CHAPTER III.3. IND-COHERENT SHEAVES ON IND-INF-SCHEMES

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Introduction

In this Chapter we will perform a construction central to this book: we will extend the assignment

$$X \rightsquigarrow \text{IndCoh}(X),$$

viewed as a functor out of the $(\infty,2)$-category of correspondences on schemes, to a functor out of the $(\infty,2)$-category of correspondences on ind-inf-schemes.

0.1. **Why does everything work so nicely?** Let us explain the mechanism of why IndCoh works out so well on ind-inf-schemes (there is no much difference between ind-inf-schemes and inf-schemes, the former being just a little more general).

0.1.1. The key assertion here is [Chapter III.2, Theorem 4.1.3]. It says that an ind-inf-scheme can be written as

$$\mathcal{X} = \underset{a \in A}{\text{colim}}\, X_a,$$

where the colimit is taken in $\text{PreStk}_{\text{laft}}$, and where the maps $X_a \overset{i_{a,b}}{\longrightarrow} X_b$ are nil-closed embeddings of schemes; in particular, they are proper.

We can (tautologically) write $\text{IndCoh}(\mathcal{X})$ as the limit

$$\underset{a \in A^{\text{op}}}{\lim} \text{IndCoh}(X_a)$$

under the functors $i^!_{a,b}$.

Hence, by [Chapter I.1, Proposition 2.5.7] we have:

$$\text{IndCoh}(\mathcal{X}) \simeq \underset{a \in A}{\text{colim}}\, \text{IndCoh}(X_a),$$

where the functors $\text{IndCoh}(X_a) \to \text{IndCoh}(X_b)$ are $(i_{a,b})^{\text{IndCoh}}_*$.

0.1.2. The latter presentation implies that a functor out of $\text{IndCoh}(\mathcal{X})$ amounts to a compatible family of functors out of the categories $\text{IndCoh}(X_a)$.

This readily implies that if $f : \mathcal{X} \to \mathcal{Y}$ is an ind-inf-schematic ind-proper map between laft prestacks, then the functor $f^!$ admits a left adjoint that satisfies base change against !-pullbacks. The latter, in turn, entails the descent property of IndCoh for point-wise surjective ind-inf-schematic ind-proper maps.

In addition, we obtain that $\text{IndCoh}(\mathcal{X})$ is compactly generated, and has a t-structure with reasonable properties.

0.2. **Direct image for IndCoh on ind-inf-schemes.** The first step in making IndCoh into a functor out of the category of correspondences is the construction the direct image part of this functor.

0.2.1. Since ind-inf-schemes are laft prestacks, we know what $\text{IndCoh}(\mathcal{X})$ is for $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$. We also know how to form the !-pullback for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$. What we do not yet know is how to form push-forwards.

The construction of push-forwards will be given as a result of the combination of Corollary 4.3.5 and Theorem 4.3.3. It amounts to the following.

As we have already mentioned, if $i : \mathcal{Y} \to \mathcal{X}$ is an ind-proper map between ind-inf-schemes, then the functor $i^! : \text{IndCoh}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{X})$ admits a left adjoint, to be denoted i^{IndCoh}_*. In particular, we can take i to be nil-closed map from a scheme.

Now, the claim is that for a map $f : \mathcal{X}_1 \to \mathcal{X}_2$ there is a uniquely defined functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$$

such that for every commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & \mathcal{X}_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X_2 & \xrightarrow{i_2} & \mathcal{X}_2, \end{array}$$

where X_1, X_2 are schemes and i_1, i_2 are nil-closed maps, we have

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}.$$

Moreover, if f is itself nil-closed, then f_*^{IndCoh} identifies with the left adjoint of $f^!$.

Essentially, the existence and uniqueness of f_*^{IndCoh} follows from the description of functors out of $\mathrm{IndCoh}(\mathcal{X})$ (in this case $\mathcal{X} = \mathcal{X}_1$) in Sect. 0.1.2. What this amounts to technically will be reviewed in Sect. 0.2.3.

0.2.2. Having defined the functor f_*^{IndCoh} for any morphism $\mathcal{X} \to \mathcal{Y}$, we can in particular take $\mathcal{Y} = \mathrm{pt}$. In this way we obtain the functor of global sections

$$\Gamma(\mathcal{X}, -)^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect}\,.$$

Here is an example of what this functor does. As we will see in Chapter IV.3, the category of inf-schemes whose reduced scheme is pt is canonically equivalent to the category of Lie algebras in Vect. For a given Lie algebra $\mathfrak{g}$, the category IndCoh on the corresponding inf-scheme $B(\mathfrak{g})$ identifies canonically with the category $\mathfrak{g}$-mod of modules over $\mathfrak{g}$.

Under this identification, the functor of global sections $\Gamma(B(\mathfrak{g}), -)^{\mathrm{IndCoh}}$ corresponds to the functor of $\mathfrak{g}$-coinvariants. (Moreover, the forgetful functor $\mathfrak{g}\text{-mod} \to \mathrm{Vect}$ is the pullback under the map $\mathrm{pt} \to B(\mathfrak{g})$.)

0.2.3. A more precise description of the construction in Sect. 0.2.1 is as follows.

We consider the category $\mathrm{indinfSch}_{\mathrm{laft}}$ and its full subcategory $\mathrm{Sch}_{\mathrm{aft}}$. We now consider the categories

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}} \subset (\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}},$$

where we restrict morphisms to be nil-closed.

Consider the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad X \mapsto \mathrm{IndCoh}(X), \quad (X_1 \overset{g}{\to} X_2) \mapsto g_*^{\mathrm{IndCoh}}.$$

We consider the operation of *left Kan extension*

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}}});$$

this a functor

$$(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

One shows (Proposition 4.1.3 and Lemma 1.4.4) that the value of the above functor on a given $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ identifies canonically with $\mathrm{IndCoh}(\mathcal{X})$.

Now, the key assertion is Theorem 4.3.3 that says that the natural transformation

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}}}) \to$$
$$\to \mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}})|_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}})$$

is an isomorphism. This theorem ensures that the functor

$$\mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}}\hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}) : \mathrm{indinfSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

takes the value $\mathrm{IndCoh}(\mathcal{X})$ on a given $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$. Being a functor, it gives rise to the sought-for functoriality of IndCoh:

$$\mathcal{X}_1 \xrightarrow{f} \mathcal{X}_2 \quad \mapsto \quad f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2).$$

0.3. **Extending to correspondences.**

0.3.1. In order to extend the functor

$$\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{aft}}} := \mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}}\hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}})$$

to a functor out of the category of correspondences, we apply the machinery of [Chapter V.2, Sect. 1]. The only thing to check is that the functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$$

thus constructed satisfy base change against the !-pullback functors under ind-proper maps. I.e., given a Cartesian diagram of objects of $\mathrm{indinfSch}_{\mathrm{laft}}$

$$\begin{array}{ccc} \mathcal{X}_1' & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}_2' & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

where g_2 (and hence g_1) is ind-proper, we have, by adjunction, natural transformations

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}$$

and

$$(g_1)_*^{\mathrm{IndCoh}} \circ (f')^! \to f^! \circ (g_2)_*^{\mathrm{IndCoh}}.$$

We show that these natural transformations are isomorphisms. Once this is done, by [Chapter V.2, Theorem 1.1.9], we obtain the desired functor

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

This is a functor from the $(\infty, 2)$-category of correspondences, whose objects are ind-inf-schemes, horizontal and vertical morphisms are arbitrary maps, and 2-morphisms are given by ind-proper maps.

0.3.2. Finally, we apply [Chapter V.2, Theorem 6.1.5] and extend the latter functor to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

I.e., it is a functor from the $(\infty, 2)$-category of correspondences, whose objects are all laft prestacks, horizontal morphisms are arbitrary maps, vertical morphisms are ind-inf-schematic maps, and 2-morphisms are ind-inf-schematic and ind-proper maps.

This is the furthest point that we can imagine that the theory of IndCoh can be extended to.

0.4. **What else is done in this Chapter?**

0.4.1. In Sect. 1 we analyze the behavior of the category IndCoh on *ind-schemes*. Some of the assertions concerning ind-schemes (such as base change) are redundant: they will be reproved for ind-inf-schemes in greater generality. We have included them in order to compare the statements (and methods of their proofs) for ind-schemes and ind-inf-schemes.

We show that for an ind-scheme $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is compactly generated and that its compact objects are of the form $i_*^{\mathrm{IndCoh}}(\mathcal{F})$, where $i : X \to \mathcal{X}$ is a closed embedding with $X \in \mathrm{Sch}_{\mathrm{aft}}$ and $\mathcal{F} \in \mathrm{Coh}(X)$. In the above formula, $i_*^{\mathrm{IndCoh}}(\mathcal{F})$ is the left adjoint to the functor $i^!$.

We show that the category $\mathrm{IndCoh}(\mathcal{X})$ has a unique t-structure, for which the above functors $i_*^{\mathrm{IndCoh}}(\mathcal{F})$ are t-exact.

We apply a left Kan extension to the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and obtain a functor

$$\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}} : \mathrm{indSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

We show that its value on a given $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ identifies canonically with $\mathrm{IndCoh}(\mathcal{X})$. This is quite a bit easier than for ind-inf-scheme because of [Chapter III.2, Corollary 1.7.5(b)]), the analog of which *fails* for ind-inf-schemes.

The construction of $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$ has the property that for an ind-proper map $f : \mathcal{X} \to \mathcal{Y}$, the corresponding functor $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$ is the left adjoint of $f^!$.

We show that a morphism $f : \mathcal{X} \to \mathcal{Y}$ between ind-schemes, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$

is left t-exact, and if f is ind-affine, then it is t-exact.

0.4.2. In Sect. 2 we establish the base change property for ind-schematic ind-proper morphisms. Namely, let

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

be a Cartesian diagram of laft prestacks.

Suppose that the vertical arrows are ind-schematic ind-proper. In this case, by adjunction we obtain a natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}$$

and we show that it is an isomorphism.

Now suppose that in the above diagram all prestacks are ind-schemes the horizontal arrows are ind-proper. Then, again by adjunction, we have a natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

and we also show that it is an isomorphism.

0.4.3. In Sect. 3 we initiate the study of IndCoh on ind-inf-schemes. The key statement is Proposition 3.1.2. It says that for an ind-inf-schematic nil-isomorphism $f : \mathcal{X} \to \mathcal{Y}$, the functor $f^!$ is:

(i) conservative;

(ii) admits a left adjoint;

(iii) the left adjoint of $f^!$ satisfies base change against !-pullbacks.

The proof of Proposition 3.1.2 relies on [Chapter III.2, Corollary 4.3.3], which is deduced from [Chapter III.2, Theorem 4.1.3] and says that an ind-inf-scheme whose underlying reduced ind-scheme is an affine scheme, can be written as a colimit of affine schemes under nil-closed maps.

From Proposition 3.1.2 we deduce the various favorable properties of IndCoh on ind-inf-schemes mentioned in Sect. 0.1.

In particular, we establish ind-proper descent for IndCoh. The statement here is that if $\mathcal{X} \to \mathcal{Y}$ is ind-inf-schematic ind-proper and point-wise surjective map between laft prestacks, then the pullback functor

$$\operatorname{IndCoh}(\mathcal{Y}) \to \operatorname{Tot}(\operatorname{IndCoh}(\mathcal{X}^\bullet))$$

is an equivalence, where $\mathcal{X}^\bullet$ is the co-simplicial prestack given by the Čech nerve of $\mathcal{X} \to \mathcal{Y}$.

0.4.4. In Sect. 4 we carry out the construction of the functoriality of IndCoh with respect to the operation of direct image, described in Sect. 0.2 above.

In Sect. 5 we carry out the construction of IndCoh as a functor out of the category of correspondences, already explained in Sect. 0.3.

In Sect. 6 we show that for in ind-inf-scheme $\mathcal{X}$, the category $\operatorname{IndCoh}(\mathcal{X})$ is canonically self-dual. I.e., there is a canonically defined identification

$$\mathbb{D}^{\text{Serre}}_{\mathcal{X}} : \operatorname{IndCoh}(\mathcal{X})^\vee \to \operatorname{IndCoh}(\mathcal{X}),$$

or equivalently

$$\mathbf{D}^{\text{Serre}}_{\mathcal{X}} : (\operatorname{IndCoh}(\mathcal{X})^c)^{\text{op}} \to \operatorname{IndCoh}(\mathcal{X})^c.$$

Under this identification, for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$, the functor *dual* to $f^{\text{IndCoh}}_* : \operatorname{IndCoh}(\mathcal{X}_1) \to \operatorname{IndCoh}(\mathcal{X}_2)$ is

$$f^! : \operatorname{IndCoh}(\mathcal{X}_2) \to \operatorname{IndCoh}(\mathcal{X}_1).$$

It follows formally that if f is ind-proper, then f^{IndCoh}_* sends $\operatorname{IndCoh}(\mathcal{X}_1)^c$ to $\operatorname{IndCoh}(\mathcal{X}_2)^c$ and

$$\mathbf{D}^{\text{Serre}}_{\mathcal{X}_2} \circ (f^{\text{IndCoh}}_*)^{\text{op}} \circ \mathbf{D}^{\text{Serre}}_{\mathcal{X}_1} \simeq f^{\text{IndCoh}}_*$$

as functors $\operatorname{IndCoh}(\mathcal{X}_1)^c \to \operatorname{IndCoh}(\mathcal{X}_2)^c$.

