that the right vertical arrow is an isomorphism implies that the left vertical arrow is an isomorphism, as required.

2.7.3. *Step 2.* We shall now carry out the induction step. Assume that

$$\leq^n Z \rightarrow \tilde{Z}_n \rightarrow \mathcal{X}$$

has been constructed.

Set

$$\tilde{Z}'_{n+1} := \tilde{Z}_n \sqcup_{\leq^n Z} \leq^{n+1} Z.$$

By [Chapter III.1, Proposition 5.4.2], the morphism  $\leq^n Z \rightarrow \leq^{n+1} Z$  has a (canonical) structure of square-zero extension. Hence, the morphism

$$\tilde{Z}_n \rightarrow \tilde{Z}'_{n+1}$$

also has a structure of square-zero extension, by an ideal  $\mathcal{J}' \in \text{QCoh}(\tilde{Z}_n)^{\heartsuit}[n+1]$ .

Since the morphism  $\leq^n Z \rightarrow \tilde{Z}_n$  is affine, we have a canonical map

$$\tilde{Z}'_{n+1} \rightarrow \mathcal{X}.$$

We need to factor the latter morphism as

$$\tilde{Z}'_{n+1} \rightarrow \tilde{Z}_{n+1} \rightarrow \mathcal{X},$$

where  $\tilde{Z}_{n+1} \in \leq^{n+1}\text{Sch}_{\text{ft}}$ , and  $\tilde{Z}_n \rightarrow \leq^n \tilde{Z}_{n+1}$  is an isomorphism.

We claim that we can find such a  $\tilde{Z}_{n+1}$  so that  $\tilde{Z}_n \rightarrow \leq^n \tilde{Z}_{n+1}$  is a square-zero extension by an ideal  $\mathcal{J} \in \text{Coh}(\tilde{Z}_n)^{\heartsuit}[n+1]$ .

This follows by the argument in Step 3 of the proof of [Chapter III.1, Theorem 9.1.2]

□(Proposition 1.7.2)

## 3. (IND)-INF-SCHEMES

Ind-inf-schemes are our primary object of study. These are the algebro-geometric spaces on which the category  $\text{IndCoh}$  is defined along with the operations of  $!$ -pullback and  $*$ -pushforward; in this respect they behave much in the same way as schemes (the main difference is the absence of  $t$ -structure); we will develop this in [Chapter III.3]. In addition, it turns out that ind-inf-schemes are well-adapted to a lot of formal differential geometry, as we shall see in [Chapter IV.4] and [Chapter IV.5].

What is surprising is that the class of ind-inf-schemes is quite large. In this section we define ind-inf-schemes and discuss some basic properties.

**3.1. The notion of (ind)-inf-scheme.** We will only define the notion of (ind)-inf-scheme, under the ‘laft’ hypothesis. One can give a definition in general, but it is more technical and currently we do not see sufficient applications for it.

3.1.1. Let  $\mathcal{X}$  be an object of  $\text{PreStk}_{\text{laft}}$ .

**Definition 3.1.2.** *We shall say that  $\mathcal{X}$  is an inf-scheme (resp., ind-inf-scheme) if:*

- $\mathcal{X}$  admits deformation theory;
- The reduced prestack  ${}^{\text{red}}\mathcal{X}$  is a reduced quasi-compact scheme (resp., ind-scheme).

We let  $\text{indinfSch}_{\text{laft}}$  (resp.,  $\text{infSch}_{\text{laft}}$ ) denote the full subcategory of  $\text{PreStk}_{\text{laft}}$  spanned by ind-inf-schemes (resp., inf-schemes). It is clear that both subcategories are closed under finite limits.

3.1.3. *Examples.*

(i) By Proposition 1.3.2, any object of  $\text{indSch}$  is an ind-inf-scheme.

(ii) Let  $\mathcal{Z}$  be an object of  $\text{PreStk}_{\text{laft}}$ . Consider the de Rham prestack  $\mathcal{Z}_{\text{dR}}$ :

$$\text{Maps}(S, \mathcal{Z}_{\text{dR}}) := \text{Maps}({}^{\text{red}}S, \mathcal{Z}).$$

If  ${}^{\text{red}}\mathcal{Z}$  is a reduced ind-scheme (resp., scheme), then  $\mathcal{Z}_{\text{dR}}$  is an ind-inf-scheme (resp., inf-scheme). Indeed,

$${}^{\text{red}}\mathcal{Z}_{\text{dR}} = {}^{\text{red}}\mathcal{Z},$$

while the cotangent complex of  $\mathcal{Z}_{\text{dR}}$  is zero.

However,  $\mathcal{Z}_{\text{dR}}$  is *not* an ind-scheme. For example, it violates condition (d) of Theorem 1.7.7.

(iii) Let  $\mathcal{Y} \rightarrow \mathcal{X}$  be a map in  $\text{PreStk}_{\text{laft}}$ . We define the formal completion of  $\mathcal{X}$  along  $\mathcal{Y}$  (or of  $\mathcal{Y}$  in  $\mathcal{X}$ ), denoted  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  to be the prestack

$$\mathcal{X} \times_{\mathcal{X}_{\text{dR}}} \mathcal{Y}_{\text{dR}}.$$

Note that  ${}^{\text{red}}\mathcal{X}_{\mathcal{Y}}^{\wedge} \simeq {}^{\text{red}}\mathcal{Y}$ .

Hence, if  ${}^{\text{red}}\mathcal{Y}$  is a reduced ind-scheme (resp., scheme), and  $\mathcal{X}$  admits deformation theory, then  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  is an ind-inf-scheme (resp., inf-scheme).

