PART III.3. IND-COHERENT SHEAVES ON IND-INF-SCHEMES

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INTRODUCTION

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1. Ind-coherent sheaves on ind-schemes

In order to develop the theory of IndCoh on ind-inf-schemes, we first need to do this for ind-schemes. The latter theory follows rather easily from one on schemes.

In this section we will mainly review the results from [GL:IndSch], Sect. 2.

1.1. Basic properties. In this subsection we will express the category IndCoh on an ind-scheme $X$ from that on schemes equipped with a closed embedding into $X$.

1.1.1. Let $\text{IndCoh}^i_{\text{indSch}_{\text{laft}}}$ denote the restriction of the functor $\text{IndCoh}^i_{\text{PreStk}_{\text{laft}}}$ to the full subcategory $\left(\text{indSch}_{\text{laft}}\right)^{\text{op}} \hookrightarrow \left(\text{PreStk}_{\text{laft}}\right)^{\text{op}}$.

In particular, for $X \in \text{indSch}_{\text{laft}}$ we have a well-defined category $\text{IndCoh}(X)$.

1.1.2. Suppose $X$ has been written as

$$X \simeq \text{conv} X' \quad X' \simeq \text{colim} X_a,$$

where $X_a \in \text{Sch}_{\text{aff}}$ with the maps $i_{a,b} : X_a \to X_b$ being closed embeddings. In this case we have:

**Proposition 1.1.3.** Under the above circumstances, $!$-restriction defines an equivalence

$$\text{IndCoh}(X) \to \lim_{a \in A^{op}} \text{IndCoh}(X^a),$$

where for $a \to b$, the corresponding functor $\text{IndCoh}(X_b) \to \text{IndCoh}(X_a)$ is $i_{a,b}^*$.

**Proof.** This follows from the convergence property of the functor $\text{IndCoh}^i_{\text{Sch}_{\text{aff}}}$, see [Book-II.2, Lemma 3.2.4 and Sect. 3.4.1].

**Remark 1.1.4.** The reason we exhibit an ind-scheme $X$ as $\text{conv}(\text{colim} X_a)$ rather than just as $\text{colim} X_a$ is that the former presentation comes up in practice more often: many ind-schemes are given in this form. The fact that the resulting prestack is indeed an ind-scheme (i.e., can be written as a colimit of schemes under closed embeddings) is [Book-III.2, Corollary 1.4.4] and is somewhat non-trivial.

1.1.5. Combining the above proposition with [GL:DG, Lemma 1.3.3], we obtain:

**Corollary 1.1.6.** For $X$ written as in (1.1), we have

$$\text{IndCoh}(X) \simeq \text{colim}_{a \in A} \text{IndCoh}(X^a),$$

where for $a \to b$, the corresponding functor $\text{IndCoh}(X_a) \to \text{IndCoh}(X_b)$ is $(i_{a,b})_*$.

**Corollary 1.1.7.** For $X \in \text{indSch}_{\text{laft}}$ and a closed embedding $i : X \to X$, the functor

$$i^*_\text{IndCoh} : \text{IndCoh}(X) \to \text{IndCoh}(X),$$

left adjoint to $i^i$, is well-defined.

For $X \in \text{indSch}_{\text{laft}}$, let $\text{Coh}(X)$ denote the full subcategory of $\text{IndCoh}(X)$ spanned by objects $i^*_\text{IndCoh}(F), \quad i : X \to X$ is a closed embedding and $F \in \text{Coh}(X)$.

From Corollary 1.1.6 we obtain:
Corollary 1.1.8. For an ind-scheme \( X \), the category \( \text{IndCoh}(X) \) is compactly generated and \( \text{IndCoh}(X)^\perp = \text{Coh}(X) \).

Proof. Follows from [DrGa, Corollary 1.9.4 and Lemma 1.9.5]. \( \square \)

1.1.9. Here is another convenient fact about the category \( \text{IndCoh}(X) \), where \( X \in \text{indSch}_{\text{left}} \).

Let \( X' \hookrightarrow X \xhookleftarrow{} X'' \) be closed embeddings.

We would like to calculate the composition \( (i')^! \circ (i'')_*^{\text{IndCoh}} : \text{IndCoh}(X'') \rightarrow \text{IndCoh}(X') \).

Let \( A \) denote the category \( (\text{Sch}_{\text{left}})_{\text{closed}} \) in \( X \), so that \( X' \) and \( X'' \) correspond to indices \( a' \) and \( a'' \), respectively. Let \( i_{a'} \) and \( i_{a''} \) denote the corresponding closed embeddings, i.e., the maps \( i' \) and \( i'' \), respectively. Let \( B \) be any category cofinal in \( A_{a' \sqcup a''/} := A_{a'}/ \times A_{a''/} \).

For \( b \in B \), let

\[
X' = X_a \xrightarrow{i_{a',b}} X_b \xleftarrow{i_{a'',b}} X_{a''} = X''
\]
denote the corresponding maps.

The next assertion results from [GL:DG, Sect. 1.3.5]:

Lemma 1.1.10. Under the above circumstances, we have a canonical isomorphism

\[
(i')^! \circ (i'')_*^{\text{IndCoh}} \simeq \colim_{b \in B} (i_{a',b})^! \circ (i_{a'',b})_*^{\text{IndCoh}}.
\]

1.2. t-structure. In this subsection we will study the naturally defined t-structure on \( \text{IndCoh}(X) \) of an ind-scheme.

1.2.1. For \( X \in \text{indSch}_{\text{left}} \) we introduce a t-structure on the category \( \text{IndCoh}(X) \) as follows:

An object \( F \in \text{IndCoh}(X) \) belongs to \( \text{IndCoh}(X)^{\geq 0} \) if and only if for every closed embedding \( i : X \rightarrow X \), where \( X \in \text{Sch}_{\text{left}} \), we have \( i^!(F) \in \text{IndCoh}(X)^{\geq 0} \).

By construction, this t-structure is compatible with filtered colimits, which by definition means that \( \text{IndCoh}(X)^{\geq 0} \) is preserved by filtered colimits.

1.2.2. We can describe this t-structure and the category \( \text{IndCoh}(X)^{\leq 0} \) more explicitly. Write

\[
\text{cl} X \simeq \colim_{a \in A} X_a,
\]

where \( X_a \in (\text{clSch}_{\text{left}})_{\text{closed}} \) in \( X \).

For each \( a \), let \( i_a \) denote the corresponding map (automatically, a closed embedding) \( X_a \rightarrow X \).

By Corollary 1.1.7, we have a pair of adjoint functors

\[
(i_a)^{\text{IndCoh}} : \text{IndCoh}(X_a) \rightleftarrows \text{IndCoh}(X) : i_a^!.
\]

Lemma 1.2.3. Under the above circumstances we have:

(a) An object \( F \in \text{IndCoh}(X) \) belongs to \( \text{IndCoh}^{\geq 0} \) if and only if for every \( a \), the object \( i_a^!(F) \in \text{IndCoh}(X_a)^{\geq 0} \).

(b) The category \( \text{IndCoh}(X)^{\leq 0} \) is generated under colimits by the essential images of the functors \( (i_a)^{\text{IndCoh}}(\text{Coh}(X_a)^{\leq 0}) \).
**Proof.** It is easy to see that for a quasi-compact DG scheme \( X \), the category \( \text{IndCoh}(X) \leq 0 \) is generated under colimits by \( \text{Coh}(\mathcal{X}) \leq 0 \). In particular, by adjunction, an object \( \mathcal{F} \in \text{IndCoh}(X) \) is coconnective if and only if its restriction to \( \mathcal{X} \) is coconnective.

Hence, in the definition of \( \text{IndCoh}(X) \geq 0 \), instead of all closed embeddings \( X \to X \), it suffices to use only those with \( X \) a classical scheme.

Note that the category \( A \) is cofinal in \( (\mathcal{X})_{\text{closed}} \). This implies point (a) of the lemma. Point (b) follows formally from point (a). \( \square \)

**1.2.4.** Suppose \( i : X \to X \) is a closed embedding of a scheme into an ind-scheme. By Corollary 1.1.7, we have a well-defined functor \( i^*_\text{IndCoh} : \text{IndCoh}(X) \to \text{IndCoh}(X) \), which is the right adjoint to \( i^! \). Since \( i^! \) is left t-exact, the functor \( i^*_\text{IndCoh} \) is right t-exact. However, we claim:

**Lemma 1.2.5.** The functor \( i^*_\text{IndCoh} \) is t-exact.

**Proof.** We need to show that for \( \mathcal{F} \in \text{IndCoh}(X) \geq 0 \), and a closed embedding \( i' : X' \to X \), we have

\[
(i')^! \circ i^*_\text{IndCoh}(\mathcal{F}) \in \text{IndCoh}(X') \geq 0.
\]

This follows from Lemma 1.1.10: in the notations of loc.cit., each of the functors \( (i_{a'', b})^*_{\text{IndCoh}} \) is t-exact (because \( i_{a'', b} \) is a closed embedding), each of the functors \( (i_{a'', b})^! \) is left t-exact (because \( i_{a'', b} \) is a closed embedding), and the category \( B \) is filtered. \( \square \)

**Corollary 1.2.6.** The subcategory \( \text{Coh}(X) = \text{IndCoh}(X)^c \) is preserved by the truncation functors.

**Proof.** Follows from Lemma 1.2.5. \( \square \)

**Corollary 1.2.7.** The t-structure on \( \text{IndCoh}(X) \) is obtained from the t-structure on \( \text{Coh}(X) \) by the procedure of [Book-II.1, Lemma 1.2.4].

**1.3.** **Recovering** \( \text{IndCoh} \) **from ind-proper maps.** The contents of this subsection are rather formal: we show that the functor \( \text{IndCoh} \) on ind-schemes can be recovered from the corresponding functor on schemes, where we restrict 1-morphisms to be proper, or even closed embeddings. This is not surprising, given the definition of ind-schemes.

1.3.1. Recall what it means for a map in PreStk to be ind-proper (resp., ind-finite, ind-nil-closed embedding, ind-closed embedding), see [Book-III.2, Definitions 1.6.7 and 1.6.11].

1.3.2. Consider the corresponding 1-full subcategories

\[
(i\text{ndSch}_{\text{laft}})_{\text{ind-closed}} \subset (i\text{ndSch}_{\text{laft}})_{\text{ind-nil-closed}} \subset (i\text{ndSch}_{\text{laft}})_{\text{ind-finite}} \subset (i\text{ndSch}_{\text{laft}})_{\text{ind-proper}}
\]

and the corresponding categories

\[
(\text{Sch}_{\text{laft}})_{\text{closed}} \subset (\text{Sch}_{\text{laft}})_{\text{nil-closed}} \subset (\text{Sch}_{\text{laft}})_{\text{finite}} \subset (\text{Sch}_{\text{laft}})_{\text{proper}}.
\]

Consider the corresponding fully faithful embeddings

\[
(\text{Sch}_{\text{laft}})_{\text{closed}} \hookrightarrow (i\text{ndSch}_{\text{laft}})_{\text{ind-closed}}, \quad (\text{Sch}_{\text{laft}})_{\text{nil-closed}} \hookrightarrow (i\text{ndSch}_{\text{laft}})_{\text{ind-nil-closed}}, \quad (\text{Sch}_{\text{laft}})_{\text{finite}} \hookrightarrow (i\text{ndSch}_{\text{laft}})_{\text{ind-finite}}
\]
and

\[(\text{indSch}_\text{aft})_{\text{proper}} \rightarrow (\text{indSch}_\text{aft})_{\text{ind-proper}}.\]

Let \(\text{IndCoh}^!_{(\text{Sch}_\text{aft})_{\text{proper}}}\) denote the functor

\[
\text{IndCoh}^!_{(\text{Sch}_\text{aft})_{\text{proper}}} : ((\text{Sch}_\text{aft})_{\text{proper}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}},
\]

and similarly, for “proper” replaced by “nil-closed”, “finite” or “closed”.

Let \(\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-proper}}}\) denote the functor

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-proper}}} : ((\text{indSch}_\text{aft})_{\text{ind-proper}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}},
\]

and similarly, for “proper” replaced by “finite”, “nil-closed” or “closed”.

1.3.3. We claim:

**Proposition 1.3.4.** The naturally defined functors

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-proper}}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{proper}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-proper}})^{\text{op}},
\]

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-finite}}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{finite}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-finite}})^{\text{op}},
\]

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-nil-closed}}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{nil-closed}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-nil-closed}})^{\text{op}},
\]

and

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-closed}}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{closed}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-closed}})^{\text{op}}.
\]

are isomorphisms.

**Proof.** By [Book-III.2, Corollary 1.7.5(b)], for \(X \in \text{indSch}_\text{aft}\), the functor

\[
(\text{Sch}_\text{aft})_{\text{closed}} \times (\text{indSch}_\text{aft})_{\text{ind-closed}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{closed}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-closed}})^{\text{op}}
\]

is cofinal.

This implies that the functor

\[
\text{IndCoh}^!_{(\text{indSch}_\text{aft})_{\text{ind-nil-closed}}} \rightarrow \text{RKE}_{((\text{Sch}_\text{aft})_{\text{closed}})^{\text{op}}}((\text{indSch}_\text{aft})_{\text{ind-nil-closed}})^{\text{op}}
\]

is an isomorphism.