1. Ind-coherent sheaves on ind-schemes

In order to develop the theory of IndCoh on ind-inf-schemes, we first need to do this for ind-schemes. The latter theory follows rather easily from one on schemes.

In this section we will mainly review the results from [GaRo1, Sect. 2].

1.1. **Basic properties.** In this subsection we will express the category IndCoh on an ind-scheme $\mathcal{X}$ from that on schemes equipped with a closed embedding into $\mathcal{X}$.

1.1.1. Let $\mathrm{IndCoh}^!_{\mathrm{indSch}_{\mathrm{laft}}}$ denote the restriction of the functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ to the full subcategory

$$(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

In particular, for $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ we have a well-defined category $\mathrm{IndCoh}(\mathcal{X})$.

1.1.2. Suppose $\mathcal{X}$ has been written as

$$\mathcal{X} \simeq {}^{\mathrm{conv}}\mathcal{X}' \quad \mathcal{X}' \simeq \underset{a\in A}{\mathrm{colim}}\, X_a, \tag{1.1}$$

where $X_a \in \mathrm{Sch}_{\mathrm{aft}}$ with the maps $i_{a,b} : X_a \to X_b$ being closed embeddings. In this case we have:

Proposition 1.1.3. *Under the above circumstances, !-restriction defines an equivalence*

$$\mathrm{IndCoh}(\mathcal{X}) \to \lim_{a\in A^{\mathrm{op}}} \mathrm{IndCoh}(X^a),$$

where for $a \to b$, the corresponding functor $\mathrm{IndCoh}(X_b) \to \mathrm{IndCoh}(X_a)$ *is* $i^!_{a,b}$.

Proof. This follows from the convergence property of the functor $\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}}$, see [Chapter II.2, Lemma 3.2.4 and Sect. 3.4.1]. □

Remark 1.1.4. The reason we exhibit an ind-scheme $\mathcal{X}$ as ${}^{\mathrm{conv}}(\underset{a\in A}{\mathrm{colim}}\, X_a)$ rather than just as $\underset{a\in A}{\mathrm{colim}}\, X_a$ is that the former presentation comes up in practice more often: many ind-schemes are given in this form. The fact that the resulting prestack is indeed an ind-scheme (i.e., can be written as a colimit of schemes under closed embeddings) is [Chapter III.2, Corollary 1.4.4] and is somewhat non-trivial.

1.1.5. Combining the above proposition with [Chapter I.1, Proposition 2.5.7], we obtain:

Corollary 1.1.6. *For $\mathcal{X}$ written as in* (1.1)*, we have*

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \underset{a\in A}{\mathrm{colim}}\, \mathrm{IndCoh}(X^a),$$

where for $a \to b$, the corresponding functor $\mathrm{IndCoh}(X_a) \to \mathrm{IndCoh}(X_b)$ *is* $(i_{a,b})^{\mathrm{IndCoh}}_*$.

Corollary 1.1.7. *For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ and a closed embedding $i : X \to \mathcal{X}$ from $X \in \mathrm{Sch}_{\mathrm{aft}}$, the functor*

$$i^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to $i^!$, is well-defined.

For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, let $\mathrm{Coh}(\mathcal{X})$ denote the full subcategory of $\mathrm{IndCoh}(\mathcal{X})$ spanned by objects

$$i^{\mathrm{IndCoh}}_*(\mathcal{F}), \quad i : X \to \mathcal{X} \text{ is a closed embedding and } \mathcal{F} \in \mathrm{Coh}(X).$$

From Corollary 1.1.6 we obtain:

Corollary 1.1.8. *For an ind-scheme $\mathcal{X}$, the category* $\mathrm{IndCoh}(\mathcal{X})$ *is compactly generated and*

$$\mathrm{IndCoh}(\mathcal{X})^c = \mathrm{Coh}(\mathcal{X}).$$

Proof. Follows from [DrGa1, Corollary 1.9.4 and Lemma 1.9.5.]. □

1.1.9. Here is another convenient fact about the category $\operatorname{IndCoh}(\mathcal{X})$, where $\mathcal{X} \in \operatorname{indSch}_{\mathrm{laft}}$. Let

$$X' \overset{i'}{\to} \mathcal{X} \overset{i''}{\leftarrow} X''$$

be closed embeddings.

We would like to calculate the composite

$$(i')^! \circ (i'')_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X'') \to \operatorname{IndCoh}(X').$$

Let A denote the category $(\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}$, so that X' and X'' correspond to indices a' and a'', respectively. Let $i_{a'}$ and $i_{a''}$ denote the corresponding closed embeddings, i.e., the maps i' and i'', respectively. Let B be any category cofinal in

$$A_{a' \sqcup a''/} := A_{a'/} \underset{A}{\times} A_{a''/}.$$

For $b \in B$, let

$$X' = X_{a'} \overset{i_{a',b}}{\longrightarrow} X_b \overset{i_{a'',b}}{\longleftarrow} X_{a''} = X''$$

denote the corresponding maps.

The next assertion results from [Ga4, Lemma 1.3.6]:

Lemma 1.1.10. *Under the above circumstances, we have a canonical isomorphism*

$$(i')^! \circ (i'')_*^{\operatorname{IndCoh}} \simeq \underset{b \in B}{\operatorname{colim}}\, (i_{a',b})^! \circ (i_{a'',b})_*^{\operatorname{IndCoh}}.$$

1.2. **t-structure.** In this subsection we will study the naturally defined t-structure on IndCoh of an ind-scheme.

1.2.1. For $\mathcal{X} \in \operatorname{indSch}_{\mathrm{laft}}$ we introduce a t-structure on the category $\operatorname{IndCoh}(\mathcal{X})$ as follows:

An object $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$ belongs to $\operatorname{IndCoh}(\mathcal{X})^{\geq 0}$ if and only if for every closed embedding $i : X \to \mathcal{X}$, where $X \in \mathrm{Sch}_{\mathrm{aft}}$, we have $i^!(\mathcal{F}) \in \operatorname{IndCoh}(X)^{\geq 0}$.

By construction, this t-structure is compatible with filtered colimits, which by definition means that $\operatorname{IndCoh}(\mathcal{X})^{\geq 0}$ is preserved by filtered colimits.

1.2.2. We can describe this t-structure and the category $\operatorname{IndCoh}(\mathcal{X})^{\leq 0}$ more explicitly. Write

$${}^{\mathrm{cl}}\mathcal{X} \simeq \underset{a \in A}{\operatorname{colim}}\, X_a,$$

where $X_a \in ({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}$.

For each a, let i_a denote the corresponding map (automatically, a closed embedding) $X_a \to \mathcal{X}$. By Corollary 1.1.7, we have a pair of adjoint functors

$$(i_a)_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X_a) \rightleftarrows \operatorname{IndCoh}(\mathcal{X}) : i_a^!.$$

Lemma 1.2.3. *Under the above circumstances we have:*

(a) *An object* $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$ *belongs to* $\operatorname{IndCoh}^{\geq 0}(\mathcal{X})$ *if and only if for every* a, *the object* $i_a^!(\mathcal{F}) \in \operatorname{IndCoh}(X_a)$ *belongs to* $\operatorname{IndCoh}(X_a)^{\geq 0}$.

(b) *The category* $\operatorname{IndCoh}(\mathcal{X})^{\leq 0}$ *is generated under colimits by the essential images of the functors* $(i_a)_*^{\operatorname{IndCoh}}\left(\operatorname{Coh}(X_a)^{\leq 0}\right)$.

Proof. It is easy to see that for a quasi-compact DG scheme X, the category $\operatorname{IndCoh}(X)^{\leq 0}$ is generated under colimits by $\operatorname{Coh}({}^{cl}X)^{\leq 0}$. In particular, by adjunction, an object $\mathcal{F} \in \operatorname{IndCoh}(X)$ is coconnective if and only if its restriction to ${}^{cl}X$ is coconnective.

Hence, in the definition of $\operatorname{IndCoh}(\mathcal{X})^{\geq 0}$, instead of all closed embeddings $X \to \mathcal{X}$, it suffices to use only those with X a classical scheme.

Note that the category A is cofinal in $({}^{cl}\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}$. This implies point (a) of the lemma. Point (b) follows formally from point (a).

□

1.2.4. Suppose $i : X \to \mathcal{X}$ is a closed embedding of a scheme into an ind-scheme. By Corollary 1.1.7, we have a well-defined functor

$$i_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(\mathcal{X}),$$

which is the right adjoint to $i^!$. Since $i^!$ is left t-exact, the functor $i_*^{\operatorname{IndCoh}}$ is right t-exact. However, we claim:

Lemma 1.2.5. *The functor $i_*^{\operatorname{IndCoh}}$ is t-exact.*

Proof. We need to show that for $\mathcal{F} \in \operatorname{IndCoh}(X)^{\geq 0}$, and a closed embedding $i' : X' \to \mathcal{X}$, we have

$$(i')^! \circ i_*^{\operatorname{IndCoh}}(\mathcal{F}) \in \operatorname{IndCoh}(X')^{\geq 0}.$$

This follows from Lemma 1.1.10: in the notations of *loc.cit.*, each of the functors $(i_{a'',b})_*^{\operatorname{IndCoh}}$ is t-exact (because $i_{a'',b}$ is a closed embedding), each of the functors $(i_{a'',b})^!$ is left t-exact (because $i_{a',b}$ is a closed embedding), and the category B is filtered.

□

Corollary 1.2.6. *The subcategory*

$$\operatorname{Coh}(\mathcal{X}) = \operatorname{IndCoh}(\mathcal{X})^c$$

is preserved by the truncation functors.

Proof. Follows from Lemma 1.2.5 and the corresponding fact for schemes. □

Corollary 1.2.7. *The t-structure on* $\operatorname{IndCoh}(\mathcal{X})$ *is obtained from the t-structure on* $\operatorname{Coh}(\mathcal{X})$ *by the procedure of* [Chapter II.1, Lemma 1.2.4].

1.3. **Recovering** IndCoh **from ind-proper maps.** The contents of this subsection are rather formal: we show that the functor IndCoh on ind-schemes can be recovered from the corresponding functor on schemes, where we restrict 1-morphisms to be proper, or even closed embeddings. This is not surprising, given the definition of ind-schemes.

1.3.1. Recall what it means for a map in PreStk to be *ind-proper* (resp., *ind-closed embedding*), see [Chapter III.2, Definitions 1.6.7 and 1.6.11].

1.3.2. Consider the corresponding 1-full subactegories

$$(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}} \subset (\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}$$

and the corresponding categories

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \subset (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}.$$

Consider the corresponding fully faithful embeddings

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}},$$

and

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}.$$

Let $\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}$ denote the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}|_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}}} : ((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and similarly, for 'proper' replaced by 'closed'.

Let $\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}}$ denote the functor

$$\mathrm{IndCoh}^!_{\mathrm{indSch}_{\mathrm{laft}}}|_{((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}}} : ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and similarly, for 'ind-proper' replaced by 'ind-closed'.

1.3.3. We claim:

Proposition 1.3.4. *For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, the functors*

$$(1.2)\quad (\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in }\mathcal{X}} \simeq (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}})_{/\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

and

$$(1.3)\quad (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}\,/\mathcal{X}} \simeq (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})_{/\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

are cofinal.

Proof. The cofinality of (1.2) is given by [Chapter III.2, Corollary 1.7.5(b)]. Since (1.3) is fully faithful, we have that the functor

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in }\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})_{/\mathcal{X}},$$

and hence (1.3), is also cofinal. □

Corollary 1.3.5. *The naturally defined functors*

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}\right),$$

and

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}}\right)$$

are isomorphisms.

Proof. The cofinality of (1.2) implies that the functor

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-closed}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}}\right)$$

is an isomorphism.

The cofinality of (1.3) implies that the functor

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}\right)$$

is an isomorphism.

□

1.4. **Direct image for** IndCoh **on ind-schemes.** In this subsection we show how to construct the functor of direct image on IndCoh for morphisms between ind-schemes.

1.4.1. Consider the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where for a morphism $f : X_1 \to X_2$ in $\mathrm{Sch}_{\mathrm{aft}}$, the functor

$$\mathrm{IndCoh}(X_1) \to \mathrm{IndCoh}(X_2)$$

is f_*^{IndCoh}, see [Chapter II.1, Sect. 2.2].

