THE CATEGORY OF SINGULARITIES AS A CRYSTAL AND GLOBAL SPRINGER FIBERS

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Abstract. We prove the 'Gluing Conjecture' on the spectral side of the categorical geometric Langlands conjecture. The key tool is the structure of crystal on the category of singularities, which allows to reduce the conjecture to the question of homological triviality of certain homotopy types. These homotopy types are obtained by gluing from a global version of Springer fibers.

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Introduction

0.1. **The goal: the Gluing Conjecture.** The goal of this paper is to prove the 'Gluing Conjecture' ([Ga3, Conjecture 9.3.7]), which constitutes one of the main steps towards the proof of the categorical geometric Langlands conjecture. We give a precise statement of the result in Sect. 4.3 of the present paper. Here we describe it more informally.

0.1.1. Let X be a smooth and complete curve, and G a reductive group over an algebraically closed ground field k of characteristic 0. We work on the *spectral side* of geometric Langlands for G, which concerns the stack LocSys_G that classifies G-local systems on X.

As was suggested in [AG], the category on the spectral side of geometric Langlands is a certain modification of the category of quasi-coherent sheaves on LocSys_G. Namely, it is the full subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_G)$, consisting of objects whose *singular support* is contained

in the global nilpotent cone. We refer the reader to [AG, Sect. 11], where the precise meaning of these words is explained.

The resulting category is denoted $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$; the categorical geometric Langlands conjecture predicts an equivalence between $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$ and the category $\text{D-mod}(\mathrm{Bun}_{\check{G}})$ of D-modules on the stack $\mathrm{Bun}_{\check{G}}$ that classifies principal $\check{G}$-bundles on X (here $\check{G}$ is the Langlands dual group of G).

The category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$ contains the usual category $\mathrm{QCoh}(\mathrm{LocSys}_G)$ of quasi-coherent sheaves as a full subcategory.

The Gluing Conjecture aims to express $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$ in terms of the categories $\mathrm{QCoh}(\mathrm{LocSys}_P)$, where P runs through the set of standard parabolic subgroups of G (including $P = G$). Essentially, the goal is to compensate for the modification

$$\mathrm{QCoh}(\mathrm{LocSys}_G) \rightsquigarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$$

by considering all parabolic subgroups of G, and working with usual quasi-coherent sheaves on the corresponding moduli stacks.

0.1.2. More precisely, for a standard parabolic P, there is a natural map

$$\mathrm{LocSys}_P \to \mathrm{LocSys}_G \tag{0.1}$$

induced by the embedding $P \hookrightarrow G$. We consider the category of *quasi-coherent sheaves on* LocSys_P*, equipped with a connection along the fibers of* (0.1); denote this category temporarily by $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$.

Below we make a brief digression to explain what exactly we mean by such a category. As this may appear too technical for an introduction, the reader may choose to skip the explanation, take the existence of a well-defined category $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$ on faith, and proceed to Sect. 0.1.4.

0.1.3. First off, it is *impossible* to makes sense of 'quasi-coherent sheaves on a stack equipped with a connection along a fibration' without resorting to derived algebraic geometry[1]. So, for the rest of the introduction, when we say 'scheme' (reps., 'algebraic stack', 'prestack'), we mean a derived scheme (reps., derived algebraic stack, prestack within derived algebraic geometry).

It is more natural to consider ind-coherent sheaves first. Given a map of prestacks $f : \mathcal{Z} \to \mathcal{Y}$, we let $\mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}}$ be the category of *ind-coherent sheaves on* $\mathcal{Z}$ *equipped with a connection along the fibers of* f, which we define to be

$$\mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} := \mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}).$$

Here $\mathrm{IndCoh}(\mathcal{W})$ is the category of ind-coherent sheaves on a prestack $\mathcal{W}$ (which is defined for any prestack *locally almost of finite type*, see [Ga1, Sect. 10]), and $\mathcal{W}_{\mathrm{dR}}$ is the de Rham prestack corresponding to a prestack $\mathcal{W}$ (see Sect. 1.3).

Pullback along the map $\mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}$ defines a functor

$$\mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Z}),$$

which can be viewed as the functor of forgetting the connection.

