PART II.1. IND-COHERENT SHEAVES ON SCHEMES

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Introduction
1. IND-COHERENT SHEAVES ON INFSCHMES

In this section we introduce the category IndCoh($X$) and study its basic properties.

1.1. Definition of the category. In this subsection we define IndCoh($X$) and the functors that connect it to the usual category QCoh($X$) of quasi-coherent sheaves.

1.1.1. For $X \in \text{Sch}_{\text{aff}}$ we consider the category QCoh($X$) and its full (but not cocomplete) subcategory Coh($X$), consisting of bounded complexes with coherent cohomologies.

We define the category IndCoh($X$) by

\[ \text{IndCoh} (X) := \text{Ind}(\text{Coh}(X)). \]

1.1.2. By construction, we have a naturally defined functor

\[ \Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X) \]

obtained by ind-extension of the tautological inclusion Coh($X$) $\hookrightarrow$ QCoh($X$).

We have:

**Lemma 1.1.3.** Assume that $X$ is a smooth classical scheme. Then $\Psi_X$ is an equivalence.

**Proof.** It is known by [TT] (see also [Ne]) that for $X$ which is a quasi-separated and quasi-compact classical scheme, QCoh($X$) $\simeq$ Ind(QCoh($X$)$_{\text{perf}}$). Now, for $X$ a regular classical scheme, we have

\[ \text{QCoh}(X)_{\text{perf}} = \text{Coh}(X), \]

as subcategories of QCoh($X$).

**Remark 1.1.4.** It is shown in [GL:IndCoh, Proposition 1.5.4] that the assertion of the above lemma is in fact “if and only if”.

1.1.5. The monoidal action of QCoh. We note:

**Proposition 1.1.6.** There exists a uniquely defined monoidal action of QCoh($X$), viewed as a monoidal category, on IndCoh($X$), such that the functor $\Psi_X$ is compactible with the QCoh($X$)-actions.

**Proof.** The action in question is obtained by ind-extension of the action of the non-cocomplete monoidal category QCoh($X$)$_{\text{perf}}$ on Coh($X$).

1.1.7. We give the following definition:

**Definition 1.1.8.** We shall say that $X \in \text{Sch}_{\text{aff}}$ is eventually coconnective if the structure sheaf $\mathcal{O}_X$ belongs to Coh($X$).

I.e., $X$ is eventually coconnective if, Zariski locally, the structure sheaf had non-zero cohomologies in finitely many degrees.

We have:

**Lemma 1.1.9.** If $X$ is eventually coconnective, the functor $\Psi_X$ admits a left adjoint, to be denoted $\Xi_X$, and this left adjoint is fully faithful.
Proof. If $X$ is eventually coconnective, we have
$$\text{QCoh}(X)_{\text{perf}} \subset \text{Coh}(X),$$
and, using the fact that $\text{QCoh}(X) \simeq \text{Ind}(\text{QCoh}(X)_{\text{perf}})$, the functor $\Xi_X$ is obtained as the ind-extension of the above embedding.

The composition $\Psi_X \circ \Xi_X$ is the ind-extension of the functor
$$\text{QCoh}(X)_{\text{perf}} \hookrightarrow \text{Coh}(X) \hookrightarrow \text{QCoh}(X),$$
and it is manifest that its map to $\text{Id}_{\text{QCoh}(X)}$ is an isomorphism. \qed

Remark 1.1.10. In [GL:IndCoh, Proposition 1.5.2] it is shown that $\Psi_X$ admits a left adjoint if and only if $X$ is eventually coconnective.

1.2. $t$-structure. Some of the most basic operations on the IndCoh category (such as the functor of direct image studied in the next section) are inherited from those on QCoh using the $t$-structures on both categories. The crucial fact is that the eventually coconnective (a.k.a., bounded below) parts of the two categories are equivalent.

The goal of this subsection is to define the $t$-structure on $\text{IndCoh}(X)$ and establish its basic properties.

1.2.1. We claim:

Proposition 1.2.2. The category $\text{IndCoh}(X)$ carries a unique $t$-structure that satisfies
$$\text{IndCoh}(X)_{\leq 0} = \{ \mathcal{F} \in \text{IndCoh}(X) \mid \Psi_X(\mathcal{F}) \in \text{QCoh}(X)_{\leq 0} \}.$$

Moreover:
(a) The functor $\Psi_X$ is $t$-exact.
(b) This $t$-structure is compatible with filtered colimits (i.e., the subcategory $\text{IndCoh}(X)_{\geq 0}$ is closed under filtered colimits).
(c) The induced functor
$$\Psi_X : \text{IndCoh}(X)_{\geq n} \to \text{QCoh}(X)_{\geq n}$$
is an equivalence for any $n$.

As a corollary we obtain:

Corollary 1.2.3. The functor $\Psi_X$ defines an equivalence $\text{IndCoh}(X)^+ \to \text{QCoh}(X)^+$.

Proof of Proposition 1.2.2. It is clear that the condition of the proposition determines the $t$-structure uniquely. To establish its properties we will use the following general assertion:

Lemma 1.2.4. Let $C_0$ be a (non-cocomplete) DG category, endowed with a $t$-structure. Then $C := \text{IndCoh}(C_0)$ carries a unique $t$-structure, which is compatible with filtered colimits, and for which the tautological inclusion $C_0 \hookrightarrow C$ is $t$-exact. Moreover:
(1) The subcategory $C_{\leq 0}$ (resp., $C_{\geq 0}$) is compactly generated under filtered colimits by $C_{0_{\leq 0}}$ (resp., $C_{0_{\geq 0}}$).
(2) Let $D$ be another DG category endowed with a $t$-structure which is compatible with filtered colimits, and let $F : C \to D$ a continuous functor. Then $F$ is $t$-exact (resp., left $t$-exact, right $t$-exact) if and only if $F|_{C_0}$ is.
We apply Lemma 1.2.4(1) to $C_0 = \text{Coh}(X)$ and obtain a (a priori different) t-structure on $\text{IndCoh}(X)$, which satisfies point (b) of the proposition. It also satisfies point (a) of the proposition, by Lemma 1.2.4(2) applied to $D = \text{QCoh}(X)$ and $F = \Psi_X$.

To show that this t-structure coincides with the one introduced earlier, it suffices to show that $\Psi_X$ is conservative when restricted to $\text{IndCoh}(X)^{\geq 0}$. Hence, it remains to show that the t-structure, given by Lemma 1.2.4, satisfies points (c) of the proposition.

To prove point (c), it is sufficient to consider the case of $n = 0$. Using Lemma 1.2.4(1), the required assertion follows from the next statement:

**Lemma 1.2.5.** The category $\text{QCoh}(X)^{\geq 0}$ is compactly generated under filtered colimits by $\text{Coh}(X)^{\geq 0}$.

1.2.6. Note that Proposition 1.2.2 implies that, as long as the functor $\Psi_X$ is not an equivalence (i.e., $X$ is not a classical smooth scheme), the category $\text{IndCoh}(X)$ is not left-complete in its t-structure. The latter means that for $F \in \text{IndCoh}(X)$, the canonical arrow

$$X \to \lim_n \tau^{\geq -n}(X)$$

is not necessarily an isomorphism.

Furthermore, we see that for any $X$, the functor $\Psi_X$ realizes $\text{QCoh}(X)$ as the left completion of $\text{IndCoh}(X)$.

1.2.7. From Proposition 1.2.2 we also obtain the following:

**Corollary 1.2.8.** The inclusion $\text{Coh}(X) \subset \text{IndCoh}(X)^c$ is an equality.

**Proof.** Since the category $\text{Coh}(X)$ compactly generates $\text{IndCoh}(X)$, the category $\text{IndCoh}(X)^c$ is the Karoubi-completion of $\text{Coh}(X)$. I.e., every object $F \in \text{IndCoh}(X)^c$ can be realized as a direct summand of an object $F' \in \text{Coh}(X)$. In particular, $F \in \text{IndCoh}(X)^+$.

The object $\Psi_X(F)$ is a direct summand of $\Psi_X(F')$. Hence, $\Psi_X(F)$, regarded as an object of $\text{QCoh}(X)$, belongs to $\text{Coh}(X)$. Let us denote this object of $\text{Coh}(X)$ by $\tilde{F}$.

Thus, we can regard $F$ and $\tilde{F}$ as objects of $\text{IndCoh}(X)^+$ such that

$$\Psi_X(F) \simeq \Psi_X(\tilde{F}).$$

Applying Proposition 1.2.2(c), we obtain that $F \simeq \tilde{F}$ as objects of $\text{IndCoh}(X)$.

\[\square\]

2. THE DIRECT IMAGE FUNCTOR

The assignment

$$X \mapsto \text{IndCoh}(X)$$

is “very functorial”. However, all of this functoriality is born from a single course: the operation of direct image, defined in this section.

2.1. Direct image for an individual morphism. In this subsection we perform the first step in developing the formalism of direct image for $\text{IndCoh}$: we define the corresponding functor for one given morphism between schemes.
2.1.1. Let \( f: X \to Y \) be a morphism in \( \text{Sch}_{	ext{aff}} \). We claim:

**Proposition 2.1.2.** There exists a uniquely defined functor
\[
f^\text{IndCoh}_* : \text{IndCoh}(X) \to \text{IndCoh}(Y),
\]
which is left t-exact, and makes the diagram
\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{\Psi_X} & \text{QCoh}(X) \\
\downarrow f^\text{IndCoh}_* & & \downarrow f_* \\
\text{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \text{QCoh}(Y)
\end{array}
\]
commute.

**Proof.** By continuity, the functor \( f^\text{IndCoh}_* \) is the ind-extension of its restriction to \( \text{Coh}(X) \subset \text{IndCoh}(X) \).

The commutative diagram in the proposition implies that
\[
\Psi_Y \circ f^\text{IndCoh}_*|_{\text{Coh}(X)} = f_*|_{\text{Coh}(X)},
\]
as functors \( \text{Coh}(X) \to \text{QCoh}(Y) \). Furthermore, \( f^\text{IndCoh}_*|_{\text{Coh}(X)} \) is a functor that takes values in \( \text{QCoh}(Y)^+ \).

Note that \( \Psi_Y|_{\text{QCoh}(Y)^+} \) is invertible by Proposition 1.2.2(c). Hence, \( f^\text{IndCoh}_*|_{\text{Coh}(X)} \) is recovered as
\[
(\Psi_Y|_{\text{QCoh}(Y)^+})^{-1} \circ (f_*|_{\text{Coh}(X)}).
\]

\[\square\]

2.1.3. Recall (see [GL:QCoh, Proposition 2.3.6]) that for \( X \in \text{Sch}_{	ext{aff}} \), the monoidal category \( \text{QCoh}(X) \) is rigid. Hence, by [GL:DG, Sect. 6], for a morphism \( f: X \to Y \), the functor
\[
f_* : \text{QCoh}(X) \to \text{QCoh}(Y)
\]
has a canonical structure of morphism in \( \text{QCoh}(Y)^\text{-mod} \), where \( \text{QCoh}(Y) \) acts on \( \text{QCoh}(X) \) via \( f^* \).

Combining the arguments proving Proposition 2.1.2 and Proposition 1.1.6, we prove the following:

**Proposition 2.1.4.** For a morphism \( f: X \to Y \), the functor
\[
f^\text{IndCoh}_* : \text{IndCoh}(X) \to \text{IndCoh}(Y)
\]
has a unique structure of 1-morphism in \( \text{QCoh}(Y)^\text{-mod} \) which makes the square
\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{\Psi_X} & \text{QCoh}(X) \\
\downarrow f^\text{IndCoh}_* & & \downarrow f_* \\
\text{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \text{QCoh}(Y)
\end{array}
\]
commute in \( \text{QCoh}(Y)^\text{-mod} \).
2.2. **Upgrading to a functor.** We now claim that the assignment
\[ X \mapsto \text{IndCoh}(X), \quad f \mapsto f^\text{IndCoh} \]
upgrades to a functor
\[ \text{Sch}_{\text{aff}} \to \text{DGCat}^{\text{cont}}, \]
(2.1)

to be denoted \( \text{IndCoh}_{\text{Sch}_{\text{aff}}} \).

Such an extension is not altogether automatic because we live in the world of higher categories. But constructing it will not be very difficult.

2.2.1. First, we recall the functor
\[ \text{QCoh}^*_{\text{Sch}_{\text{aff}}} : (\text{Sch}_{\text{aff}})^{\text{op}} \to \text{DGCat}^{\text{cont}}, \]
see [GL:QCoh, ???].