Recall the notation

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} = \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} : (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and consider also the corresponding functor

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}} : (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Denote

$$\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}} := \mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})\hookrightarrow(\mathrm{indSch}_{\mathrm{laft}})}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})}),$$

and let

$$\mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}},\ \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}$$

denote its restriction to the corresponding 1-full subcategories.

The same proof as that of Corollary 1.3.5 gives:

Proposition 1.4.2. *The natural maps*

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}\hookrightarrow(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \to \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}},$$

and

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}\hookrightarrow(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}}) \to \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}$$

are isomorphisms.

1.4.3. Recall from [Chapter I.1, Sect. 2.4] the notion of two functors obtained from each other by passing to adjoints.

Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between ∞-categories. Let $\Phi_1 : \mathbf{C}_1 \to \mathrm{DGCat}_{\mathrm{cont}}$ be a functor such that for every $\mathbf{c}_1' \to \mathbf{c}_1''$, the corresponding functor

$$\Phi_1(\mathbf{c}_1') \to \Phi_1(\mathbf{c}_1'')$$

admits a right adjoint. Let $\Psi_1 : \mathbf{C}_1^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ be the resulting functor given by taking the right adjoints.

Let Φ_2 and Ψ_2 be the left (resp., right) Kan extension of Φ_1 (resp., Ψ_1) along F (resp., F^{op}). The following is a particular case of [Chapter V.2, Proposition 2.2.7]:

Lemma 1.4.4. *Under the above circumstances, the functor Ψ_2 is obtained from Φ_2 by taking right adjoints.*

1.4.5. We apply Lemma 1.4.4 in the following situation:

$$\mathbf{C}_1 := (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}},\ \mathbf{C}_2 := (\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}},$$

and F to be the natural embedding. We take

$$\Phi_1 := \mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} \text{ and } \Psi_1 := \mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}.$$

These two functors are obtained from one another by passage to adjoints, by the definiton of the functor $\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}$, see [Chapter II.1, Corollary 5.1.12].

Corollary 1.4.6. *The functor*

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}} : ((\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is obtained from the functor

$$\mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}}} : (\mathrm{indSch}_{\mathrm{laft}})_{\text{ind-proper}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

by passing to right adjoints.

1.4.7. By the above corollary and Proposition 1.4.2, for $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ there is a canonical isomorphism

$$\mathrm{IndCoh}^!_{\mathrm{indSch}_{\mathrm{laft}}}(\mathcal{X}) \simeq \mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}(\mathcal{X}),$$

and by definition the left hand side is $\mathrm{IndCoh}(\mathcal{X})$. Thus, given a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ in $\mathrm{indSch}_{\mathrm{laft}}$, the functor $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$ gives a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2),$$

which, by definition of the functor $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$, agrees with the previously defined IndCoh direct image functor when restricted to $\mathrm{Sch}_{\mathrm{aft}}$. Furthermore, by Corollary 1.4.6, if f is ind-proper, then f_*^{IndCoh} is the left adjoint of $f^!$. In particular, for a closed embedding

$$X \overset{i}{\to} \mathcal{X}$$

of a scheme $X \in \mathrm{Sch}_{\mathrm{aft}}$ into an ind-scheme $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, the corresponding functor i_*^{IndCoh} agrees with that of Corollary 1.1.7.

1.4.8. We can now make the following observation pertaining to the behavior of the t-structure with respect to direct images:

Lemma 1.4.9. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map of ind-schemes. Then the functor f_*^{IndCoh} is left t-exact. Furthermore, if f is ind-affine, then it is t-exact.*

Proof. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\geq 0}$. We wish to show that $f_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathcal{X}_2)^{\geq 0}$. By Corollary 1.2.7, we can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\geq 0}$ where

$$i_1 : X_1 \to \mathcal{X}_1$$

is a closed embedding of a scheme. Now, let

$$X_1 \overset{g}{\to} X_2 \overset{i_2}{\to} \mathcal{X}_2$$

be a factorization of $f \circ i_1$, where i_2 is a closed embedding of a scheme. Thus, it suffices to show that the functor

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}$$

is left t-exact. However, $(i_2)_*^{\mathrm{IndCoh}}$ is t-exact by Lemma 1.2.5, while $g_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ is left t-exact, since g is a map between schemes.

Now, suppose that f is ind-affine. In this case, we wish to show that f_*^{IndCoh} is also right t-exact. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\leq 0}$. We can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\leq 0}$ where $i_1 : X_1 \to \mathcal{X}_1$ is a closed embedding. In the notation as above, it suffices to show that

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}$$

is t-exact.

By Lemma 1.2.5, $(i_2)_*^{\mathrm{IndCoh}}$ is t-exact. Hence, it suffices to show that g_*^{IndCoh} is t-exact. However, g is an affine map between schemes, and the assertion follows.

□

2. Proper base change for ind-schemes

Base change for IndCoh is a crucial property needed for its definition as a functor out of the category of correspondences. In this section we make two (necessary) preparatory steps, establishing base change for morphisms between ind-schemes.

2.1. 1st version.

2.1.1. Recall the notion of *ind-schematic* map in PreStk, see [Chapter III.2, Defintion 1.6.5(a)].

Let

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

be a Cartesian diagram in $\mathrm{PreStk}_{\mathrm{laft}}$ with f being ind-schematic and ind-proper. We claim:

Proposition 2.1.2. *The functors $f^!$ and $(f')^!$ admit left adjoints, to be denoted f_*^{IndCoh} and $(f')_*^{\mathrm{IndCoh}}$, respectively. Moreover, the natural transformation*

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}, \tag{2.1}$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

2.1.3. We begin by reviewing the setting of [Chapter I.1, Lemma 2.6.4]:

Let $G : \mathbf{C}_2 \to \mathbf{C}_1$ be a functor between ∞-categories. Let A be a category of indices, and suppose we are given an A-family of commutative diagrams

$$\begin{array}{ccc} \mathbf{C}_1^a & \xleftarrow{i_1^a} & \mathbf{C}_1 \\ {\scriptstyle G^a}\uparrow & & \uparrow{\scriptstyle G} \\ \mathbf{C}_2^a & \xleftarrow{i_2^a} & \mathbf{C}_2. \end{array}$$

Assume that for each $a \in A$, the functor G^a admits a left (resp. right) adjoint F^a. Furthermore, assume that for each map $a' \to a''$ in A, the diagram

$$\begin{array}{ccc} \mathbf{C}_1^{a''} & \xleftarrow{i_1^{a',a''}} & \mathbf{C}_1^{a'} \\ {\scriptstyle F^{a''}}\downarrow & & \downarrow{\scriptstyle F^{a'}} \\ \mathbf{C}_2^{a''} & \xleftarrow{i_2^{a',a''}} & \mathbf{C}_2^{a'}, \end{array}$$

which a priori commutes up to a natural transformation, actually commutes.

Finally, assume that the functors

$$\mathbf{C}_1 \to \lim_{a \in A} \mathbf{C}_1^a \text{ and } \mathbf{C}_2 \to \lim_{a \in A} \mathbf{C}_2^a$$

are equivalences.

In the above situation, [Chapter I.1, Lemma 2.6.4] says:

Lemma 2.1.4. *The functor G admits a left (resp. right) adjoint F, and for every $a \in A$, the diagram*

$$\begin{array}{ccc} \mathbf{C}_1^{a} & \xleftarrow{i_1^{a}} & \mathbf{C}_1 \\ {\scriptstyle F^{a}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{C}_2^{a} & \xleftarrow{i_2^{a}} & \mathbf{C}_2, \end{array}$$

which a priori commutes up to a natural transformation, commutes.

2.1.5. To show that (2.1) is an isomorphism, it suffices to show that it becomes an isomorphism after composing with $f^!$ for every map $f : S \to \mathcal{X}_2'$, with $S \in \text{Sch}_{\text{aft}}$. Therefore, we can assume that $\mathcal{X}_2' = X_2' \in \text{Sch}_{\text{aft}}$.

Now, we will apply Lemma 2.1.4 to the following situation. Let $\mathbf{C}_1 := \text{IndCoh}(\mathcal{X}_1)$, $\mathbf{C}_2 := \text{IndCoh}(\mathcal{X}_2)$ and let A be the category $(\text{Sch}_{\text{aft}})_{/\mathcal{X}_2}$. For each $Z \in (\text{Sch}_{\text{aft}})_{/\mathcal{X}_2}$, let

$$\mathbf{C}_2^a := \text{IndCoh}(Z), \quad \mathbf{C}_1^a := \text{IndCoh}(Z \underset{\mathcal{X}_2}{\times} \mathcal{X}_1).$$

Now, since X_2' is in particular an object of $(\text{Sch}_{\text{aft}})_{/\mathcal{X}_2}$, by Lemma 2.1.4, the assertion of Proposition 2.1.2 reduces to the case when $\mathcal{X}_2 = X_2 \in \text{Sch}_{\text{aft}}$ and $\mathcal{X}_2' = X_2' \in \text{Sch}_{\text{aft}}$. In this case $\mathcal{X}_1, \mathcal{X}_1' \in \text{indSch}_{\text{laft}}$ and the left adjoints exist by Corollary 1.4.6.

2.1.6. We have

$$\mathcal{X}_1 \simeq \underset{a \in A}{\text{colim}}\, X_a,$$

where $X_a \in \text{Sch}_{\text{aft}}$ and $i_a : X_a \to \mathcal{X}_1$ are closed embeddings.

Set

$$X_a' := X_2' \underset{X_2}{\times} X_a.$$

We have:

$$\mathcal{X}_1' \simeq \underset{a \in A}{\text{colim}}\, X_a',$$

Let i_a' denote the corresponding closed embedding $X_a' \to \mathcal{X}_1'$, and let g_a denote the map $X_a' \to X_a$.

Note that the maps $f \circ i_a : X_a \to X_2$ and $f' \circ i_a' : X_a' \to X_2'$ are proper, by assumption.

2.1.7. By Corollary 1.1.6, we have:

$$\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1)} \simeq \underset{a\in A}{\mathrm{colim}}\, (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^! \text{ and } \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}'_1)} \simeq \underset{a\in A}{\mathrm{colim}}\, (i'_a)_*^{\mathrm{IndCoh}} \circ (i'_a)^!.$$

Hence, we can rewrite the left-hand side in (2.1) as

$$\underset{a\in A}{\mathrm{colim}}\, (f')_*^{\mathrm{IndCoh}} \circ (i'_a)_*^{\mathrm{IndCoh}} \circ (i'_a)^! \circ g_1^!,$$

and the right-hand side as

$$\underset{a\in A}{\mathrm{colim}}\, g_2^! \circ f_*^{\mathrm{IndCoh}} \circ (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^!.$$

It follows from the construction that the map in (2.1) is given by a compatible system of maps for each $a \in A$

$$\begin{aligned}(f')_*^{\mathrm{IndCoh}} \circ (i'_a)_*^{\mathrm{IndCoh}} \circ (i'_a)^! \circ g_1^! \simeq (f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ (g_1 \circ i'_a)^! \simeq \\ (f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ (i_a \circ g_a)^! \simeq (f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ g_a^! \circ i_a^! \to \\ \to g_2^! \circ (f \circ i_a)_*^{\mathrm{IndCoh}} \circ i_a^! \simeq g_2^! \circ f_*^{\mathrm{IndCoh}} \circ (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^!,\end{aligned}$$

where the arrow

$$(f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ g_a^! \to g_2^! \circ (f \circ i_a)_*^{\mathrm{IndCoh}}$$

is base change for the Cartesian square

$$\begin{array}{ccc} X'_a & \xrightarrow{g_a} & X_a \\ {\scriptstyle f'\circ i'_a}\downarrow & & \downarrow{\scriptstyle f\circ i_a} \\ X'_2 & \xrightarrow{g_2} & X_2. \end{array}$$

Hence, the required isomorphism follows from proper base change in the case of schemes, see [Chapter II.2, Proposition 3.1.4(b)].

□

2.1.8. In the sequel we will need the following corollary of Proposition 2.1.2:

Corollary 2.1.9. *Let $\mathcal{X}' \to \mathcal{X}$ be a map of ind-schemes. Then the functor*

$$\mathrm{IndCoh}(\mathcal{X}') \to \lim_{(Z \xrightarrow{f} \mathcal{X}) \in ((\mathrm{Sch}_{\mathrm{aft}})_{\text{proper over } \mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(Z \underset{\mathcal{X}}{\times} \mathcal{X}')$$

is an equivalence.