The other two isomorphisms follow from the fact that the functors

\[
(\text{Sch}_\text{aft})_{\text{closed}} \times (\text{indSch}_\text{aft})_{\text{ind-closed}} 
\rightarrow (\text{Sch}_\text{aft})_{\text{nil-closed}} \times (\text{indSch}_\text{aft})_{\text{ind-nil-closed}} 
\rightarrow (\text{Sch}_\text{aft})_{\text{finite}} \times (\text{indSch}_\text{aft})_{\text{ind-finite}} 
\rightarrow (\text{Sch}_\text{aft})_{\text{proper}} \times (\text{indSch}_\text{aft})_{\text{ind-proper}}
\]

are fully faithful. □

1.4. **Direct image for IndCoh on ind-schemes.** In this subsection we show how to construct the functor of direct image on IndCoh for morphisms between ind-schemes.
1.4.1. We now consider the functor
\[ \text{IndCoh}_{\text{Sch}^aft}: \text{Sch}^aft \to \text{DGCat}_{\text{cont}}, \]
where for a morphism \( f: X_1 \to X_2 \) in \( \text{Sch}^aft \), the functor
\[ \text{IndCoh}(X_1) \to \text{IndCoh}(X_2) \]
is \( f^\ast \text{IndCoh} \), see [Book-II.1, Sect. 2.2].

Recall the notation
\[ \text{IndCoh}(\text{Sch}^aft)_\text{proper} = \text{IndCoh}_{\text{Sch}^aft}|_{(\text{Sch}^aft)_\text{proper}}: (\text{Sch}^aft)_\text{proper} \to \text{DGCat}_{\text{cont}}, \]
and let consider also the corresponding functors
\[ \text{IndCoh}(\text{Sch}^aft)_\text{finite}, \text{IndCoh}(\text{Sch}^aft)_\text{nil-closed}, \text{IndCoh}(\text{Sch}^aft)_\text{nil-closed} \to \text{DGCat}_{\text{cont}}. \]

Denote
\[ \text{IndCoh}(\text{indSch}^aft)_\text{ind-proper} := \text{LKE}_{(\text{Sch}^aft)_\text{proper}}(\text{IndCoh}(\text{Sch}^aft)_\text{proper}), \]
and let
\[ \text{IndCoh}(\text{indSch}^aft)_\text{ind-nil-closed}, \text{IndCoh}(\text{indSch}^aft)_\text{ind-finite} \]
and
\[ \text{IndCoh}(\text{indSch}^aft)_\text{ind-closed} \]
denote its restriction to the corresponding 1-full subcategories.

The same proof as in the case of Proposition 1.3.4 gives:

**Proposition 1.4.2.** The natural maps
\[ \text{LKE}_{(\text{Sch}^aft)_\text{proper}}(\text{IndCoh}(\text{Sch}^aft)_\text{proper}) \to \text{IndCoh}(\text{indSch}^aft)_\text{ind-proper}, \]
\[ \text{LKE}_{(\text{Sch}^aft)_\text{finite}}(\text{IndCoh}(\text{Sch}^aft)_\text{finite}) \to \text{IndCoh}(\text{indSch}^aft)_\text{ind-nil-closed}, \]
\[ \text{LKE}_{(\text{Sch}^aft)_\text{nil-closed}}(\text{IndCoh}(\text{Sch}^aft)_\text{nil-closed}) \to \text{IndCoh}(\text{indSch}^aft)_\text{ind-nil-closed} \]
and
\[ \text{LKE}_{(\text{Sch}^aft)_\text{closed}}(\text{IndCoh}(\text{Sch}^aft)_\text{closed}) \to \text{IndCoh}(\text{indSch}^aft)_\text{ind-closed} \]
are isomorphisms.

1.4.3. Recall from [Funct, Sect. 11.2] the notion of two functors obtained from each other by passing to adjoints. The following is a particular case of [Funct, Corollary 13.2.8]:

Let \( F: C_1 \to C_2 \) be a functor between \( \infty \)-categories. Let \( \Phi_1: C_1 \to \text{DGCat}_{\text{cont}} \) be a functor such that for every \( c'_1 \to c''_1 \), the corresponding functor
\[ \Phi_1(c'_1) \to \Phi_1(c''_1) \]
admits a right adjoint. Let \( \Psi_1: C_1^{\text{op}} \to \text{DGCat}_{\text{cont}} \) be the resulting functor given by taking the right adjoints.

Let \( \Phi_2 \) and \( \Psi_2 \) be the left (resp., right) Kan extension of \( \Phi_1 \) (resp., \( \Psi_1 \)) along \( F \) (resp., \( F^{\text{op}} \)). We have:

**Lemma 1.4.4.** Under the above circumstances, the functor \( \Psi_2 \) is obtained from \( \Phi_2 \) by taking right adjoints.
1.4.5. We apply Lemma 1.4.4 in the following situation:

\[ \mathbf{C}_1 := \text{(Sch\textsubscript{af}t\textsubscript{aft}))} \text{proper}, \quad \mathbf{C}_2 := \text{(indSch\textsubscript{aft\textsubscript{aft})} ind-proper}, \]

and \( F \) to be the natural embedding. We take

\[ \Phi_1 := \text{IndCoh(Sch\textsubscript{aft\textsubscript{aft})} proper} \quad \text{and} \quad \Psi_1 := \text{IndCoh(Sch\textsubscript{aft\textsubscript{aft})} proper}. \]

These two functors are obtained from one another by passage to adjoints, by the definition of the functor \( \text{IndCoh(Sch\textsubscript{aft\textsubscript{aft})} proper}) \), see [Book-II.1, Corollary 5.1.12].

**Corollary 1.4.6.** The functor

\[ \text{IndCoh((indSch\textsubscript{aft\textsubscript{aft)}) ind-proper})^{op} \rightarrow \text{DGCat}_{\text{cont}} \]

is obtained from the functor

\[ \text{IndCoh(indSch\textsubscript{aft\textsubscript{aft)}) ind-proper} : \text{(indSch\textsubscript{aft\textsubscript{aft)}) ind-proper} \rightarrow \text{DGCat}_{\text{cont}} \]

by passing to right adjoints.

**Remark 1.4.7.** Let us informally explain the concrete meaning of this corollary. Namely, it says that for any morphism \( f : X_1 \rightarrow X_2 \) between objects of \( \text{indSch\textsubscript{aft\textsubscript{aft)}} \), there is a well-defined functor

\[ f^*_{\text{IndCoh}} : \text{IndCoh}(X_1) \rightarrow \text{IndCoh}(X_2). \]

Moreover, when \( f \) is ind-proper, this functor is the left adjoint of the functor \( f^! \).

1.4.8. We can now make the following observation pertaining to the behavior of the t-structure with respect to direct images:

**Lemma 1.4.9.** Let \( f : X_1 \rightarrow X_2 \) be a map of ind-schemes. Then the functor \( f^*_{\text{IndCoh}} \) is left t-exact. Furthermore, if \( f \) is ind-affine, then it is t-exact.

**Proof.** Let \( \mathcal{F} \in \text{IndCoh}(X_1)^{\geq 0} \). We wish to show that \( f^*_{\text{IndCoh}}(\mathcal{F}) \in \text{IndCoh}(X_2)^{\geq 0} \). By Corollary 1.2.7, we can assume that \( \mathcal{F} = (i_1)^*_{\text{IndCoh}}(\mathcal{F}_1) \) for \( \mathcal{F}_1 \in \text{IndCoh}(X_1)^{\geq 0} \) where

\[ i_1 : X_1 \rightarrow X_1 \]

is a closed embedding of a scheme. Now, let

\[ X_1 \xrightarrow{g} X_2 \xrightarrow{i_2} X_2 \]

be a factorization of \( f \circ i_1 \), where \( i_2 \) is a closed embedding of a scheme. Thus, it suffices to show that the functor

\[ f^*_{\text{IndCoh}} \circ (i_1)^*_{\text{IndCoh}} \simeq (i_2)^*_{\text{IndCoh}} \circ g^*_{\text{IndCoh}} \]

is left t-exact. However, \( (i_2)^*_{\text{IndCoh}} \) is t-exact by Lemma 1.2.5, while \( g^*_{\text{IndCoh}}(\mathcal{F}_1) \) is left t-exact, since \( g \) is a map between schemes.

Now, suppose that \( f \) is ind-affine. In this case, we wish to show that \( f^*_{\text{IndCoh}} \) is also right t-exact. Let \( \mathcal{F} \in \text{IndCoh}(X_1)^{\leq 0} \). We can assume that \( \mathcal{F} = (i_1)^*_{\text{IndCoh}}(\mathcal{F}_1) \) for \( \mathcal{F}_1 \in \text{IndCoh}(X_1)^{\leq 0} \) where \( i_1 : X_1 \rightarrow X_1 \) is a closed embedding. In the above notations, it suffices to show that

\[ f^*_{\text{IndCoh}} \circ (i_1)^*_{\text{IndCoh}} \simeq (i_2)^*_{\text{IndCoh}} \circ g^*_{\text{IndCoh}} \]

is t-exact.

By Lemma 1.2.5, \( (i_2)^*_{\text{IndCoh}} \) is t-exact. Hence, it suffices to show that \( g^*_{\text{IndCoh}} \) is t-exact. However, \( g \) is an affine map between schemes, and the assertion follows.
2. PROPER BASE CHANGE FOR IND-SCHEMES

Base change for IndCoh is a crucial property needed for its definition as a functor out of the category of correspondences. In this section we make two (necessary) preparatory steps, establishing base change for morphisms between ind-schemes.

2.1. 1st version.

2.1.1. Recall the notion of ind-schematic map in PreStk, see [Book-III.2, Definition 1.6.5(a)].

Let

\[ \begin{array}{ccc}
X' & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2,
\end{array} \]

be a Cartesian diagram in PreStk with \( f \) being ind-schematic and ind-proper. We claim:

**Proposition 2.1.2.** The functors \( f^! \) and \( (f')^! \) admit left adjoints, to be denoted \( f_*^{\text{IndCoh}} \) and \( (f')_*^{\text{IndCoh}} \), respectively. The natural transformation

\[ (f')_*^{\text{IndCoh}} \circ g_1 \overset{\sim}{\cong} g_2 \circ f_*^{\text{IndCoh}}, \]

arising by adjunction from

\[ g_1 \circ f^! \overset{\sim}{\cong} (f')^! \circ g_2, \]

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

2.1.3. We begin by making the following observation. Let \( G : C_2 \to C_1 \) be a functor between \( \infty \)-categories. Let \( A \) be a category of indices, and suppose we are given an \( A \)-family of commutative diagrams

\[ \begin{array}{ccc}
C'_1 & \xleftarrow{i''_1} & C_1 \\
\uparrow G_a & & \uparrow G \\
C'_2 & \xleftarrow{i''_2} & C_2.
\end{array} \]

Assume that for each \( a \in A \), the functor \( G_a \) admits a left adjoint \( F^a \). Furthermore, assume that for each map \( a' \to a'' \) in \( A \), the diagram

\[ \begin{array}{ccc}
C'_1 & \xleftarrow{i''_{1,a''}} & C'_1 \\
\downarrow F^{a''} & & \downarrow F^{a'} \\
C'_2 & \xleftarrow{i''_{2,a''}} & C'_2,
\end{array} \]

which a priori commutes up to a natural transformation, actually commutes.

Finally, assume that the functors

\[ C_1 \to \lim_{a \in A} C'_1 \text{ and } C_2 \to \lim_{a \in A} C'_2. \]

are equivalences.

Under the above circumstances we have:
Lemma 2.1.4. The functor $G$ admits a left adjoint, and for every $a \in A$, the diagram

$$
\begin{array}{ccc}
C_1 & \xleftarrow{i_a^2} & C_1 \\
F & \downarrow & F \\
C_2 & \xleftarrow{i_a} & C_2,
\end{array}
$$

which a priori commutes up to a natural transformation, commutes.

2.1.5. Applying Lemma 2.1.4, we obtain that the assertion of Proposition 2.1.2 reduces to the case when $X_2 = X_2' \in \text{Sch}_{aft}$ and $X_2' = X_2' \in \text{Sch}_{aft}$. In this case $X_1, X_1' \in \text{indSch}_{aft}$.

Write

$$X_1 \simeq \colim_{a \in A} X_a,$$

where $X_a \in \text{Sch}_{aft}$ and $i_a : X_a \to X_1$ are closed embeddings.

Set

$$X_1' := X_2' \times_{X_2} X_a.$$

We have:

$$X_1' \simeq \colim_{a \in A} X_1'.$$

Let $i_a'$ denote the corresponding closed embedding $X_a' \to X_1'$, and let $g_a$ denote the map $X_a' \to X_a$.

Note that the maps $f \circ i_a : X_a \to X_2$ and $f' \circ i_a' : X_a' \to X_2'$ are proper, by assumption.