3.1.4. We give the following definition:

**Definition 3.1.5.** *Let  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  be a morphism in  $\text{PreStk}_{\text{laft}}$ . We shall say that  $f$  is (ind)-inf-schematic if its base change by an affine scheme yields an (ind)-inf-scheme.*

**3.2. Properties of (ind)-inf-schemes.**

3.2.1. By [Chapter III.1, Proposition 8.2.2(a)] we have:

**Corollary 3.2.2.** *Let  $\mathcal{X} \in \text{indinfSch}$ . Then  $\mathcal{X}$  satisfies Nisnevich descent.*

*Remark 3.2.3.* According to [Chapter III.1, Remark 8.2.3], any object of  $\text{indinfSch}$  satisfies étale descent.

3.2.4. We now claim:

**Lemma 3.2.5.** *Any ind-inf-scheme  $\mathcal{X}$  can be exhibited as a filtered colimit in  $\text{PreStk}$*

$$\text{colim}_{\alpha} \mathcal{X}_{\alpha},$$

where  $\mathcal{X}_{\alpha} \in \text{infSch}_{\text{laft}}$  and the maps  $X_{\alpha_1} \rightarrow X_{\alpha_2}$  are ind-closed embeddings.

*Proof.* Write  ${}^{\text{red}}\mathcal{X}$  as a filtered colimit in  ${}^{\text{red}}\text{PreStk}$

$$\text{colim}_{\alpha} X_{\alpha}, \quad X_{\alpha} \in {}^{\text{red}}\text{Sch}_{\text{ft}}.$$

Let  $\mathcal{X}_{\alpha}$  be the formal completion of  $X_{\alpha}$  in  $\mathcal{X}$ , i.e.,

$$\mathcal{X}_{\alpha} = (X_{\alpha})_{\text{dR}} \times_{\mathcal{X}_{\text{dR}}} \mathcal{X}.$$

This gives the desired presentation.  $\square$

**3.3. Ind-inf-schemes vs. ind-schemes.** In this subsection we discuss various conditions that guarantee that a given (ind)-inf-scheme is in fact an (ind)-scheme.

3.3.1. First we observe:

**Lemma 3.3.2.** *Let  $\mathcal{X}' \rightarrow \mathcal{X}$  be a map in  $\text{PreStk}_{\text{laft}}$  with  $\mathcal{X}$  an ind-inf-scheme (resp., ind-scheme). Then  $\mathcal{X}'$  is an ind-inf-scheme (resp., ind-scheme) if and only if for every  $S \in (<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$ , the base change  $S \times_{\mathcal{X}} \mathcal{X}'$  is an ind-inf-scheme (resp., ind-scheme).*

3.3.3. *When is an ind-inf-scheme an ind-scheme?* A partial answer to this question is provided by Theorem 1.3.12. Here is a more algorithmic answer:

**Corollary 3.3.4.** *An object  $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$  belongs to  $\text{indSch}_{\text{laft}}$  if and only if:*

- For any  $(S, x : S \rightarrow \mathcal{X}) \in ({}^{\text{red}}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$ , we have  $T_x^*({}^{\text{red}}\mathcal{X}/\mathcal{X}) \in \text{Pro}(\text{QCoh}(S))^{\leq -1}$ ;
- For any  $(S, x : S \rightarrow \mathcal{X}) \in ({}^{\text{red}}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$ , the object  $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(S)^{\heartsuit})$  is given by a surjective system.

*Proof.* This is a restatement of Theorem 1.7.7.  $\square$

The above assertion has a number of corollaries that will be useful in the sequel:

**Corollary 3.3.5.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a nil-isomorphism, where  $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$  and  $\mathcal{Y} \in \text{indSch}_{\text{laft}}$ . Assume that for every  $(S, x : S \rightarrow \mathcal{X}) \in ({}^{\text{red}}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$  we have:*

- $T_x^*(\mathcal{X}/\mathcal{Y}) \in \text{Pro}(\text{QCoh}(S))^{\leq 0}$ ;
- The object  $H^0(T_x^*(\mathcal{X}/\mathcal{Y})) \in \text{Pro}(\text{QCoh}(S)^{\heartsuit})$  is given by a surjective system.

Then  $\mathcal{X} \in \text{indSch}_{\text{laft}}$ .

*Proof.* We claim that  $\mathcal{X}$  satisfies the conditions of Corollary 3.3.4. We need to show that for every  $(S, x : S \rightarrow \mathcal{X}) \in ({}^{\text{red}}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$  we have:

- $T_x^*({}^{\text{red}}\mathcal{X}/\mathcal{X}) \in \text{Pro}(\text{QCoh}(S))^{\leq -1}$ ;
- $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(S)^{\heartsuit})$  is given by a surjective system.

Consider the fiber sequence

$$T_x^*(\mathcal{X}/\mathcal{Y}) \rightarrow T_x^*(\text{red}\mathcal{X}/\mathcal{Y}) \rightarrow T_x^*(\text{red}\mathcal{X}/\mathcal{X})$$

in  $\text{Pro}(\text{QCoh}(S)^-)$ .

Since  $\mathcal{Y}$  is an ind-scheme and  $\text{red}\mathcal{X} \simeq \text{red}\mathcal{Y}$ , we have

$$T_x^*(\text{red}\mathcal{X}/\mathcal{Y}) \in \text{Pro}(\text{QCoh}(S)^{\leq -1}).$$

Hence,  $T_x^*(\text{red}\mathcal{X}/\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^{\leq -1})$ , as desired.