Proof. The statement of the corollary is equivalent to the fact that for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}')$, the map

$$\underset{(Z \xrightarrow{f} \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{proper over } \mathcal{X}}}{\mathrm{colim}} f'^{\mathrm{IndCoh}}_* \circ f'^!(\mathcal{F}) \to \mathcal{F} \tag{2.2}$$

is an isomorphism, where

$$Z \underset{\mathcal{X}}{\times} \mathcal{X}' =: Z' \xrightarrow{f'} \mathcal{X}'.$$

Note that base change (i.e., Proposition 2.1.2) implies the projection formula, so for each $Z' \xrightarrow{f'} \mathcal{X}'$ as above, the natural map

$$f'^{\mathrm{IndCoh}}_* \circ f'^!(\mathcal{F}) \to f'^{\mathrm{IndCoh}}_* \circ f'^!(\omega_{\mathcal{X}}) \overset{!}{\otimes} \mathcal{F}$$

is an isomorphism.

Hence, the map in (2.2) is obtained by tensoring with $\mathcal{F}$ from the map

$$\underset{(Z\xrightarrow{f}\mathcal{X})\in(\mathrm{Sch}_{\mathrm{aft}})_{\text{proper over }\mathcal{X}}}{\mathrm{colim}} f'^{\mathrm{IndCoh}}_{*}\circ f'^{!}(\omega_{\mathcal{X}'})\to\omega_{\mathcal{X}'},$$

and therefore it is sufficient to check that the latter map is an isomorphism.

However, again by base change, the latter map identifies with the pullback under $\mathcal{X}'\to\mathcal{X}$ of the map

$$\underset{(Z\xrightarrow{f}\mathcal{X})\in(\mathrm{Sch}_{\mathrm{aft}})_{\text{proper over }\mathcal{X}}}{\mathrm{colim}} f^{\mathrm{IndCoh}}_{*}\circ f^{!}(\omega_{\mathcal{X}})\to\omega_{\mathcal{X}},$$

while the latter is an isomorphism by Corollary 1.3.5.

□

2.2. **2nd version.**

2.2.1. Now, let

$$\begin{array}{ccc}\mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1\\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f}\\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2\end{array}$$

be a Cartesian diagram in $\mathrm{indSch}_{\mathrm{laft}}$, such that the map g_2, and hence g_1, is ind-proper.

From the isomorphism

$$(g_2)^{\mathrm{IndCoh}}_{*}\circ(f')^{\mathrm{IndCoh}}_{*}\simeq f^{\mathrm{IndCoh}}_{*}\circ(g_1)^{\mathrm{IndCoh}}_{*},$$

we obtain, by adjunction, a natural transformation:

$$(f')^{\mathrm{IndCoh}}_{*}\circ g_1^{!}\to g_2^{!}\circ f^{\mathrm{IndCoh}}_{*}. \tag{2.3}$$

We claim:

Proposition 2.2.2. *The map* (2.3) *is an isomorphism.*

The rest of this subsection is devoted to the proof of the proposition.

2.2.3. First, suppose that f is ind-proper. In this case, the map (2.3) equals the map (2.1). Hence, it is an isomorphism by Proposition 2.1.2.

2.2.4. We have

$$\mathcal{X}'_2\simeq\underset{a\in A}{\mathrm{colim}}\,X'_{2,a},$$

where $X'_{2,a}\in\mathrm{Sch}_{\mathrm{aft}}$ and the maps $i_{2,a}:X'_{2,a}\to\mathcal{X}'_2$ are closed embeddings. Therefore, it suffices to show that (2.3) becomes an isomorphism after composing both sides with $i^{!}_{2,a}$ for every a. Thus, we can assume without loss of generality that $\mathcal{X}'_2=X'_2\in\mathrm{Sch}_{\mathrm{aft}}$.

Furthermore, by Corollary 1.1.6, we need to show that (2.3) becomes an isomorphism after precomposing both sides with the functor $(i_1)^{\mathrm{IndCoh}}_{*}$ for a closed embedding $i_1:X_1\to\mathcal{X}_1$ with

$X_1 \in \mathrm{Sch}_{\mathrm{aft}}$. Consider the commutative diagram

$$\begin{array}{ccc} X_1' & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ \mathcal{X}_1' & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X_2' & \xrightarrow{g_2} & \mathcal{X}_2', \end{array}$$

where both squares are Cartesian. Since i_1 is ind-proper, we have that base change holds for the top square. Hence, to show that (2.3) becomes an isomorphism after precomposing with $(i_1)_*^{\mathrm{IndCoh}}$, we need to show that base change holds for the outer square. In particular, we reduce to the case when $\mathcal{X}_1 = X_1 \in \mathrm{Sch}_{\mathrm{aft}}$.

2.2.5. By [Chapter III.2, Corollary 1.7.5(b)], we can factor the map $X_1 \to \mathcal{X}_2$ as a composition

$$X_1 \to \widetilde{X}_1 \to \mathcal{X}_2,$$

where $\widetilde{X}_1 \in \mathrm{Sch}_{\mathrm{aft}}$ and $\widetilde{X}_1 \to \mathcal{X}_2$ is a closed embedding (and in particular schematic). We have the diagram

$$\begin{array}{ccc} X_1' & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \widetilde{X}_1' & \longrightarrow & \widetilde{X}_1 \\ \downarrow & & \downarrow \\ X_2' & \longrightarrow & \mathcal{X}_2 \end{array}$$

where all the squares are Cartesian. The top square is a Cartesian square in $\mathrm{Sch}_{\mathrm{aft}}$ and hence satisfies base change by [Chapter II.1, Proposition 5.2.1]. In the bottom square, the map $\widetilde{X}_1 \to \mathcal{X}_2$ is ind-proper, and hence it satisfies base change by the above. Hence the outer square satisfies base change as desired. □

3. IndCoh on (ind)-inf-schemes

In this section we begin the development of the theory of IndCoh on ind-inf-schemes. We will essentially bootstrap it from IndCoh on ind-schemes, using *nil base change*.

3.1. **Nil base change.** As just mentioned, nil base change is a crucial property of the category IndCoh. Its proof relies on the structural results on inf-schemes from [Chapter III.2, Sect. 4].

3.1.1. Recall the notion of an *ind-inf-schematic* map in PreStk, see [Chapter III.2, Definition 3.1.5].

We will show:

Proposition 3.1.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$, and assume that f is an inf-schematic nil-isomorphism.*

(a) *The functor $f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1)$ admits a left adjoint, to be denoted f_*^{IndCoh}.*

(b) *The functor $f^!$ is conservative.*

(c) *For a Cartesian daigram*

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

the natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

3.1.3. *Proof of Proposition 3.1.2.* Using Lemma 2.1.4, we reduce the assertion to the case when $\mathcal{X}_2 = X_2 \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, for points (a) and (b), and further to the case when $\mathcal{X}'_2 = X'_2 \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ for point (c).

In this case $\mathcal{X}_1$ has the property that ${}^{\mathrm{red}}\mathcal{X}_1 = X_1 \in {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. By [Chapter III.2, Corollary 4.3.3], we can write

$$\mathcal{X}_1 \simeq \underset{a \in A}{\mathrm{colim}}\, X_{1,a},$$

where A is the category

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}_1} \underset{({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/X_1}}{\times} \{X_1\},$$

and the colimit is taken in the category $\mathrm{PreStk}_{\mathrm{laft}}$.

In particular, for every a, the resulting map $X_{1,a} \to X_2$ is a nil-isomorphism, and hence, proper. Moreover, for every morphism $a' \to a''$, the corresponding map

$$i_{a',a''} : X_{1,a'} \to X_{1,a''}$$

is also a nil-isomorphism and, in particular, is proper.

We have

$$\mathrm{IndCoh}(\mathcal{X}_1) \simeq \lim_{a \in A^{\mathrm{op}}} \mathrm{IndCoh}(X_{1,a}),$$

and the fact that $f^!$ is conservative follows from the fact that each $(f \circ i_a)^!$ is conservative.

Using [Chapter I.1, Proposition 2.5.7], we can therefore rewrite

$$\mathrm{IndCoh}(\mathcal{X}_1) \simeq \underset{a \in A}{\mathrm{colim}}\, \mathrm{IndCoh}(X_{1,a}), \tag{3.1}$$

where the colimit is taken with respect to the functors

$$(i_{a',a''})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_{1,a'}) \to \mathrm{IndCoh}(X_{1,a''}).$$

Now, the left adjoint to f is given by the compatible collection of functors

$$(f \circ i_a)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_{1,a}) \to \mathrm{IndCoh}(X_2).$$

Thus, it remains to establish the base change property. However, the latter follows by repeating the argument in Sects. 2.1.6-2.1.7.

□

3.2. **Basic properties.** We will now use nil base change to establish some basic properties of the category IndCoh on an ind-inf-scheme.

3.2.1. First, as a corollary of Proposition 3.1.2 we obtain:

Corollary 3.2.2. *Let* $\mathcal{X}$ *be an object of* $\text{indinfSch}_{\text{laft}}$. *Then the category* $\text{IndCoh}(\mathcal{X})$ *is compactly generated.*

Proof. Consider the canonical map $i : {}^{\text{red}}\mathcal{X} \to \mathcal{X}$. The category $\text{IndCoh}({}^{\text{red}}\mathcal{X})$ is compactly generated by Corollary 1.1.8. Now, Proposition 3.1.2 implies that the essential image of $i_*^{\text{IndCoh}}(\text{IndCoh}({}^{\text{red}}\mathcal{X})^c)$ compactly generates $\text{IndCoh}(\mathcal{X})$.

□

3.2.3. Let

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

be a Cartesian diagram in $\text{PreStk}_{\text{laft}}$ with f being ind-inf-schematic and ind-proper. We claim:

Proposition 3.2.4. *The functors* $f^!$ *and* $(f')^!$ *admit left adjoints, to be denoted* f_*^{IndCoh} *and* $(f')_*^{\text{IndCoh}}$, *respectively. The natural transformation*

$$(f')_*^{\text{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\text{IndCoh}}, \tag{3.2}$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

Proof. By Lemma 2.1.4, the assertion of the proposition reduces to the case when $\mathcal{X}_2$ and (resp., both $\mathcal{X}_2$ and $\mathcal{X}'_2$) belong to $\text{Sch}^{\text{aff}}_{\text{aft}}$. Denote these objects by X_2 and X'_2, respectively. In this case $\mathcal{X}_1$ (resp., both $\mathcal{X}_1$ and $\mathcal{X}'_1$) belong to $\text{indinfSch}_{\text{laft}}$. The existence of the left adjoint f_*^{IndCoh} (and therefore also $(f')_*^{\text{IndCoh}}$) follows, using [Chapter III.2, Corollary 4.1.4] and Proposition 2.1.2, by the same argument as Proposition 3.1.2(a).

Now, let $\mathcal{X}_0$ be any object of $\text{indSch}_{\text{laft}}$ endowed with a nil-isomorphism to $\mathcal{X}_1$; e.g., $\mathcal{X}_0 = {}^{\text{red}}\mathcal{X}_1$. Set

$$\mathcal{X}'_0 := X'_2 \underset{X_2}{\times} \mathcal{X}_0,$$

and consider the diagram

$$\begin{array}{ccc} \mathcal{X}'_0 & \xrightarrow{g_0} & \mathcal{X}_0 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X'_2 & \xrightarrow{g_2} & X_2, \end{array}$$

in which both squares are Cartesian.

By Proposition 3.1.2(b), it suffices to prove the assertion for the top square and for the outer square.

Now, the assertion for the outer square is given by Proposition 2.1.2, and for the top square by Proposition 3.1.2(c).

□

3.3. **Descent for ind-inf-schematic ind-proper maps.**

3.3.1. Let $\mathcal{X}^\bullet$ be a groupoid simplicial object in $\mathrm{PreStk}_{\mathrm{laft}}$, see [Lu1, Definition 6.1.2.7]. Denote by

$$p_s, p_t : \mathcal{X}^1 \rightrightarrows \mathcal{X}^0 \tag{3.3}$$

the corresponding maps.

We form a co-simplicial category $\mathrm{IndCoh}(\mathcal{X}^\bullet)^!$ using the !-pullback functors, and consider its totalization $\mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$. Consider the functor of evaluation on 0-simplices:

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)) \to \mathrm{IndCoh}(\mathcal{X}^0).$$

3.3.2. We claim:

Proposition 3.3.3. *Suppose that the maps p_s, p_t in (3.3) are ind-inf-schematic and ind-proper. Then:*
(a) *The functor ev^0 admits a left adjoint and the adjoint pair*

$$\mathrm{IndCoh}(\mathcal{X}^0) \rightleftarrows \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$$

is monadic. Furthermore, the resulting monad on $\mathrm{IndCoh}(\mathcal{X}^0)$, viewed as a plain endo-functor, is canonically isomorphic to $(p_t)_*^{\mathrm{IndCoh}} \circ (p_s)^!$.
(b) *Suppose that $\mathcal{X}^\bullet$ is the Čech nerve of a map $f : \mathcal{X}^0 \to \mathcal{Y}$, where f is ind-schematic and ind-proper. Assume also that f is surjective at the level of k-points. Then the resulting map*

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$$

is an equivalence.