2.1.6. By Corollary 1.1.6, we have:

$$\text{Id}_{\text{IndCoh}(X_1)} \simeq \colim_{a \in A} (i_a)_*^{\text{IndCoh}} \circ (i_a')!^{\text{IndCoh}}$$

and

$$\text{Id}_{\text{IndCoh}(X_1')} \simeq \colim_{a \in A} (i_a')_*^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}}.$$

Hence, we can rewrite the left-hand side in (2.1) as

$$\colim_{a \in A} (f')_*^{\text{IndCoh}} \circ (i_a')_*^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}} \circ g_a!,$$

and the right-hand side as

$$\colim_{a \in A} g_a! \circ f_*^{\text{IndCoh}} \circ (i_a)_*^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}}.$$

It follows from the construction that the map in (2.1) is given by a compatible system of maps for each $a \in A$

$$(f')_*^{\text{IndCoh}} \circ (i_a')_*^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}} \circ g_a! \simeq (f' \circ i_a')_*^{\text{IndCoh}} \circ (g_1 \circ i_a')!^{\text{IndCoh}} \simeq (f' \circ i_a')_*^{\text{IndCoh}} \circ (i_a \circ g_a)! \simeq (f' \circ i_a')_*^{\text{IndCoh}} \circ g_a! \circ i_a! \rightarrow g_2! \circ (f \circ i_a)_*^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}} \circ (i_a)!^{\text{IndCoh}},$$

where the arrow

$$(f' \circ i_a')_*^{\text{IndCoh}} \circ g_a! \rightarrow g_2! \circ (f \circ i_a)_*^{\text{IndCoh}}$$

is base change for the Cartesian square

$$
\begin{array}{ccc}
X_a' & \xrightarrow{g_a} & X_a \\
f' \circ i_a' \downarrow & \ & \downarrow f \circ i_a \\
X_2' & \xrightarrow{g_2} & X_2.
\end{array}
$$
Hence, the required isomorphism follows from proper base change in the case of schemes, see [Book-II.2, Proposition 3.1.4(b)].

\[\Box\]

### 2.2. 2nd version.

#### 2.2.1. Let now

\[
\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
f' & \downarrow & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2
\end{array}
\]

be a Cartesian diagram in indSch_{\text{laft}}, and assume now that the map \(g_2\), and hence \(g_1\), are ind-proper.

From the isomorphism

\[
(g_2)_s^{\text{IndCoh}} \circ (f')_*^{\text{IndCoh}} \simeq f_*^{\text{IndCoh}} \circ (g_1)_s^{\text{IndCoh}},
\]

by adjunction, we obtain a natural transformation:

\[\text{(2.2)} \quad (f')_*^{\text{IndCoh}} \circ g'_1 \to g'_2 \circ f_*^{\text{IndCoh}}.\]

We claim:

**Proposition 2.2.2.** The map (2.2) is an isomorphism.

The rest of this subsection is devoted to the proof of the proposition.

#### 2.2.3. By Corollary 1.1.6, we need to show that (2.2) becomes an isomorphism after precomposing both sides with the functor \((i_1)_s^{\text{IndCoh}}\) for a closed embedding \(X_1 \to X_1\) with \(X_1 \in \text{Sch}_{\text{aft}}\).

This allows to assume that \(X_1 = X_1 \in \text{Sch}_{\text{aft}}\).

By [Book-III.2, Corollary 1.7.5(b)], we can factor the map \(X_1 \to X_2\) as a composition

\[X_1 \to \tilde{X}_1 \to X_2,\]

where \(\tilde{X}_1 \in \text{Sch}_{\text{aft}}\) and \(\tilde{X}_1 \to X_2\) is a closed embedding.

Hence, we can consider separately the following two cases:

(I) The case when \(X_1 = X_1 \in \text{Sch}_{\text{aft}}\) and \(X_1 \to X_2\) is a closed embedding.

(II) The case when \(X_1 = X_1 \in \text{Sch}_{\text{aft}}\) and \(X_2 = X_2 \in \text{Sch}_{\text{aft}}\).

#### 2.2.4. We note that that in Case (I), the map \(f\) is ind-proper. Moreover, the map (2.2) equals the map (2.1). Hence, it is an isomorphism by Proposition 2.1.

#### 2.2.5. The proof of Case (II) essentially repeats the argument in Sects. 2.1.5-2.1.6. We include it for completeness.

Write

\[X'_2 \simeq \colim_{a \in A} X'_{2,a},\]

where the maps \(i_{2,a} : X'_{2,a} \to X'_2\) are closed embeddings.

In this case

\[X'_1 \simeq \colim_{a \in A} X'_{1,a},\]

where

\[X'_{1,a} \simeq X'_{2,a} \times_{X'_2} X_1.\]
For every \( a \) denote by \( f_a : X'_{1,a} \to X'_{2,a} \) and by \( i_{1,a} : X'_{1,a} \to X' \) the corresponding closed embedding.

Note that the compositions
\[
g_2 \circ i_{2,a} : X'_{2,a} \to X_2 \quad \text{and} \quad g_1 \circ i_{1,a} : X'_{1,a} \to X_1
\]
are proper, by assumption.

2.2.6. By Corollary 1.1.6, we have:
\[
\text{Id}_{\text{IndCoh}(X'_{2})} \simeq \colim_{a \in A}(i_{2,a})^!_{\text{IndCoh}} \circ (i_{2,a})^! \quad \text{and} \quad \text{Id}_{\text{IndCoh}(X'_{1})} \simeq \colim_{a \in A}(i_{1,a})^!_{\text{IndCoh}} \circ (i_{1,a})^!.
\]

Hence, we can rewrite the left-hand side in (2.2) as
\[
\colim_{a \in A}(f'_a)^!_{\text{IndCoh}} \circ (i_{1,a})^!_{\text{IndCoh}} \circ (i_{1,a})^! \circ g_1^!
\]
and the right-hand side as
\[
\colim_{a \in A}(i_{2,a})^!_{\text{IndCoh}} \circ (i_{2,a})^! \circ g_2^! \circ f^!_{\text{IndCoh}}.
\]

It follows from the construction that the map in (2.2) is given by a compatible system of maps for each \( a \in A \):
\[
(f_a)^!_{\text{IndCoh}} \circ (i_{1,a})^!_{\text{IndCoh}} \circ (i_{1,a})^! \circ g_1^! \simeq (i_{2,a})^!_{\text{IndCoh}} \circ (f_a)^!_{\text{IndCoh}} \circ (i_{1,a})^! \circ g_1^! \simeq
\]
\[
(z_{i_{2,a}}^!_{\text{IndCoh}} \circ (f_a)^!_{\text{IndCoh}} \circ (g_1 \circ i_{1,a})^! \to (i_{2,a})^!_{\text{IndCoh}} \circ (g_2 \circ i_{2,a})^! \circ f^!_{\text{IndCoh}} \simeq
\]
\[
(z_{i_{2,a}}^!_{\text{IndCoh}} \circ (i_{2,a})^! \circ g_2^! \circ f^!_{\text{IndCoh}},
\]
where the map
\[
(f_a)^!_{\text{IndCoh}} \circ (g_1 \circ i_{1,a})^! \to (g_2 \circ i_{2,a})^! \circ f^!_{\text{IndCoh}}
\]
is base change for the Cartesian square
\[
\begin{array}{ccc}
X'_{1,a} & \xrightarrow{g_1 \circ i_{1,a}} & X_1 \\
\downarrow f_a & & \downarrow f \\
X'_{2,a} & \xrightarrow{g_2 \circ i_{2,a}} & X_2.
\end{array}
\]

Hence, the required isomorphism follows from proper base change for schemes, see [Book-II.1, Proposition 5.2.1].

\[\square\]

2.3. Groupoids and descent.

2.3.1. Let \( \mathcal{X}^\bullet \) be a groupoid simplicial object in \( \text{PreStk}_{\text{lf}} \), see [Lu0], Definition 6.1.2.7. Denote by
\[
(2.3)
\]
\[
p_s, p_t : \mathcal{X}^1 \twoheadrightarrow \mathcal{X}^0
\]
the corresponding maps.

We form a co-simplicial category \( \text{IndCoh}(\mathcal{X}^\bullet)^! \) using the \(!\)-pullback functors, and consider its totalization \( \text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet)^!) \). Consider the functor of evaluation on 0-simplices:
\[
ev^0 : \text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet)) \to \text{IndCoh}(\mathcal{X}^0).
\]
2.3.2. We claim:

**Proposition 2.3.3.** Assume that the maps $p_s, p_t$ in (2.3) are ind-schematic and ind-proper.

(a) The functor $ev^0$ admits a left adjoint. The resulting monad on $\text{IndCoh}(X^0)$, viewed as a plain endo-functor, is canonically isomorphic to $(p_t)^{\text{IndCoh}} \circ (p_s)^!$. The adjoint pair

$$\text{IndCoh}(X^0) \rightleftarrows \text{Tot}(\text{IndCoh}(X^0)^!)$$

is monadic.

(b) Suppose that $X^\bullet$ is the Čech nerve of a map $f : X^0 \to Y$, where $f$ is ind-schematic and ind-proper. Assume also that $f$ is surjective at the level of $k$-points. Then the resulting map

$$\text{IndCoh}(Y) \to \text{Tot}(\text{IndCoh}(X^\bullet)^!)$$

is an equivalence.

*Proof.* Follows by repeating the argument of [Book-II.1, Proposition 7.2.2].

3. IndCoh on (ind)-inf-schemes

In this section we begin the development of the theory of IndCoh on ind-inf-schemes. We will essentially bootstrap it from IndCoh on ind-schemes, using nil base change.

3.1. Nil base change. As was just mentioned, nil base change is a crucial property of the category IndCoh. Its proof relies on the structural results on inf-schemes from [Book-III.2, Sect. 4].

3.1.1. Recall the notion of inf-schematic map in PreStk, see [Book-III.2, Definition 3.1.6].

We are going to show:

**Proposition 3.1.2.** Let $f : X_1 \to X_2$ be a map in PreStk_{f,aft}, and assume that $f$ is an inf-schematic nil-isomorphism.

(a) The functor $f^! : \text{IndCoh}(X_2) \to \text{IndCoh}(X_1)$ admits a left adjoint, to be denoted $f_*^{\text{IndCoh}}$.

(b) The functor $f^!$ is conservative.

(c) For a Cartesian diagram

$$\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2,
\end{array}$$

the natural transformation

$$(f'_*)^{\text{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\text{IndCoh}},$$

arising by adjunction from

$$g_1^! \circ f^! \simeq (f'_*)^! \circ g_2^!,$$

is an isomorphism.
3.1.3. Proof of Proposition 3.1.2. Using Lemma 2.1.4, we reduce the assertion to the case when \( X_2 = X_2 \in \text{Sch}^{\text{aff}} \), for points (a) and (b), and further to the case when \( X'_2 = X'_2 \in \text{Sch}^{\text{aff}} \) for point (c).

In this case \( X_1 \) has the property that \( \text{red}X_1 = X_1 \in \text{redSch}^{\text{aff}} \). By [Book-III.2, Corollary 4.3.4], we can write

\[
X_1 \simeq \colim_{a \in A} X_{1,a},
\]

where \( A \) is the category

\[
(S\text{ch}^{\text{aff}})/X_1 \times \{X_1\},
\]

and the colimit is taken in the category PreStk_{laft}.

In particular, for every \( a \), the resulting map \( X_{1,a} \to X_2 \) is a nil-isomorphism, and in particular proper. Moreover, for every morphism \( a' \to a'' \), the corresponding map

\[
i_{a',a''} : X_{1,a'} \to X_{1,a''}
\]

is also a nil-isomorphism and is proper.

We have:

\[
\text{IndCoh}(X_1) \simeq \lim_{a \in A^{op}} \text{IndCoh}(X_{1,a}).
\]

The fact that \( f^! \) is conservative follows from the fact that each \( (f \circ i_a)^! \) is conservative.

Using [GL:DG, Lemma 1.3.3], we can therefore rewrite

\[
(3.1) \quad \text{IndCoh}(X_1) \simeq \colim_{a \in A} \text{IndCoh}(X_{1,a}),
\]

where the colimit is taken with respect to the functors

\[
(i_{a',a''})_{\text{IndCoh}}^* : \text{IndCoh}(X_{1,a'}) \to \text{IndCoh}(X_{1,a''}).
\]

Now, the left adjoint to \( f \) is given by the compatible collection of functors

\[
(f \circ i_a)_{\text{IndCoh}} : \text{IndCoh}(X_{1,a}) \to \text{IndCoh}(X_2).
\]

Thus, it remains to establish the base change property. However, the latter follows by repeating the argument in Sects. 2.1.5-2.1.6 □

3.2. Basic properties. We will now use nil base change to establish some basic properties of the category IndCoh on an ind-inf-scheme.

3.2.1. First, as a corollary of Proposition 3.1.2 we obtain:

**Corollary 3.2.2.** Let \( \mathcal{X} \) be an object of indinfSch_{laft}. Then the category \( \text{IndCoh}(\mathcal{X}) \) is compactly generated.

**Proof.** Consider the canonical map \( i : \text{red} \mathcal{X} \to \mathcal{X} \). The category \( \text{IndCoh}(\text{red} \mathcal{X}) \) is compactly generated by Corollary 1.1.8. Now, Proposition 3.1.2 implies that the essential image of \( i_{\text{IndCoh}}^*(\text{IndCoh}(\text{red} \mathcal{X})) \) compactly generates \( \text{IndCoh}(\mathcal{X}) \). □
3.2.3. Let
\[
\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2,
\end{array}
\]
be a Cartesian diagram in $\text{PreStk}_{\text{laft}}$ with $f$ being ind-inf-schematic and ind-nil-proper (see [Book-III.2, Definition 1.6.11] for what this means). We claim:

**Proposition 3.2.4.** The functors $f^!$ and $(f')^!$ admit left adjoints, to be denoted $f^!_{\text{IndCoh}}$ and $(f')^!_{\text{IndCoh}}$, respectively. The natural transformation
\[
(f')^!_{\text{IndCoh}} \circ g_1^! \rightarrow g_2^! \circ f^!_{\text{IndCoh}},
\]
arriving by adjunction from
\[
g_1^! \circ f' \simeq (f')^! \circ g_2^!,
\]
is an isomorphism.