Applying [Funct, Sect. 11.2], we obtain a functor
\[ \text{QCoh}^*_{\text{Sch}_{\text{aff}}} : \text{Sch}_{\text{aff}} \to \text{DGCat}^{\text{cont}}, \]
obtained from \( \text{QCoh}^*_{\text{Sch}_{\text{aff}}} \) by passage to right adjoints.

2.2.2. Now, we claim:

**Proposition 2.2.3.** There exists a uniquely defined functor
\[ \text{IndCoh}_{\text{Sch}_{\text{aff}}} : \text{Sch}_{\text{aff}} \to \text{DGCat}^{\text{cont}}, \]
equipped with a natural transformation
\[ \Psi_{\text{Sch}_{\text{aff}}} : \text{IndCoh}_{\text{Sch}_{\text{aff}}} \to \text{QCoh}_{\text{Sch}_{\text{aff}}}, \]
which at the level of objects and 1-morphisms is given by the assignment
\[ X \mapsto \text{IndCoh}(X), \quad f \mapsto f^\text{IndCoh}. \]

The rest of this subsection is devoted to the proof of this proposition.

2.2.4. Consider the following \((\infty, 1)\)-categories:

- \( \text{DGCat}^{+\text{cont}} \) and \( \text{DGCat}^{t\text{cont}} \):

The category \( \text{DGCat}^{+\text{cont}} \) consists of non-cocomplete DG categories \( \mathbf{C} \), endowed with a t-structure, such that \( \mathbf{C} = \mathbf{C}^+ \). We also require that \( \mathbf{C}^{\geq 0} \) contain filtered colimits and that the embedding \( \mathbf{C}^{\geq 0} \hookrightarrow \mathbf{C} \) commute with filtered colimits. As 1-morphisms we take those exact functors \( F : \mathbf{C}_1 \to \mathbf{C}_2 \) that are left t-exact up to a finite shift, and such that \( F|_{\mathbf{C}_1^{\geq 0}} \) commutes with filtered colimits. The higher categorical structure is uniquely determined by the requirement that the forgetful functor
\[ \text{DGCat}^{+\text{cont}} \to \text{DGCat} \]
be 1-fully faithful.

The category \( \text{DGCat}^{t\text{cont}} \) consists of cocomplete DG categories \( \mathbf{C} \), endowed with a t-structure, such that \( \mathbf{C}^{\geq 0} \) is closed under filtered colimits, and such that \( \mathbf{C} \) is compactly generated by objects from \( \mathbf{C}^+ \). As 1-morphisms we allow those exact functors \( F : \mathbf{C}_1 \to \mathbf{C}_2 \) that are continuous and left t-exact up to a finite shift. The higher categorical structure is uniquely determined by the requirement that the forgetful functor
\[ \text{DGCat}^{t\text{cont}} \to \text{DGCat}_{\text{cont}} \]
be 1-fully faithful.
We have a naturally defined functor
\[ \text{DGCat}_\text{cont} \to \text{DGCat}_\text{cont}^{+}, \quad C \mapsto C^{+}. \]

**Lemma 2.2.5.** The functor (2.2) is 1-fully faithfull.

2.2.6. We will use the following general assertion. Let \( T : \mathcal{D}' \to \mathcal{D} \) be a functor between \((\infty, 1)\)-categories, which is 1-fully faithful. Let \( \mathcal{I} \) be another \((\infty, 1)\)-category, and let
\[ i \mapsto F'(i), \]
be an assignment, such that the assignment
\[ i \mapsto T \circ F'(i) \]
has been extended to a functor \( F : \mathcal{I} \to \mathcal{D} \).

**Lemma 2.2.7.** Suppose that for every \( \alpha \in \text{Maps}_\mathcal{I}(i_1, i_2) \), the point \( F(\alpha) \in \text{Maps}_\mathcal{D}(F(i_1), F(i_2)) \) lies in the connected component corresponding to the image of
\[ \text{Maps}_\mathcal{D}(F'(i_1), F'(i_2)) \to \text{Maps}_\mathcal{D}(F(i_1), F(i_2)). \]
Then there exists a unique extension of (2.3) to a functor \( F' : \mathcal{I} \to \mathcal{D} \) equipped with an isomorphism \( T \circ F' \simeq F \).

Let now \( F'_1 \) and \( F'_2 \) be two assignments as in (2.3), satisfying the assumption of Lemma 2.2.7. Let us be given an assignment
\[ i \mapsto \psi'_i \in \text{Maps}_\mathcal{D}(F'_1(i), F'_2(i)). \]

**Lemma 2.2.8.** Suppose that the assignment
\[ i \mapsto T(\psi'_i) \in \text{Maps}_\mathcal{D}(F_1(i), F_2(i)) \]
has been extended to a natural transformation \( \psi : F_1 \to F_2 \). Then there exists a unique extension of (2.4) to a natural transformation \( \psi' : F'_1 \to F'_2 \) equipped with an isomorphism \( T \circ \psi' \simeq \psi \).

2.2.9. We are now ready to prove Proposition 2.2.3:

**Step 1.** We start with the functor
\[ \text{QCoh}_{\text{Sch}_\text{aff}} : \text{Sch}_\text{aff} \to \text{DGCat}_{\text{cont}}, \]
and consider
\[ \mathcal{I} = \text{Sch}_\text{aff}, \quad \mathcal{D} = \text{DGCat}_{\text{cont}}, \quad \mathcal{D}' = \text{DGCat}_{\text{cont}}^{+}, \quad F = \text{QCoh}_{\text{Sch}_\text{aff}}, \]
and the assignment
\[ (X \in \text{Sch}_\text{aff}) \mapsto (\text{QCoh}(X) \in \text{DGCat}_{\text{cont}}^{+}). \]

Applying Lemma 2.2.7, we obtain a functor
\[ \text{QCoh}_{\text{Sch}_\text{aff}}^+ : \text{Sch}_\text{aff} \to \text{DGCat}_{\text{cont}}^{+}. \]

**Step 2.** Note that Proposition 2.1.2 defines a functor
\[ \text{IndCoh}_{\text{Sch}_\text{aff}}^+ : \text{Sch}_\text{aff} \to \text{DGCat}_{\text{cont}}^{+}, \]
and the natural transformation
\[ \Psi_{\text{Sch}_\text{aff}} : \text{IndCoh}_{\text{Sch}_\text{aff}}^+ \to \text{QCoh}_{\text{Sch}_\text{aff}}^+ \]
at the level of objects and 1-morphisms.
Since the functor $DGCat_{\text{cont}} \to DGCat_{\text{cont}}$ is 1-fully faithful, by Lemmas 2.2.7 and 2.2.8, the existence and uniqueness of the pair $(\text{IndCoh}_{\text{Sch}^{\text{aff}}}, \Psi_{\text{Sch}^{\text{aff}}})$ with a fixed behavior on objects and 1-morphisms, is equivalent to that of $(\text{IndCoh}^{+}_{\text{Sch}^{\text{aff}}}, \Psi^{+}_{\text{Sch}^{\text{aff}}})$.

**Step 3.** By Lemma 2.2.5, combined with Lemmas 2.2.7 and 2.2.8, we obtain that the existence and uniqueness of the pair $(\text{IndCoh}^{+}_{\text{Sch}^{\text{aff}}}, \Psi^{+}_{\text{Sch}^{\text{aff}}})$, with a fixed behavior on objects and 1-morphisms is equivalent to the existence and uniqueness of the pair

$$(\text{IndCoh}^{+}_{\text{Sch}^{\text{aff}}}, \Psi^{+}_{\text{Sch}^{\text{aff}}}),$$

obtained by composing with the functor (2.2).

The latter, however, is given by

$$\text{IndCoh}^{+}_{\text{Sch}^{\text{aff}}} := \text{QCoh}^{+}_{\text{Sch}^{\text{aff}}} \text{ and } \Psi^{+}_{\text{Sch}^{\text{aff}}} := \text{Id}.$$

□

3. The functor of “usual” inverse image

We now construct another piece of functoriality in the assignment $X \mapsto \text{IndCoh}(X)$, namely, the functor of $\ast$-pullback.

Unlike the case of QCoh, its role in the theory is rather limited—a “more important” functor is that of $!$-pullback. However, the $\ast$-pullback is a necessary step in the construction of the $!$-pullback, which is why we discuss it.

3.1. Inverse image with respect to eventually coconnective morphisms. Unlike the case of QCoh, the functor of $\ast$-pullback on IndCoh is not defined for all maps of schemes, but only for *eventually coconnective* ones. In this subsection we give the corresponding construction.

3.1.1. Let $f : X \to Y$ be a morphism in $\text{Sch}^{\text{aff}}$.

**Definition 3.1.2.** We shall say that $f$ is eventually coconnective if the functor

$$f^{\ast} : \text{QCoh}(Y) \to \text{QCoh}(X)$$

sends $\text{Coh}(Y) \subset \text{QCoh}(Y)$ to $\text{QCoh}(X)^{+}$.

It is easy to see that if $f$ is eventually coconnective, then it sends $\text{Coh}(Y)$ to $\text{Coh}(X)$: indeed, for any morphism $f$, the functor $f^{\ast}$ sends objects of $\text{QCoh}(Y)^{-}$ with coherent cohomologies to objects with a similar property on $X$.

In addition, we have the following fact, established in [GL:IndCoh, Lemma 3.4.2]:

**Lemma 3.1.3.** The following conditions are equivalent:

(a) $f$ is eventually coconnective;

(b) $f$ is of finite Tor amplitude, i.e., is left t-exact up to a finite cohomological shift.

**Corollary 3.1.4.** The class of eventually coconnective morphisms is stable under base change.
3.1.5. Let \( f : X \to Y \) be eventually coconnective. Ind-extending the functor
\[
f^* : \text{Coh}(Y) \to \text{Coh}(X)
\]
we obtain a functor
\[
f^{\text{IndCoh}*} : \text{IndCoh}(Y) \to \text{IndCoh}(X),
\]
which makes the diagram
\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{\Psi_X} & \text{QCoh}(X) \\
\uparrow f^{\text{IndCoh}*} & & \uparrow f^* \\
\text{IndCoh}(Y) & \xrightarrow{\Psi_Y} & \text{QCoh}(Y)
\end{array}
\]
commute.

We have:

**Proposition 3.1.6.** The functor \( f^{\text{IndCoh}*} \) is a left adjoint to \( f_*^{\text{IndCoh}} \).

**Proof.** It is sufficient to construct a functorial isomorphism
\[
\text{Maps}_{\text{IndCoh}(X)}(f^{\text{IndCoh}*}(\mathcal{F}_Y), \mathcal{F}_X) \cong \text{Maps}_{\text{IndCoh}(Y)}(\mathcal{F}_Y, f_*^{\text{IndCoh}}(\mathcal{F}_X))
\]
for \( \mathcal{F}_Y \in \text{Coh}(Y) \) and \( \mathcal{F}_X \in \text{Coh}(X) \). However, by construction, the left-hand side in (3.1) is
\[
\text{Maps}_{\text{Coh}(X)}(f^*(\mathcal{F}_Y), \mathcal{F}_X) \cong \text{Maps}_{\text{QCoh}(X)}(f^*(\mathcal{F}_Y), \mathcal{F}_X),
\]
while the right-hand side maps isomorphically by the functor \( \Psi_Y \) to
\[
\text{Maps}_{\text{QCoh}(Y)}(\mathcal{F}_Y, f_*(\mathcal{F}_X)).
\]
Now, (3.1) follows from the \( (f^*, f_*) \)-adjunction on QCoh. \( \square \)

**Remark 3.1.7.** It is shown in [GL:IndCoh], that the functor \( f_*^{\text{IndCoh}} \) admits a left adjoint if and only if the morphism \( f \) is eventually coconnective.

3.1.8. Note that by [GL:DG, Sect. 6, Adjoint Functors], the functor \( f^{\text{IndCoh}*} \) carries a canonical structure of morphism in \( \text{QCoh}(Y)\text{-mod} \). It is easy to see that this is the same structure as obtained by ind-extending the structure of compatibility with the action of \( \text{QCoh}(Y)\text{perf} \) on
\[
f^* : \text{Coh}(Y) \to \text{Coh}(X).
\]

3.1.9. Let \( \text{Sch}_{\text{event-coconn}} \subset \text{Sch}_{\text{aff}} \) be the 1-full subcategory, where we restrict 1-morphisms to maps that are eventually coconnective.