Proof. Follows by repeating the argument of [Chapter II.1, Proposition 7.2.2]. □

3.3.4. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ recall the category $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}$, see [Chapter III.1, Sect. 4.3.6]. From Proposition 3.3.3 and [Chapter III.1, Corollary 4.4.2] we obtain:

Corollary 3.3.5. *Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$, which is ind-inf-schematic, ind-proper and surjective at the level of k-points. Then the pullback functor*

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{Y})^-)^{\mathrm{fake}}_{\mathrm{laft}} \to \mathrm{Tot}\left(\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X}^\bullet/\mathcal{Y})^-)^{\mathrm{fake}}_{\mathrm{laft}}\right)$$

is an equivalence.

3.3.6. *Descent for maps.* Let $f : \mathcal{X} \to \mathcal{Y}$ be an inf-schematic nil-isomorphism in $\mathrm{PreStk}_{\mathrm{laft}}$. We then have:

Proposition 3.3.7. *For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, the natural map*

$$\mathrm{Tot}(\mathrm{Maps}(\mathcal{X}^\bullet, \mathcal{Z})) \to \mathrm{Maps}(\mathcal{Y}, \mathcal{Z})$$

is an isomorphism.

Proof. The statement automatically reduces to the case when $\mathcal{Y} = Y \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Furthermore, by [Chapter III.2, Corollary 4.3.4], we can further assume that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

The assertion of the proposition is evident if Y is reduced: in this case the simplicial object $X^\bullet$ is split. Hence, by [Chapter III.1, Proposition 5.4.2], by induction, it suffices to show that if the assertion holds for a given Y and we have a square-zero extension $Y \hookrightarrow Y'$ by means of $\mathcal{F} \in \mathrm{Coh}(Y)^{\leq 0}$, then the assertion holds also for $X' \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\text{nil-isom to } Y'}$.

Set $X := X' \underset{Y'}{\times} Y'$. By [Chapter III.1, Proposition 5.3.2], the map

$$X^\bullet \hookrightarrow X'^\bullet$$

has a structure of simplicial object in the category of square-zero extensions.

By the induction hypothesis, it is enough to show that for a given map $z : Y \to \mathcal{Z}$, the map

$$\text{Maps}(Y', \mathcal{Z}) \underset{\text{Maps}(Y,\mathcal{Z})}{\times} \{z\} \to \text{Tot}(\text{Maps}(X'^{\bullet}, \mathcal{Z})) \underset{\text{Tot}(\text{Maps}(X^{\bullet},\mathcal{Z}))}{\times} \{z\} \tag{3.4}$$

is an isomorphism.

Since $\mathcal{Z}$ admits deformation theory, the left-hand side in (3.4) is canonically isomorphic to the groupoid of null-homotopies of the composition

$$T^*_z(\mathcal{Z}) \to T^*(Y) \to \mathcal{F}.$$

We rewrite the right-hand side in (3.4) as the totalization of the simplicial space

$$\text{Maps}(X'^{\bullet}, \mathcal{Z}) \underset{\text{Maps}(X^{\bullet},\mathcal{Z})}{\times} \{z\}.$$

The above simplicial groupoid identifies with that of null-homotopies of the composition

$$T^*_{z\circ f^{\bullet}}(\mathcal{Z}) \to T^*(X^{\bullet}) \to f^{\bullet}(\mathcal{F}),$$

where $f^{\bullet}$ denotes the map $X^{\bullet} \to Y$.

Now, the desired property follows the the descent property of $\text{Pro}(\text{QCoh}(-)^-)^{\text{fake}}_{\text{laft}}$, see Corollary 3.3.5 above.

□

3.4. **t-structure for ind-inf-schemes.** The category IndCoh on an ind-inf-scheme also possesses a t-structure. However, it has less favorable properties than in the case of ind-schemes.

3.4.1. Let $\mathcal{X}$ be an ind-inf-scheme. We define a t-structure on the category $\text{IndCoh}(\mathcal{X})$ by declaring that an object $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$ belongs to $\text{IndCoh}(\mathcal{X})^{\geq 0}$ if and only if $i^!(\mathcal{F})$ belongs to $\text{IndCoh}({}^{\text{red}}\mathcal{X})^{\geq 0}$, where

$$i : {}^{\text{red}}\mathcal{X} \to \mathcal{X}$$

is the canonical map.

Equivalently, we let $\text{IndCoh}(\mathcal{X})^{\leq 0}$ be generated under colimits by the essential image of $\text{IndCoh}({}^{\text{red}}\mathcal{X})^{\leq 0}$ under i_*^{IndCoh}.

It is easy to see that if f is an ind-finite map, then the functor $f^!$ is left t-exact.

3.4.2. Suppose that $\mathcal{X}$ is actually an ind-scheme. We claim that the t-structure defined above, when we view $\mathcal{X}$ as a mere ind-inf-scheme, coincides with one for $\mathcal{X}$ considered as an ind-scheme of Sect. 1.2. This follows from the next lemma:

Lemma 3.4.3. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a nil-isomorphism of ind-schemes. Then for $\mathcal{F} \in \text{IndCoh}(\mathcal{X}_2)$ we have:*

$$\mathcal{F} \in \text{IndCoh}(\mathcal{X}_2)^{\geq 0} \Leftrightarrow f^!(\mathcal{F}) \in \mathcal{F} \in \text{IndCoh}(\mathcal{X}_1)^{\geq 0}.$$

Proof. The $\Rightarrow$ implication is tautological. For the $\Leftarrow$ implication, by the definition of the t-structure on $\text{IndCoh}(\mathcal{X}_2)$, we can assume that $\mathcal{X}_2 = X_2 \in \text{Sch}_{\text{aft}}$ and $\mathcal{X}_1 = X_1 \in \text{Sch}_{\text{aft}}$; i.e. f is a nil-isomorphism of schemes $X_1 \to X_2$.

By the definition of the t-structure on $\text{IndCoh}(X_2)$ and adjunction, it suffices to show that $\text{Coh}(X_2)^{\leq 0}$ is generated by the essential image of $\text{Coh}(X_1)^{\leq 0}$ under f_*^{IndCoh}, which is obvious.

□

Corollary 3.4.4. *Let* $\mathcal{X}$ *be an object of* $\mathrm{indinfSch}_{\mathrm{laft}}$.

(a) *For* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, *we have* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ *if and only if for every nil-closed map* $f : X \to \mathcal{X}$ *with* $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *we have*

$$f^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}.$$

(b) *The category* $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$ *is generated under colimits by the essential images of the categories* $\mathrm{IndCoh}(X)^{\leq 0}$ *for* $f : X \to \mathcal{X}$ *with* $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *and* f *nil-closed.*

3.4.5. As mentioned above, if f is an ind-finite map $\mathcal{X}_1 \to \mathcal{X}_2$ of ind-inf-schemes, then the functor $f^!$ is left t-exact. By adjunction, this implies that the functor f_*^{IndCoh} is right t-exact.

Lemma 3.4.6. *Let* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *be an* ind-finite and ind-schematic *map between ind-inf-schemes. Then the functor* f_*^{IndCoh} *is t-exact.*

Proof. We only have to prove that f_*^{IndCoh} is left t-exact. By Proposition 3.1.2(c), the assertion reduces to the case when $\mathcal{X}_2 = \mathrm{indSch}_{\mathrm{laft}}$. In the latter case, $\mathcal{X}_1$ is an ind-scheme, and the assertion follows from the fact that the functor f_*^{IndCoh} for a map of ind-schemes is left t-exact, by Lemma 1.4.9. □

Remark 3.4.7. It is easy to see that the assertion of the lemma is false without the assumption that f be ind-schematic, see Sect. 0.2.2

4. The direct image functor for ind-inf-schemes

In this section we construct the direct image functor on IndCoh for maps between ind-inf-schemes. The idea is that one can bootstrap it from the case of maps that are nil-closed embeddings, while for the latter the sought-for procedure is obtained as left/right Kan extension from the case of schemes.

4.1. **Recovering from nil-closed embeddings.**

In this subsection we show that if we take IndCoh on the category of schemes, with morphisms restricted to nil-closed maps, then its right Kan extension to ind-inf-schemes recovers the usual IndCoh.

4.1.1. Consider the fully faithful embeddings

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow \mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}\,.$$

Denote

$$\mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{laft}}} := \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}|_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}}}.$$

Since

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{RKE}_{(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}})$$

is an isomorphism, the map

$$\mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{laft}}} \to \mathrm{RKE}_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}) \tag{4.1}$$

is an isomorphism.

4.1.2. Let

$$(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}} \subset \text{indinfSch}_{\text{laft}}$$

denote the 1-full subcategory, where we restrict 1-morphisms to be nil-closed.

Denote

$$\text{IndCoh}^!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}} := \text{IndCoh}^!_{\text{indinfSch}_{\text{laft}}}|_{((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}.$$

From the isomorphism (4.1), we obtain a canonically defined map

$$\text{IndCoh}^!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}} \to \text{RKE}_{((\text{Sch}_{\text{aft}})_{\text{nil-closed}})^{\text{op}} \hookrightarrow ((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}(\text{IndCoh}^!_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}}). \tag{4.2}$$

We will prove:

Proposition 4.1.3. *The map* (4.2) *is an isomorphism.*

Proof. We need to show that for $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$, the functor

$$\text{IndCoh}(\mathcal{X}) \to \lim_{Z \in ((\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}})^{\text{op}}} \text{IndCoh}(Z)$$

is an equivalence.

However, this follows from [Chapter III.2, Corollary 4.1.4], since the functor IndCoh takes colimits in $\text{PreStk}_{\text{laft}}$ to limits.

□

4.1.4. For the sequel we will need the following observation:

Corollary 4.1.5. *Let $\mathcal{X}' \to \mathcal{X}$ be a map of ind-inf-schemes. Then the functor*

$$\text{IndCoh}(\mathcal{X}') \to \lim_{Z \in ((\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}})^{\text{op}}} \text{IndCoh}(Z \underset{\mathcal{X}}{\times} \mathcal{X}')$$

is an equivalence.

Proof. Same as that of Corollary 2.1.9. □

4.2. **Recovering from nil-isomorphisms.** The material in this subsection is not needed for the sequel and is included for the sake of completeness.

4.2.1. Let

$$(\text{indinfSch}_{\text{laft}})_{\text{nil-isom}} \subset \text{indinfSch}_{\text{laft}} \text{ and } (\text{indSch}_{\text{laft}})_{\text{nil-isom}} \subset \text{indSch}_{\text{laft}}$$

denote the 1-full subcategories, where we restrict 1-morphisms to be nil-isomorphisms.

Denote also

$$\text{IndCoh}^!_{(\text{indinfSch}_{\text{laft}})_{\text{nil-isom}}} := \text{IndCoh}^!_{\text{indinfSch}_{\text{laft}}}|_{((\text{indinfSch}_{\text{laft}})_{\text{nil-isom}})^{\text{op}}}$$

and

$$\text{IndCoh}^!_{(\text{indSch}_{\text{laft}})_{\text{nil-isom}}} := \text{IndCoh}^!_{\text{indSch}_{\text{laft}}}|_{((\text{indSch}_{\text{laft}})_{\text{nil-isom}})^{\text{op}}}$$

4.2.2. From Proposition 4.1.3 we deduce:

Corollary 4.2.3. *The natural map*

$$\mathrm{IndCoh}^!_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-isom}}} \to$$
$$\to \mathrm{RKE}_{((\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-isom}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-isom}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-isom}}})$$

is an isomorphism.

Proof. By Corollary 1.3.5 and Proposition 4.1.3, the map

$$\mathrm{IndCoh}^!_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}} \to$$
$$\to \mathrm{RKE}_{((\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-closed}})^{\mathrm{op}} \hookrightarrow ((\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-closed}}})$$

is an isomorphism.

Hence, it remains to show that for $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$, the restriction map

$$\lim_{\mathcal{Y} \in ((\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-closed in } \mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Y}) \to \lim_{\mathcal{Y} \in ((\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-isom to } \mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Y})$$

is an isomorphism.

We claim that the map

$$(\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-isom to } \mathcal{X}} \to (\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-closed in } \mathcal{X}}$$

is cofinal. Indeed, it admits a left adjoint, given by sending an object

$$(\mathcal{Y} \to \mathcal{X}) \in (\mathrm{indSch}_{\mathrm{laft}})_{\text{nil-closed in } \mathcal{X}}$$

$$\mathcal{Y} \underset{{}^{\mathrm{red}}\mathcal{Y}}{\sqcup} {}^{\mathrm{red}}\mathcal{X} \to \mathcal{X},$$

where the push-out is taken in the category $\mathrm{PreStk}_{\mathrm{laft}}$.

□

4.3. **Constructing the direct image functor.** In this subsection we finally construct the direct image functor. The crucial assertion is Theorem 4.3.3, which says that this functor is the 'right one'.