**Proof.** By Lemma 2.1.4, the assertion of the proposition reduces to the case when $X_2$ and (resp., both $X_2$ and $X'_2$) belong to $\text{Sch}_{\text{aff}}$. Denote these objects by $X_2$ and $X'_2$, respectively. In this case $X_1$ (resp., both $X_1$ and $X'_1$) belong to $\text{indSch}_{\text{aff}}$.

Let $X_0$ be any object of $\text{indSch}_{\text{aff}}$ endowed with a nil-isomorphism to $X_1$; e.g., $X_0 = \text{red}X_1$. Set
\[
X'_0 := X'_2 \times X_0,
\]
and consider the diagram
\[
\begin{array}{ccc}
X'_0 & \xrightarrow{g_0} & X_0 \\
\downarrow i' & & \downarrow i \\
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2,
\end{array}
\]
in which both squares are Cartesian.

By Proposition 3.1.2(b), it is sufficient to prove the assertion for the top square and for the outer square.

Now, the assertion for the outer square is given by Proposition 2.1.2, and for the top square by Proposition 3.1.2(c).

\[\square\]

3.3. Ind-nil-proper descent.

3.3.1. By the same logic as in Proposition 2.3.3 we obtain:

**Proposition 3.3.2.** Let $\mathcal{X}^\bullet$ be a groupoid simplicial object in $\text{PreStk}_{\text{laft}}$ where the maps
\[
p_s, p_t : \mathcal{X}^1 \rightarrow \mathcal{X}^0
\]
are ind-inf-schematic and ind-nil-proper.

(a) The functor
\[
ev^0 : \text{IndCoh}(\mathcal{X}^\bullet) \rightarrow \text{IndCoh}(\mathcal{X}^0)
\]
admits a left adjoint. The resulting monad on $\text{IndCoh}(X^0)$, viewed as a plain endo-functor, is canonically isomorphic to $(p_3)^!_{\text{IndCoh}} \circ (p_4)^!$. The adjoint pair

\[
\text{IndCoh}(X^0) \Rightarrow \text{Tot}(\text{IndCoh}(X^\bullet))^!
\]

is monadic.

(b) Suppose that $X^\bullet$ is the Čech nerve of a map $f : X^0 \to Y$, where $f$ is ind-inf-schematic and ind-nil-proper. Assume also that $f$ is surjective at the level of $k$-points. Then the resulting map

\[
\text{IndCoh}(Y) \to \text{Tot}(\text{IndCoh}(X^\bullet)^!)\]

is an equivalence.

3.3.3. For $X \in \text{PreStk}_{\text{left}}$ recall the category $\text{Pro}((\text{QCoh}(X)^{-})_{\text{fake}}$, see [Book-III.1, Sect. 4.3.1]. From Proposition 3.3.2 and [Book-III.1, Corollary 4.3.7] we obtain:

**Corollary 3.3.4.** Let $X \to Y$ be a map in $\text{PreStk}_{\text{left}}$, which is ind-inf-schematic, inf-proper and surjective at the level of $k$-points. Then the pullback functor

\[
\text{Pro}((\text{QCoh}(Y)^{-})_{\text{fake}} \to \text{Tot}((\text{Pro}((\text{QCoh}(X^\bullet/Y)^{-})_{\text{fake}})
\]

is an equivalence.

3.3.5. **Descent for maps.** Let $f : Y \to X$ be an ind-nil-schematic and ind-nil-proper map between objects of $\text{PreStk}_{\text{left}}$, which is surjective at the level of $k$-points. We claim:

**Proposition 3.3.6.** For $Z \in \text{PreStk}_{\text{left-def}}$, the natural map

\[
\text{Tot}(\text{Maps}(Y^\bullet, Z)) \to \text{Maps}(X, Z)
\]

is an isomorphism.

**Proof.** The statement automatically reduces to the case when $X = X \in <\infty\text{Sch}_{\text{aff}}$. Furthermore, by [Book-III.2, Corollary 4.1.5], we can assume that $Y = Y \in <\infty\text{Sch}_{\text{aff}}$.

The assertion of the proposition is evident if $X$ is reduced: in this case the simplicial object $Y^\bullet$ is split. Hence, by [Book-III.1, Proposition 5.4.2], by induction, it suffices to show that if the assertion holds for a given $X$ and we have a square-zero extension $X \hookrightarrow X'$ by means of $\mathcal{F} \in \text{Coh}(X)^{\leq 0}$, then the assertion holds also for $Y' \in (\text{Sch}_{\text{aff}})_{\text{nil-isom to } X'}$.

Set $Y := Y' \times X'$. By [Book-III.1, Proposition 5.3.2], the map

\[
Y^\bullet \hookrightarrow Y'^\bullet
\]

has a structure of simplicial object in the category of square-zero extensions.

By the induction hypothesis, it is enough to show that for a given map $z : X \to Z$, the map

\[
\text{Maps}(X', Z)_{\text{Maps}(X, Z)} \times \{z\} \to \text{Tot}(\text{Maps}(Y'^\bullet, Z))_{\text{Tot}(\text{Maps}(Y^\bullet, Z))} \times \{z\}
\]

is an isomorphism.

Since $Z$ admits deformation theory, the left-hand side in (3.3) is canonically isomorphic to the groupoid of null-homotopies of the composition

\[
T_z^*(Z) \rightarrow T^*(X) \rightarrow \mathcal{F}.
\]

We rewrite the right-hand side in (3.3) as the totalization of the simplicial space

\[
\text{Maps}(Y'^\bullet, Z)_{\text{Maps}(Y^\bullet, Z)} \times \{f\}.
\]
The above simplicial groupoid identifies with that of null-homotopies of the composition
\[ T_{\ast z} \circ T^\ast (Y) \rightarrow f^\ast (F), \]
where \( f^\ast \) denotes the map \( Y \rightarrow X \).

Now, the desired property follows the descent property of \( \text{Pro}(\text{QCoh}(-)^\text{fake}) \), see Corollary 3.3.4 above.

3.4. t-structure for ind-inf-schemes. The category \( \text{IndCoh} \) on an ind-inf-scheme also possesses a t-structure. However, it has less favorable properties than in the case of ind-schemes.

3.4.1. Let \( X \) be an ind-inf-scheme. We define a t-structure on the category \( \text{IndCoh}(X) \) by declaring that an object \( F \in \text{IndCoh}(X) \) belongs to \( \text{IndCoh}(X)_{\geq 0} \) if and only if \( i^! (F) \in \text{IndCoh}(\text{red}X)_{\geq 0} \), where
\[ i : \text{red}X \rightarrow X \]
is the canonical map.

Equivalently, we let \( \text{IndCoh}(X)_{\leq 0} \) be generated under colimits by the essential image of \( \text{IndCoh}(\text{red}X)_{\leq 0} \) under \( i^!_{\text{IndCoh}} \).

It is easy to see that if \( f \) is a nil-closed map, then functor \( f^! \) is left t-exact.

3.4.2. Assume for a moment that \( X \) is actually an ind-scheme. We claim that the t-structure defined above, when we view \( X \) as a mere ind-inf-scheme, coincides with one for \( X \) considered as an ind-scheme of Sect. 1.2. This follows from the next lemma:

Lemma 3.4.3. Let \( f : X_1 \rightarrow X_2 \) be a nil-isomorphism of ind-schemes. Then for \( F \in \text{IndCoh}(X_2) \) we have:
\[ F \in \text{IndCoh}(X_2)^{\geq 0} \iff f^!(F) \in \text{IndCoh}(X_1)^{\geq 0}. \]

Proof. The \( \Rightarrow \) implication is tautological. For the \( \Leftarrow \) implication, by the definition of the t-structure on \( \text{IndCoh}(X_2) \), we can assume that \( X_2 = X_2 \in \text{Sch}_{\text{aff}} \) and \( X_1 = X_1 \in \text{Sch}_{\text{aff}} \). That is, \( f \) is a nil-isomorphism of schemes \( X_1 \rightarrow X_2 \).

By the definition of the t-structure on \( \text{IndCoh}(X_2) \) and adjunction, it suffices to show that \( \text{Coh}(X_2)^{\leq 0} \) is generated by the essential image of \( \text{Coh}(X_1)^{\leq 0} \) under \( f^!_{\text{IndCoh}} \), which is obvious.

Corollary 3.4.4. Let \( X \) be an object of \( \text{indinfSch}_{\text{aff}} \).
(a) For \( F \in \text{IndCoh}(X) \), we have \( F \in \text{IndCoh}(X)^{\geq 0} \) if and only if for every nil-closed map \( f : X \rightarrow X \) with \( X \in \text{redSch}_{\text{aff}} \) we have
\[ f^!(F) \in \text{IndCoh}(X)^{\geq 0}. \]
(b) The category \( \text{IndCoh}(X)^{\leq 0} \) is generated under colimits by the essential images of the categories \( \text{IndCoh}(X)^{\leq 0} \) for \( f : X \rightarrow X \) with \( X \in \text{redSch}_{\text{aff}} \) and \( f \) nil-closed.
3.4.5. As we have seen above, if \( f \) is a nil-closed map \( X_1 \to X_2 \) of ind-inf-schemes, then the functor \( f^! \) is left t-exact. By adjunction, this implies that the functor \( f^!_{\text{IndCoh}} \) is right t-exact.

**Lemma 3.4.6.** Let \( f : X_1 \to X_2 \) be a nil-closed and ind-schematic map between ind-inf-schemes. Then the functor \( f^!_{\text{IndCoh}} \) is t-exact.

**Proof.** We only have to prove that \( f^!_{\text{IndCoh}} \) is left t-exact. By Proposition 3.1.2(c), the assertion reduces to the case when \( X_2 = \text{indSch}_{\text{laft}} \). In the latter case, \( X_1 \) is an ind-scheme, and the assertion follows from the fact that the functor \( f^!_{\text{IndCoh}} \) for a map of ind-schemes is left t-exact, by Lemma 1.4.9. \( \square \)

**Remark 3.4.7.** It is easy to see that the assertion of the lemma is false without the assumption that \( f \) be ind-schematic.

### 4. The direct image functor for ind-inf-schemes

In this section we construct the direct image functor on IndCoh for maps between ind-inf-schemes. The idea is that one can bootstrap it from the case of maps that are nil-closed embeddings, while for the latter the sought-for procedure is obtained as left/right Kan extension from the case of schemes.

#### 4.1. Recovering from nil-closed embeddings

In this subsection we show that if we take IndCoh on the category of schemes, with morphisms restricted to nil-closed maps, then its right Kan extension to ind-inf-schemes recovers the usual IndCoh.

4.1.1. Consider the fully faithful embeddings

\[
\text{Sch}_{\text{aff}} \hookrightarrow \text{Sch}_{\text{aff}} \hookrightarrow \text{indinfSch}_{\text{laft}} \hookrightarrow \text{PreStk}_{\text{laft}}.
\]

Denote

\[
\text{IndCoh}^!_{\text{indinfSch}_{\text{laft}}} := \text{IndCoh}^!_{\text{PreStk}_{\text{laft}}} |_{((\text{indinfSch}_{\text{laft}})^{\text{op}})}.
\]

Since

\[
\text{IndCoh}^!_{\text{Sch}_{\text{aff}}} \to \text{RKE}_{((\text{Sch}_{\text{aff}})^{\text{op}} \hookrightarrow ((\text{Sch}_{\text{aff}})^{\text{op}}))^{\text{op}}}((\text{IndCoh}^!_{\text{Sch}_{\text{aff}}})^{\text{op}})
\]

is an isomorphism, the map

\[
(4.1) \quad \text{IndCoh}^!_{\text{indinfSch}_{\text{laft}}} \to \text{RKE}_{((\text{Sch}_{\text{aff}})^{\text{op}} \hookrightarrow ((\text{indinfSch}_{\text{laft}})^{\text{op}}) \to ((\text{IndCoh}^!_{\text{Sch}_{\text{aff}}})^{\text{op}})}
\]

is an isomorphism.

4.1.2. Let

\[
((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}} \subset \text{indinfSch}_{\text{laft}}
\]

denote the 1-full subcategory, where we restrict 1-morphisms to be nil-closed.

Denote

\[
\text{IndCoh}^!_{((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}} := \text{IndCoh}^!_{\text{indinfSch}_{\text{laft}}} |_{((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}.
\]

From the isomorphism (4.1), we obtain a canonically defined map

\[
(4.2) \quad \text{IndCoh}^!_{((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}} \to \text{RKE}_{((\text{Sch}_{\text{aff}})_{\text{nil-closed}})^{\text{op}} \hookrightarrow ((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}((\text{IndCoh}^!_{(\text{Sch}_{\text{aff}})_{\text{nil-closed}}})^{\text{op}})
\]

We will prove:

**Proposition 4.1.3.** The map (4.2) is an isomorphism.
Proof. We need to show that for \( \mathcal{X} \in \text{indinfSch}_{\text{laft}} \), the functor

\[
\text{IndCoh}(\mathcal{X}) \to \lim_{Z \in ((\text{Sch}_{\text{laft}})_{\text{nil-closed in } \mathcal{X}})^{op}} \text{IndCoh}(Z)
\]
is an equivalence.