By [Funct, Sect. 11.2], combining Propositions 2.2.3 and 3.1.6, we obtain:

**Corollary 3.1.10.** The assignment
\[
X \rightsquigarrow \text{IndCoh}(X), \quad f \rightsquigarrow f^{\text{IndCoh}*}
\]
canonicaly extends to a functor, to be denoted \( \text{IndCoh}_{\text{(Sch}_{\text{aff}})\text{-event-coconn}}}^{(\text{Sch}_{\text{aff}})\text{-event-coconn}} : ((\text{Sch}_{\text{aff}})\text{-event-coconn})^{\text{op}} \to \text{DGCat}_{\text{cont}},
\]

obtained from
\[
\text{IndCoh}_{\text{Sch}_{\text{aff}}}((\text{Sch}_{\text{aff}})\text{-event-coconn})
\]
by adjunction.

3.2. **Base change for eventually coconnective morphisms.** An important property of the \( * \)-pullback (and one which is crucial for the construction of the !-pullback) is base change. It closely mimics the corresponding phenomenon for QCoh.
3.2.1. Let

\[
\begin{array}{c}
X_1 \xrightarrow{g_X} X_2 \\
f_1 \downarrow \downarrow f_2 \\
Y_1 \xrightarrow{g_Y} Y_2
\end{array}
\]

be a Cartesian diagram in Sch aft.

Suppose that \(f_2\) is eventually coconnective. By Corollary 3.1.4, the morphism \(f_1\) is also eventually coconnective. Then the isomorphism of functors

\[
(g_Y)_*^{\text{IndCoh}} \circ (f_1)_*^{\text{IndCoh}} \simeq (f_2)_*^{\text{IndCoh}} \circ (g_X)_*^{\text{IndCoh}}
\]

gives rise to a natural transformation.

(3.2) \((f_2)^{\text{IndCoh},*} \circ (g_Y)_*^{\text{IndCoh}} \rightarrow (g_X)_*^{\text{IndCoh}} \circ (f_1)^{\text{IndCoh},*}\).

**Proposition 3.2.2.** The map (3.2) is an isomorphism.

**Proof.** It is enough to show that

(3.3) \((f_2)^{\text{IndCoh},*} \circ (g_Y)_*^{\text{IndCoh}}(\mathcal{F}) \rightarrow (g_X)_*^{\text{IndCoh}} \circ (f_1)^{\text{IndCoh},*}(\mathcal{F})\)

is an isomorphism for \(\mathcal{F} \in \text{Coh}(Y_1)\).

In this case both sides of (3.3) belong to \(\text{IndCoh}(X_2)^+\). By Proposition 1.2.2, it is therefore sufficient to show that the map

(3.4) \(\Psi_{X_2} \circ (f_2)^{\text{IndCoh},*} \circ (g_Y)_*^{\text{IndCoh}} \rightarrow \Psi_{X_2} \circ (g_X)_*^{\text{IndCoh}} \circ (f_1)^{\text{IndCoh},*}\)

is an isomorphism.

We have:

\[
\Psi_{X_2} \circ (f_2)^{\text{IndCoh},*} \circ (g_Y)_*^{\text{IndCoh}} \simeq (f_2)^* \circ \Psi_{Y_2} \circ (g_Y)_*^{\text{IndCoh}} \simeq (f_2)^* \circ (g_Y)_* \circ \Psi_{Y_1}
\]

and

\[
\Psi_{X_2} \circ (g_X)_*^{\text{IndCoh}} \circ (f_1)^{\text{IndCoh},*} \simeq (g_X)_* \circ \Psi_{X_1} \circ (f_1)^{\text{IndCoh},*} \simeq (g_X)_* \circ (f_1)^* \circ \Psi_{Y_1}.
\]

Now, it follows from the construction of the \((f_\text{IndCoh},*, f_\text{IndCoh})\)-adjunctions, that the map in (3.4) corresponds to the map

\((f_2)^* \circ (g_Y)_* \circ \Psi_{Y_1} \rightarrow (g_X)_* \circ (f_1)^* \circ \Psi_{Y_1},\)

obtained from the \((f^*, f_*)\)-adjunctions.

Hence, (3.4) is an isomorphism by base change for QCoh.

\[\square\]

3.3. **Tensoring up.** In this section we study the following question: given a map \(f : X \rightarrow Y\), how closely can we approximate \(\text{IndCoh}(X)\) from knowing \(\text{IndCoh}(Y)\) and the QCoh categories on both schemes.

In the process we will come across several *convergence-type* assertions, that are of significant technical importance: some maps that are isomorphisms in QCoh are much less obviously so in the IndCoh context.
3.3.1. Let \( f : X \to Y \) be an eventually coconnective map. Regarding the functor \( f^{\text{IndCoh,*}} \) as a map in \( \text{Qcoh}(Y)-\text{mod} \), we obtain a functor

\[
\text{IndCoh}_{\text{IndCoh,*}} : \text{Qcoh}(X) \otimes_{\text{Qcoh}(Y) \text{IndCoh}(Y)} \to \text{IndCoh}(X).
\]

We claim:

**Proposition 3.3.2.** The functor (3.5) is fully faithful.

**Remark 3.3.3.** It is shown in [GL:IndCoh, Proposition 4.4.9] that the functor (3.5) is an equivalence when \( f \) is smooth. In Proposition 4.1.6, we will prove this in the case when \( f \) is an open embedding.

3.3.4. Note that for a diagram of \((\infty, 1)\)-categories

\[
\begin{array}{ccc}
D_1 & \xrightarrow{T} & D_2 \\
\downarrow & & \downarrow \\
C & \xrightarrow{F_1} & D_2 \\
& \downarrow & \\
& & F_2
\end{array}
\]

if the functors \( F_1 \) and \( F_2 \) admit right adjoints, we have a natural transformation of the resulting endo-functors of \( C \):

\[
F_1^R \circ F_1 \to F_2^R \circ F_2.
\]

Furthermore, if \( T \) is fully faithful, then the above natural transformation is an isomorphism.

From here we obtain that the functor \( (\text{Id}_{\text{Qcoh}(X)} \otimes f^{\text{IndCoh,*}}) \) gives rise to a map of endo-functors of \( \text{IndCoh}(Y) \):

\[
(f_*(\mathcal{O}_X) \otimes -) \simeq ((f_* \circ f^*) \otimes \text{Id}_{\text{IndCoh}(Y)}) \to f_*^{\text{IndCoh}} \circ f^{\text{IndCoh,*}},
\]

where \( f_*(\mathcal{O}_X) \otimes - \) denotes the functor of action of \( f_* \mathcal{O}_X \in \text{Qcoh}(Y) \) on \( \text{IndCoh}(Y) \).

Furthermore, from Proposition 3.3.2 we obtain:

**Corollary 3.3.5.** The map (3.6) is an isomorphism.

3.3.6. The rest of the subsection is devoted to the proof of Proposition 3.3.2. We note that the left-hand side in (3.5) is compactly generated by objects of the form

\[
\mathcal{E}_X \otimes \mathcal{F}_Y \in \text{Qcoh}(X) \otimes_{\text{Qcoh}(Y) \text{IndCoh}(Y)} \text{IndCoh}(Y),
\]

where \( \mathcal{E}_X \in \text{Qcoh}(X)_{\text{perf}} \) and \( \mathcal{F}_Y \in \text{Coh}(Y) \). Moreover, the functor \( (\text{Id}_{\text{Qcoh}(X)} \otimes f^{\text{IndCoh,*}}) \) sends these objects to compact objects in \( \text{IndCoh}(X) \).

Hence, it is enough to show that for \( \mathcal{E}_X^1, \mathcal{E}_X^2 \) and \( \mathcal{F}_Y^1, \mathcal{F}_Y^2 \) as above, the map

\[
\text{Maps}_{\text{Qcoh}(X)} \otimes_{\text{Qcoh}(Y) \text{IndCoh}(Y)} (\mathcal{E}_X^1 \otimes \mathcal{F}_Y^1, \mathcal{E}_X^2 \otimes \mathcal{F}_Y^2) \to
\]

\[
\to \text{Maps}_{\text{IndCoh}(Y)} (\mathcal{E}_X^1 \otimes f^{\text{IndCoh,*}}(\mathcal{F}_Y^1), \mathcal{E}_X^2 \otimes f^{\text{IndCoh,*}}(\mathcal{F}_Y^2))
\]

is an isomorphism, where in the right-hand side \( \otimes \) denotes the action of \( \text{Qcoh} \) on \( \text{IndCoh} \).

We can rewrite the map in (3.7) as

\[
\text{Maps}_{\text{Qcoh}(X)} \otimes_{\text{Qcoh}(Y) \text{IndCoh}(Y)} (\mathcal{O}_X \otimes \mathcal{F}_Y^1, \mathcal{E}_X \otimes \mathcal{F}_Y^2) \to
\]

\[
\to \text{Maps}_{\text{IndCoh}(X)} (\mathcal{O}_X \otimes f^{\text{IndCoh,*}}(\mathcal{F}_Y^1), \mathcal{E}_X \otimes f^{\text{IndCoh,*}}(\mathcal{F}_Y^2)),
\]
where $E_X \simeq E_X^2 \otimes (E_X^1)^\vee$.

Furthermore, we rewrite the map in (3.8) as

$$\text{Maps}_{\text{IndCoh}(Y)}(\mathcal{F}_Y^1, \mathcal{F}_Y^2) \rightarrow \text{Maps}_{\text{IndCoh}(Y)}(\mathcal{F}_Y^1, f^\text{IndCoh}_*(E \otimes f^\text{IndCoh,*}(\mathcal{F}_Y^2))).$$

I.e., we are reduced to showing that the following version of the projection formula:

**Proposition 3.3.7.** For an eventually coconnective map, the natural transformation between the functors

$$\text{QCoh}(X) \times \text{IndCoh}(Y) \to \text{IndCoh}(Y),$$

that sends $E_X \in \text{QCoh}(X)$ and $F_Y \in \text{IndCoh}(Y)$ to the map

$$(3.9) \quad f_*(E_X) \otimes F_Y \to f^\text{IndCoh}_*(E_X \otimes f^\text{IndCoh,*}(F_Y)),$$

is an isomorphism.

**Remark 3.3.8.** Note that there is another kind of projection formula, that encodes the compatibility of $f^\text{IndCoh}_*$ with the monoidal action of $\text{QCoh}(Y)$, and which holds tautologically for any morphism $f$, see Proposition 2.1.4:

For $E_Y \in \text{QCoh}(Y)$ and $F_X \in \text{IndCoh}(X)$ we have:

$$f^\text{IndCoh}_*(f^*(E_Y) \otimes F_X) \simeq E_Y \otimes f^\text{IndCoh}_*(F_X).$$

**3.3.9. Proof of Proposition 3.3.7.** It is enough to prove the isomorphism (3.9) holds for $E_X \in \text{QCoh}(X)^\text{perf}$ and $F_Y \in \text{Coh}(Y)$.

We also note that the map (3.9) becomes an isomorphism after applying the functor $\Psi_Y$, by the usual projection formula for $\text{QCoh}$. For $E_X \in \text{QCoh}(X)^\text{perf}$ and $F_Y \in \text{Coh}(Y)$ we have

$$f^\text{IndCoh}_*(E_X \otimes f^\text{IndCoh,*}(F_Y)) \in \text{IndCoh}(Y)^+. $$

Hence, by Proposition 1.2.2, it suffices to show that in this case

$$f_*(E_X) \otimes F_Y \in \text{IndCoh}(Y)^+. $$

We note that the object $f_*(E_X) \in \text{QCoh}(Y)^b$ is of bounded Tor dimension. The required fact follows from the next general observation:

**Lemma 3.3.10.** For $X \in \text{Sch}_{\text{fin}}$ and $E \in \text{QCoh}(X)^b$, whose Tor dimension is bounded on the left by an integer $n$, the functor

$$E \otimes - : \text{IndCoh}(X) \to \text{IndCoh}(X) $$

has a cohomological amplitude bounded on the left by $n$.

3.3.11. **Proof of Lemma 3.3.10.** We need to show that the functor $E \otimes -$ sends $\text{IndCoh}(X)^{\geq 0}$ to $\text{IndCoh}(X)^{\geq -n}$. By Lemma 1.2.4(1), it is sufficient to show that this functor sends $\text{Coh}(X)^{\geq 0}$ to $\text{IndCoh}(X)^{\geq -n}$. By cohomological devissage, the latter is equivalent to sending $\text{Coh}(X)^\vee$ to $\text{IndCoh}(X)^{\geq -n}$.