Consider now the fiber sequence

$$T_{f \circ x}^*(\mathcal{Y}) \rightarrow T_x^*(\mathcal{X}) \rightarrow T_x^*(\mathcal{X}/\mathcal{Y}),$$

and the corresponding exact sequence

$$H^0(T_{f \circ x}^*(\mathcal{Y})) \rightarrow H^0(T_x^*(\mathcal{X})) \rightarrow H^0(T_x^*(\mathcal{X}/\mathcal{Y})) \rightarrow 0.$$

From here we obtain that  $H^0(T_x^*(\mathcal{X})) \in \text{Pro}(\text{QCoh}(S)^\heartsuit)$  is given by a surjective system, since both  $H^0(T_{f \circ x}^*(\mathcal{Y}))$  and  $H^0(T_x^*(\mathcal{X}/\mathcal{Y}))$  are.  $\square$

#### 4. (IND)-INF-SCHEMES AND NIL-CLOSED EMBEDDINGS

The results of this section are of crucial technical importance. We prove two types of results. One is about approximating (ind)-inf-schemes by schemes; this is needed for the development of  $\text{IndCoh}$  on (ind)-inf-schemes. The other is about recovering an (ind)-inf-scheme as a prestack (i.e., a presheaf on the category of affine schemes) from its restriction to a much smaller subcategory of test schemes; this is needed for the study of formal moduli problems in [Chapter IV.1].

**4.1. Exhibiting ind-inf-schemes as colimits.** In this subsection we show that an (ind)-inf-scheme  $\mathcal{X}$  is isomorphic to the colimit of schemes equipped with a nil-closed map into  $\mathcal{X}$ .

4.1.1. For  $\mathcal{X} \in \text{PreStk}$  let

$$\text{PreStk}_{\text{nil-closed in } \mathcal{X}} \subset \text{PreStk}/\mathcal{X}$$

be the full subcategory spanned by objects  $f : \mathcal{X}' \rightarrow \mathcal{X}$  for which  $f$  is nil-closed.

We will use a similar notation for full subcategories of  $\text{PreStk}$ , e.g.,

$$\text{Sch}_{\text{nil-closed in } \mathcal{X}} \subset \text{Sch}/\mathcal{X},$$

etc.

4.1.2. We have the following assertion (cf. Corollary 1.7.5(a') in the case of ind-schemes):

**Theorem 4.1.3.** *Let  $\mathcal{X}$  be an object of  $\text{indinfSch}_{\text{laft}}$ . Then the map*

$$\text{colim}_{Z \in (\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

where the colimit is taken in  $\text{PreStk}$ , is an isomorphism.

Evaluating the two sides in Theorem 4.1.3 on  $<^\infty \text{Sch}_{\text{ft}}^{\text{aff}}$ , we obtain:

**Corollary 4.1.4.** *Let  $\mathcal{X}$  be an object of  $\text{indinfSch}_{\text{laft}}$ . Then the map*

$$\text{colim}_{Z \in (\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

where the colimit is taken in  $\text{PreStk}_{\text{laft}}$ , is an isomorphism.

With future applications in mind, let us state the following particular case of Corollary 4.1.4 separately:

**Corollary 4.1.5.** *Let  $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$  be such that  $\text{red}\mathcal{X} = \mathcal{X}_0$  is ind-affine. Then the functor*

$$(\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle)_{/\mathcal{X}} \times_{(\text{redSch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}_0}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } \mathcal{X}_0} \hookrightarrow (\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle)_{/\mathcal{X}}$$

is cofinal.

*Remark 4.1.6.* We note that the analog of Corollary 1.7.5(b) *fails* for ind-inf-schemes. I.e., it is *not* true that the inclusion

$$(\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}} \hookrightarrow (\text{Sch}_{\text{aft}})_{/\mathcal{X}}$$

is cofinal.

The rest of this subsection is devoted to the proof of Theorem 4.1.3.

4.1.7. *Step 0.* For  $(S, x) \in (\text{clSch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}}$  consider the category  $\text{Factor}(x, \text{nil-closed, ft, cl})$  of factorizations

$$S \rightarrow Z \rightarrow \mathcal{X},$$

where  $Z \in (\text{clSch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}$ . In Steps 1-6 we will show that this category is contractible.

Since  $\text{cl}\mathcal{X} \in \text{PreStk}_{\text{lft}}$ , it is easy to see that we can assume that  $S \in \text{clSch}_{\text{ft}}^{\text{aff}}$ .

4.1.8. *Step 1.* Denote  $S_0 := \text{red}S$  and  $x_0 := x|_{S_0}$ . Consider the category

$$\text{Factor}(x_0, \text{nil-closed, red})$$

of factorizations

$$S_0 \rightarrow Z_0 \rightarrow \mathcal{X},$$

where  $Z_0 \in (\text{redSch}_{\text{ft}})_{\text{closed in } \text{red}\mathcal{X}}$ .

The category  $\text{Factor}(x_0, \text{nil-closed, red})$  is contractible, since  $\text{red}\mathcal{X}$  is a (reduced) ind-scheme locally of finite type.

We have a functor

$$(4.1) \quad \text{Factor}(x, \text{nil-closed, ft, cl}) \rightarrow \text{Factor}(x_0, \text{nil-closed, red}), \quad Z \mapsto \text{red}Z,$$

and it is enough to show that (4.1) is a homotopy equivalence.

We note that (4.1) is a co-Cartesian fibration via the formation of push-outs. Hence, it is enough to show that the fibers of (4.1) are contractible.

4.1.9. *Step 2.* For an object  $S_0 \rightarrow Z_0 \rightarrow \mathcal{X}$  of  $\text{Factor}(x_0, \text{nil-closed, red})$ . The fiber of (4.1) over this object is described as follows.