4.3.1. Consider again the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where for a morphism $f : X_1 \to X_2$, the functor $\mathrm{IndCoh}(X_1) \to \mathrm{IndCoh}(X_2)$ is f_*^{IndCoh}.

Recall the notation:

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}}} := \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}}}.$$

Denote

$$\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}} := \mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}),$$

Note that by Proposition 1.4.2, the restriction of $\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}}$ to $\mathrm{IndCoh}_{\mathrm{infSch}_{\mathrm{laft}}}$ identifies canonically with $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$.

4.3.2. Denote

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}} := \mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}}|_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}.$$

We have a canonical map

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed}}}) \to \to \mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}. \tag{4.3}$$

We claim:

Theorem 4.3.3. *The map* (4.3) *is an isomorphism.*

4.3.4. Note that by combining Lemma 1.4.4 and Theorem 4.3.3, we obtain:

Corollary 4.3.5. *The functors*

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}} \text{ and } \mathrm{IndCoh}^!_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{nil-closed}}}$$

are obtained from one another by passing to adjoints.

Remark 4.3.6. The concrete meaning of the combination of the above corollary and Theorem 4.3.3 is the following. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism between objects of $\mathrm{indinfSch}_{\mathrm{laft}}$. Then the claim is that we have a well-defined functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2),$$

which tautologically agrees with the previously constructed IndCoh direct image functor when $\mathcal{X}_1, \mathcal{X}_2 \in \mathrm{indSch}_{\mathrm{laft}}$.

Furthermore, if f is nil-closed, then f_*^{IndCoh} is the left adjoint of $f^!$.

Remark 4.3.7. Given an ind-proper map

$$f : \mathcal{X} \to \mathcal{Y}$$

in $\mathrm{indinfSch}_{\mathrm{laft}}$, we have defined two functors that we called f_*^{IndCoh}; namely, one is the functor given by $\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}}$ and the other is the right adjoint of $f^!$. A priori, these two functors are unrelated.

However, this abuse of notation will be justified in Corollary 5.2.3, where we will establish a canonical identification of these functors. Namely, we will show that the assertion of Theorem 4.3.3 and therefore Corollary 4.3.5 can be strengthened by replacing the class of nil-closed morphisms by that of ind-proper ones.

Remark 4.3.8. In what follows, for $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = \mathrm{pt}$, we shall also use the notation

$$\Gamma(\mathcal{X}, -)^{\mathrm{IndCoh}}$$

for the functor $(p_{\mathcal{X}})_*^{\mathrm{IndCoh}}$, where $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{pt}$ is the projection.

4.4. **Proof of Theorem 4.3.3.** Before we begin the proof, let us explain why the proof of Theorem 4.3.3 is more involved than that of Proposition 4.1.3.

The reason is that in Proposition 4.1.3, we could test the equivalence via affine schemes, and the latter we did using [Chapter III.2, Theorem 4.1.3]. Now, for Theorem 4.3.3 affine schemes are not enough. We could have gotten away cheaply if we knew that for (a not necessarily affine) scheme $Z \in \mathrm{Sch}_{\mathrm{aft}}$ and a map $Z \to \mathcal{X}$, the category of its factorizations as

$$Z \to Z' \to \mathcal{X},$$

where $Z' \to \mathcal{X}$ is a nil-isomorphism, is contractible. However, the latter fact is simply *not true* (see [Chapter III.2, Remark 4.1.6]).

4.4.1. *Step -1.* We need to show that for $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$, the functor

$$\underset{Z \in (\text{Sch}_{\text{aft}})_{\text{nil-closed in } \mathcal{X}}}{\text{colim}} \text{IndCoh}(Z) \to \underset{Y \in (\text{Sch}_{\text{aft}})_{/\mathcal{X}}}{\text{colim}} \text{IndCoh}(Y)$$

is an equivalence.

The convergence property of the IndCoh functor allows to replace Sch_{aft} by ${}^{<\infty}\text{Sch}_{\text{ft}}$. Thus, we need to show that the functor

$$\underset{Z \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}}{\text{colim}} \text{IndCoh}(Z) \to \underset{Y \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{/\mathcal{X}}}{\text{colim}} \text{IndCoh}(Y)$$

is an equivalence.

4.4.2. *Step 0.* Consider the commutative diagram

$$\begin{array}{ccc} \underset{Y \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{/\mathcal{X}}}{\text{colim}} \text{IndCoh}(Y) & \longleftarrow & \underset{S \in ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{X}}}{\text{colim}} \text{IndCoh}(S) \\ \uparrow & & \uparrow \\ \underset{Z \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}}{\text{colim}} \text{IndCoh}(Z) & \longleftarrow & \underset{(S \to Z \to \mathcal{X}) \in \mathbf{C}}{\text{colim}} \text{IndCoh}(S), \end{array} \tag{4.4}$$

where $\mathbf{C}$ is the category of $S \to Z \to \mathcal{X}$, with $S \in {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}}$ and

$$(Z \to \mathcal{X}) \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}.$$

We will show that the horizontal arrows and the right vertical arrow in this diagram are equivalences. This will prove that the left vertical arrow is also an equivalence.

4.4.3. *Step 1.* Consider the functor $\mathbf{C} \to ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{X}}$, appearing in the right vertical arrow in (4.4). We claim that it is cofinal, which would prove that the right vertical arrow in (4.4) is an equivalence.

We note that the functor $\mathbf{C} \to ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{X}}$ is a Cartesian fibration. Hence, the fact that it is cofinal is equivalent to the fact that it has contractible fibers.

The fiber over a given object $S \to \mathcal{X}$ is the category of factorizations

$$S \to Z \to \mathcal{X}, \quad (Z \to \mathcal{X}) \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}.$$

This category is contractible by [Chapter III.2, Theorem 4.1.3].

4.4.4. *Step 2.* Consider the functor $\mathbf{C} \to ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}$, appearing in the bottom horizontal arrow in (4.4). It is a co-Cartesian fibration.

Hence, in order to show that this arrow in the diagram is an equivalence, it suffices to show that for a given $Z \in ({}^{<\infty}\text{Sch}_{\text{ft}})_{\text{nil-closed in } \mathcal{X}}$, the functor

$$\underset{S \in ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{/Z}}{\text{colim}} \text{IndCoh}(S) \to \text{IndCoh}(Z) \tag{4.5}$$

is an equivalence.

We have the following assertion, proved below:

Proposition 4.4.5. *The functor* $\text{IndCoh}_{\text{Sch}_{\text{aft}}}$*, regarded as a presheaf on* Sch_{aft} *with values in* $(\text{DGCat}_{\text{cont}})^{\text{op}}$ *satisfies Zariski descent.*

This proposition readily implies that (4.5) is an equivalence (for a general statement along these lines see [Ga1, Proposition 6.4.3]; here we apply it to ${}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \subset {}^{<\infty}\text{Sch}_{\text{ft}}$.)

4.4.6. *Step 3.* To treat the top horizontal arrow in (4.4), we consider the category $\mathbf{D}$ of

$$S \to Y \to \mathcal{X}, \quad S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \quad Y \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}},$$

and functor

$$\underset{(S\to Y\to \mathcal{X})\in \mathbf{D}}{\mathrm{colim}} \mathrm{IndCoh}(S) \to \underset{S\in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(S). \tag{4.6}$$

We note that the functor (4.6) is an equivalence, because the corresponding forgetful functor $\mathbf{D} \to ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}$ is cofinal (it is a Cartesian fibration with contractible fibers).

Hence, it remains to show that the composition

$$\underset{(S\to Y\to \mathcal{X})\in \mathbf{D}}{\mathrm{colim}} \mathrm{IndCoh}(S) \to \underset{Y\in ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(S)$$

of (4.6) with the top horizontal arrow in (4.4) is an equivalence.

The above functor corresponds to the co-Cartesian fibration $\mathbf{D} \to ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}$. Hence, it suffices to show that for a fixed $Y \in ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}$, the functor

$$\underset{S\in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}}{\mathrm{colim}} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(Y)$$

is an equivalence.

However, this follows as in Step 2 from Proposition 4.4.5.

4.4.7. *Proof of Proposition 4.4.5.* The assertion of the proposition is equivalent to the following. Let $X \in \mathrm{Sch}_{\mathrm{aft}}$ be covered by two opens U_1 and U_2. Denote

$$U_1 \overset{j_1}{\hookrightarrow} X,\ U_2 \overset{j_2}{\hookrightarrow} X,\ U_1 \cap U_2 \overset{j_{12}}{\hookrightarrow} X,$$
$$U_1 \cap U_2 \overset{j_{12,1}}{\hookrightarrow} U_1,\ U_1 \cap U_2 \overset{j_{12,1}}{\hookrightarrow} U_2.$$

Then the claim is that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(U_1\cap U_2) & \xrightarrow{(j_{12,1})^{\mathrm{IndCoh}}_*} & \mathrm{IndCoh}(U_1) \\ {\scriptstyle (j_{12,2})^{\mathrm{IndCoh}}_*}\downarrow & & \downarrow {\scriptstyle (j_1)^{\mathrm{IndCoh}}_*} \\ \mathrm{IndCoh}(U_1) & \xrightarrow{(j_2)^{\mathrm{IndCoh}}_*} & \mathrm{IndCoh}(X) \end{array}$$

is a push-out square in $\mathrm{DGCat}_{\mathrm{cont}}$.

In other words, given $\mathbf{C} \in \mathrm{DGCat}$ and a triple of continuous functors

$$F_1 : \mathrm{IndCoh}(U_1) \to \mathbf{C},\ F_2 : \mathrm{IndCoh}(U_2) \to \mathbf{C},\ F_{12} : \mathrm{IndCoh}(U_1 \cap U_2) \to \mathbf{C}$$

endowed with isomorphisms

$$F_1 \circ (j_{12,1})^{\mathrm{IndCoh}}_* \simeq F_{12} \simeq F_2 \circ (j_{12,2})^{\mathrm{IndCoh}}_*,$$

we need to show that this data comes from a uniquely defined functor

$$F : \mathrm{IndCoh}(X) \to \mathbf{C}.$$

The sought-for functor F is recovered as follows: for $\mathcal{F} \in \mathrm{IndCoh}(X)$, we have

$$F(\mathcal{F}) = F_1(j_1^!(\mathcal{F})) \underset{F_{12}(j_{12}^!(\mathcal{F}))}{\times} F_2(j_2^!(\mathcal{F})),$$

where the maps $F_i(j_i^!(\mathcal{F})) \to F_{12}(j_{12}^!(\mathcal{F}))$ are given by

$$F_i(j_i^!(\mathcal{F})) \to F_i((j_{12,i})^{\mathrm{IndCoh}}_* \circ j_{12,i}^! \circ j_i^!(\mathcal{F})) = F_i((j_{12,i})^{\mathrm{IndCoh}}_* \circ j_{12}^!(\mathcal{F}) \simeq F_{12}(j_{12}^!(\mathcal{F})).$$

□

4.5. **Base change.** As in the case of ind-schemes, there are two types of base change isomorphism for ind-proper inf-schematic maps. The first is given by Proposition 3.2.4. Here we will prove the second.

4.5.1. Let

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

be a Cartesian diagram of objects of $\text{indinfSch}_{\text{laft}}$ such that g_2 is an ind-closed embedding. Note that in this case, the right adjoint of $(g_2)_*^{\text{IndCoh}}$ is $g_2^!$ (and similarly for g_1).

From the isomorphism

$$(g_2)_*^{\text{IndCoh}} \circ (f')_*^{\text{IndCoh}} \simeq f_*^{\text{IndCoh}} \circ (g_1)_*^{\text{IndCoh}},$$

we obtain, by adjunction, a natural transformation:

$$(f')_*^{\text{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\text{IndCoh}}. \tag{4.7}$$

We claim:

Proposition 4.5.2. *The map* (4.7) *is an isomorphism.*

Proof. Let $\mathcal{X}_0 := {}^{\text{red}}\mathcal{X}_1$. Set

$$\mathcal{X}'_0 := \mathcal{X}'_1 \underset{\mathcal{X}_1}{\times} \mathcal{X}_0,$$

and consider the diagram

$$\begin{array}{ccc} \mathcal{X}'_0 & \xrightarrow{g_0} & \mathcal{X}_0 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

in which both squares are Cartesian.

Since the functor $i^!$ is conservative (by Proposition 3.1.2(b)), its left adjoint i_*^{IndCoh} generates the target. Hence, it suffices to show that the outer square and the top square each satisfy base change of Proposition 4.5.2.

Note that base change for the top square is given by Proposition 3.1.2(c).