Let $i$ denote the closed embedding $i!X =: X' \to X$. The functor $i^\text{IndCoh}_*$ induces an equivalence $\text{Coh}(X')^\vee \to \text{Coh}(X)^\vee$. So, it is enough to show that for $F' \in \text{Coh}(X')^\vee$, we have

$$F \otimes i^\text{IndCoh}_*(F') \in \text{IndCoh}(X)^{\geq -n}.$$ 

We have:

$$E \otimes i^\text{IndCoh}_*(F') \simeq i^\text{IndCoh}_*(i^*(E) \otimes F').$$

Note that the functor $i_*^{\text{IndCoh}}$ is t-exact (since $i_*$ is), and $i^*(E)$ has Tor dimension bounded by the same integer $n$.

This reduces the assertion of the lemma to the case when $X$ is classical. Further, by Proposition 4.2.4 (which will be proved independently later), the statement is Zariski local, so we can assume that $X$ is affine.

In the latter case, the assumption on $E$ implies that it can be represented by a complex of flat $\mathcal{O}_X$-modules that lives in the cohomological degrees $\geq -n$. This reduces the assertion further to the case when $E$ is a flat $\mathcal{O}_X$-module in degree 0. In this case we claim that the functor $E \otimes - : \text{IndCoh}(X) \to \text{IndCoh}(X)$
is t-exact.

The latter follows from Lazard’s lemma: such an $E$ is a filtered colimit of locally free $\mathcal{O}_X$-modules $E'$, while for each such $E'$, the functor $E' \otimes - : \text{IndCoh}(X) \to \text{IndCoh}(X)$ is by definition the ind-extension of the functor $E' \otimes - : \text{Coh}(X) \to \text{Coh}(X)$, and the latter is t-exact.

4. Open embeddings

The behavior of direct and inverse image functors for open embeddings is, obviously, an important piece of information about $\text{IndCoh}$.

4.1. Restriction to an open. In this subsection we show that the behavior of $\text{IndCoh}$ with respect to open embeddings is “exactly the same” as that of $\text{Qcoh}$.

4.1.1. Let now $j: \tilde{X} \hookrightarrow X$ be an open embedding. We claim:

**Proposition 4.1.2.** The functor $j_*^{\text{IndCoh}} : \text{IndCoh}(\tilde{X}) \to \text{IndCoh}(X)$ is fully faithful.

**Proof.** We need to show that the co-unit of the adjunction

$$j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}} \to \text{Id}_{\text{IndCoh}(\tilde{X})}$$
is an isomorphism.

Since the functors in question are continuous, it is enough to check that

$$j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}}(\mathcal{F}) \to \mathcal{F}$$
is an isomorphism for $\mathcal{F} \in \text{Coh}(X)$. However, in this case both $j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}}(\mathcal{F})$ and $\mathcal{F}$ belong to $\text{IndCoh}(X)^+$, so by Proposition 1.2.2, it is sufficient to check that

$$\Psi_X \circ j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}} \to \Psi_X$$
is an isomorphism.

However,

$$\Psi_X \circ j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}} \simeq j_*^{\text{IndCoh}} \circ \Psi_X \circ j_*^{\text{IndCoh}} \simeq j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}} \circ \Psi_X,$$
and it follows from the construction of the $(f_*^{\text{IndCoh}}, f_*^{\text{IndCoh}})$-adjunction that the resulting map

$$j_*^{\text{IndCoh}} \circ j_*^{\text{IndCoh}} \circ \Psi_X \to \Psi_X$$
comes from the co-unit of the \((j^*, j_*)\)-adjunction. Therefore, it is an isomorphism, as
\[ j^* \circ j_* \to \text{Id}_{\text{QCoh}(X)} \]
is an isomorphism.

4.1.3. The next assertion follows immediately from Lemma 1.2.4:

**Lemma 4.1.4.** For an open embedding \(j\), the functor \(j^{\text{IndCoh},*}\) is t-exact.

4.1.5. Finally, let us recall the functor (3.5). We claim:

**Proposition 4.1.6.** Assume that \(f\) is an open embedding. Then the functor (3.5) is an equivalence.

**Proof.** We already know that the functor in question is fully faithful. Hence, it remains to show that its essential image generates the target category. But this follows from Proposition 4.1.2. \(\Box\)

4.2. **Zariski descent.** In this subsection we will show that \(\text{IndCoh}\) can be glued locally from a Zariski cover, in a way completely parallel to \(\text{QCoh}\).

4.2.1. Let now \(f : U \to X\) be a Zariski cover, i.e., \(U\) is the disjoint union of open subschemes of \(X\), whose union is all of \(X\). Let \(U^*\) be the Čech nerve of \(f\). The functors of \((\text{IndCoh},*)\)-pullback define a cosimplicial category
\[ \text{IndCoh}(U^*) , \]
which is augmented by \(\text{IndCoh}(X)\).

We claim:

**Proposition 4.2.2.** The functor
\[ \text{IndCoh}(X) \to \text{Tot}(\text{IndCoh}(U^*)) \]
is an equivalence.

**Proof.** The usual argument reduces the assertion of the proposition to the following. Let \(X = U_1 \cup U_2; U_{12} = U_1 \cap U_2\). Let
\[ U_1 \overset{j_1}{\to} X, U_2 \overset{j_2}{\to} X, U_{12} \overset{j_{12}}{\to} X, U_{12} \overset{j_{12,1}}{\to} U_1, U_{12} \overset{j_{12,2}}{\to} U_2 \]
denote the corresponding open embeddings.

We need to show that the functor
\[ \text{IndCoh}(X) \to \text{IndCoh}(U_1) \times_{\text{IndCoh}(U_{12})} \text{IndCoh}(U_1) \]
that sends \(\mathcal{F} \in \text{IndCoh}(X)\) to the datum of
\[ \{ j_1^{\text{IndCoh},*}(\mathcal{F}_1), j_2^{\text{IndCoh},*}(\mathcal{F}_2), j_{12,1}^{\text{IndCoh},*}(\mathcal{F}_1 ) \simeq j_{12,2}^{\text{IndCoh},*}(\mathcal{F}_2 ) \simeq j_{12}^{\text{IndCoh},*}(\mathcal{F} ) \} \]
is an equivalence.

We construct a right adjoint functor
\[ \text{IndCoh}(U_1) \times_{\text{IndCoh}(U_{12})} \text{IndCoh}(U_1) \to \text{IndCoh}(X) \]
by sending
\[ \{ \mathcal{F}_1 \in \text{IndCoh}(U_1), \mathcal{F}_2 \in \text{IndCoh}(U_2), \mathcal{F}_{12} \in \text{IndCoh}(U_{12}), j_{12,1}^{\text{IndCoh},*}(\mathcal{F}_1 ) \simeq \mathcal{F}_{12} \simeq j_{12,2}^{\text{IndCoh},*}(\mathcal{F}_2 ) \} \]
to

\[ \ker \left( (j_1)_* \text{IndCoh} (\mathcal{F}_1) \oplus (j_2)_* \text{IndCoh} (\mathcal{F}_1) \right) \to (j_{12})_* \text{IndCoh} (\mathcal{F}_{12}) \],

where the maps \((j_i)_* \text{IndCoh}(\mathcal{F}_i) \to (j_{12})_* \text{IndCoh}(\mathcal{F}_{12})\) are

\[ (j_i)_* \text{IndCoh}(\mathcal{F}_i) \to (j_{12})_* \text{IndCoh} \circ (j_{12,i})_* \text{IndCoh} \circ (j_{12,i})_* \text{IndCoh} \sim (j_{12})_* \text{IndCoh}(\mathcal{F}_{12}). \]

It is straightforward to see from Propositions 4.1.2 and 3.2.2 that the composition

\[ \text{IndCoh}(U_1) \times \text{IndCoh}(U_1) \to \text{IndCoh}(U_1) \times \text{IndCoh}(U_1) \]

is canonically isomorphic to the identity functor.

To prove that the composition

\[ \text{IndCoh}(X) \to \text{IndCoh}(U_1) \times \text{IndCoh}(U_1) \to \text{IndCoh}(X) \]

is also isomorphic to the identity functor, it is sufficient to show that for \( \mathcal{F} \in \text{IndCoh}(X) \), the canonical map from it to

\[ (4.1) \quad \ker \left( (j_1)_* \text{IndCoh} \circ (j_1)_* \text{IndCoh} \circ (\Psi_X(\mathcal{F})) \oplus (j_2)_* \text{IndCoh} \circ (j_2)_* \text{IndCoh} \circ (\Psi_X(\mathcal{F})) \right) \to (j_{12})_* \text{IndCoh} \circ j_{12}^{-}\text{IndCoh}(X) \]

is an isomorphism.

Since all functors in question are continuous, it is sufficient to do so for \( \mathcal{F} \in \text{Coh}(X) \). In this case, both sides of (4.1) belong to \( \text{IndCoh}(X)^+ \). So, it is enough to prove that the map in question becomes an isomorphism after applying the functor \( \Psi_X \). However, in this case we are dealing with the map

\[ \Psi_X(\mathcal{F}) \to \ker \left( (j_1)_* \circ (j_1)_* (\Psi_X(\mathcal{F})) \oplus (j_2)_* \circ (j_2)_* (\Psi_X(\mathcal{F})) \right) \to (j_{12})_* j_{12}^{-}\text{IndCoh}(X), \]

which is known to be an isomorphism. \( \square \)

4.2.3. We also have

**Proposition 4.2.4.** Let \( f : U \to X \) be a Zariski cover. Then \( \mathcal{F} \in \text{IndCoh}(X) \) belongs to \( \text{IndCoh}(X)^{\leq 0} \) (resp., \( \text{IndCoh}(X)^{\geq 0} \)) if and only if \( f^{\text{IndCoh}*}(\mathcal{F}) \) does.

**Proof.** The “only if” direction for both statements follows from Lemma 4.1.4.

For the “if” direction, assuming that \( f^*(\mathcal{F}) \in \text{IndCoh}(X)^{\leq 0} \), it is sufficient to show that \( \Psi_X(\mathcal{F}) \in \text{QCoh}(X)^{\leq 0} \), and the assertion follows from the corresponding assertion for QCoh.

If \( f^{\text{IndCoh}*}(\mathcal{F}) \in \text{IndCoh}(X)^{\geq 0} \), the assertion follows from the construction of the inverse functor

\[ \text{Tot} (\text{IndCoh}(U^\bullet)) \to \text{IndCoh}(X). \]

\( \square \)
5. Proper maps

If until now, the theory of IndCoh has run in parallel to (and was inherited from that of) QCoh, in this section we will come across to the main piece of difference between the two. This piece of difference is the functor of !-pullback, studied in this section.

5.1. The !-pullback. In this subsection we introduce the functor of !-pullback for proper maps. Its extension for arbitrary maps between schemes is the subject of [Book-II.2, Sect. 3].

5.1.1. Let \( f: X \to Y \) be a map in Sch_{af}. We recall the following definition:

**Definition 5.1.2.** The map \( f \) is said to be proper (resp., closed embedding) if the corresponding map \( \cl X \to \cl Y \) has this property.

5.1.3. Let \( f: X \to Y \) be a proper map. We claim:

**Lemma 5.1.4.** The functor \( f^!_{\text{IndCoh}}: \text{IndCoh}(X) \to \text{IndCoh}(Y) \) sends \( \text{Coh}(X) \subset \text{IndCoh}(X) \) to \( \text{Coh}(Y) \subset \text{IndCoh}(Y) \).

**Proof.** By the construction of \( f^!_{\text{IndCoh}} \), it is sufficient to show that the functor \( f_*: \text{QCoh}(X) \to \text{QCoh}(Y) \) sends \( \text{Coh}(X) \subset \text{QCoh}(X) \) to \( \text{Coh}(Y) \subset \text{QCoh}(Y) \).

First, we note that the assertion holds when \( f \) is a closed embedding.

In general, by the devissage with respect to the t-structure, it is sufficient to show that for \( F \in \text{Coh}(X) \), we have \( f_*(F) \in \text{Coh}(Y) \).

Let \( i \) denote the canonical closed embedding \( \cl X \hookrightarrow X \). The functor \( i_* \) is an equivalence \( \text{Coh}(\cl X) \to \text{Coh}(X) \). Hence, \( F = i_*(F') \) for \( F' \in \text{Coh}(\cl X) \). This reduces the assertion to the case when \( X \) is classical.