Let  $Z'$  denote the push-out

$$S \sqcup_{S_0} Z_0.$$

(Note, however, that  $Z'$  is *not necessarily* of finite type.)

Since  $S$  was assumed of finite type, the map  $S_0 \rightarrow S$  is a nilpotent embedding (in fact, a finite succession of square-zero extensions). Since the morphism  $S_0 \rightarrow Z_0$  is affine, by [Chapter III.1, Corollary 7.2.3], we obtain a canonically defined map  $Z' \rightarrow \mathcal{X}$ .

The sought-for fiber is the category of factorizations of the above map  $Z' \rightarrow \mathcal{X}$  as

$$Z' \rightarrow Z \rightarrow \mathcal{X},$$

where  $Z \in \text{clSch}_{\text{ft}}$  and  $\text{red}Z = Z_0$ .

4.1.10. *Step 3.* We will prove the following general assertion. Suppose that  $Z'_0 \rightarrow Z'$  can be written as a finite succession of square-zero extensions, with  $Z'_0, Z' \in (\text{clSch}_{\text{qc}})_{/X}$  and  $Z'_0 \in \text{clSch}_{\text{ft}}$ .

Let  $\mathbf{C}(x)$  denote the category of factorizations of the map  $x : Z' \rightarrow X$  as

$$Z' \rightarrow Z \rightarrow X,$$

where  $Z \in \text{clSch}_{\text{ft}}$  and  $Z'_0 \rightarrow Z$  a nil-isomorphism.

We claim that  $\mathbf{C}(x)$  is contractible.

4.1.11. *Step 4.* Suppose that  $Z'_0 \rightarrow Z'$  can be written as a succession of  $m$  square-zero extensions. We will argue by induction on  $m$ .

If  $m = 1$ , the required assertion is proved by repeating [Chapter III.1, Proof of Theorem 9.1.2, Steps 2-3].

Let us carry out the induction step. Choose an intermediate extension

$$Z'_0 \rightarrow Z'_{\frac{1}{2}} \rightarrow Z',$$

and let  $x_{\frac{1}{2}}$  denote the map  $Z'_{\frac{1}{2}} \rightarrow X$ .

Let  $\mathbf{C}(x_{\frac{1}{2}})$  denote the corresponding category of factorizations of  $x_{\frac{1}{2}}$ . By the induction hypothesis, we can assume that  $\mathbf{C}(x_{\frac{1}{2}})$  is contractible.

Let  $\mathbf{D}$  denote the category of commutative diagrams

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \uparrow & & \uparrow \\ Z'_{\frac{1}{2}} & \longrightarrow & Z_{\frac{1}{2}} \end{array}$$

in  $(\text{clSch}_{\text{qc}})_{Z'_0/} / X$  with  $Z, Z_{\frac{1}{2}} \in \text{clSch}_{\text{ft}}$ , and where all the maps are nil-isomorphisms.

We have the forgetful functors

$$\mathbf{C}(x_{\frac{1}{2}}) \leftarrow \mathbf{D} \rightarrow \mathbf{C}(x).$$

We will show that both these functors are homotopy equivalences. This will prove that  $\mathbf{C}(x)$  is contractible.

4.1.12. *Step 5.* The functor  $\mathbf{D} \rightarrow \mathbf{C}(x)$  is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it is enough to show that it has contractible fibers.

However, for an object  $(Z' \rightarrow Z \rightarrow X) \in \mathbf{C}(x)$ , the fiber in question has a final point: take  $Z_{\frac{1}{2}} = Z$ .

4.1.13. *Step 6.* The functor  $\mathbf{D} \rightarrow \mathbf{C}(x_{\frac{1}{2}})$  is a co-Cartesian fibration via the formation of push-outs. Hence, in order to show that it is a homotopy equivalence, it is enough to show that it has contractible fibers.

For an object

$$(Z'_{\frac{1}{2}} \rightarrow Z_{\frac{1}{2}} \rightarrow X) \in \mathbf{C}(x_{\frac{1}{2}}),$$

set

$$\tilde{Z}' := Z' \sqcup_{Z'_{\frac{1}{2}}} Z_{\frac{1}{2}}.$$

Let  $\tilde{x}$  denote the resulting map  $\tilde{Z}' \rightarrow \mathcal{X}$ . The fiber in question is the category  $\mathbf{C}(\tilde{x})$ . This category is contractible by the induction hypothesis, applied to the nil-isomorphism

$$Z_{\frac{1}{2}} \rightarrow \tilde{Z}'.$$

4.1.14. *Step 7.* Now, let  $(S, x)$  be an arbitrary object of  $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ . Let us show that the category  $\mathrm{Factor}(x, \mathrm{nil}\text{-closed}, \mathrm{aft})$  of factorizations

$$S \rightarrow Z \rightarrow \mathcal{X},$$

where  $Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil}\text{-closed}}$  in  $\mathcal{X}$  is contractible.

Denote  $S_0 = {}^{\mathrm{cl}}S$  and  $x_0 = x|_{S_0}$ . Consider the functor

$$\mathrm{Factor}(x, \mathrm{nil}\text{-closed}, \mathrm{aft}) \rightarrow \mathrm{Factor}(x_0, \mathrm{nil}\text{-closed}, \mathrm{ft}, \mathrm{cl}), \quad Z \mapsto Z_0 := {}^{\mathrm{cl}}Z.$$

This functor is a coCartesian fibration via the formation of push-outs. Since we already know that  $\mathrm{Factor}(x_0, \mathrm{nil}\text{-closed}, \mathrm{cl})$  is contractible, it suffices to show that the fibers of the above functor are contractible. The latter is established by repeating the argument of [Chapter III.1, Theorem 9.1.4].