This reduces the assertion of the proposition to the case when $\mathcal{X}_1$ is a classical reduced ind-scheme. In this case the map f factors as

$$\mathcal{X}_1 \to \mathcal{X}_{3/2} \to \mathcal{X}_2,$$

where $\mathcal{X}_{3/2} = {}^{\mathrm{red}}\mathcal{X}_2$. Set $\mathcal{X}'_{3/2} := \mathcal{X}'_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_{3/2}$, and consider the diagram

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_0} & \mathcal{X}_1 \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{X}'_{3/2} & \xrightarrow{g_3} & \mathcal{X}_{3/2} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

in which both squares are Cartesian.

It is sufficient to show that the two inner squares each satisfy base change of Proposition 4.5.2. For the bottom square, this is given by Proposition 3.1.2(c).

We note now that $\mathcal{X}_{3/2}$ and $\mathcal{X}'_{3/2}$ are *ind-schemes*. Hence, base change for the top square, this is given by Proposition 2.2.2.

□

5. Extending the formalism of correspondences to inf-schemes

In this section we will take the formalism of IndCoh to what (in our opinion) is its ultimate domain of definition: the category of correspondences, where the objects are all prestacks (locally almost of finite type), pullbacks are taken with respect to any maps, push-forwards are taken with respect to ind-inf-schematic maps, and adjunctions hold for ind-proper maps.

5.1. **Set-up for extension.** As a first step, we will consider the category of correspondences, where the objects are ind-inf-schemes, pullbacks and push-forwards are taken with respect to any maps, and adjunctions are for nil-closed maps. We will construct the required functor by the Kan extension procedure from [Chapter V.2, Theorem 1.1.9].

5.1.1. We consider the category $\mathrm{indinfSch}_{\mathrm{laft}}$ with the following three classes of morphisms

$$vert = \mathrm{all},\ horiz = \mathrm{all},\ adm = \text{nil-closed}\,.$$

Let

$$\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\mathrm{all;all}}$$

be the resulting $(\infty, 2)$-category of correspondences.

5.1.2. Consider also the category

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}},$$

and the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

constructed in [Chapter II.2, Theorem 2.1.4].

We restrict it along

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{laft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}},$$

and obtain a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}\,.$$

Note that by [Chapter V.1, Theorem 4.1.3], this restriction does not lose any information.

We wish to extend the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\text{all;all}}}$ to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

along the tautological functor

$$\mathrm{Sch}_{\mathrm{laft}} \to \mathrm{indinfSch}_{\mathrm{laft}}.$$

We will apply [Chapter V.2, Theorem 1.1.9] to obtain this extension. In the present context, the conditions of [Chapter V.2, Sect. 1.1.6] are satisfied for the following reasons:

Condition (1) is satisfied by Proposition 4.5.2;

Condition (2) is satisfied by by Theorem 4.3.3;

Condition (3) is satisfied by by Proposition 3.2.4;

Condition (4) is satisfied by Proposition 4.1.3.

Condition (*) is satisfied by Corollary 4.1.5.

5.1.3. Applying [Chapter V.2, Theorem 1.1.9], we obtain:

Theorem 5.1.4. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

whose restriction along

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\text{all;all}} \to \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}}$$

identifies with $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\text{all;all}}}$.

Moreover, the restrictions of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}}}$ *to*

$$(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}} \text{ and } \mathrm{indinfSch}_{\mathrm{laft}}$$

identify, respectively, with

$$\mathrm{IndCoh}^{!}_{\mathrm{indinfSch}_{\mathrm{laft}}} \text{ and } \mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}}.$$

5.2. **Adding adjunctions for ind-proper morphsisms.** In this subsection we will extend the functor from the previous subsection, where we include adjunctions for ind-proper maps.

5.2.1. Consider now the $(\infty, 2)$-category

$$\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}},$$

where we enlarge the class of 2-morphisms to that of ind-proper maps.

Consider the 2-fully faithful functor

$$\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{nil-closed}}_{\text{all;all}} \to \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}}.$$

We are going to prove:

Theorem 5.2.2. *There exists a unique extension of the functor* $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{proper}}_{\text{all;all}}}$ *to a functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}.$$

As a formal corollary, using [Chapter V.1, Theorem 3.2.2], we obtain:

Corollary 5.2.3. *The functors*

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}} := \mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}}|_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}}$$

and

$$\mathrm{IndCoh}^!_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}} := \mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{laft}}}|_{((\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}})^{\mathrm{op}}}$$

are obtained from each other by passing to adjoints.

5.2.4. Let us explain the concrete content of Theorem 5.2.2 and Corollary 5.2.3.

First, Corollary 5.2.3 says that if $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an ind-proper map between ind-inf-schemes, then the functor f_*^{IndCoh} is the left adjoint of $f^!$.

Next, let

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

be a Cartesian diagram in $\mathrm{indinfSch}_{\mathrm{laft}}$.

Theorem 5.1.4 says that we have a *canonical isomorphism*

$$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!. \tag{5.1}$$

If f is ind-proper, then the morphism $\leftarrow$ in (5.1) is obtained by adjunction from the (iso)morphism

$$(f')^! \circ g_2^! \simeq g_1^! \circ f^!.$$

If g_2 is ind-proper, then the morphism $\to$ in (5.1) is obtained by adjunction from the (iso)morphism

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}}.$$

In particular, a generalization of Proposition 4.5.2 holds with 'nil-closed' replaced by 'ind-proper'.

If the ind-inf-schemes in the above diagram are schemes, then the isomorphism (5.1) equals one defined a priori in this case by [Chapter II.2, Corollary 3.1.4].

5.2.5. Note that by combining Corollary 5.2.3 with Lemma 1.4.4, we obtain:

Corollary 5.2.6.

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \to \mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}}$$

is an isomorphism.

5.3. **Proof of Theorem 5.2.2.**

5.3.1. *The case of ind-schemes.* Consider the category $\operatorname{indSch}_{\mathrm{laft}}$ with the following three classes of morphisms

$$vert = \text{all},\ horiz = \text{all},\ adm = \text{ind-proper}\,.$$

We claim that we have the following result:

Theorem 5.3.2. *There exists a uniquely defined functor*

$$\operatorname{IndCoh}_{\operatorname{Corr}(\operatorname{indSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}}} : (\operatorname{indSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}} \to \operatorname{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

whose restriction along

$$\operatorname{Corr}(\operatorname{Sch}_{\mathrm{aft}})^{\text{proper}}_{\text{all;all}} \to \operatorname{Corr}(\operatorname{indSch}_{\mathrm{laft}})^{\text{ind-proper}}_{\text{all;all}}$$

identifies canonically with $\operatorname{IndCoh}_{\operatorname{Corr}(\operatorname{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\text{all;all}}}$.

Proof. This follows from [Chapter V.2, Theorem 1.1.9] applied to the functor

$$\operatorname{Sch}_{\mathrm{aft}} \to \operatorname{indSch}_{\mathrm{laft}}\,.$$

Here the conditions of [Chapter V.2, Sect. 1.1.6] are satisfied for the following reasons:

Condition (1) holds by Proposition 2.2.2;

Condition (2) holds by Corollary 1.3.5;

Condition (3) holds by Proposition 2.1.2;

Condition (4) holds by Proposition 1.4.2.

Condition (*) holds by by Corollary 2.1.9.

□

Remark 5.3.3. The difference between the case of ind-schemes and that of inf-schemes that the fact that the map

$$\operatorname{LKE}_{(\operatorname{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\operatorname{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}}(\operatorname{IndCoh}_{(\operatorname{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \to \operatorname{IndCoh}_{(\operatorname{indinfSch}_{\mathrm{laft}})_{\text{ind-proper}}}$$

is an isomorphism only follows a posteriori from Theorem 5.2.2, while the corresponding fact for ind-schemes, i.e., the isomorphism

$$\operatorname{LKE}_{(\operatorname{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\operatorname{indSch}_{\mathrm{laft}})_{\text{ind-proper}}}(\operatorname{IndCoh}_{(\operatorname{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \simeq \operatorname{IndCoh}_{(\operatorname{indSch}_{\mathrm{laft}})_{\text{ind-proper}}}$$

is given by Corollary 1.3.5.

5.3.4. We are going to deduce Theorem 5.2.2 from Theorem 5.1.4 by applying [Chapter V.1, Theorem 4.1.3]. Thus, we need to check that the inclusion

$$\text{nil-closed} \subset \text{ind-proper}$$

satisfies the condition of [Chapter V.1, Sect. 4.1.2].

That is, we consider a ind-proper morphism

$$f : \mathcal{X}_1 \to \mathcal{X}_2$$

of objects of $\operatorname{indinfSch}_{\mathrm{laft}}$, and the Cartesian square:

$$\begin{array}{ccc} \mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1 & \xrightarrow{p_2} & \mathcal{X}_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}_1 & \xrightarrow{f} & \mathcal{X}_2. \end{array}$$

The diagonal map

$$\Delta_{\mathcal{X}_1/\mathcal{X}_2} : \mathcal{X}_1 \to \mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1$$

is nil-closed. Hence, from the $((\Delta_{\mathcal{X}_1/\mathcal{X}_2})_*^{\mathrm{IndCoh}}, (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^!$-adjunction, we obtain a natural transformation

$$(\Delta_{\mathcal{X}_1/\mathcal{X}_2})_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^! \to \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1)}.$$

By composing, the latter natural transformation gives rise to

$$(5.2)\quad \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1)} \simeq (\mathrm{id}_{\mathcal{X}_1})_*^{\mathrm{IndCoh}} \circ (\mathrm{id}_{\mathcal{X}_1})^! \simeq$$
$$\simeq (p_1)_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^! \circ p_2^! \to (p_1)_*^{\mathrm{IndCoh}} \circ p_2^! \simeq f^! \circ f_*^{\mathrm{IndCoh}},$$

where the last isomorphism is due to the existence of the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}$, see Sect. 5.2.4.

We need to show that the natural transformation (5.2) is the unit of an adjunction. I.e., that for $\mathcal{F}_1 \in \mathrm{IndCoh}(\mathcal{X}_1)$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(\mathcal{X}_2)$, the map

$$(5.3)\quad \mathrm{Maps}(f_*^{\mathrm{IndCoh}}(\mathcal{F}_1), \mathcal{F}_2) \to \mathrm{Maps}(f^! \circ f_*^{\mathrm{IndCoh}}(\mathcal{F}_1), f^!(\mathcal{F}_2)) \simeq$$
$$\simeq \mathrm{Maps}((p_1)_*^{\mathrm{IndCoh}} \circ p_2^!(\mathcal{F}_1), f^!(\mathcal{F}_2)) \to \mathrm{Maps}(\mathcal{F}_1, f^!(\mathcal{F}_2))$$

is an isomorphism.

We note that by Theorem 5.1.4 the map (5.3) is the unit for the $(f_*^{\mathrm{IndCoh}}, f^!)$ adjunction, when f is nil-closed.

5.3.5. Note that the natural transformation (5.2) is defined for any map f which is *nil-separated*, i.e., one for which $\Delta_{\mathcal{X}_1/\mathcal{X}_2}$ is nil-closed.

Let $g : \mathcal{X}_0 \to \mathcal{X}_1$ be another nil-separated map between objects of $\mathrm{indinfSch}_{\mathrm{laft}}$. Diagram chase implies:

Lemma 5.3.6. *For $\mathcal{F}_0 \in \mathrm{IndCoh}(\mathcal{X}_0)$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(\mathcal{X}_2)$, the diagram*

$$\begin{array}{ccc}
\mathrm{Maps}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), f^!(\mathcal{F}_2)) & \xrightarrow{(5.3)} & \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2)) \\
{\scriptstyle (5.3)}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{Maps}(f_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), \mathcal{F}_2) & \xrightarrow{(5.3)} & \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2))
\end{array}$$

commutes.

5.3.7. Let us take $\mathcal{X}_0 := {}^{\mathrm{red}}\mathcal{X}_1$ and g to be the canonical embedding. By Proposition 3.1.2, it is sufficient to show that (5.3) is an isomorphism for $\mathcal{F}_1$ of the form $g_*^{\mathrm{IndCoh}}(\mathcal{F}_0)$ for $\mathcal{F}_0 \in \mathrm{IndCoh}(\mathcal{X}_0)$.

Using Lemma 5.3.6, and the fact that the map

$$\mathrm{Maps}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), f^!(\mathcal{F}_2)) \to \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2))$$

is an isomorphism in this case, since g is nil-closed, we obtain that it is sufficient to show that (5.3) is an isomorphism, when the initial map f is replaced by $f \circ g$.

Thus, in proving that (5.3) is an isomorphism, we can assume that $\mathcal{X}_1$ is a reduced ind-scheme.

5.3.8. Let us now factor f as

$$\mathcal{X}_1 \to \mathcal{X}_{3/2} \to \mathcal{X}_2,$$

where $\mathcal{X}_{3/2} := {}^{\mathrm{red}}\mathcal{X}_2$. Applying Lemma 5.3.6 again, we obtain that it is enough to show that (5.3) is an isomorphism for f replaced by $\mathcal{X}_1 \to \mathcal{X}_{3/2}$ and $\mathcal{X}_{3/2} \to \mathcal{X}_2$ separately.