We factor the map \( f: X \to Y \) as \( X \to \cl Y \hookrightarrow Y \). Since \( i \) is a closed embedding, we have reduced the assertion to the case when \( Y \) is classical as well. In the latter case, the assertion is well-known.

**Remark 5.1.6.** The continuity of the functor \( f^!_{\text{IndCoh}} \) is the raison d'être of the category IndCoh, and its main difference from QCoh.

5.1.5. The above lemma implies that the functor \( f^!_{\text{IndCoh}} \) sends \( \text{IndCoh}(X)^c \) to \( \text{IndCoh}(Y)^c \).

Hence, \( f^!_{\text{IndCoh}} \) admits a continuous right adjoint, to be denoted \( f^!: \text{IndCoh}(X) \to \text{IndCoh}(Y) \).

**Remark 5.1.7.** By [GL:DG, Sect. 6, Adjoint Functors], we obtain that the functor \( f^! \) has a natural structure of 1-morphism in \( \text{QCoh}(Y)\text{-mod} \).
5.1.8. Note that the functor $f^\text{IndCoh}_*$ is right t-exact, up to a finite shift. Hence, the functor $f_!$ is left t-exact up to a finite shift. In particular, $f_!$ maps IndCoh($Y$)$^+$ to IndCoh($X$)$^+$.

Let $f^{\text{QCoh}!}$ denote the not necessarily continuous right adjoint to $f_*$. It also has the property that it maps QCoh($Y$)$^+$ to QCoh($X$)$^+$.

**Lemma 5.1.9.** The diagram

\[
\begin{array}{ccc}
\text{IndCoh}(X)^+ & \xleftarrow{f'_!} & \text{IndCoh}(Y)^+ \\
\Psi_X & & \Psi_Y \\
\text{QCoh}(X)^+ & \xleftarrow{f^{\text{QCoh}!}_*} & \text{QCoh}(Y)^+ \\
\end{array}
\]

obtained by passing to right adjoints along the horizontal arrows in

\[
\begin{array}{ccc}
\text{IndCoh}(X)^+ & \xrightarrow{f^\text{IndCoh}_*} & \text{IndCoh}(Y)^+ \\
\Psi_X & & \Psi_Y \\
\text{QCoh}(X)^+ & \xrightarrow{f_*} & \text{QCoh}(Y)^+ \\
\end{array}
\]

commutes.

**Proof.** Follows from the fact that the vertical arrows are equivalences, by Proposition 1.2.2. \qed

**Remark 5.1.10.** It is not in general true that the diagram

\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xleftarrow{f'_!} & \text{IndCoh}(Y) \\
\Psi_X & & \Psi_Y \\
\text{QCoh}(X) & \xleftarrow{f^{\text{QCoh}!}_*} & \text{QCoh}(Y) \\
\end{array}
\]

obtained by passing to right adjoints along the horizontal arrows in

\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{f^\text{IndCoh}_*} & \text{IndCoh}(Y) \\
\Psi_X & & \Psi_Y \\
\text{QCoh}(X) & \xrightarrow{f_*} & \text{QCoh}(Y) \\
\end{array}
\]

commutes.

For example, take $X = \text{pt} = \text{Spec}(k)$, $Y = \text{Spec}(k[t]/t^2)$ and $0 \neq \mathcal{F} \in \text{IndCoh}(Y)$ be in the kernel of the functor $\Psi_Y$. Then $\Psi_X \circ f'_!(\mathcal{F}) \neq 0$. Indeed, $\Psi_X$ is an equivalence, and $f'_!$ is conservative, see Corollary 6.1.5.

5.1.11. Let $(\text{Sch}_{\text{aff}})_{\text{proper}}$ be a 1-full subcategory of $\text{Sch}_{\text{aff}}$ when we restrict 1-morphisms to be proper maps. By [Funct, Sect. 11.2], we obtain:

**Corollary 5.1.12.** There exists a canonically defined functor

\[
\text{IndCoh}^!_{(\text{Sch}_{\text{aff}})_{\text{proper}}} : ((\text{Sch}_{\text{aff}})_{\text{proper}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}},
\]

obtained from

\[
\text{IndCoh}((\text{Sch}_{\text{aff}})_{\text{proper}}) := \text{IndCoh}((\text{Sch}_{\text{aff}})_{\text{proper}})_{((\text{Sch}_{\text{aff}})_{\text{proper}})^{\text{op}}}
\]

by passing to right adjoints.
5.2. **Base change for proper maps.** A crucial property of the !-pullback is *base change* against the *-direct image. We establish it in this subsection.

5.2.1. Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{g_X} & X_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
Y_1 & \xrightarrow{g_Y} & Y_2
\end{array}
\]

be a Cartesian diagram in Sch_{aff}, with the vertical maps being proper.

The isomorphism of functors

\[
(f_2)^{\text{IndCoh}}_* \circ (g_X)^{\text{IndCoh}}_* \simeq (g_Y)^{\text{IndCoh}}_* \circ (f_1)^{\text{IndCoh}}_*
\]

gives rise to a natural transformation:

\[
(5.1) \quad (g_X)^{\text{IndCoh}}_* \circ f_1^! \to f_2^! \circ (g_Y)^{\text{IndCoh}}_*.
\]

We will prove:

**Proposition 5.2.2.** The map (5.1) is an isomorphism.

**Proof.** Since all functors involved are continuous, it is enough to show that the map

\[
(g_X)_*^{\text{IndCoh}} \circ f_1^!(\mathcal{F}) \to f_2^! \circ (g_Y)^{\text{IndCoh}}_* (\mathcal{F})
\]

is an isomorphism for \( \mathcal{F} \in \text{Coh}(Y_1) \). Hence, it is enough to show that (5.1) is an isomorphism when restricted to IndCoh(Y_1)^+.

By Lemma 5.1.9 and Proposition 1.2.2, this reduces the assertion to showing that the natural transformation

\[
(5.2) \quad (g_X)_* \circ f_1^{\text{Coh},!} \to f_2^{\text{Coh},!} \circ (g_Y)_*
\]

is an isomorphism for the functors

\[
\begin{array}{ccc}
\text{Qcoh}(X_1)^+ & \xrightarrow{(g_X)_*} & \text{Qcoh}(X_2)^+ \\
\uparrow{f_1^{\text{Coh},!}} & & \uparrow{f_2^{\text{Coh},!}} \\
\text{Qcoh}(Y_1)^+ & \xrightarrow{(g_Y)_*} & \text{Qcoh}(Y_2)^+,
\end{array}
\]

where the natural transformation comes from the commutative diagram

\[
\begin{array}{ccc}
\text{Qcoh}(X_1)^+ & \xrightarrow{(g_X)_*} & \text{Qcoh}(X_2)^+ \\
\downarrow{(f_1)_*} & & \downarrow{(f_2)_*} \\
\text{Qcoh}(Y_1)^+ & \xrightarrow{(g_Y)_*} & \text{Qcoh}(Y_2)^+
\end{array}
\]

by passing to right adjoint along the vertical arrows.

We consider the commutative diagram

\[
\begin{array}{ccc}
\text{Qcoh}(X_1) & \xrightarrow{(g_X)_*} & \text{Qcoh}(X_2) \\
\downarrow{(f_1)_*} & & \downarrow{(f_2)_*} \\
\text{Qcoh}(Y_1) & \xrightarrow{(g_Y)_*} & \text{Qcoh}(Y_2),
\end{array}
\]
and the diagram
\[
\begin{array}{ccc}
\text{QCoh}(X_1) & \xrightarrow{(g_X)_*} & \text{QCoh}(X_2) \\
\uparrow_{\text{f}_1^{\text{QCoh},!}} & & \uparrow_{\text{f}_2^{\text{QCoh},!}} \\
\text{QCoh}(Y_1) & \xrightarrow{(g_Y)_*} & \text{QCoh}(Y_2),
\end{array}
\]

obtained by passing to right adjoints along the vertical arrows. (Note, however, that the functors involved are no longer continuous).

We claim that the resulting natural transformation
\[
(g_X)_* \circ \text{f}_1^{\text{QCoh},!} \to (g_Y)_* \circ (f_2)_*
\]

between the functors
\[
\text{QCoh}(Y_1) \rightleftarrows \text{QCoh}(X_2)
\]
is an isomorphism. This would imply that (5.2) is an isomorphism by restricting to the eventually coconnective subcategory.

To prove that (5.3) is an isomorphism, we note that this map is obtained by passing to right adjoints in the natural transformation
\[
(g_Y)^* \circ (f_2)_* \to (f_1)_* \circ (g_X)^*
\]
as functors
\[
\text{QCoh}(X_2) \rightleftarrows \text{QCoh}(Y_1)
\]
in the commutative diagram
\[
\begin{array}{ccc}
\text{QCoh}(X_1) & \xleftarrow{(g_X)^*} & \text{QCoh}(X_2) \\
\downarrow_{(f_1)_*} & & \downarrow_{(f_2)_*} \\
\text{QCoh}(Y_1) & \xleftarrow{(g_Y)^*} & \text{QCoh}(Y_2).
\end{array}
\]

Now, (5.4) is an isomorphism by the usual base change for QCoh. Hence, (5.3) is an isomorphism as well.

\[
\square
\]

5.3. Pullback compatibility. The !-pullback for arbitrary maps between schemes will be defined in such a way that it is the !-pullback for proper morphisms, and the *-pullback for open embeddings.

Hence, if we want that !-pullback to be well-defined, a certain compatibility must take place, when we decompose a morphism in two different ways as a composition of a proper morphism and an open embedding. A basic case of such compatibility is established in this subsection.

5.3.1. Let
\[
\begin{array}{ccc}
X_1 & \xrightarrow{g_X} & X_2 \\
\downarrow_{f_1} & & \downarrow_{f_2} \\
Y_1 & \xrightarrow{g_Y} & Y_2
\end{array}
\]
be a Cartesian diagram in Sch_{coh}, with the vertical maps being proper, and horizontal maps being eventually coconnective.

We start with the base change isomorphism
\[
(f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh},*} \simeq g_Y^{\text{IndCoh},*} \circ (f_2)_*^{\text{IndCoh}}
\]
of Proposition 3.2.2, and by the \((f_*^{\text{IndCoh}}, f!^{\text{IndCoh}})\)-adjunction obtain a map
\[
y_*^{\text{IndCoh}} \circ f_2^! \to f_1^! \circ g_*^{\text{IndCoh}}.
\]

Remark 5.3.2. Note that one can get another map
\[
y_*^{\text{IndCoh}} \circ f_2^! \to f_1^! \circ g_*^{\text{IndCoh}}
\]
namely, via the \((g_*^{\text{IndCoh}}, g_*^{\text{IndCoh}})\)-adjunction from the isomorphism
\[
f_2^! \circ (g_Y)^{*}_{\text{IndCoh}} \simeq (g_X)^{*}_{\text{IndCoh}} \circ f_1^!
\]
of Proposition 5.2.2. A diagram chase shows that the map (5.6) is canonically the same as (5.5).

5.3.3. We are going to prove:

**Proposition 5.3.4.** Suppose that \(g_Y\) (and hence \(g_X\)) are open embeddings. Then the map (5.5) is an isomorphism.

Remark 5.3.5. It is shown in [GL:IndCoh, Proposition 7.1.6] that the map (5.5) is an isomorphism for any eventually coconnective \(g_Y\).

**Proof.** By Proposition 4.1.2, it suffices to show that the induced map
\[
(g_X)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \circ f_2^! \to (g_X)^{*}_{\text{IndCoh}} \circ f_1^! \circ g_*^{\text{IndCoh}}
\]
is an isomorphism.

Using Proposition 5.2.2, we have
\[
(g_X)^{*}_{\text{IndCoh}} \circ f_1^! \circ g_*^{\text{IndCoh}} \simeq f_2^! \circ (g_Y)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}}.
\]

Hence, we need to show that the map
\[
(g_X)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \circ f_2^! \to f_2^! \circ (g_Y)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}}
\]
is an isomorphism.

Consider the diagram
\[
\begin{array}{ccc}
(g_X)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \circ f_2^! & \xrightarrow{(5.7)} & f_2^! \circ (g_Y)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \\
\uparrow & & \uparrow \\
(f_2^!) & \xrightarrow{\text{id}} & (f_2^!)
\end{array}
\]
(5.8)
where the vertical maps are given by the unit of the \((g_*^{\text{IndCoh}}, g_*^{\text{IndCoh}})\)-adjunction.