4.2. **A construction of ind-inf-schemes.** Our current goal is to prove a partial converse to Theorem 4.1.3, which will give rise to a procedure for explicitly constructing ind-inf-schemes.

4.2.1. We start with an object  $\mathcal{X}_0 \in {}^{\mathrm{red}}\mathrm{indSch}_{\mathrm{ft}}$ . In this subsection we will assume that  $\mathcal{X}_0$  is ind-affine.

Let  $\mathcal{X}_{\mathrm{nil}\text{-closed}}$  be a presheaf on the category

$${}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \times_{\mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}}} ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{closed}} \text{ in } \mathcal{X}_0,$$

where the functor  ${}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \rightarrow \mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}}$  is  $S \mapsto {}^{\mathrm{red}}S$ .

4.2.2. We impose the following assumptions on  $\mathcal{X}_{\mathrm{nil}\text{-closed}}$ :

- The restriction of  $\mathcal{X}_{\mathrm{nil}\text{-closed}}$  to the full subcategory

$$\begin{aligned} ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{closed}} \text{ in } \mathcal{X}_0 &= \mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}} \times_{\mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}}} ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{closed}} \text{ in } \mathcal{X}_0 \subset \\ &\subset {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \times_{\mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}}} ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{closed}} \text{ in } \mathcal{X}_0 \end{aligned}$$

takes the value  $* \in \mathrm{Spc}$ .

- For a push-out diagram

$$Z_1 \sqcup_Z Z'$$

in  ${}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \times_{\mathrm{redSch}_{\mathrm{ft}}^{\mathrm{aff}}} ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{closed}} \text{ in } \mathcal{X}_0$ , where  $Z \hookrightarrow Z'$  has a structure of square-zero extension, the resulting map

$$\mathcal{X}_{\mathrm{nil}\text{-closed}}(Z_1 \sqcup_Z Z') \rightarrow \mathcal{X}_{\mathrm{nil}\text{-closed}}(Z_1) \times_{\mathcal{X}_{\mathrm{nil}\text{-closed}}(Z)} \mathcal{X}_{\mathrm{nil}\text{-closed}}(Z')$$

is an isomorphism (cf. characterization of deformation theory in [Chapter III.1, Corollary 6.3.6]).

4.2.3. Let  $\mathcal{X}$  denote the left Kan extension of  $\mathcal{X}_{\text{nil-closed}}$  under the fully faithful embedding

$$\left( <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } \mathcal{X}_0} \right)^{\text{op}} \hookrightarrow \left( <^\infty \text{Sch}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} (\text{redSch}^{\text{aff}})_{/\mathcal{X}_0} \right)^{\text{op}}.$$

Note that

$$<^\infty \text{Sch}^{\text{aff}} \times_{\text{redSch}^{\text{aff}}} (\text{redSch}^{\text{aff}})_{/\mathcal{X}_0} = (<^\infty \text{Sch}^{\text{aff}})_{/(\mathcal{X}_0)_{\text{dR}}}.$$

Thus, we can view  $\mathcal{X}$  as an object of  ${}^{\text{conv}}\text{PreStk}$  mapping to  $(\mathcal{X}_0)_{\text{dR}}$ . By construction,  $\mathcal{X}$  belongs to

$$\text{PreStk}_{\text{laft}} \subset {}^{\text{conv}}\text{PreStk},$$

and  $\text{red}\mathcal{X}$  is canonically isomorphic to  $\mathcal{X}_0$ .

4.2.4. We are going to prove:

**Theorem 4.2.5.** *Under the above circumstances  $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$ .*

By combining with Corollary 4.1.5, we obtain:

**Corollary 4.2.6.** *The assignments*

$$\mathcal{X}_{\text{nil-closed}} \rightsquigarrow \mathcal{X}$$

and

$$\mathcal{X} \rightsquigarrow \mathcal{X} |_{<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } \mathcal{X}_0}}$$

define mutually inverse equivalences between

$$(\text{indinfSch}_{\text{laft}})_{/(\mathcal{X}_0)_{\text{dR}}} \times_{(\text{redindSch}_{\text{laft}})_{/\mathcal{X}_0}} *$$

and the category of presheaves on  $<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } \mathcal{X}_0}$ , satisfying the assumptions of Sect. 4.2.2.

The rest of the subsection is devoted to the proof of Theorem 4.2.5.

4.2.7. *Step 1.* We only have to show that  $\mathcal{X}$  admits deformation theory. Since  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ , by [Chapter III.1, Corollary 7.2.6], it suffices to check the following:

Let  $S \rightarrow Z$  be a map in  $(<^\infty \text{Sch}^{\text{aff}})_{/\mathcal{X}}$ , where  $\text{red}Z \rightarrow \mathcal{X}_0$  is a closed embedding and  $Z \in <^\infty \text{Sch}_{\text{ft}}^{\text{aff}}$ . We need to show that for a map  $S \hookrightarrow S'$ , equipped with a structure of square-zero extension, and such that  $S' \in <^\infty \text{Sch}^{\text{aff}}$ , the map

$$\text{Maps}_{Z'}(Z', \mathcal{X}) \rightarrow \text{Maps}_{S'}(S', \mathcal{X})$$

is an isomorphism, where  $Z' := Z \sqcup_S S'$ .

Fix a point  $x' \in \text{Maps}_{S'}(S', \mathcal{X})$ . We need to show that the groupoid

$$(4.2) \quad \text{Maps}_{Z'}(Z', \mathcal{X}) \times_{\text{Maps}_{S'}(S', \mathcal{X})} \{x'\}$$

is contractible.