However, this follows from [Book-III.2, Corollary 4.1.4], since the functor \( \text{IndCoh} \) takes colimits in \( \text{PreStk}_{\text{laft}} \) to limits.

\[\square\]

4.2. Recovering from nil-isomomorphisms. The material in this subsection is not needed for the sequel and is included for the sake of completeness.

4.2.1. Let

\( (\text{indinfSch}_{\text{laft}})_{\text{nil-isom}} \subset \text{indinfSch}_{\text{laft}} \) and \( (\text{indSch}_{\text{laft}})_{\text{nil-isom}} \subset \text{indSch}_{\text{laft}} \)
denote the 1-full subcategories, where we restrict 1-morphisms to be nil-isomomorphisms.

Denote also

\[
\text{IndCoh}!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-isom}}} := \text{IndCoh}!_{\text{indinfSch}_{\text{laft}}} \mid ((\text{indinfSch}_{\text{laft}})_{\text{nil-isom}})^{op}
\]
and

\[
\text{IndCoh}!_{(\text{indSch}_{\text{laft}})_{\text{nil-isom}}} := \text{IndCoh}!_{\text{indSch}_{\text{laft}}} \mid ((\text{indSch}_{\text{laft}})_{\text{nil-isom}})^{op}
\]

4.2.2. From Proposition 4.1.3 we deduce:

Corollary 4.2.3. The natural map

\[
\text{IndCoh}!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-isom}}} \to \text{RKE}((\text{indSch}_{\text{laft}})_{\text{nil-isom}})^{op}\to ((\text{indinfSch}_{\text{laft}})_{\text{nil-isom}})^{op} (\text{IndCoh}!_{(\text{Sch}_{\text{laft}})_{\text{nil-isom}}})
\]
is an isomorphism.

Proof. By Proposition 1.3.4 and Proposition 4.1.3, the map

\[
\text{IndCoh}!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}} \to \text{RKE}((\text{indSch}_{\text{laft}})_{\text{nil-closed}})^{op}\to ((\text{indinfSch}_{\text{laft}})_{\text{nil-isom}})^{op} (\text{IndCoh}!_{(\text{Sch}_{\text{laft}})_{\text{nil-closed}}})
\]
is an isomorphism.

Hence, it remains to show that for \( \mathcal{X} \in \text{indinfSch}_{\text{laft}} \), the restriction map

\[
\lim_{\mathcal{Y} \in ((\text{indSch}_{\text{laft}})_{\text{nil-closed in } \mathcal{X}})^{op}} \text{IndCoh}(\mathcal{Y}) \to \lim_{\mathcal{Y} \in ((\text{indSch}_{\text{laft}})_{\text{nil-isom to } \mathcal{X}})^{op}} \text{IndCoh}(\mathcal{Y})
\]
is an isomorphism.

We claim that the map

\( (\text{indSch}_{\text{laft}})_{\text{nil-isom to } \mathcal{X}} \to (\text{indSch}_{\text{laft}})_{\text{nil-closed in } \mathcal{X}} \)
is cofinal. Indeed, it admits a left adjoint, given by sending an object

\( (\mathcal{Y} \to \mathcal{X}) \in (\text{indSch}_{\text{laft}})_{\text{nil-closed in } \mathcal{X}} \)

\[
\mathcal{Y} \underset{\text{red} \mathcal{Y}}{\sqcup} \mathcal{X} \to \mathcal{X},
\]
where the push-out is taken in the category \( \text{PreStk}_{\text{laft}} \).

\[\square\]

4.3. Constructing the direct image functor. In this subsection we finally construct the direct image functor. The crucial assertion is Theorem 4.3.2, which says that this functor is the “right one”.

4.3.1. Consider again the functor
\[ \text{IndCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}}, \]
where for a morphism \( f : X_1 \to X_2 \), the functor \( \text{IndCoh}(X_1) \to \text{IndCoh}(X_2) \) is \( f_!^{\text{IndCoh}} \).

Recall the notation:
\[ \text{IndCoh}_{\text{Sch}_{\text{aft}}} |_{\text{nil-closed}} := \text{IndCoh}_{\text{Sch}_{\text{aft}}} | (\text{Sch}_{\text{aft}})_{\text{nil-closed}}. \]

Denote
\[ \text{IndCoh}_{\text{indinfSch}_{\text{aft}}} := LK_{\text{Sch}_{\text{aft}}} \hookrightarrow \text{indinfSch}_{\text{aft}}(\text{IndCoh}_{\text{Sch}_{\text{aft}}}), \]
and denote
\[ \text{IndCoh}_{\text{indinfSch}_{\text{aft}}} |_{\text{nil-closed}} := \text{IndCoh}_{\text{indinfSch}_{\text{aft}}} | (\text{indinfSch}_{\text{aft}})_{\text{nil-closed}}. \]

We have a canonical map
\[ (4.3) \quad LK_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}} \hookrightarrow (\text{indinfSch}_{\text{aft}})_{\text{nil-closed}} \to \text{IndCoh}_{(\text{indinfSch}_{\text{aft}})_{\text{nil-closed}}}. \]

We claim:

**Theorem 4.3.2.** *The map (4.3) is an isomorphism.*

4.3.3. Note that by combining Lemma 1.4.4 and Theorem 4.3.2, we obtain:

**Corollary 4.3.4.** *The functors
\[ \text{IndCoh}_{(\text{indinfSch}_{\text{aft}})_{\text{nil-closed}}} \quad \text{and} \quad \text{IndCoh}_{! (\text{indinfSch}_{\text{aft}})_{\text{nil-closed}}} \]
are obtained from one another by passing to adjoints.*

The upshot of Theorem 4.3.2 and Corollary 4.3.4 is that for a morphism \( f : X_1 \to X_2 \) we have a well-defined functor
\[ f_!^{\text{IndCoh}} : \text{IndCoh}(X_1) \to \text{IndCoh}(X_2), \]
and when \( f \) is nil-closed, the above functor is the left adjoint to \( f_! \).

In what follows, for \( X_1 = X \) and \( X_2 = \text{pt} \), we shall also use the notation
\[ \Gamma(X, -)^{\text{IndCoh}} \]
for the functor \( (p_X)_*^{\text{IndCoh}} \), where \( p_X : X \to \text{pt} \) is the projection.

**Remark 4.3.5.** In the sequel, we will show that the assertion of Corollary 4.3.4 can be strengthened by replacing the class of nil-closed morphisms by that of ind-nil-proper ones.

4.4. **Proof of Theorem 4.3.2.**

4.4.1. *Step -1.* We need to show that the functor
\[ \text{colim}_{Z \in (\text{Sch}_{\text{aft}})_{\text{nil-closed}} \subset X} \text{IndCoh}(Z) \to \text{colim}_{Y \in (\text{Sch}_{\text{aft}})_{\text{nil-closed}} \setminus X} \text{IndCoh}(Y) \]
is an equivalence.

First, the convergence property of the IndCoh functor allows to replace \( \text{Sch}_{\text{aft}} \) by \( \prec \text{Sch}_{\text{aft}} \).

Thus, we need to show that the functor
\[ \text{colim}_{Z \in \prec \text{Sch}_{\text{aft}}_{\text{nil-closed}} \subset X} \text{IndCoh}(Z) \to \text{colim}_{Y \in \prec \text{Sch}_{\text{aft}}_{\text{nil-closed}} \setminus X} \text{IndCoh}(Y) \]
is an equivalence.
4.4.2. **Step 0.** Consider the commutative diagram

\[
\begin{array}{ccc}
\colim_{Y \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X} \mathrm{IndCoh}(Y) & \leftrightarrow & \colim_{S \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X} \mathrm{IndCoh}(S) \\
\uparrow & & \uparrow \\
\colim_{Z \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) \text{nil-closed in } X} \mathrm{IndCoh}(Z) & \leftrightarrow & \colim_{(S \to Z \to X) \in \mathcal{C}} \mathrm{IndCoh}(S),
\end{array}
\]

where \( \mathcal{C} \) is the category of \( S \to Z \to X \), with \( S \in \prec \infty \mathrm{Sch}_{\mathrm{aff}} \) and 
\( (Z \to X) \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) \text{nil-closed in } X \).

We will show that the horizontal arrows and the right vertical arrow in this diagram are equivalences. This will prove that the left vertical arrow is also an equivalence.

4.4.3. **Step 1.** Consider the functor \( \mathcal{C} \to (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X \), appearing in the right vertical arrow in (4.4). We claim that it is cofinal, which would prove that the right vertical arrow in (4.4) is an equivalence.

We note that the functor \( \mathcal{C} \to (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X \) is a Cartesian fibration. Hence, the fact that it is cofinal is equivalent to the fact that it has contractible fibers.

The fiber over a given object \( S \to X \) is the category of factorizations
\( S \to Z \to X \), \( (Z \to X) \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) \text{nil-closed in } X \).

This category is contractible by [Book-III.2, Proposition 4.1.3].

4.4.4. **Step 2.** Consider the functor \( \mathcal{C} \to (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) \text{nil-closed in } X \), appearing in the bottom horizontal arrow in (4.4). It is a co-Cartesian fibration.

Hence, in order to show that this arrow in the diagram is an equivalence, it suffices to show that for a given \( Z \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) \text{nil-closed in } X \), the functor

\[
\colim_{S \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / Z} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(Z)
\]

is an equivalence.

We have the following assertion, proved below:

**Proposition 4.4.5.** *The functor \( \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aff}}} \), regarded as a presheaf on \( \mathrm{Sch}_{\mathrm{aff}} \) with values in \((DGCat_{\mathrm{cont}})^{\text{op}} \), satisfies Zariski descent.*

Now, the fact that (4.5) is an equivalence (for any \( Z \in \prec \infty \mathrm{Sch}_{\mathrm{aff}} \)) follows from [GL:IndCoh, Proposition 6.4.3], applied to
\( \prec \infty \mathrm{Sch}_{\mathrm{aff}} \subset \prec \infty \mathrm{Sch}_{\mathrm{aff}} \).

4.4.6. **Step 3.** To treat the top horizontal arrow in (4.4), we consider the category \( \mathcal{D} \) of

\( S \to Y \to X, \quad S \in \prec \infty \mathrm{Sch}_{\mathrm{aff}}, \quad Y \in \prec \infty \mathrm{Sch}_{\mathrm{aff}}, \)

and functor

\[
\colim_{(S \to Y \to X) \in \mathcal{D}} \mathrm{IndCoh}(S) \to \colim_{S \in (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X} \mathrm{IndCoh}(S).
\]

We note that the functor (4.6) is an equivalence, because the corresponding forgetful functor \( \mathcal{D} \to (\prec \infty \mathrm{Sch}_{\mathrm{aff}}) / X \) is cofinal (it is a Cartesian fibration with contractible fibers).
Hence, it remains to show that the composition of (4.6) with the top horizontal arrow in (4.4), i.e.,
\[
\colim_{(S \to Y \to X) \in \mathcal{D}} \IndCoh(S) \to \colim_{Y \in \langle \infty \text{Sch}_{af} \rangle / X} \IndCoh(S)
\]
is an equivalence.

The above functor corresponds to the co-Cartesian fibration \( \mathcal{D} \to \langle \infty \text{Sch}_{af} \rangle / X \). Hence, it suffices to show that for a fixed \( Y \in \langle \infty \text{Sch}_{af} \rangle / X \), the functor
\[
\colim_{S \in \langle \infty \text{Sch}_{af} \rangle / Y} \IndCoh(S) \to \IndCoh(Y)
\]
is an equivalence.

However, this follows as in Step 2 from Proposition 4.4.5.

4.4.7. Proof of Proposition 4.4.5. The assertion of the proposition is equivalent to the following. Let \( X \in \text{Sch}_{af} \) be covered by two opens \( U_1 \) and \( U_2 \). Denote
\[
U_1 \overset{j_1}{\to} X, \quad U_2 \overset{j_2}{\to} X, \quad U_1 \cap U_2 \overset{j_12}{\to} X,
\]
\[
U_1 \cap U_2 \overset{j_121}{\to} U_1, \quad U_1 \cap U_2 \overset{j_121}{\to} U_2.
\]

Then the claim is that the diagram
\[
\begin{array}{ccc}
\IndCoh(U_1 \cap U_2) & \overset{(j_{12,1})_*^{\IndCoh}}{\longrightarrow} & \IndCoh(U_1) \\
\downarrow^{(j_{12,2})_*^{\IndCoh}} & & \downarrow^{(j_1)_*^{\IndCoh}} \\
\IndCoh(U_1) & \overset{(j_2)_*^{\IndCoh}}{\longrightarrow} & \IndCoh(X)
\end{array}
\]
is a push-out square in \( \text{DGCat}_{\text{cont}} \).