Suppose now that $\mathcal{Z}$ is a quasi-smooth algebraic stack (a.k.a. derived locally complete intersection); see [AG, Sect. 8.1] for the definition. For example, $\mathcal{Z} = \mathrm{LocSys}_P$ is quasi-smooth. We

[1] Unless some very stringent smoothness conditions are satisfied, such as the map being smooth and schematic.

then define the full subcategory $\mathrm{QCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} \subset \mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}}$ of quasi-coherent sheaves on $\mathcal{Z}$ equipped with a connection along the fibers of f by the condition that it fits into the following pullback diagram of categories:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} & \longrightarrow & \mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathcal{Z}) & \xrightarrow{\Xi_{\mathcal{Z}}} & \mathrm{IndCoh}(\mathcal{Z}). \end{array}$$

Here $\Xi_{\mathcal{Z}}$ is the tautological functor of embedding QCoh into IndCoh of [Ga1, Sect. 1.5] (extended to algebraic stacks in [Ga1, Sect. 11.7.3]). Note that the essential image of $\Xi_{\mathcal{Z}}$ is the full subcategory of objects with zero singular support, see [AG, Corollary 8.2.8].

0.1.4. Returning to the situation of LocSys_G, the assignment

$$P \rightsquigarrow \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$$

can be viewed as a diagram of categories, indexed by the poset $\mathrm{Par}(G)$ of standard parabolics of G.

Hence, we can talk about the category

$$\mathrm{Glue}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}, P \in \mathrm{Par}(G)), \tag{0.2}$$

obtained by *gluing* the categories $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$. The definition of the operation of gluing is reminded in Sect. 4.1.

The name 'gluing' is motivated by the following example: given a stratified topological space $Y = \underset{a\in A}{\cup} Y_a$ for a finite poset A, there is an equivalence between the category $\mathrm{Shv}(Y)$ of sheaves on Y and the glued category $\mathrm{Glue}(\mathrm{Shv}(Y_a), a \in A)$, see Example 4.1.7.

Informally, an object of (0.2) is a collection of objects

$$\mathcal{F}_P \in \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G} \quad \text{for all } P \in \mathrm{Par}(G),$$

plus a homotopy-coherent system of compatibility maps (but not necessarily isomorphisms)

$$\mathcal{F}_{P_2}|_{\mathrm{LocSys}_{P_1}} \to \mathcal{F}_{P_1} \quad \text{for all } P_1 \subset P_2.$$

0.1.5. For every P, pullback defines a functor

$$\mathrm{IndCoh}(\mathrm{LocSys}_G) \to \mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}.$$

Consider the composition

$$\begin{aligned} \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G) \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_G) \to \\ \to \mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G} \to \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}, \end{aligned}$$

where the last arrow is the right adjoint to the inclusion

$$\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G} \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}.$$

As $P \in \mathrm{Par}(G)$ varies, we obtain a functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G) \to \mathrm{Glue}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}, P \in \mathrm{Par}(G)). \tag{0.3}$$

The Gluing Conjecture reads:

Conjecture 0.1.6. *The functor* (0.3) *is* fully faithful.

As was mentioned earlier, the goal of the present paper is to prove this conjecture.

0.2. **The automorphic side of Langlands duality.** Let us now explain the counterpart of the Gluing Conjecture on the automorphic side of the categorical geometric Langlands conjecture.

As the contents of this subsection play a motivational role only, the reader may skip it and proceed to Sect. 0.3.

0.2.1. On the automorphic side of the categorical Langlands conjecture, we are dealing with the category D-mod($\mathrm{Bun}_{\check{G}}$). As explained in [Ga3, Sect. 8], the category is equipped with the functor of *extended Whittaker coefficient*

$$\text{W-coeff}^{\mathrm{ext}}_{\check{G},\check{G}} : \text{D-mod}(\mathrm{Bun}_{\check{G}}) \to \mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G}),$$

where $\mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G})$ is the *extended Whittaker category*.

Recall that the category $\mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G})$ is obtained by gluing:

$$\mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G}) \simeq \mathrm{Glue}(\mathrm{Whit}(\check{G},\check{P}), P \in \mathrm{Par}(G)),$$

where for a parabolic P, we denote by $\mathrm{Whit}(\check{G},\check{P})$ the P-degenerate Whittaker category (see [Ga3, Sect. 7]).

For example, for $P = G$, the category $\mathrm{Whit}(\check{G},\check{G})$ is the *usual* (that is, non-degenerate) Whittaker category of [Ga3, Sect. 5], and for $P = B$, the category $\mathrm{Whit}(\check{G},\check{B})$ is the *principal series* category $\mathrm{I}(\check{G},\check{B})$ of [Ga3, Sect. 6].

0.2.2. In [Ga3] there were formulated several 'quasi-theorems'[2] that jointly provide a canonically defined fully faithful functor

$$\mathrm{Glue}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}, P \in \mathrm{Par}(G)) \hookrightarrow \mathrm{Glue}(\mathrm{Whit}(\check{G},\check{P}), P \in \mathrm{Par}(G)) \simeq \mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G}).$$

Assuming the quasi-theorems hold, we obtain a diagram

$$\begin{array}{ccc} \mathrm{Glue}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}, P \in \mathrm{Par}(G)) & \longrightarrow & \mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G}) \\ {\scriptstyle (0.3)}\uparrow & & \uparrow{\scriptstyle \text{W-coeff}^{\mathrm{ext}}_{\check{G},\check{G}}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G) & & \text{D-mod}(\mathrm{Bun}_{\check{G}}). \end{array} \tag{0.4}$$

The categorical Langlands conjecture claims that there exists an equivalence

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G) \to \text{D-mod}(\mathrm{Bun}_{\check{G}})$$

complementing (0.4) to a commutative diagram.