4.2.8. *Step 2.* Let  $\mathbf{C}$  be the category of factorizations of the given map  $S \rightarrow \mathcal{X}$  as

$$S \rightarrow Z_1 \rightarrow \mathcal{X},$$

where  $\text{red}Z_1 \rightarrow \mathcal{X}_0$  is a closed embedding, and  $Z_1 \in {}^{<\infty}\text{Sch}_{\text{ft}}^{\text{aff}}$ . By the construction of  $\mathcal{X}$  as a left Kan extension, the category  $\mathbf{C}$  is contractible.

Let  $\mathbf{C}'$  be the category whose objects are commutative diagrams

$$\begin{array}{ccccc} S' & \longrightarrow & Z'_1 := Z_1 \sqcup_S S' & \longrightarrow & \mathcal{X} \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ S & \longrightarrow & Z_1 & \longrightarrow & \mathcal{X}, \end{array}$$

where the bottom row is an object of  $\mathbf{C}$ .

We have a natural forgetful functor  $\mathbf{C}' \rightarrow \mathbf{C}$ . We claim that this functor is a co-Cartesian fibration in groupoids, such that every edge in  $\mathbf{C}$  induces a homotopy equivalence between the fibers. The claim will be proved in Step 6.

Since the category  $\mathbf{C}$  is contractible, we obtain that for any  $(S \rightarrow Z_1 \rightarrow \mathcal{X}) \in \mathbf{C}$  the map

$$\mathbf{C}' \times_{\mathbf{C}} \{(S \rightarrow Z_1 \rightarrow \mathcal{X})\} \rightarrow \mathbf{C}'$$

is a homotopy equivalence. In particular, we can take  $(S \rightarrow Z_1 \rightarrow \mathcal{X})$  to be the original map  $(S \rightarrow Z \rightarrow \mathcal{X})$ .

(Note that

$$\mathbf{C}' \times_{\mathbf{C}} \{(S \rightarrow Z \rightarrow \mathcal{X})\}$$

identifies with the groupoid (4.2), whose contractibility we want to establish.)

4.2.9. *Step 3.* Let  $\text{Factor}(x')$  denote the category of factorizations of  $x'$  as

$$S' \rightarrow W \rightarrow \mathcal{X},$$

with  $W \in {}^{<\infty}\text{Sch}_{\text{ft}}^{\text{aff}}$  and  $\text{red}W \rightarrow \mathcal{X}_0$  being a closed embedding. This category is contractible by definition.

We claim that there is a canonical functor

$$\text{Factor}(x') \rightarrow \mathbf{C}'.$$

Indeed, for an object of  $\text{Factor}(x')$  as above, consider the composed map

$$S \rightarrow S' \rightarrow W,$$

and set  $W' := W \sqcup_S S'$ . The extension  $W \hookrightarrow W'$  splits by construction.

We regard

$$S \rightarrow W \rightarrow \mathcal{X}$$

as an object of  $\mathbf{C}$ . And we regard the composition

$$W' \rightarrow W \rightarrow \mathcal{X}$$

as an object of  $\mathbf{C}'$  over it.

4.2.10. *Step 4.* Let  $\mathbf{D}$  denote the category where an object is given by the following data:

- A square-zero extension  $Z \hookrightarrow \tilde{Z}'$  with  $\tilde{Z}' \in <^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}}$ ;
- A map  $Z' \rightarrow \tilde{Z}'$  in the category of square-zero extensions of  $Z$ .
- A map  $\tilde{x}' : \tilde{Z}' \rightarrow \mathcal{X}$ , extending  $x'$  and compatible with the restriction to  $Z$ .

We have a natural functor  $\mathbf{D} \rightarrow \text{Maps}_{Z'}(Z', \mathcal{X}) \times_{\text{Maps}_{S'}(S', \mathcal{Y})} \{x'\}$ , and we claim that this functor is a homotopy equivalence.

Indeed, note that the scheme  $Z'$  can be written as a filtered limit of the  $\tilde{Z}'$ 's, taken over the category of square-zero extensions

$$Z \hookrightarrow \tilde{Z}', \quad \tilde{Z}' \in \leq^n \text{Sch}_{\text{ft}}^{\text{aff}},$$

where  $n$  is such that  $Z, Z' \in \leq^n \text{Sch}^{\text{aff}}$ . Hence, our assertion follows from the fact that  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ , and hence takes filtered limits in  $\leq^n \text{Sch}^{\text{aff}}$  to colimits.

Note also that we have a naturally defined functor

$$\mathbf{D} \rightarrow \text{Factor}(x')$$

that sends  $\tilde{Z}'$  to  $W$ .

4.2.11. *Step 5.* We have a *non-commuting* diagram of categories:

$$(4.3) \quad \begin{array}{ccc} \text{Maps}_{Z'}(Z', \mathcal{X}) \times_{\text{Maps}_{S'}(S', \mathcal{X})} \{x'\} & \longrightarrow & \mathbf{C}' \\ \uparrow & & \uparrow \\ \mathbf{D} & \longrightarrow & \text{Factor}(x'). \end{array}$$

However, we claim that the two resulting maps

$$\mathbf{D} \rightrightarrows \mathbf{C}'$$

are homotopic. Indeed, the two functors send an object of  $\mathbf{D}$  as above to

$$\begin{array}{ccccc} S' & \longrightarrow & Z' & \longrightarrow & \mathcal{X} \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ S & \longrightarrow & Z & \longrightarrow & \mathcal{X}, \end{array}$$

(for the clockwise circuit)

$$\begin{array}{ccccc} S' & \longrightarrow & \tilde{Z}' \sqcup_S S' & \longrightarrow & \mathcal{X} \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ S & \longrightarrow & \tilde{Z}' & \xrightarrow{\tilde{x}'} & \mathcal{X}, \end{array}$$

(for the anti-clockwise circuit), respectively. The required homotopy is provided by the map  $Z \rightarrow \tilde{Z}'$ .