We will prove:

**Lemma 5.3.6.** The diagram (5.8) commutes.

Let us finish the proof of the proposition modulo this lemma.

By Corollary 3.3.2, for \(\mathcal{F} \in \text{IndCoh}(Y_2)\), we have canonical isomorphisms
\[
(g_X)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \circ f_2^!(\mathcal{F}) \simeq (g_X)^{*}_{\text{IndCoh}} \circ (\mathcal{O}_{X_1}) \otimes f_2^!(\mathcal{F})
\]
and
\[
(g_Y)^{*}_{\text{IndCoh}} \circ g_*^{\text{IndCoh}} \circ (\mathcal{F}) \simeq (g_Y)^{*}_{\text{IndCoh}} \circ (\mathcal{O}_{Y_1}) \otimes \mathcal{F},
\]
where \(\otimes\) denotes the action of QCoh on IndCoh.
By the compatibility of the action of QCoh with the !-pullback, we have
\[ f_2' \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh},*}(\mathcal{F}) \simeq f_2'((g_Y)_*(\mathcal{O}_{Y1})) \otimes f_2'((\mathcal{F}) \simeq (g_X)_*(\mathcal{O}_{X1}) \otimes f_2'((\mathcal{F})). \]

Hence, the map (5.7), evaluated on \( \mathcal{F} \), can be thought of as an endomorphism, to be denoted by \( \phi \), of
\[ (g_X)_*(\mathcal{O}_{X1}) \otimes f_2'((\mathcal{F}). \]

Furthermore, by Lemma 5.3.6, the map
\[ f_2'((\mathcal{F}) \simeq \mathcal{O}_{X1} \otimes f_2'((\mathcal{F}) \to (g_X)_*(\mathcal{O}_{X1}) \otimes f_2'((\mathcal{F}) \xrightarrow{\phi} (g_X)_*(\mathcal{O}_{X1}) \otimes f_2'((\mathcal{F}) \]
is canonically homotopic to the tautological map
\[ f_2'((\mathcal{F}) \to (g_X)_*(\mathcal{O}_{X1}) \otimes f_2'((\mathcal{F}). \]

We claim that this implies that \( \phi \) is itself canonically homotopic to the identity. Indeed, for any \( \mathcal{F}', \mathcal{F}'' \in \text{IndCoh}(\mathcal{X}_2) \), the map
\[ \text{Maps}((g_X)_*(\mathcal{O}_{X1}) \otimes \mathcal{F}', (g_X)_*(\mathcal{O}_{X1}) \otimes \mathcal{F}'') \to \text{Maps}(\mathcal{F}', (g_X)_*(\mathcal{O}_{X1}) \otimes \mathcal{F}'') \]
identifies by Corollary 3.3.5 with the map
\[ \text{Maps}((g_X)_*^{\text{IndCoh}} \circ (g_X)_*^{\text{IndCoh},*}(\mathcal{F}'), (g_X)_*^{\text{IndCoh}} \circ (g_X)_*^{\text{IndCoh},*}(\mathcal{F}'')) \to \text{Maps}(\mathcal{F}', (g_X)_*^{\text{IndCoh}} \circ (g_X)_*^{\text{IndCoh},*}(\mathcal{F}'')), \]
and the latter is an isomorphism by Proposition 4.1.2.

5.3.7. Proof of Lemma 5.3.6. We remark that the reason that the proof is not completely straightforward is that the natural map in the isomorphism
\[ (f_1)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh},*} \simeq g_Y^{\text{IndCoh},*} \circ (f_2)_*^{\text{IndCoh}}, \]
goes from right to left. So, the actual arrow involved in the construction of the map (5.5) is somewhat implicit.

By definition, the map \( \uparrow \) followed by \( \to \) in (5.8) is the composition
\[ (g_X)_*^{\text{IndCoh}} \circ f_1' \circ (f_1)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh},*} \circ f_2' \longrightarrow (g_X)_*^{\text{IndCoh}} \circ f_1' \circ g_Y^{\text{IndCoh},*} \circ (f_2)_*^{\text{IndCoh}} \circ f_2' \]
\[ f_2' \]
\[ f_2' \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh},*} \]
\[ (g_X)_*^{\text{IndCoh}} \circ f_1' \circ g_Y^{\text{IndCoh},*} \]
where the horizontal arrow is given by the inverse of the isomorphism
\[ g_Y^{\text{IndCoh},*} \circ (f_2)_*^{\text{IndCoh}} \to (f_1)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh},*}. \]
Note that the composed map $\rightarrow\downarrow\downarrow$ in the above diagram can be rewritten as
\[
(g_X)_*^{\text{IndCoh}} \circ f_1^* \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^* \longrightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^*
\]
\[
\downarrow \\
\quad \\
\downarrow
\]
\[
f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*
\]
where now the second map is again given by the inverse to (5.9). Hence, we need to establish the commutativity of the diagram
\[
(g_X)_*^{\text{IndCoh}} \circ f_1^* \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^* \longrightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^*
\]
\[
\downarrow \\
(g_X)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^* \\
\downarrow \\
f_2^*
\]
\[
f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*
\]
Consider the map
\[
(5.10)
f_2^* \rightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*
\]
equal to the composition
\[
f_2^* \rightarrow f_2^* \circ (f_2)_*^{\text{IndCoh}} \circ f_2^* \rightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*,
\]
where both maps are given by the units of the corresponding adjunctions.

Now, diagram chase shows that both of the following diagrams commute:
\[
(g_X)_*^{\text{IndCoh}} \circ f_1^* \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^* \longrightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ (f_1)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^*
\]
\[
\downarrow \\
(g_X)_*^{\text{IndCoh}} \circ g_X^{\text{IndCoh,*}} \circ f_2^* \\
\downarrow \\
f_2^* \quad \text{id}
\]
\[
(5.10)
\quad \\
f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*
\]
and
\[
f_2^* \quad \text{(5.10)} \\
\quad \text{id}
\]
\[
f_2^* \rightarrow f_2^* \circ (g_Y)_*^{\text{IndCoh}} \circ g_Y^{\text{IndCoh,*}} \circ (f_2)_*^{\text{IndCoh}} \circ f_2^*
\]
Together, this implies the required commutativity. \[\square\]

6. Closed embeddings

The behavior of IndCoh with respect to closed embedding is “better” than that of QCoh. The main point of difference is that the direct image functor under a closed embedding for IndCoh preserves compactness, which is not the case for QCoh.
6.1. **Category with support.** In this subsection we study the full subcategory of \( \text{IndCoh}(X) \) corresponding with objects “with support” on a given closed subscheme.

6.1.1. Let \( X \) be an object of \( \text{Sch}_{afn} \), and let \( i: Z \to X \) be a closed embedding. Let \( j: U \to X \) be the complementary open.

We let \( \text{IndCoh}(X)_Z \) denote the full subcategory of \( \text{IndCoh}(X) \) equal to \( \ker \left( j^{\text{IndCoh,*}} : \text{IndCoh}(X) \to \text{IndCoh}(U) \right) \).

Note that the embedding \( \text{IndCoh}(X)_Z \hookrightarrow \text{IndCoh}(Z) \) admits a right adjoint given by sending \( F \) to \( \ker(F \to j_{\ast} \text{IndCoh}_{\ast} \circ j^{\text{IndCoh,*}}(F)) \), which by Corollary 3.3.5 is the same as \( \ker(\mathcal{O}_X \to j_{\ast}(\mathcal{O}_U)) \otimes F \).

6.1.2. We claim:

**Proposition 6.1.3.**

(a) The subcategory \( \text{IndCoh}(X)_Z \subset \text{IndCoh}(X) \) is compatible with the t-structure (i.e., is preserved by the truncation functors).

(b) The subcategory \( \text{IndCoh}(X)_Z \subset \text{IndCoh}(X) \) is generated by the essential image of the functor \( i_{\ast}^{\text{IndCoh}} : \text{IndCoh}(Z) \to \text{IndCoh}(X) \).

(c) The functor \( i_{\ast} : \text{IndCoh}(X) \to \text{IndCoh}(Z) \) is conservative, when restricted to \( \text{IndCoh}(X)_Z \).

(d) The category \( \text{IndCoh}(X)_Z \) identifies with the ind-completion of \( \text{Coh}(X)_Z := \ker \left( j_{\ast} : \text{Coh}(X) \to \text{Coh}(U) \right) \).

**Proof.** Point (a) follows from the fact that the functor \( j^{\text{IndCoh,*}} \) is t-exact.

Let us prove point (b). The category \( \text{IndCoh}(X) \) is generated by \( \text{IndCoh}(X)^{\dagger} \). The description of the right adjoint to \( \text{IndCoh}(X)_Z \hookrightarrow \text{IndCoh}(Z) \) implies that the same is true for \( \text{IndCoh}(X)_Z \).

For every \( F \in \text{IndCoh}(X)^{\dagger} \), the map

\[
\colim_n \tau^{\leq n}(F) \to F,
\]

is an isomorphism, by Proposition 1.2.2 and because the corresponding fact is true in \( \text{Q}\text{Coh}^{\dagger} \).

Hence, by point (a), every \( F \in \text{IndCoh}(X)^{\dagger}_Z \) is a colimit of cohomologically bounded objects from \( \text{IndCoh}(X)_Z \). This implies that \( \text{IndCoh}(X)_Z \) is generated by \( \text{IndCoh}(X)_Z^{\dagger} = \text{Coh}(X)_Z^{\dagger} \).

Now, it is easy to see that every object \( F \in \text{Coh}(X)_Z^{\dagger} \) has a filtration

\[
F = \bigcup_n F_n
\]

with \( F_n/F_{n-1} \in i_{\ast}(\text{Coh}(Z)_{\dagger}) \). This proves point (b).

Point (c) follows from point (b) by adjunction.
To prove point (d), it suffices to note that the objects from $\text{Coh}(X_Z)$ are compact in $\text{IndCoh}(X_Z)$ (because they are compact in $\text{IndCoh}(X)$) and that they generate $\text{IndCoh}(X_Z)$, by point (b).

6.1.4. As a corollary we obtain:

**Corollary 6.1.5.** Let $f : X' \to X$ be a nil-isomorphism, i.e., a map such that $\text{red}X' \to \text{red}X$ is an isomorphism. Then the functor $f^!$ is conservative. Equivalently, the essential image of $\text{IndCoh}(X')$ under $f_*^{\text{IndCoh}}$ generates $\text{IndCoh}(X)$.

**Proof.** We can assume that $X' = \text{red}X$, so $f$ is also a closed embedding. In this case the assertion of the corollary follows from Proposition 6.1.3(b).

6.2. **A conservativeness result for proper maps.** The main result established in this subsection, Proposition 6.2.2, is of technical significance.

6.2.1. We shall now use Proposition 6.1.3 to prove the following:

**Proposition 6.2.2.** Let $f : X \to Y$ be a proper map, which is surjective at the level of geometric points. Then the functor $f^! : \text{IndCoh}(Y) \to \text{IndCoh}(X)$ is conservative.

The rest of this subsection is devoted to the proof of the proposition.

6.2.3. By Corollary 6.1.5 we can assume that both $X$ and $Y$ are classical and reduced. We argue by Noetherian induction, assuming that the statement is true for all proper closed subschemes of $Y$. We need to show that the essential image of $\text{IndCoh}(X)$ under $f_*^{\text{IndCoh}}$ generates $\text{IndCoh}(Y)$.

By Proposition 6.1.3(a), it is sufficient to show that $Y$ contains an open subscheme $\overset{\circ}{Y} \subset Y$ such that for $\overset{\circ}{X} := f^{-1}(\overset{\circ}{Y})$, the essential image of $\text{IndCoh}(\overset{\circ}{X})$ under $(f|_{\overset{\circ}{X}})_*^{\text{IndCoh}}$ generates $\text{IndCoh}(\overset{\circ}{Y})$.

Since $Y$ is classical and reduced, it contains a non-empty open smooth subscheme, which we take to be $\overset{\circ}{Y}$. By Lemma 1.1.3 and Lemma 1.1.9, we are reduced to showing the following:

**Lemma 6.2.4.** Let $f : X \to Y$ be a proper morphism of classical schemes with $Y$ smooth. Then the essential image of $\text{Qcoh}(X)$ under $f_*$ generates $\text{Qcoh}(Y)$.