For the map $\mathcal{X}_{3/2} \to \mathcal{X}_2$, the assertion follows from the fact that the map in question is nil-closed.

5.3.9. Hence, we are further reduced to the case when f is a ind-proper map between ind-schemes. However, in this case, the required isomorphism follows from Theorem 5.3.2: it follows by [Chapter V.1, Theorem 4.1.3] from the existence of the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

whose restriction to $\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}$ is isomorphic to

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{infSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}|_{\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}.$$

□

5.4. **Extending to prestacks.** In this subsection, we will finally extend the formalism to the category of correspondences that has all laft prestacks as objects.

5.4.1. Consider the category $\mathrm{PreStk}_{\mathrm{laft}}$, and the three classes of morphisms

$$\mathrm{indinfsch},\ \mathrm{all},\ \mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper},$$

where 'indinfsch' stands for the class of ind-inf-schematic morphisms, and 'ind-proper' for the class of morphisms that are ind-proper.

Consider the tautological embedding

$$\mathrm{indinfSch}_{\mathrm{laft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

It satisfies the conditions of [Chapter V.2, Theorem 6.1.5], with respect to the classes

$$(\mathrm{all}, \mathrm{all}, \mathrm{ind\text{-}proper}) \to (\mathrm{indinfsch}, \mathrm{all}, \mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}).$$

Now, consider the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

and the corresponding functor

$$\mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{laft}}} : (\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Clearly, the map

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{RKE}_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{laft}}})$$

is an isomorphism.

5.4.2. Hence, by [Chapter V.2, Theorem 6.1.5], from Theorem 5.1.4, we obtain the following theorem, which is *for us* the ultimate version of the formalism of ind-coherent sheaves:

Theorem 5.4.3. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

equipped with isomorphisms

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}|_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}} \simeq \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$$

and

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}|_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} \simeq \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}},$$

where the latter two isomorphisms are compatible in a natural sense.

5.4.4. The concrete meaning of Theorem 5.4.3 is analogous to that of Theorem 5.1.4, with the difference that we can now consider the direct image functor f_*^{IndCoh} when f is an ind-inf-schematic map

$$f : \mathcal{X}_1 \to \mathcal{X}_2,$$

with $\mathcal{X}_1, \mathcal{X}_2$ being objects of $\mathrm{PreStk}_{\mathrm{laft}}$, and not necessarily ind-inf-schemes. The functor f_*^{IndCoh} satisfies base change for Cartesian squares

$$\begin{array}{ccc} \mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

with vertical maps being ind-inf-schematic:

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}.$$

Moreover, for f ind-inf-schematic and ind-proper, the functor f_*^{IndCoh} is the left adjoint of $f^!$. In this case the base change isomorphism comes by adjunction from

$$(f')^! \circ g_2^! \simeq g_1^! \circ f^!.$$

If g_2 is ind-inf-schematic and ind-proper, the base change isomorphism comes by adjunction from

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ (g_2)_*^{\mathrm{IndCoh}}.$$

5.5. **Open embeddings.** The formalism of Theorem 5.4.3 contains the $(f_*^{\mathrm{IndCoh}}, f^!)$ adjunction for f proper.

However, it does not explicitly contain the $(f^!, f_*^{\mathrm{IndCoh}})$-adjunction for f which is an open embedding. In this subsection we will show that the latter follows automatically.

5.5.1. Let $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}$ denote the restriction of the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}$$

to

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}.$$

We regard it as a functor of $(\infty, 1)$-categories

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Consider the $(\infty, 2)$-category $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}}$.

5.5.2. We claim:

Proposition 5.5.3. *There exists a unique extension of* $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}$ *to a functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}.$$

Proof. We start with the three classes of 1-morphisms in $\mathrm{PreStk}_{\mathrm{laft}}$

$$\mathrm{indinfsch},\ \mathrm{all},\ \mathrm{isom},$$

and enlarge it to

$$\mathrm{indinfsch},\ \mathrm{all},\ \mathrm{open}.$$

This enlargement satisfies the assumptions of [Chapter V.2, Sect. 6.1.1]. Hence, if the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}}}$ exists, then it is unique.

Furthermore, to prove the existence, it is sufficient to do so for the pair of categories

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{open;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}},$$

and the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{open;all}}} := \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}|_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})\mathrm{open;all}}.$$

To construct the sought-for functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}$$

we proceed as follows.

We start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

and we recall that by construction (see [Chapter II.2, Sect. 2.1.2]), it extends to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}.$$

Now, the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}}}$$

is obtained from $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{open;all}}}$ by [Chapter V.2, Theorem 6.1.5] for the functor

$$\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

□

6. Self-duality and multiplicative structure of IndCoh on ind-inf-schemes

In this section we will show how the formalism of IndCoh as a functor out of the category of correspondences of ind-inf-schemes defines Serre duality on ind-inf-schemes. This is parallel to [Chapter II.2, Sect. 4].

6.1. **The multiplicative structure.** In this subsection we discuss a canonical symmetric monoidal structure on IndCoh.

6.1.1. Recall that the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$$

is endowed with a symmetric monoidal structure, see [Chapter II.2, Theorem 4.1.2]. Hence, the same is true for its restriction

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}.$$

Applying [Chapter V.3, Proposition 3.3.3], we obtain:

Corollary 6.1.2. *The functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$$

carries a unique symmetric monoidal structure extending one on $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}$.

Applying [Chapter V.3, Proposition 3.1.2], from Corollary 6.1.2, we obtain:

Corollary 6.1.3. *The functor*

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$$

carries a unique symmetric monoidal structure extending one on $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$.

6.2. **Duality.** In this subsection we show that the symmetric monoidal structure on IndCoh gives rise to Serre duality. The idea is that an ind-inf-scheme $\mathcal{X}$ is canonically self-dual as an object of the category of correspondences equipped with its natural monoidal structure.

6.2.1. By restricting the functor $\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$ to

$$\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}} \subset \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}},$$

we obtain a symmetric monoidal structure on the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

As in [Chapter II.2, Theorem 4.2.5], we deduce:

Theorem 6.2.2. *We have a commutative diagram of functors*

$$\begin{array}{ccc}
(\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}})^{\mathrm{op}} & \xrightarrow{\left(\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}}\right)^{\mathrm{op}}} & \left(\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\right)^{\mathrm{op}} \\
\varpi \downarrow & & \downarrow \text{dualization} \\
\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}} & \xrightarrow{\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}}} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}.
\end{array}$$

As in [Chapter II.2, Sect. 4.2.2], the functor ϖ is the natural anti-equivalence on the category $\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}$ corresponding to interchanging the roles of vertical and horizontal arrows. The right vertical arrow is the functor of passage to the dual category.

6.2.3. Let us explain the concrete meaning of Theorem 6.2.2. This is parallel to [Chapter II.2, Sect. 4.2.6].

For an individual object $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ it says that there is a natural self-duality data on the category $\mathrm{IndCoh}(\mathcal{X})$, i.e.,

$$\mathbf{D}^{\mathrm{Serre}}_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X})^{\vee} \simeq \mathrm{IndCoh}(\mathcal{X}). \tag{6.1}$$

Furthermore, for a map $f : \mathcal{X}_1 \to \mathcal{X}_2$, there is a canonical identification

$$f^! \simeq (f^{\mathrm{IndCoh}}_*)^{\vee}. \tag{6.2}$$

6.2.4. Below we shall write down explicitly the unit and counit functors

$$\epsilon_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect} \text{ and } \mu_{\mathcal{X}} : \mathrm{Vect} \to \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X})$$

that define the identification (6.1).

Remark 6.2.5. We observe that the fact that the functors $\epsilon_{\mathcal{X}}$ and $\mu_{\mathcal{X}}$ do indeed define an isomorphism (6.1) is easy to check directly. I.e., this does not require the full statement of Theorem 6.2.2.

6.2.6. The pairing

$$\epsilon_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect}$$

is the composition

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\Delta^!} \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{(p_{\mathcal{X}})_*^{\mathrm{IndCoh}}} \mathrm{Vect}\,.$$

Here $p_{\mathcal{X}}$ is the map $\mathcal{X} \to \mathrm{pt}$, so $(p_{\mathcal{X}})_*^{\mathrm{IndCoh}} \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)$. The first map is an isomorphism due to the fact that $\mathrm{IndCoh}(\mathcal{X})$ is dualizable as a DG category.

The unit functor

$$\mu_{\mathcal{X}} : \mathrm{Vect} \to \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X})$$

is the composition

$$\mathrm{Vect} \xrightarrow{p_{\mathcal{X}}^!} \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\Delta_*^{\mathrm{IndCoh}}} \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}).$$

One can explicitly verify that $(\epsilon_{\mathcal{X}}, \mu_{\mathcal{X}})$ specified above define an identification

$$\mathrm{IndCoh}(\mathcal{X})^{\vee} \simeq \mathrm{IndCoh}(\mathcal{X})$$

by calculating the composition

$$\mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\mathrm{Id} \otimes \mu_{\mathcal{X}}} \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\epsilon_{\mathcal{X}} \otimes \mathrm{Id}} \mathrm{IndCoh}(\mathcal{X}).$$

Indeed, it can be calculated via the commutative diagram

$$\begin{array}{ccccc}
\mathrm{IndCoh}(\mathcal{X}) & \xleftarrow{\Delta^!} & \mathrm{IndCoh}(\mathcal{X}\times\mathcal{X}) & \xleftarrow{p_1^!} & \mathrm{IndCoh}(\mathcal{X}) \\
{\scriptstyle \Delta_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}\times\Delta)_*^{\mathrm{IndCoh}}} & & \\
\mathrm{IndCoh}(\mathcal{X}\times\mathcal{X}) & \xleftarrow[(\Delta\times\mathrm{id})^!]{} & \mathrm{IndCoh}(\mathcal{X}\times\mathcal{X}\times\mathcal{X}) & & \\
{\scriptstyle (p_2)_*^{\mathrm{IndCoh}}}\downarrow & & & & \\
\mathrm{IndCoh}(\mathcal{X}), & & & &
\end{array}$$

and the base chase isomorphism (5.1) isomorphs it with the identity functor. The other composition is calculated in the same way by symmetry.

6.2.7. For the sake of completeness, let us explicitly perform the calculation that defines an identification (6.2).

We can think of both functors as given by objects of

$$\mathrm{IndCoh}(\mathcal{X}_1) \otimes \mathrm{IndCoh}(\mathcal{X}_2) \simeq \mathrm{IndCoh}(\mathcal{X}_1 \times \mathcal{X}_2)$$

and diagram chase shows that both are given by the object

$$(\Gamma_f)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}_1}),$$

where $\Gamma_f : \mathcal{X}_1 \to \mathcal{X}_1 \times \mathcal{X}_2$ is the graph of f, and

$$\omega_{\mathcal{X}_1} := p^!_{\mathcal{X}_1}(k).$$

6.2.8. The datum of self-duality

$$\mathbf{D}^{\mathrm{Serre}}_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X})$$

is equivalent to that of an equivalence

$$(\mathrm{IndCoh}(\mathcal{X})^c)^{\mathrm{op}} \to \mathrm{IndCoh}(\mathcal{X})^c.$$

We shall refer to the above functor as 'Serre duality' on $\mathcal{X}$, and denote it by $\mathbb{D}^{\mathrm{Serre}}_{\mathcal{X}}$.

From Theorem 5.1.4, isomorphism (6.2) and [Chapter I.1, Proposition 7.3.5], we obtain:

Corollary 6.2.9. *For an ind-proper map $f : \mathcal{X}_1 \to \mathcal{X}_2$ of ind-inf-schemes, we have a commutative diagram*

$$\begin{array}{ccc} (\mathrm{IndCoh}(\mathcal{X}_1)^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Serre}}_{\mathcal{X}_1}} & \mathrm{IndCoh}(\mathcal{X}_1)^c \\ {\scriptstyle (f^{\mathrm{IndCoh}}_*)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle f^{\mathrm{IndCoh}}_*} \\ (\mathrm{IndCoh}(\mathcal{X}_2)^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Serre}}_{\mathcal{X}_2}} & \mathrm{IndCoh}(\mathcal{X}_2)^c. \end{array}$$

6.3. Convolution categories and algebras.

6.3.1. As in [Chapter II.2, Sect. 4.1.5], from Corollary 6.1.3 we obtain that the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}$$

also carries a canonical right-lax symmetric monoidal structure.

6.3.2. This allows to extend the formalism in [Chapter II.2, Sect. 5] by replacing

–the class of schematic quasi-compact maps by the class of ind-inf-schematic maps;

-the class of schematic and proper maps by the class of maps that are ind-inf-schematic and ind-proper.