That is to say, given \( C \in \text{DGCat} \) and a triple of continuous functors
\[
F_1 : \IndCoh(U_1) \to C, \quad F_2 : \IndCoh(U_2) \to C, \quad F_{12} : \IndCoh(U_1 \cap U_2) \to C
\]
endowed with isomorphisms
\[
F_1 \circ (j_{12,1})_*^{\IndCoh} \simeq F_{12} \simeq F_2 \circ (j_{12,2})_*^{\IndCoh},
\]
we need to show that this data comes from a uniquely defined functor
\[
F : \IndCoh(X) \to C.
\]

The sought-for functor \( F \) is recovered as follows: for \( \mathcal{F} \in \IndCoh(X) \), we have
\[
F(\mathcal{F}) = F_1((j_1)_!(\mathcal{F})) \times_{F_{12}((j_{12})_!(\mathcal{F}))} F_2((j_2)_!(\mathcal{F}))
\]
where the maps \( F_i((j_i)_!(\mathcal{F})) \to F_{12}((j_{12})_!(\mathcal{F})) \) are given by
\[
F_i((j_i)_!(\mathcal{F})) \to F_i((j_{12,1})_*^{\IndCoh} \circ j_{12,1} \circ j_i^!(\mathcal{F})) = F_i((j_{12,1})_*^{\IndCoh} \circ j_{12}^!(\mathcal{F}) \simeq F_{12}((j_{12})_!(\mathcal{F})).
\]

□

4.5. Base change. As in the case of ind-schemes, there are two types of base change isomorphism for nil-closed inf-schematic maps.
4.5.1. One has to do with a Cartesian diagram

\[
\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2
\end{array}
\]

be a Cartesian diagram in \text{PreStk}\text{\text_{\text{laft}}} with \(f\) being inf-schematic and nil-closed. We claim that

\textbf{Proposition 4.5.2.} \textit{The natural transformation}

\begin{equation}
(f')_! \text{IndCoh} \circ g_1^! \to g_2^! \circ f_\text{IndCoh},
\end{equation}

\textit{arrising by adjunction from}

\[g_1^! \circ f' \simeq (f')_! \circ g_2^!,\]

\textit{is an isomorphism.}

\textbf{Proof.} This is a particular case of Proposition 3.2.4. \qed

4.5.3. Let now

\[
\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2
\end{array}
\]

be a Cartesian diagram of objects of indinfSch\text{\text_{\text{laft}}} with \(g_2\) being nil-closed.

From the isomorphism

\[g_2^! \circ (f')_! \text{IndCoh} \simeq (g_1^! \circ f'_*) \text{IndCoh},\]

by adjunction, we obtain a natural transformation:

\begin{equation}
(f')_! \text{IndCoh} \circ g_1^! \to g_2^! \circ f_\text{IndCoh}.
\end{equation}

We claim:

\textbf{Proposition 4.5.4.} \textit{The map} (4.8) \textit{is an isomorphism.}

\textbf{Proof.} Let \(X_0 := \text{red}X_1\). Set

\[X'_0 := X'_1 \times X_0,
\]

and consider the diagram

\[
\begin{array}{ccc}
X'_0 & \xrightarrow{g_0} & X_0 \\
\downarrow & & \downarrow \\
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2
\end{array}
\]

in which both squares are Cartesian.

By Proposition 3.1.2(b), it suffices to show that the outer square and the top square each satisfy base change. Now, base change for the top square is given by Proposition 3.1.2(c).

This reduces the assertion of the proposition to the case when \(X_1\) is a classical reduced ind-scheme. In this case the map \(f\) factors as

\[X_1 \to X_{3/2} \to X_2,
\]
where $X_{3/2} = \text{red} X_2$. Set $X'_{3/2} := X'_2 \times X_{3/2}$, and consider the diagram

\[
\begin{array}{ccc}
X'_1 & \overset{g_0}{\longrightarrow} & X_1 \\
\downarrow & & \downarrow \\
X'_{3/2} & \overset{g_3}{\longrightarrow} & X_{3/2} \\
f' \downarrow & & \downarrow f \\
X'_2 & \overset{g_2}{\longrightarrow} & X_2,
\end{array}
\]

in which both squares are Cartesian.

It is sufficient to show that the two inner squares each satisfy base change. Now, for the top square, this is given by Proposition 2.2.2. For the bottom square, this is given by Proposition 3.1.2(c).

\[\square\]

Remark 4.5.5. Let $f : X_1 \to X_2$ be a ind-nil-proper map between ind-inf-schemes. On the one hand, we have a well-defined functor

\[f^\IndCoh_* : \IndCoh(X_1) \to \IndCoh(X_2).\]

On the other hand, by Proposition 3.2.4, we have the functor

\[(f^!)_L : \IndCoh(X_1) \to \IndCoh(X_2),\]

left adjoint to $f^!$.

We will ultimately show that there is a canonical isomorphism $f^\IndCoh_* \simeq (f^!)_L$, see Sect. 5.2.

5. Extending the formalism of correspondences to inf-schemes

In this section we will take the formalism of IndCoh to what (in our opinion) is its ultimate domain of definition: the category of correspondences, where the objects are all prestacks (locally almost of finite type), pullbacks are taken with respect to any maps, push-forwards are taken with respect to ind-inf-schematic maps, and adjunctions hold for ind-nil-proper maps.

5.1. Set-up for extension. As a first step, we will consider the category of correspondences, where the objects are ind-inf-schemes, pullbacks and push-forwards are taken with respect to any maps, and adjunctions are for nil-closed maps. We will construct the required functor by a 2-categorical left Kan extension procedure from [Funct, Theorem 13.4.4].

5.1.1. We consider the category $\text{indinfSch}_{\text{left}}$ with the following three classes of morphisms

\[\text{vert} = \text{all, horiz} = \text{all, adm} = \text{nil-closed}.\]

Let

\[(\text{indinfSch}_{\text{left}})^{\text{nil-closed}}_{\text{corr:all:all}}\]

be the resulting $(\infty, 2)$-category of correspondences.
5.1.2. Consider also the category 
\[(\text{Sch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all},\]
and the functor
\[(\text{IndCoh}_{\text{Sch}_{\text{aff}}}^{\text{nil-closed}})^{\text{proper}}_{\text{corr:all;all}} : (\text{Sch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all} \to \text{DGCat}^{2}\text{-Cat}_{\text{cont}},\]
obtained from the functor \((\text{IndCoh}_{\text{Sch}_{\text{aff}}}^{\text{proper}})_{\text{corr:all;all}}\) of [Book-II.2, Theorem 2.1.4] by restriction along
\[(\text{Sch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all} \to (\text{Sch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all}^{\text{proper}}.\]

We wish to extend the functor \((\text{IndCoh}_{\text{Sch}_{\text{aff}}}^{\text{nil-closed}})_{\text{corr:all;all}}\) to a functor
\[(\text{IndCoh}_{\text{indinfSch}_{\text{aff}}}^{\text{nil-closed}})_{\text{corr:all;all}} : (\text{indinfSch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all} \to \text{DGCat}^{2}\text{-Cat}_{\text{cont}},\]
along the tautological functor
\[\text{Sch}_{\text{aff}} \to \text{indinfSch}_{\text{aff}}.\]

We refer the reader to [Funct, Sect. 13.4.2], where sufficient conditions for the existence and canonicity of such an extension are formulated. In the present context, conditions (1), (2) and (3) amount to to Proposition 4.1.3 (for property (1)), Theorem 4.3.2 (for property (2)), and Propositions 4.5.2 and 4.5.4 (for property (3)).

5.1.3. Applying [Funct, Theorem 13.4.4], we obtain:

**Theorem 5.1.4.** There exists a canonically defined functor
\[\text{IndCoh}_{\text{indinfSch}_{\text{aff}}}^{\text{nil-closed}} : (\text{indinfSch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all} \to \text{DGCat}^{2}\text{-Cat}_{\text{cont}},\]
whose restrictions to \(\text{indinfSch}_{\text{aff}}^{\text{op}}\) and \(\text{indinfSch}_{\text{aff}}\)
identify with
\[\text{IndCoh}_{\text{indinfSch}_{\text{aff}}}^{\uparrow} \text{indinfSch}_{\text{aff}} \text{and IndCoh}_{\text{indinfSch}_{\text{aff}}}^{\text{nil-closed}}\]
respectively, and such that the restriction of \(\text{IndCoh}_{\text{indinfSch}_{\text{aff}}}^{\text{nil-closed}}\) under
\[(\text{Sch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all} \to (\text{indinfSch}_{\text{aff}})_{\text{nil-closed}}\text{corr:all;all}\]
identifies canonically with \(\text{IndCoh}_{\text{Sch}_{\text{aff}}}^{\text{nil-closed}}\).

5.1.5. Let us explain the concrete content of the above theorem.

Let
\[
\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2
\end{array}
\]
be a Cartesian diagram of objects of \(\text{indinfSch}_{\text{aff}}\).

Theorem 5.1.4 says that we have a **canonical isomorphism**
\[(5.1) \quad g_2^\dagger \circ (fX)^\dagger \simeq (fY)^\dagger \circ g_1^\dagger.\]

Moreover, this isomorphism coincides with the base change isomorphism of (4.7) (resp., (4.8)), when \(fX\) (resp., \(g_2\)) is nil-closed.

Moreover, if the ind-inf-schemes in the above diagram are schemes, then the isomorphism (5.1) equals one defined a priori in this case by [Book-II.2, Corollary 3.1.4].
Remark 5.1.6. In Sect. 5.2 we will show that the isomorphism (5.1) comes from an adjunction one of the maps $f$ or $g_2$ is ind-nil-proper (but not necessarily nil-closed).

5.2. **Enlarging 2-morphisms to ind-nil-proper maps.** In this subsection we will extend the functor from the previous subsection, where we include adjunctions for ind-nil-proper maps.

5.2.1. Consider now the $(\infty, 2)$-category

$$\text{ind-nil-proper} \cap \text{all;all},$$

where we enlarge the class of 2-morphisms to that of ind-nil-proper maps.

Consider the 2-fully faithful functor

$$\text{ind-nil-proper} \cap \text{all;all} \rightarrow \text{ind-nil-proper} \cap \text{all;all},$$

We are going to prove:

**Theorem 5.2.2.** There exists a unique extension of the functor functor

$$\text{IndCoh} : \text{ind-nil-proper} \cap \text{all;all} \rightarrow \text{DGCat}_{\text{2-Cat}}.$$

There exists a unique isomorphism

$$\text{IndCoh} : \text{ind-nil-proper} \cap \text{all;all} \simeq \text{IndCoh} : \text{nil-closed} \cap \text{all;all},$$

compatible with the isomorphism

$$\text{IndCoh} : \text{nil-closed} \cap \text{all;all} \simeq \text{IndCoh} : \text{proper} \cap \text{all;all}.$$

5.2.3. By [Funct, Theorem 12.2.2], from Theorem 5.2.2, we obtain:

**Corollary 5.2.4.**

(a) The functors

$$\text{IndCoh} \cap \text{ind-nil-proper} := \text{IndCoh} \cap \text{ind-nil-proper},$$

and

$$\text{IndCoh} : \text{ind-nil-proper} \cap \text{all;all} \simeq \text{IndCoh} \cap \text{nil-closed} \cap \text{all;all},$$

are obtained from each other by passing to adjoints.

(b) Let

$$\begin{array}{ccc}
X'_1 & \xrightarrow{g_1} & X_1 \\
\downarrow f' & & \downarrow f \\
X'_2 & \xrightarrow{g_2} & X_2,
\end{array}$$

be a Cartesian diagram of objects of indinfSch with $g_2$ being ind-nil-proper. Then the natural transformation

$$(f')_* \text{IndCoh} \circ g_1^! \rightarrow g_2^! \circ f_* \text{IndCoh},$$

obtained by adjunction from the isomorphism,

$$(g_2)_* \text{IndCoh} \circ (f')_* \text{IndCoh} \simeq f_* \text{IndCoh} \circ (g_1)_* \text{IndCoh},$$

is an isomorphism.

The rest of this subsection is devoted to the proof of Theorem 5.2.2.
5.2.5. The case of ind-schemes. Consider the category indSch laft with the following three classes of morphisms

\[ \text{vert} = \text{all}, \quad \text{horiz} = \text{all}, \quad \text{adm} = \text{ind-nil-proper}. \]

We claim that we have the following result:

**Theorem 5.2.6.** There exists a canonically defined functor

\[ \operatorname{IndCoh}_{\operatorname{indSch}_{\text{laft}}^{\text{ind-nil-proper}}} : (\operatorname{indSch}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr};\text{all};\text{all}} \to \operatorname{DGCat}^{2}\text{-Cat}_{\text{cont}}, \]

whose restrictions to

\[ \operatorname{indSch}_{\text{laft}}^{\text{op}} \text{ and } \operatorname{indSch}_{\text{laft}} \]

identify with

\[ \operatorname{IndCoh}_{\operatorname{indSch}_{\text{laft}}}^{\text{op}} \text{ and } \operatorname{IndCoh}_{\operatorname{indSch}_{\text{laft}}}, \]

respectively, and such that the restriction of \( \operatorname{IndCoh}_{(\operatorname{indSch}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr};\text{all};\text{all}}} \) under

\( (\operatorname{Sch}_{\text{laft}})^{\text{proper}}_{\text{corr};\text{all};\text{all}} \to (\operatorname{indSch}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr};\text{all};\text{all}} \)

identifies canonically with \( \operatorname{IndCoh}_{(\operatorname{Sch}_{\text{laft}})^{\text{nil-closed}}_{\text{corr};\text{all};\text{all}}} \). Furthermore, there is a canonical isomorphism between the restrictions

\[ \operatorname{IndCoh}_{(\operatorname{indSch}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr};\text{all};\text{all}}} \mid (\operatorname{indSch}_{\text{laft}})^{\text{nil-closed}}_{\text{corr};\text{all};\text{all}} \simeq \operatorname{IndCoh}_{(\operatorname{indSch}_{\text{laft}})^{\text{ind-nil-proper}}_{\text{corr};\text{all};\text{all}}} \mid (\operatorname{Sch}_{\text{laft}})^{\text{nil-closed}}_{\text{corr};\text{all};\text{all}}. \]

**Proof.** This follows from [Funct, Theorem 13.4.4] applied to the functor

\( \operatorname{Sch}_{\text{laft}} \to \operatorname{indSch}_{\text{laft}}, \)

using Proposition 1.3.4, Proposition 1.4.2, and Propositions 2.1.2 and 2.2.2.