**Proof of Lemma 6.2.4.** Let $\mathcal{C} \subset \text{Qcoh}(Y)$ be the full subcategory generated by the essential image of $\text{Qcoh}(X)$ under $f_*$. Note that $\mathcal{C}$ is a monoidal ideal, since the functor $f_*$ respects the action of $\text{Qcoh}(Y)$. Consider $\mathcal{E} := f_*(\mathcal{O}_X) \in \mathcal{C}$. This is an object of $\text{Qcoh}(Y)$, whose fiber at every geometric point of $Y$ is non-zero.

However, it is easy to see that for any Noetherian classical scheme $Y$ and $\mathcal{E} \in \mathcal{C} \subset \text{Qcoh}(Y)$ with the above properties, we have

$$\mathcal{C} = \text{Qcoh}(Y).$$
6.3. **Products.** It is known that for a pair of quasi-compact schemes $X_1$ and $X_2$, tensor product defines an equivalence of DG categories

$$\text{QCoh}(X_1) \otimes \text{QCoh}(X_2) \rightarrow \text{QCoh}(X_1 \times X_2).$$

In this subsection we will establish a similar assertion for $\text{IndCoh}$. This is not altogether tautological: for example the validity of this fact relies on the assumption that the ground field $k$ be perfect.

6.3.1. Let $X_1$ and $X_2$ be two objects of $\text{Sch}_\mathfrak{a}$. We claim:

**Lemma 6.3.2.** There exists a uniquely defined functor

\[ \mathfrak{M} : \text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2) \rightarrow \text{IndCoh}(X_1 \times X_2) \]

that preserves compactness and makes the diagram

\[ \begin{array}{ccc}
\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2) & \xrightarrow{\mathfrak{M}} & \text{IndCoh}(X_1 \times X_2) \\
\psi_{X_1} \times \psi_{X_2} & \downarrow & \downarrow \psi_{X_1 \times X_2} \\
\text{QCoh}(X_1) \otimes \text{QCoh}(X_2) & \xrightarrow{\mathfrak{M}} & \text{QCoh}(X_1 \times X_2)
\end{array} \]

commute.

**Proof.** The anticlock-wise composition sends the compact generators of the category

$$\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2)$$

(i.e., objects of the form $\mathcal{F}_1 \otimes \mathcal{F}_2$ for $\mathcal{F}_i \in \text{Coh}(X_i)$) to $\text{QCoh}(X_1 \times X_2)^{+}$. Hence, it sends all of $(\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2))^c$ to $\text{Coh}(X_1 \times X_2)^{+}$.

The sought-for functor is the ind-extension of

$$\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2) \xrightarrow{\mathfrak{M} \otimes \mathfrak{M}} \text{IndCoh}(X_1 \times X_2)^{+}.$$

This functor preserves compactness since the objects

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \text{Coh}(X_1 \times X_2)^{+}, \quad \forall \mathcal{F}_i \in \text{Coh}(X_i)$$

belong to $\text{Coh}(X_1 \times X_2)$. \hfill \Box

6.3.3. We now claim:

**Proposition 6.3.4.**

(a) The functor (6.1) is fully faithful.

(b) If the ground field $k$ is perfect, then (6.1) is an equivalence.

**Proof.** Since the functor (6.1) preserves compactness, for point (a) it is sufficient to show that for $\mathcal{F}_i, \mathcal{G}_i \in \text{Coh}(X_i), \ i = 1, 2$, the map

$$\text{Maps}_{\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2)}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{G}_1 \otimes \mathcal{G}_2) \rightarrow \text{Maps}_{\text{IndCoh}(X_1 \times X_2)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \mathcal{G}_1 \boxtimes \mathcal{G}_2)$$

is an isomorphism.

We have a commutative diagram

\[ \begin{array}{ccc}
\text{Maps}_{\text{IndCoh}(X_1)}(\mathcal{F}_1, \mathcal{G}_1) & \rightarrow & \text{Maps}_{\text{IndCoh}(X_1 \times X_2)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \mathcal{G}_1 \boxtimes \mathcal{G}_2) \\
\downarrow & & \downarrow \\
\text{Maps}_{\text{IndCoh}(X_1 \times X_2)}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{G}_1 \otimes \mathcal{G}_2) & \rightarrow & \text{Maps}_{\text{IndCoh}(X_1 \times X_2)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \mathcal{G}_1 \boxtimes \mathcal{G}_2)
\end{array} \]
where the vertical arrows are isomorphisms, by the coherence assumption.

Hence, it remains to show that
\[
\text{Maps}_{\text{QCoh}(X_1)}(\mathcal{F}', \mathcal{F}''_1) \otimes \text{Maps}_{\text{QCoh}(X_2)}(\mathcal{F}', \mathcal{F}''_2) \rightarrow \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\mathcal{F}'_1 \boxtimes \mathcal{F}'_2, \mathcal{F}''_1 \boxtimes \mathcal{F}''_2)
\]
is an isomorphism. This is not immediate since the objects \( \mathcal{F}'_i \in \text{QCoh}(S_i) \) are not compact. To circumvent this, we proceed as follows.

It is enough to show that
\[
\tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1)}(\mathcal{F}', \mathcal{F}''_1) \otimes \text{Maps}_{\text{QCoh}(X_2)}(\mathcal{F}', \mathcal{F}''_2) \right) \rightarrow \tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\mathcal{F}'_1 \boxtimes \mathcal{F}'_2, \mathcal{F}''_1 \boxtimes \mathcal{F}''_2) \right)
\]
is an isomorphism for any fixed \( n \).

Choose \( \alpha_1 : \tilde{\mathcal{F}}_1' \rightarrow \mathcal{F}_1' \) (resp., \( \alpha_2 : \tilde{\mathcal{F}}_2' \rightarrow \mathcal{F}_2' \)) with \( \tilde{\mathcal{F}}_1' \) (resp., \( \tilde{\mathcal{F}}_2' \)) in \( \text{QCoh}(X_1)_{\text{perf}} \) (resp., \( \text{QCoh}(X_2)_{\text{perf}} \)), such that
\[
\text{Cone}(\alpha_1) \in \text{QCoh}(X_1)^{\leq -N} \quad \text{and} \quad \text{Cone}(\alpha_2) \in \text{QCoh}(X_2)^{\leq -N}
\]
for \( N \gg 0 \).

By choosing \( N \) large enough, we can ensure that
\[
\tau^{\leq m} \left( \text{Maps}_{\text{QCoh}(S_i)}(\mathcal{F}', \mathcal{F}''_1) \right) \rightarrow \tau^{\leq m} \left( \text{Maps}_{\text{QCoh}(S_i)}(\tilde{\mathcal{F}}', \mathcal{F}''_1) \right)
\]
is an isomorphism for a given integer \( m \), which in turn implies that
\[
\tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1)}(\mathcal{F}', \mathcal{F}''_1) \otimes \text{Maps}_{\text{QCoh}(X_2)}(\mathcal{F}', \mathcal{F}''_2) \right) \rightarrow \tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\tilde{\mathcal{F}}', \mathcal{F}''_1 \otimes \mathcal{F}''_2) \right)
\]
and
\[
\tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\mathcal{F}_1' \boxtimes \mathcal{F}_2', \mathcal{F}''_1 \boxtimes \mathcal{F}''_2) \right) \rightarrow \tau^{\leq n} \left( \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\tilde{\mathcal{F}}_1' \boxtimes \tilde{\mathcal{F}}_2', \mathcal{F}''_1 \boxtimes \mathcal{F}''_2) \right)
\]
are isomorphisms.

Hence, it is enough to show that
\[
\text{Maps}_{\text{QCoh}(X_1)}(\tilde{\mathcal{F}}', \mathcal{F}''_1) \otimes \text{Maps}_{\text{QCoh}(X_2)}(\tilde{\mathcal{F}}', \mathcal{F}''_2) \rightarrow \text{Maps}_{\text{QCoh}(X_1 \times X_2)}(\tilde{\mathcal{F}}_1' \boxtimes \tilde{\mathcal{F}}_2', \mathcal{F}''_1 \boxtimes \mathcal{F}''_2)
\]
is an isomorphism. But this follows from the fact that the factor
\[
(6.2) \quad \text{QCoh}(X_1) \boxtimes \text{QCoh}(X_2) \rightarrow \text{QCoh}(X_1 \times X_2)
\]
is an equivalence.

This finishes the proof of point (a).

To prove point (b), we have to show that the essential image of the functor (6.1) generates \( \text{IndCoh}(X_1 \times X_2) \). By Corollary 6.1.5 we can assume that both \( X_1 \) and \( X_2 \) are classical and reduced.

We argue by Noetherian induction, assuming that the statement is true for all proper closed subschemes \( X_i' \subset X_i \).

By Proposition 6.1.3(b), it is sufficient to show that \( X_1 \) and \( X_2 \) contain non-empty open subschemes \( X_i' \subset X_i \), for which the statement of the proposition is true.
Since $X_i$ are classical and reduced, we can take $\tilde{X}_i$ to be a non-empty open smooth subscheme of $X_i$. Note that the assumption that $k$ be perfect implies that $\tilde{X}_1 \times \tilde{X}_2$ is also smooth. Now, the assertion follows from Lemma 1.1.3 and the fact that (6.2) is an equivalence.

6.3.5. Upgrading to a functor. Consider the category $\text{Sch}^{aft}$ as endowed with a symmetric monoidal structure given by Cartesian product. We consider the category $\text{DGCat}^{\text{cont}}$ also as a symmetric monoidal $\infty$-category with respect to the operation of tensor product.

First we recall:

**Proposition 6.3.6.** The functor

$$\text{QCoh}^*_{\text{Sch}^{aft}} : (\text{Sch}^{aft})^{\text{op}} \to \text{DGCat}^{\text{cont}}$$

has a natural symmetric monoidal structure.

**Proof.** Note that the functor $\text{QCoh}^*_{\text{Sch}^{aft}}$ upgrades to a functor

$$(\text{Sch}^{aft})^{\text{op}} \to \text{DGCat}^{\text{cont}}_{\text{SymMon}} = \text{ComAlg}(\text{DGCat}^{\text{cont}}).$$

We have the following general construction. Let $\mathcal{O}$ be a symmetric monoidal category, and let

$$F : \mathcal{I}^{\text{op}} \to \text{ComAlg}(\mathcal{O})$$

be a functor, where $\mathcal{I}$ is a category with finite products. Then $F$, when viewed as a functor $F : \mathcal{I}^{\text{op}} \to \mathcal{O}$, has a natural structure of right-lax symmetric monoidal functor, where the symmetric monoidal structure on $\mathcal{I}$ is given by Cartesian product. Informally, for $i_1, i_2 \in \mathcal{I}$, we have the map

$$F(i_1) \otimes F(i_2) \to F(i_1 \times i_2)$$

where $\pi_i$ are the two projections $i_1 \times i_2 \to i_i$, and the map

$$F(i_1 \times i_2) \otimes F(i_1 \times i_2) \to F(i_1 \times i_2)$$

is given by the commutative algebra structure on $F(i_1 \times i_2)$.

Hence, we obtain that $\text{QCoh}^*_{\text{Sch}^{aft}}$ has a canonical right-lax symmetric monoidal structure. The resulting maps

$$\text{QCoh}(X_1) \times \text{QCoh}(X_2) \to \text{QCoh}(X_1 \times X_2)$$

are given by the operation of external tensor product.

The fact that the above right-lax symmetric monoidal structure is strict is the fact that the maps (6.2) are equivalences for $X_i \in \text{Sch}^{aft}$ by [GL:QCoh, Proposition 1.4.4].

6.3.7. Passing to adjoints, by [Funct, Adjunction and Symmetric Monoidal Structures], we obtain that the functor

$$\text{QCoh}^*_{\text{Sch}^{aft}} : \text{Sch}^{aft} \to \text{DGCat}^{\text{cont}}$$

also has a natural symmetric monoidal structure.

Now, arguing as in Proposition 2.2.3 we show:

**Proposition 6.3.8.** There exists a unique symmetric monoidal structure on the functor

$$\text{IndCoh}_{\text{Sch}^{aft}} : \text{Sch}^{aft} \to \text{DGCat}^{\text{cont}}$$

and the functor $\Psi_{\text{Sch}^{aft}}$, which at the level of objects is given by Lemma 6.3.2.
6.4. **Convergence.** In this subsection we will establish another crucial property of IndCoh that distinguishes it from QCoh. Namely, we will show that for a given scheme, its category IndCoh can be recovered from IndCoh on the \( n \)-coconnective truncations of this scheme.