Note that the clockwise circuit in (4.3) is a homotopy equivalence. Hence, we obtain that  $\text{Maps}_{Z'}(Z', \mathcal{X}) \times_{\text{Maps}_{S'}(S', \mathcal{Y})} \{x'\}$  is a retract of  $\text{Factor}(x')$ . Therefore, it is contractible, as required.

4.2.12. *Step 6.* It suffices to show that whenever  $Z_1 \rightarrow Z_2$  is a map in

$${}^{<\infty}\text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } \mathcal{X}_0},$$

and  $Z_1 \hookrightarrow Z'_1$  is a square-zero extension with  $Z'_1 \in {}^{<\infty}\text{Sch}^{\text{aff}}$ , then for

$$Z'_2 := Z_2 \sqcup_{Z_1} Z'_1,$$

the map

$$\text{Maps}(Z'_2, \mathcal{X}) \rightarrow \text{Maps}(Z_2, \mathcal{X}) \times_{\text{Maps}(Z_1, \mathcal{X})} \text{Maps}(Z'_1, \mathcal{X})$$

is an isomorphism.

As in Step 3, we write

$$Z'_1 \simeq \lim_{\alpha \in A} Z'_{1,\alpha},$$

where  $A$  is a filtered category, and  $Z_1 \hookrightarrow Z'_{1,\alpha}$  are square-zero extensions with  $Z'_{1,\alpha} \in \leq^n \text{Sch}_{\text{aft}}^{\text{aff}}$  for some  $n$ . Then

$$Z'_2 \simeq \lim_{\alpha \in A} Z'_{2,\alpha},$$

where

$$Z'_{2,\alpha} := Z_2 \sqcup_{Z_1} Z'_{1,\alpha}.$$

The required assertion now follows from the fact  $\mathcal{X}$  belongs to  $\text{PreStk}_{\text{laft}}$ , and hence takes filtered limits in  $\leq^n \text{Sch}_{\text{aft}}^{\text{aff}}$  to colimits.

**4.3. Exhibiting inf-schemes as colimits.** In this subsection we will adapt Theorem 4.1.3 to the case of inf-schemes. Namely, we will show that in this case we can replace the word ‘nil-closed’ by ‘nil-isomorphism’.

4.3.1. For  $\mathcal{X} \in \text{PreStk}$  let

$$\text{PreStk}_{\text{nil-isom to } \mathcal{X}} \subset \text{PreStk}/\mathcal{X}$$

be the full subcategory spanned by objects  $f : \mathcal{X}' \rightarrow \mathcal{X}$  for which  $f$  is a nil-isomorphism.

We will use a similar notation for full subcategories of  $\text{PreStk}$ , e.g.,

$$\text{Sch}_{\text{nil-isom to } \mathcal{X}} \subset \text{Sch}/\mathcal{X},$$

etc.

We claim (compare with Proposition 1.8.5 in the case of nil-schematic ind-schemes):

**Proposition 4.3.2.** *Let  $\mathcal{X}$  be an object of  $\text{infSch}_{\text{laft}}$ . Then the inclusion*

$$(\text{Sch}_{\text{aft}})_{\text{nil-isom to } \mathcal{X}} \hookrightarrow (\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}}$$

*is cofinal.*

*Proof.* It is enough to show that the embedding in question admits a left adjoint. Given an object

$$(Z \rightarrow \mathcal{X}) \in (\text{Sch}_{\text{aft}}^{\text{aff}})_{\text{nil-closed in } \mathcal{X}},$$

we note that since  $Z \in \text{Sch}_{\text{aft}}^{\text{aff}}$ , the map  $\text{red}Z \rightarrow Z$  is a nilpotent embedding.

Now, the value of the left adjoint in question is given by sending

$$(Z \rightarrow \mathcal{X}) \in (\text{Sch}_{\text{aft}}^{\text{aff}})_{\text{nil-closed in } \mathcal{X}}$$

to

$$Z \sqcup_{\text{red}Z} \text{red}\mathcal{X},$$

which maps to  $\mathcal{X}$  using [Chapter III.1, Corollary 7.2.3]. □

Combining with Theorem 4.1.3, we obtain (compare with Corollary 1.8.6(a') in the case of nil-schematic ind-schemes):

**Corollary 4.3.3.** *Let  $\mathcal{X}$  be an object of  $\text{infSch}_{\text{laft}}$ . Then the map*

$$\text{colim}_{Z \in (\text{Sch}_{\text{aft}})_{\text{nil-isom to } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

*induces an isomorphism, when the colimit is taken in either  $\text{PreStk}$  or  $\text{PreStk}_{\text{laft}}$ .*

Note that Corollary 4.3.3 admits the following corollary:

**Corollary 4.3.4.** *Let  $\mathcal{X} \in \text{infSch}_{\text{laft}}$  be such that  $\text{red}\mathcal{X} = X_0 \in \text{redSch}_{\text{ft}}^{\text{aff}}$ . Then the functor*

$$({}^{<\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{X}} \times_{(\text{redSch}_{\text{ft}}^{\text{aff}})_{/X_0}} \{X_0\} \hookrightarrow ({}^{<\infty}\text{Sch}^{\text{aff}})_{/\mathcal{X}}$$