The last assertion of the theorem follows from the canonicity of the construction in [Funct, Theorem 13.4.4].

\( \square \)

**Remark 5.2.7.** The difference between the case of ind-schemes and that of inf-schemes that it is not true (at least, not obviously so) that properties (1) and (2) of [Funct, Sect. 13.4.2] are satisfied for the functors

\[ \operatorname{IndCoh}_{(\operatorname{Sch}_{\text{laft}})^{\text{proper}}_{\text{corr};\text{all};\text{all}}} : ((\operatorname{Sch}_{\text{laft}})^{\text{proper}}_{\text{corr};\text{all};\text{all}})^{\text{op}} \to \operatorname{DGCat}_{\text{cont}}^{\text{op}} \]

and

\[ \operatorname{IndCoh}_{(\operatorname{Sch}_{\text{laft}})^{\text{proper}}_{\text{corr};\text{all};\text{all}}} : (\operatorname{Sch}_{\text{laft}})^{\text{proper}}_{\text{corr};\text{all};\text{all}} \to \operatorname{DGCat}_{\text{cont}} \]

with respect to the functor

\( \operatorname{Sch}_{\text{laft}} \to \operatorname{indInfSch}_{\text{laft}}, \)

while this is true for

\( \operatorname{Sch}_{\text{laft}} \to \operatorname{indSch}_{\text{laft}}. \)
5.2.8. We are going to deduce Theorem 5.2.2 from Theorem 5.1.4 by applying [Funct, Theorem 14.1.3]. Thus, we need to check that the inclusion
\[ \text{nil-closed} \subset \text{ind-nil-proper} \]
satisfies the condition of [Funct, Sect. 14.1.2].

That is, we consider a \( \text{inf-proper} \) morphism
\[ f : X_1 \to X_2 \]
of objects of \( \text{indinfSch} \), and the Cartesian square:
\[
\begin{array}{ccc}
X_1 \times X_2 & \to & X_1 \\
p_1 \downarrow & & \downarrow f \\
X_1 & \to & X_2.
\end{array}
\]

The diagonal map
\[ \Delta_{X_1/X_2} : X_1 \to X_1 \times X_2 \]
is nil-closed. Hence, from the \( \text{(} \Delta_{X_1/X_2} \text{)}_{\text{IndCoh}}, (\Delta_{X_1/X_2})^! \text{-adjunction} \), we obtain a natural transformation
\[ (\Delta_{X_1/X_2})_{\text{IndCoh}} \circ (\Delta_{X_1/X_2})^! \to \text{Id}_{\text{IndCoh}(X_1 \times X_2)}. \]

By composing, the latter natural transformation gives rise to
\[ (\Delta_{X_1/X_2})_{\text{IndCoh}} \circ (\Delta_{X_1/X_2})^! \circ p_2 = (\text{Id}_{\text{IndCoh}}) \circ f^! \circ f_! = f^! \circ f_! \text{IndCoh}. \]

We need to show that the natural transformation (5.2) is the unit of an adjunction. I.e., that for \( F_1 \in \text{IndCoh}(X_1) \) and \( F_2 \in \text{IndCoh}(X_2) \), the map
\[ \text{Maps}(f_!^{\text{IndCoh}}(F_1), F_2) \to \text{Maps}(f^! \circ f_!^{\text{IndCoh}}(F_1), f^!(F_2)) \]
is an isomorphism.

We note that by Theorem 5.1.4 the map (5.3) is the unit for the \( (f_!^{\text{IndCoh}}, f^!) \) adjunction, when \( f \) is nil-closed.

5.2.9. Note that the natural transformation (5.2) is defined for any map \( f \) which is \( \text{nil-separated} \), i.e., one for which \( \Delta_{X_1/X_2} \) is nil-closed.

Let \( g : X_0 \to X_1 \) be another nil-separated map between objects of \( \text{indinfSch} \). Diagram chase implies:

**Lemma 5.2.10.** For \( F_0 \in \text{IndCoh}(X_0) \) and \( F_2 \in \text{IndCoh}(X_2) \), the diagram
\[
\begin{array}{ccc}
\text{Maps}(g_!^{\text{IndCoh}}(F_0), f^!(F_2)) & \to & \text{Maps}(F_0, g^! \circ f^!(F_2)) \\
\uparrow & & \uparrow \text{id} \\
\text{Maps}(f_!^{\text{IndCoh}} \circ g_!^{\text{IndCoh}}(F_0), F_2) & \to & \text{Maps}(F_0, g^! \circ f^!(F_2))
\end{array}
\]
commutes.
5.2.11. Let us take $X_0 := \text{red} X_1$ and $g$ to be the canonical embedding. By Proposition 3.1.2, it is sufficient to show that (5.3) is an isomorphism for $\mathcal{F}_1$ of the form $g_*^\text{IndCoh}(\mathcal{F}_0)$ for $\mathcal{F}_0 \in \text{IndCoh}(X_0)$.

Using Lemma 5.2.10, and the fact that the map 
$$\text{Maps}(g_*^\text{IndCoh}(\mathcal{F}_0), f^! \mathcal{F}_2) \to \text{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2))$$
is an isomorphism in this case, since $g$ is nil-closed, we obtain that it is sufficient to show that (5.3) is an isomorphism, when the initial map $f$ is replaced by $f \circ g$.

I.e., in proving that (5.3) is an isomorphism, we can assume that $X_1$ is a reduced ind-scheme.

5.2.12. Let us now factor $f$ as $X_1 \to X_{3/2} \to X_2$, where $X_{3/2} := \text{red} X_2$. Applying Lemma 5.2.10 again, we obtain that it is enough to show that (5.3) is an isomorphism for $f$ replaced by $X_1 \to X_{3/2}$ and $X_{3/2} \to X_2$ separately.

For the map $X_{3/2} \to X_2$, the assertion follows from the fact that the map in question is nil-closed.

5.2.13. Hence, we are further reduced to the case when $f$ is a ind-nil-proper map between ind-schemes. However, in this case, the required isomorphism follows from Theorem 5.2.6: it follows by [Funct, Theorem 14.1.3] from the existence of the functor

$$\text{IndCoh}_{\text{indinfsch}, \text{all}, \text{ind-nil-proper}} : (\text{indinfsch}, \text{all}, \text{ind-nil-proper}) \to \text{DGCat}^{2 \text{-Cat}_{\text{cont}}},$$

whose restriction to $(\text{indinfsch}, \text{nil-closed}, \text{all}, \text{all}, \text{all})$ is isomorphic to

$$\text{IndCoh}_{\text{indinfsch}, \text{nil-closed}, \text{all}, \text{all}, \text{all}(\text{indinfsch}, \text{nil-closed}, \text{all}, \text{all})}. \tag{\text{5.3}}$$

5.3. Extending to prestacks. In this subsection we will finally extend the formalism to the category of correspondences that has all “laft” prestacks as objects.

5.3.1. Consider now the category $\text{PreStk}_{\text{laft}}$, and the three classes of morphisms

$$\text{indinfsch}, \text{all}, \text{indinfsch} \& \text{ind-nil-proper},$$

where “indinfsch” stands for the class of ind-inf-schematic morphisms, and “ind-nil-proper” for the class of morphisms that are ind-nil-proper.

Consider the tautological embedding

$$\text{indinfsch} \hookrightarrow \text{PreStk}_{\text{laft}}.$$

It satisfies the conditions of [Funct, Horizontal Extension], with respect to the classes

$$(\text{all}, \text{all}, \text{ind-nil-proper}) \to (\text{indinfsch}, \text{all}, \text{indinfsch} \& \text{ind-nil-proper}).$$

Consider now the functor

$$\text{IndCoh}_{\text{indinfsch}, \text{all}, \text{all}, \text{ind-nil-proper}} : (\text{indinfsch}, \text{all}, \text{all}, \text{ind-nil-proper}) \to \text{DGCat}^{2 \text{-Cat}_{\text{cont}}},$$

and the corresponding functor

$$\text{IndCoh}_{\text{indinfsch}, \text{all}, \text{all}, \text{ind-nil-proper}} : (\text{indinfsch})^{\text{op}} \to \text{DGCat}^{\text{cont}_{\text{op}}}. \tag{\text{5.4}}$$

Clearly, the map

$$\text{IndCoh}_{\text{PreStk}_{\text{laft}}} \to \text{RKE}((\text{indinfsch})^{\text{op}} \to (\text{PreStk}_{\text{laft}})^{\text{op}}(\text{IndCoh}_{\text{indinfsch}}^{\text{op}}))$$

is an isomorphism.
is an isomorphism.

5.3.2. Hence, by [Funct, Horizontal Extension], from Theorem 5.2.2, we obtain the following theorem, which is for us the ultimate version of the formalism of ind-coherent sheaves:

**Theorem 5.3.3.** There exists a uniquely defined functor

\[ \text{IndCoh}_{\text{(PreStk_{left})}_{\text{corr}:\text{indinfsch} \& \text{ind-nil-proper}}^{\text{all}}} : (\text{PreStk_{left}})^{\text{indinfsch} \& \text{ind-nil-proper}} \rightarrow \text{DGCat}^{\text{2-Cat}}, \]

equipped with isomorphisms

\[ \text{IndCoh}_{\text{(PreStk_{left})}_{\text{corr}:\text{indinfsch} \& \text{ind-nil-proper}}^{\text{all}}} \mid (\text{PreStk_{left}})_{\text{op}} \cong \text{IndCoh}_{\text{PreStk_{left}}} \]

and

\[ \text{IndCoh}_{\text{(PreStk_{left})}_{\text{corr}:\text{indinfsch} \& \text{ind-nil-proper}}^{\text{all}}} \mid (\text{indinfsch_{left}})_{\text{corr}:\text{ind-nil-proper}, \text{all}} \cong \text{IndCoh}_{(\text{indinfsch_{left}})_{\text{corr}:\text{all}, \text{all}}} \]

where the latter two isomorphisms are compatible in a natural sense.

5.3.4. The concrete meaning of Theorem 5.3.3 is analogous to that of Theorem 5.1.4, with the difference that we can now consider the direct image functor \( f^* \text{IndCoh} \) when \( f \) is an ind-inf-schematic map

\[ f : X_1 \rightarrow X_2, \]

with \( X_1, X_2 \) being objects of PreStk_{left}, and not necessarily ind-inf-schemes. The functor \( f^* \text{IndCoh} \) satisfies base change for Cartesian squares

\[
\begin{array}{ccc}
X'_1 \xrightarrow{g_1} & X_1 \\
| \downarrow f' & \downarrow f & \downarrow f \\
X'_2 \xrightarrow{g_2} & X_2,
\end{array}
\]

with vertical maps being ind-inf-schematic:

\[ (f')^* \text{IndCoh} \circ g_1^* \cong g_2^* \circ f^* \text{IndCoh}. \]

Moreover, for \( f \) ind-inf-schematic and ind-nil-proper, the functor \( f^* \text{IndCoh} \) is the left adjoint of \( f^! \). In this case the base change isomorphism comes by adjunction from

\[ (f^!)^* \circ g_1^* \cong g_1^* \circ f^! \]

If \( g_2 \) is ind-inf-schematic and ind-nil-proper, the base change isomorphism comes by adjunction from

\[ f^* \text{IndCoh} \circ (g_1)^* \text{IndCoh} \cong (f')^* \text{IndCoh} \circ (g_2)^* \text{IndCoh}. \]

5.4. **Open embddings.** The formalism of Theorem 5.3.3 contains \( (f^* \text{IndCoh}, f^!) \) adjunction for \( f \) proper.

However, it does not explicitly contain the \( (f', f^* \text{IndCoh}) \)-adjunction for \( f \) which is an open embedding. In this subsection we will show that the latter follows automatically.
5.4.1. Let $\text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:indinfsch;all}}$ denote the restriction of the functor $\text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:indinfsch;all}}^{\text{indinfsch & ind-nil-proper}}$ to $\text{PreStk}_{\text{left}} \text{corr:indinfsch;all} \subset \text{PreStk}_{\text{left}} \text{corr:indinfsch;all}^{\text{indinfsch & ind-nil-proper}}$.

We consider it as a functor of $(\infty, 1)$-categories $\text{PreStk}_{\text{left}} \text{corr:indinfsch;all} \rightarrow \text{DGCat}_{\text{cont}}$.

Consider the $(\infty, 2)$-category $\text{PreStk}_{\text{left}} \text{corr:open;all}$.

5.4.2. We claim:

**Proposition 5.4.3.** There exists a unique extension of $\text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:indinfsch;all}}$ to a functor $\text{PreStk}_{\text{left}} \text{corr:open;all} \rightarrow \text{DGCat}_{\text{cont}}^{2\text{-Cat}} {\text{2-op}}$.

**Proof.** We start with the three classes of 1-morphisms in $\text{PreStk}_{\text{left}}$:

- indinfsch, all, isom,

and enlarge it to $\text{indinfsch, all, open}$.

This enlargement satisfies the assumptions of [Funct, Sect. 14.1.1]. Hence, if the functor $\text{PreStk}_{\text{left}} \text{corr:indinfsch;all}$ exists, then it is unique.