6.4.1. Let \( X \) be an object of \( \text{Sch}_{\text{aff}} \). For each \( n \) let \( i_n \) denote the closed embedding \( \leq n X \to X \), and for \( n_1 \leq n_2 \), let \( i_{n_1,n_2} \) denote the closed embedding \( \leq n_1 X \to \leq n_2 X \).

By [GL:DG, Lemma 1.3.3], we have a canonical equivalence
\[
\text{colim}_n \text{IndCoh}(\leq n X) \simeq \lim_n \text{IndCoh}(\leq n X),
\]
where in the left-hand side the transition functors are \((i_{n_1,n_2})^\ast_{\text{IndCoh}}\), and in the right-hand side \((i_{n_1,n_2})^!\).

The functors \( i_n^! \) define a functor
\[
(6.3) \quad \text{IndCoh}(X) \to \lim_n \text{IndCoh}(\leq n X),
\]
whose left adjoint
\[
(6.4) \quad \text{colim}_n \text{IndCoh}(\leq n X) \to \text{IndCoh}(X)
\]
is given by the compatible family of functors \((i_n)_\ast^{\text{IndCoh}}\).

6.4.2. We are going to establish the following property of the category IndCoh:

**Proposition 6.4.3.** The functors (6.3) and (6.4) are mutually inverse equivalences.

**Proof.** First, we note that Corollary 6.1.5 shows that the functor (6.3) is conservative. It is clear that the functor (6.4) sends compact objects to compact ones.

Hence, it remains to prove the following: for \( F_1, F_2 \in \text{Coh}(\leq 0 X) \simeq \text{Coh}(X) \)
and \( k \in \mathbb{N} \), the map
\[
\text{colim}_n \text{Maps}_{\text{Coh}(\leq n X)}(F_1, F_2[k]) \to \text{Maps}_{\text{Coh}(X)}((i_n)_\ast(F_1), (i_n)_\ast(F_2))
\]
is an isomorphism.

We will prove more generally the following:

**Lemma 6.4.4.** For
\( F_1 \in (\text{Q Coh}(\leq n X))^{\leq 0}, F_2 \in (\text{Q Coh}(\leq n X))^{\geq -k} \),
the map
\[
(6.5) \quad \text{Maps}_{\text{Q Coh}(\leq n X)}(F_1, F_2) \to \text{Maps}_{\text{Q Coh}(X)}((i_n)_\ast(F_1), (i_n)_\ast(F_2))
\]
is an isomorphism for \( n \geq k \).
6.4.5. **Proof of Lemma 6.4.4.** We rewrite the right-hand side as
\[ \text{Maps}_{\text{QCoh}(\leq n X)}((i_n)^* \circ (i_n)_*(\mathcal{F}_1), \mathcal{F}_2), \]
and we claim that
\[ \text{Cone}((i_n)^* \circ (i_n)_*(\mathcal{F}_1) \to \mathcal{F}_1) \in (\text{QCoh}(\leq n X))^{\leq -n-1}, \]
which is equivalent to
\[ \text{Cone}((i_n)_* \circ (i_n)^* \circ (i_n)_*(\mathcal{F}_1) \to (i_n)_*(\mathcal{F}_1)) \in \text{QCoh}(X)^{\leq -n-1}, \]
and further equivalent to
\[ \text{Cone}((i_n)_*(\mathcal{F}_1) \to (i_n)_* \circ (i_n)^* \circ (i_n)_*(\mathcal{F}_1)) \in \text{QCoh}(X)^{\leq -n}. \]
In fact, we claim that for \( \mathcal{F} \in \text{QCoh}(X)^{\leq 0}, \)
\[ \text{Cone}(\mathcal{F} \to (i_n)_* \circ (i_n)^*(\mathcal{F})) \in \text{QCoh}(X)^{\leq -n}. \]
Indeed,
\[ \text{Cone}(\mathcal{F} \to (i_n)_* \circ (i_n)^*(\mathcal{F})) \simeq \text{Cone}(0_X \to (i_n)_*(0_{\leq n X}) \otimes \mathcal{F}), \]
and the assertion follows.

\[ \square \]

7. **Groupoids and descent**

In this section we will show that the category IndCoh satisfies descent with respect to proper surjective maps. We will later strengthen this to show that IndCoh satisfies h-descent.

7.1. **The Beck-Chevalley condition.** The Beck-Chevalley condition provides a general framework when the totalization of a given co-simplicial category can be described as co-modules over a co-monad acting on the category of 0-simplices.

7.1.1. Let us recall the following general framework.

Let \( \mathbf{C}^\bullet \) be a co-simplicial \( \infty \)-category. Consider the corresponding category \( \mathbf{C}^{\bullet+1} \), and the co-simplicial functor
\[ \mathbf{C}^{\bullet+1} \leftarrow \mathbf{C}^\bullet : \delta^\bullet. \]

Note that the co-simplicial category \( \mathbf{C}^{\bullet+1} \) is augmented and split by \( \mathbf{C}^0 \). Hence, we have a canonical equivalence
\[ \mathbf{C}^0 \simeq \text{Tot}(\mathbf{C}^{\bullet+1}), \]
so that the composed functor
\[ \mathbf{C}^0 \simeq \text{Tot}(\mathbf{C}^{\bullet+1}) \xrightarrow{\text{Tot}(\delta^\bullet)} \text{Tot}(\mathbf{C}^\bullet) \]
identifies with \( \text{ev}_0^{\mathbf{C}^\bullet} \).

Furthermore, the functor
\[ \mathbf{C}^0 \simeq \text{Tot}(\mathbf{C}^{\bullet+1}) \xrightarrow{\text{ev}_0^{\mathbf{C}^{\bullet+1}}} \mathbf{C}^1 \]
identifies with \( p_\ast \), where \( p_\ast, p_t \) are the two functors
\[ \mathbf{C}^0 \xrightarrow{\ast} \mathbf{C}^1. \]
7.1.2. Recall the following definition:

**Definition 7.1.3.** We shall say that \( C^\bullet \) satisfies the Beck-Chevalley condition if for each \( n \) the functor

\[
C^{n+1} \leftarrow C^n : s^n
\]

admits a left adjoint (to be denoted by \( t^n \)), and for every map \([m] \to [n]\) in \( \Delta \), the diagram

\[
\begin{array}{ccc}
C^{n+1} & \xrightarrow{t^n} & C^n \\
\uparrow & & \uparrow \\
C^{m+1} & \xrightarrow{t^m} & C^m,
\end{array}
\]

that a priori commutes up to a natural transformation, actually commutes.

We have:

**Lemma 7.1.4.** Suppose that \( C^\bullet \) satisfies the Beck-Chevalley condition. Then:

(a) The functor

\[
C^0 \leftarrow \text{Tot}(C^\bullet) : ev^0_C
\]

admits a left adjoint.

(b) The monad

\[
ev^0_C \circ (ev^0_C)^L,
\]

viewed as an endo-functor of \( C^0 \), identifies with \((p_t)^L \circ p_s\), where \((p_t)^L\) is the left adjoint of \( p^t \).

(c) The adjoint pair

\[
(ev^0_C)^L : C^0 \rightleftarrows \text{Tot}(C^\bullet) : ev^0_C
\]

is monadic.

**Proof.** The Beck-Chevalley condition implies that the simplex-wise left adjoints \( t^n \) form a co-simplicial functor

\[
t^\bullet : C^{\bullet+1} \leftarrow C^\bullet.
\]

In particular, we obtain a pair of adjoint functors

\[
\text{Tot}(t^\bullet) : \text{Tot}(C^{\bullet+1}) \rightleftarrows \text{Tot}(C^\bullet) : \text{Tot}(s^\bullet),
\]

that commute with evaluation on \( n \)-simplices for every \( n \).

Note also that \( s^0 \simeq p^t \) and so \( t^0 \simeq (p^t)^L \). Now, the required assertion concerning

\[
ev^0_{C^{\bullet}} \circ (ev^0_{C^{\bullet}})^L
\]

follows from the commutative diagram

\[
\begin{array}{ccc}
C^1 & \xrightarrow{t^0} & C^0 \\
\uparrow_{ev^0_{C^{\bullet+1}}} & & \uparrow_{ev^0_{C^{\bullet}}} \\
C^0 & \xrightarrow{\text{Tot}(t^\bullet)} & \text{Tot}(C^\bullet);
\end{array}
\]

Finally, it is easy to see that the functor \( ev^0_{C^{\bullet}} \) is conservative and commutes with \( ev^0_{C^{\bullet}} \)-split geometric realizations. Hence, it satisfies the conditions of the Barr-Beck-Lurie theorem, and therefore is monadic.

\( \square \)

7.2. **Proper descent.** We will now prove proper descent for IndCoh.
7.2.1. Let $X^\bullet$ be a groupoid simplicial object in $\text{Sch}_{\text{af}}$ (see [Lu0], Definition 6.1.2.7 for the notion of groupoid in the context of $\infty$-categories).

Denote by

$$p_s, p_t : X^1 \Rightarrow X^0$$

the corresponding maps. Let us assume that the map $p_s$ (and hence also $p_t$) is proper.

We form a co-simplicial category $\text{IndCoh}(X^\bullet)^!$ using the !-pullback functors, and consider its totalization $\text{Tot}(\text{IndCoh}(X^\bullet)^!)$. Consider the functor of evaluation on 0-simplices:

$$\text{ev}^0 : \text{Tot}(\text{IndCoh}(X^\bullet)) \to \text{IndCoh}(X^0).$$

**Proposition 7.2.2.**

(a) Then functor $\text{ev}^0$ admits a left adjoint. The resulting monad on $\text{IndCoh}(X^0)$, viewed as an endo-functor, is canonically isomorphic to $(p_t)^\text{IndCoh} \circ (p_s)^!$. The adjoint pair

$$\text{IndCoh}(X^0) \rightleftarrows \text{Tot}(\text{IndCoh}(X^\bullet)^!)$$

is monadic.

(b) Suppose that $X^\bullet$ is the Čech nerve of a map $f : X^0 \to Y$, where $f$ proper. Assume also that $f$ is surjective at the level of geometric points. Then the resulting map

$$\text{IndCoh}(Y) \to \text{Tot}(\text{IndCoh}(X^\bullet)^!)$$

is an equivalence.

**Remark 7.2.3.** Note that the fact that $\text{ev}^0$ admits a left adjoint follows from [GL:DG, Lemma 1.3.3].

Indeed, the maps in $\text{IndCoh}(X^\bullet)^!$ admit left adjoints, and we can interpret $\text{Tot}(\text{IndCoh}(X^\bullet)^!)$ as the geometric realization of the corresponding simplicial category $\text{IndCoh}(X^\bullet)$, with the left adjoint to $\text{ev}^0$ being the corresponding tautological functor

$$\text{IndCoh}(X^0) \to [\text{IndCoh}(X^\bullet)].$$

7.2.4. **Proof of Proposition 7.2.2(a).** By Lemma 7.1.4 we need to show that the co-simplicial category $\text{IndCoh}(X^\bullet)^!$ satisfies the Beck-Chevalley condition. However, this follows immediately from Proposition 5.2.2. □

7.2.5. **Proof of Proposition 7.2.2(b).** Consider the adjoint pair

$$f^! : \text{IndCoh}(X^0) \rightleftarrows \text{IndCoh}(Y) : f^\ast.$$

By Proposition 6.2.2, the functor $f^!$ is conservative, and continuous. Hence, the above pair is monadic.

The composition

$$\text{IndCoh}(Y) \to \text{Tot}(\text{IndCoh}(X^\bullet)) \to \text{IndCoh}(X^0)$$

identifies with the functor $f^!$. Hence, it remains to show that the map of the corresponding monads

$$\text{ev}^0 \circ (\text{ev}^0)^L \to f^! \circ f^\ast$$

induces an isomorphism at the level of the underlying endo-functors of $\text{IndCoh}(X^0)$.

By Proposition 7.2.2(a), the left-hand side identifies with

$$(p_t)^\text{IndCoh} \circ p_s^!.$$
Furthermore, it follows from the construction that the resulting map

\[(p_t)^{\text{IndCoh}} \circ p_! \rightarrow f^! \circ f_!^{\text{IndCoh}}\]

is the base change morphism of Proposition 5.2.2 for the Cartesian diagram

\[
\begin{array}{ccc}
X^1 & \xrightarrow{p_!} & X^0 \\
\downarrow p' & & \downarrow f \\
X^0 & \xrightarrow{f} & Y.
\end{array}
\]

Hence, the required isomorphism follows from Proposition 5.2.2.

\[\square\]
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REFERENCES