*is cofinal.*

4.3.5. We now claim (compare with Corollary 1.8.6(a'') in the case of nil-schematic ind-schemes):

**Proposition 4.3.6.** *For  $\mathcal{X} \in \text{infSch}_{\text{laft}}$  the category  $(\text{Sch}_{\text{aft}})_{\text{nil-isom to } \mathcal{X}}$  is sifted.*

*Proof.* We need to show that for a pair of nilpotent embeddings  $f_1 : Z_1 \rightarrow \mathcal{X}$  and  $f_2 : Z_2 \rightarrow \mathcal{X}$ , the category of

$$(Z_1 \xrightarrow{g_1} Z, Z_2 \xrightarrow{g_2} Z, Z \xrightarrow{f} \mathcal{X}, f_1 \sim f \circ g_1, f_2 \sim f \circ g_2)$$

is contractible.

We claim, however, that the category in question admits an initial object, namely

$$Z := Z_1 \sqcup_{\text{red}\mathcal{X}} Z_2,$$

see [Chapter III.1, Corollary 7.2.3]. □

**4.4. A construction of inf-schemes.** In this subsection we will consider a version of Proposition 4.4.5 for inf-schemes. This version will be crucial for our study of formal moduli problems in [Chapter IV.1].

4.4.1. We start with an object  $X_0 \in \text{redSch}_{\text{ft}}^{\text{aff}}$ , and let  $\mathcal{X}_{\text{nil-isom}}$  be a presheaf on the category

$${}^{<\infty}\text{Sch}_{\text{aft}} \times_{\text{redSch}_{\text{ft}}} \{X_0\},$$

where the functor  ${}^{<\infty}\text{Sch}_{\text{aft}} \rightarrow \text{redSch}_{\text{ft}}$  is  $S \mapsto \text{red}S$ .

4.4.2. We impose the following two conditions:

- $\mathcal{X}_{\text{nil-isom}}(X_0) = *$ .
- For a push-out diagram

$$Z_1 \sqcup_Z Z'$$

in  ${}^{<\infty}\text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}} \{X_0\}$ , where  $Z \hookrightarrow Z'$  has a structure of square-zero extension, the resulting map

$$\mathcal{X}_{\text{nil-isom}}(Z_1 \sqcup_Z Z') \rightarrow \mathcal{X}_{\text{nil-isom}}(Z_1) \times_{\mathcal{X}_{\text{nil-isom}}(Z)} \mathcal{X}_{\text{nil-isom}}(Z')$$

is an isomorphism (cf. remark following [Chapter III.1, Definition 6.1.2]).

4.4.3. Let  $\mathcal{X}$  denote the left Kan extension of  $\mathcal{X}_{\text{nil-isom}}$  under the fully faithful embedding

$$\left( <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} \{X_0\} \right)^{\text{op}} \hookrightarrow \left( <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{/X_0} \right)^{\text{op}}.$$

Thus, we can view  $\mathcal{X}$  as an object of  ${}^{\text{conv}}\text{PreStk}$  mapping to  $(X_0)_{\text{dR}}$ . By construction  $\mathcal{X}$  belongs to

$$\text{PreStk}_{\text{lft}} \subset {}^{\text{conv}}\text{PreStk},$$

and  $\text{red}\mathcal{X}$  is canonically isomorphic to  $X_0$ .

4.4.4. We are going to prove:

**Proposition 4.4.5.** *Under the above circumstances  $\mathcal{X} \in \text{infSch}_{\text{lft}}$ .*

Combining with Corollary 4.3.4, we obtain:

**Corollary 4.4.6.** *The assignments*

$$\mathcal{X}_{\text{nil-isom}} \rightsquigarrow \mathcal{X} \text{ and } \mathcal{X} \rightsquigarrow \mathcal{X} |_{<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} \{X_0\}}$$

define mutually inverse equivalences between

$$(\text{infSch}_{\text{lft}})_{/(X_0)_{\text{dR}}} \times_{(\text{redindSch}_{\text{lft}})_{/X_0}} *$$

and the category of presheaves on  $<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} \{X_0\}$ , satisfying the two assumptions of Sect. 4.4.2.

4.4.7. *Proof of Proposition 4.4.5.* Let  $\mathcal{X}_{\text{ind-closed}}$  denote the presheaf on the category

$$<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } X_0}$$

equal to the left Kan extension of  $\mathcal{X}_{\text{nil-isom}}$  under the fully faithful embedding

$$\left( <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} \{X_0\} \right)^{\text{op}} \hookrightarrow \left( <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } X_0} \right)^{\text{op}}.$$

Now, by Theorem 4.2.5, it is sufficient to show that  $\mathcal{X}_{\text{ind-closed}}$  satisfies the conditions of Sect. 4.2.2.

Note, however, that the functor

$$<^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} \{X_0\} \hookrightarrow <^\infty \text{Sch}_{\text{ft}}^{\text{aff}} \times_{\text{redSch}_{\text{ft}}^{\text{aff}}} (\text{redSch}_{\text{ft}}^{\text{aff}})_{\text{closed in } X_0}$$

admits a left adjoint, given by

$$Z \mapsto Z \sqcup_{\text{red}Z} X_0.$$

Hence, the value of  $\mathcal{X}_{\text{ind-closed}}$  on  $Z \rightarrow \mathcal{X}$  can be calculated as

$$\mathcal{X}_{\text{nil-isom}}(Z \sqcup_{\text{red}Z} X_0).$$

This implies the required condition on  $\mathcal{X}_{\text{ind-closed}}$ , since the above left adjoint preserves push-outs.