Furthermore, to prove the existence, it is sufficient to do so for the pair of categories $\text{PreStk}_{\text{left}} \text{corr:open;all} \subset \text{PreStk}_{\text{left}} \text{corr:open;all}$, and the functor

\[ \text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:open;all}} \coloneqq \text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:indinfsch;all}} |_{\text{PreStk}_{\text{left}} \text{corr:open;all}}. \]

To construct the sought-for functor

\[ \text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:open;all}}^{\text{open}} : \text{PreStk}_{\text{left}} \text{corr:open;all} \rightarrow \text{DGCat}_{\text{cont}}^{2\text{-Cat}} {\text{2-op}} \]

we proceed as follows.

We start with the functor

\[ \text{IndCoh}_{\text{Sch}_{\text{left}} \text{corr:open;all}} : \text{Sch}_{\text{left}} \text{corr:open;all} \rightarrow \text{DGCat}_{\text{cont}}^{2\text{-Cat}} , \]

and we recall that by construction (see [Book-II.2, Sect. 2.1.2]), it extends to a functor

\[ \text{IndCoh}_{\text{Sch}_{\text{left}} \text{corr:open;all}}^{\text{open}} : \text{Sch}_{\text{left}} \text{corr:open;all} \rightarrow \text{DGCat}_{\text{cont}}^{2\text{-Cat}} {\text{2-op}} . \]

Now, the functor $\text{IndCoh}_{\text{PreStk}_{\text{left}} \text{corr:open;all}}^{\text{open}}$ is obtained from $\text{IndCoh}_{\text{Sch}_{\text{left}} \text{corr:open;all}}^{\text{open}}$ by [Funct, Horizontal Extension] for the functor $\text{Sch}_{\text{left}} \leftrightarrow \text{PreStk}_{\text{left}}$. □
6. Self-duality and multiplicative structure of IndCoh on ind-inf-schemes

In this section we will show how the formalism of IndCoh as a functor out of the category of correspondences of ind-inf-schemes defines Serre duality on ind-inf-schemes. This is parallel to [Book-II.2, Sect. 4].

6.1. The multiplicative structure. In this subsection we discuss a canonical symmetric monoidal structure IndCoh.

6.1.1. Recall that the functor

\[ \text{IndCoh}_{(\text{Sch}_{\text{nft}})^{\text{proper}}}^{\text{proper}} : (\text{Sch}_{\text{nft}})^{\text{proper}} \rightarrow \text{DGCat}^{2-\text{Cat}} \]

is endowed with a symmetric monoidal structure, see [Book-II.2, Theorem 4.1.2]. Hence, the same is true for its restriction

\[ \text{IndCoh}_{(\text{Sch}_{\text{nft}})^{\text{nil-closed}}}^{\text{nil-closed}} : (\text{Sch}_{\text{nft}})^{\text{nil-closed}} \rightarrow \text{DGCat}^{2-\text{Cat}}. \]

Applying [Funct, Ind-extension of multiplicative structure], we obtain:

Corollary 6.1.2. The functor

\[ \text{IndCoh}_{(\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}}}^{\text{ind-nil-proper}} : (\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}} \rightarrow \text{DGCat}^{2-\text{Cat}} \]

carries a unique right-lax symmetric monoidal structure extending one on \( \text{IndCoh}_{(\text{Sch}_{\text{nft}})^{\text{nil-closed}}}^{\text{nil-closed}} \).

6.1.3. Finally, applying [Funct, on cost extension of multiplicative structure], from Corollary 6.1.2, we obtain:

Corollary 6.1.4. The functor

\[ \text{IndCoh}_{(\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}}}^{\text{ind-nil-proper}} : (\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}} \rightarrow \text{DGCat}^{2-\text{Cat}} \]

carries a unique right-lax symmetric monoidal structure extending one on \( \text{IndCoh}_{(\text{Sch}_{\text{nft}})^{\text{proper}}}^{\text{proper}} \).

6.1.5. By Corollary 3.2.2, for \( X \in \text{indinfSch}_{\text{nft}} \), the category \( \text{IndCoh}(X) \) is compactly generated, and in particular dualizable. Hence, by repeating the argument of [GL:QCoh, Proposition 1.4.4], for any \( Y \in \text{PreStk}_{\text{nft}} \), the functor

\[ \text{IndCoh}(X) \otimes \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X \times Y) \]

is an equivalence.

From here we obtain:

Corollary 6.1.6. The right-lax symmetric monoidal structure on the functor

\[ \text{IndCoh}_{(\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}}}^{\text{ind-nil-proper}} : (\text{indinfSch}_{\text{nft}})^{\text{ind-nil-proper}} \rightarrow \text{DGCat}^{2-\text{Cat}} \]

is strict.

6.2. Duality. In this subsection we show that the symmetric monoidal structure on IndCoh gives rise to Serre duality. The idea is that an ind-inf-scheme \( X \) is canonically self-dual as an object of the category of correspondences equipped with its natural monoidal structure.
6.2.1. By restricting the functor \( \text{IndCoh}_{(\text{indinfSch}_{\text{left}})_{\text{corr;all;all}}} \) to
\( \text{ind-nil-proper} \),
we obtain a symmetric monoidal structure on the functor
\( \text{IndCoh}_{(\text{indinfSch}_{\text{left}})_{\text{corr;all;all}}} : (\text{indinfSch}_{\text{left}})_{\text{corr;all;all}} \to \text{DGCat}_{\text{cont}}. \)
As in [Book-II.2, Theorem 4.2.5], we deduce:

**Theorem 6.2.2.** We have a commutative diagram of functors
\[
\begin{array}{ccc}
((\text{indinfSch}_{\text{left}})_{\text{corr;all;all}})^{\text{op}} & \xrightarrow{(\text{IndCoh}_{(\text{indinfSch}_{\text{left}})_{\text{corr;all;all}}} )^{\text{op}}} & \left(\text{DGCat}_{\text{cont}}^{\text{dualizable}}\right)^{\text{op}} \\
\cong & \Downarrow & \downarrow \text{dualization} \\
(\text{indinfSch}_{\text{left}})_{\text{corr;all;all}} & \xrightarrow{\text{IndCoh}_{(\text{indinfSch}_{\text{left}})_{\text{corr;all;all}}}} & \text{DGCat}_{\text{cont}}^{\text{dualizable}}.
\end{array}
\]

As in [Book-II.2, Sect. 4.2.2], the functor \( \varpi \) is the natural anti-equivalence on the category \( (\text{indinfSch}_{\text{left}})_{\text{corr;all;all}} \) corresponding to interchanging the roles of vertical and horizontal arrows. The right vertical arrow is the functor of passage to the dual category.

6.2.3. Let us explain the concrete meaning of Theorem 6.2.2. This is parallel to [Book-II.2, Sect. 4.2.6].

For an individual object \( X \in \text{indinfSch}_{\text{left}} \), it says that there is a natural self-duality data on the category \( \text{IndCoh}(X) \), i.e.,
\[
D_{\text{Serre}}^X : \text{IndCoh}(X)^{\vee} \simeq \text{IndCoh}(X).
\]
Furthermore, for a map \( f : X_1 \to X_2 \), there is a canonical identification
\[
f^! \simeq (f^*_{\text{IndCoh}})^{\vee}.
\]

6.2.4. Below we shall write down explicitly the unit and counit functors
\[
\epsilon_X : \text{IndCoh}(X) \otimes \text{IndCoh}(X) \to \text{Vect} \quad \text{and} \quad \mu_X : \text{Vect} \to \text{IndCoh}(X) \otimes \text{IndCoh}(X)
\]
that define the identification (6.1).

**Remark 6.2.5.** We observe that the fact that the functors \( \epsilon_X \) and \( \mu_X \) do indeed define an isomorphism (6.1) is easy to check directly. I.e., this does not require the full statement of Theorem 6.2.2.

6.2.6. The pairing
\[
\epsilon_X : \text{IndCoh}(X) \otimes \text{IndCoh}(X) \to \text{Vect}
\]
is the composition
\[
\text{IndCoh}(X) \otimes \text{IndCoh}(X) \simeq \text{IndCoh}(X \times X) \xrightarrow{\Delta^!} \text{IndCoh}(X) \xrightarrow{(p_X)^{\text{IndCoh}}} \text{Vect}.
\]
Here \( p_X \) is the map \( X \to \text{pt} \), so \((p_X)^{\text{IndCoh}} \simeq \Gamma^{\text{IndCoh}}(X, -)\). The first map is an isomorphism due to the fact that \( \text{IndCoh}(X) \) is dualizable as a DG category.

The unit functor
\[
\mu_X : \text{Vect} \to \text{IndCoh}(X) \otimes \text{IndCoh}(X)
\]
is the composition
\[
\text{Vect} \xrightarrow{p_X^!} \text{IndCoh}(X) \xrightarrow{\Delta^{\text{IndCoh}}} \text{IndCoh}(X \times X) \simeq \text{IndCoh}(X) \otimes \text{IndCoh}(X).
\]
One can explicitly verify that \((\epsilon_X, \mu_X)\) specified above define an identification
\[
\text{IndCoh}(\mathcal{X})^\vee \simeq \text{IndCoh}(\mathcal{X})
\]
by calculating the composition
\[
\text{IndCoh}(\mathcal{X}) \overset{\text{Id} \otimes \mu_X}{\longrightarrow} \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \overset{\epsilon_X \otimes \text{Id}}{\longrightarrow} \text{IndCoh}(\mathcal{X}).
\]
Indeed, it can be calculated via the commutative diagram
\[
\begin{array}{c}
\text{IndCoh}(\mathcal{X}) \downarrow \Delta_{\text{IndCoh}} \leftarrow \text{IndCoh}(\mathcal{X} \times \mathcal{X}) \downarrow (\text{id} \times \Delta)^!_{\text{IndCoh}} \\
\text{IndCoh}(\mathcal{X} \times \mathcal{X}) \downarrow (\Delta \times \text{id})^! \leftarrow \text{IndCoh}(\mathcal{X} \times \mathcal{X} \times \mathcal{X}) \downarrow (p_2)^!_{\text{IndCoh}} \leftarrow \text{IndCoh}(\mathcal{X}),
\end{array}
\]
and the base chase isomorphism (5.1) isomorphs it with the identity functor. The other composition is calculated in the same way by symmetry.

6.2.7. For the sake of completeness, let us explicitly perform the calculation that defines an identification (6.2).

We can think of both functors as given by objects of
\[
\text{IndCoh}(\mathcal{X}_1) \otimes \text{IndCoh}(\mathcal{X}_2) \simeq \text{IndCoh}(\mathcal{X}_1 \times \mathcal{X}_2)
\]
and diagram chase shows that both are given by the object
\[
(\Gamma_f)_* \text{IndCoh}(\omega_{\mathcal{X}_1}),
\]
where \(\Gamma_f : \mathcal{X}_1 \to \mathcal{X}_1 \times \mathcal{X}_2\) is the graph of \(f\), and
\[
\omega_{\mathcal{X}_1} := p_{\mathcal{X}_1}^!(k).
\]

6.2.8. The datum of self-duality
\[
\mathcal{D}^\text{Serre}_\mathcal{X} : \text{IndCoh}(\mathcal{X})^\vee \simeq \text{IndCoh}(\mathcal{X})
\]
is equivalent to that of an equivalence
\[
(\text{IndCoh}(\mathcal{X})^c)^\text{op} \to \text{IndCoh}(\mathcal{X})^c.
\]
We shall refer to the above functor as “Serre duality” on \(\mathcal{X}\), and denote it by \(\mathcal{D}^\text{Serre}_\mathcal{X}\).

From Theorem 5.1.4, isomorphism (6.2) and [GL:DG], Lemma 2.3.3, we obtain:

**Corollary 6.2.9.** For an ind-nil-proper map \(f : \mathcal{X}_1 \to \mathcal{X}_2\) of ind-inf-schemes, we have a commutative diagram
\[
\begin{array}{ccc}
(\text{IndCoh}(\mathcal{X}_1)^c)^\text{op} & \xrightarrow{\mathcal{D}^\text{Serre}_{\mathcal{X}_1}} & \text{IndCoh}(\mathcal{X}_1)^c \\
(f_*^\text{IndCoh})^\text{op} \downarrow & & \downarrow f_*^\text{IndCoh} \\
(\text{IndCoh}(\mathcal{X}_2)^c)^\text{op} & \xrightarrow{\mathcal{D}^\text{Serre}_{\mathcal{X}_2}} & \text{IndCoh}(\mathcal{X}_2)^c.
\end{array}
\]

6.3. Convolution categories and algebras.
6.3.1. As in [Book-II.2, Sect. 4.1.5], from Corollary 6.1.4 we obtain that the functor
\[ \text{IndCoh}_{(\text{PreStk}_{\text{left}})_{\text{corr}}; \text{indinfsc} \& \text{ind-nil-proper}}} : (\text{PreStk}_{\text{left}})_{\text{corr}}; \text{indinfsc}; \text{all} \rightarrow \text{DGCat}_{\text{cont}}^{2\text{-Cat}} \]
also carries a canonical right-lax symmetric monoidal structure.

6.3.2. This allows to extend the formalism in [Book-II.2, Sect. 5] by replacing
- the class of schematic quasi-compact maps by the class of ind-inf-schematic maps;
- the class of schematic and proper maps by the class of maps that are ind-inf-schematic and ind-nil-proper.
References

[DrGa] V. Drinfeld and D. Gaitsgory, *Compact generation*.


[Book-II.1]

[Book-II.2]

[Book-III.1]

[Book-III.2]

[InfSch]