In [Ga3], the following strategy for proving the categorical Langlands conjecture is suggested. First, one would show that the vertical arrows of (0.4) are fully faithful. Then, one would identify the essential images of $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G)$ and D-mod($\mathrm{Bun}_{\check{G}}$) in $\mathrm{Whit}^{\mathrm{ext}}(\check{G},\check{G})$ by using some explicit generators of both categories.

[2] By 'quasi-theorems' we mean plausible statements within reach of current methods.

0.2.3. Thus, one of the key steps in the proof of the categorical Langlands conjecture is to show that the vertical arrows in (0.4) are fully faithful. At this point, we do not know whether the functor W-coeff$^{\mathrm{ext}}_{G,\check{G}}$ (the right vertical arrow of the diagram) is fully faithful for an arbitrary group G; for $G = \mathrm{GL}_n$, it is a theorem, established in [Be].

On the other hand, the full faithfulness of the functor (0.3) (the left vertical arrow) is precisely the Gluing Conjecture, which we prove in the present paper.

0.3. **The methods: crystal structure.** We derive the Gluing Conjecture from a topological statement. Informally, the key idea is to study both sides 'microlocally.' The word 'microlocally' refers here to the correspondence between ind-coherent sheaves on a quasi-smooth scheme (or a stack) Y and conical subsets in the 'scheme of singularities' $\mathrm{Sing}(Y)$. We then relate certain categories obtained by gluing categories of ind-coherent (and quasi-coherent) sheaves to homotopy types obtained by gluing conical subsets of schemes of singularities. In particular, this applies to the category (0.2): as a result, the Gluing Conjecture reduces to homological triviality of certain homotopy types. Let us provide some details.

0.3.1. In [AG, Sect. 2.3], we explain how to associate to a quasi-smooth scheme Y a classical scheme of singularities $\mathrm{Sing}(Y)$ equipped with a $\mathbb{G}_m$-action. The scheme $\mathrm{Sing}(Y)$ measures how far Y is from being smooth.

The main construction of the paper [AG] assigns to an object $\mathcal{F} \in \mathrm{IndCoh}(Y)$ its singular support, denoted $\mathrm{SingSupp}(\mathcal{F})$, which is a conical Zariski-closed subscheme of $\mathrm{Sing}(Y)$.

It is technically easier for us to work with the category of singularities $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$ instead of $\mathrm{IndCoh}(Y)$, where

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) := \mathrm{IndCoh}(Y)/\mathrm{QCoh}(Y).$$

To an object $\mathcal{F} \in \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$ we can attach its singular support $\mathbb{P}\,\mathrm{SingSupp}(\mathcal{F})$, which is now a closed subscheme of the projectivization $\mathbb{P}\,\mathrm{Sing}(Y)$ of $\mathrm{Sing}(Y)$, see also [Ste].

A key observation, articulated in Sect. 1 of the present paper is that the assignment

$$\mathcal{F} \rightsquigarrow \mathbb{P}\,\mathrm{SingSupp}(\mathcal{F})$$

can be upgraded to a certain categorical structure: $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$ is in fact a *crystal of categories* over $\mathbb{P}\,\mathrm{Sing}(Y)$ (Theorem 1.4.2). Here is a reformulation of this statement:

Theorem 0.3.2. *There exists a canonical action of the (symmetric) monoidal category* $\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))$ *on* $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$.

In other words, this theorem says that $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ can be 'localized' onto $\mathbb{P}\,\mathrm{Sing}(Z)$.

0.3.3. The Gluing Conjecture concerns categories of sheaves with connection along fibers of a morphism. Let us define versions of the categories $\mathrm{IndCoh}(Z)$, $\mathrm{QCoh}(Z)$, and $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ for sheaves with connection.

Let $f : Z \to Y$ be a map of schemes. Consider the category

$$\mathrm{IndCoh}(Z)_{\mathrm{conn}\,/Y} := \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

introduced above.

In Sect. 3.1 (Proposition 3.1.2), we show that this category identifies with

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y),$$

and therefore contains

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Y)$$

as a full subcategory.

We are interested in the quotient $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$, which can be thought of as a version of the category of singularities.

Assume now that Y is quasi-smooth. In this case, we show in Sect. 3.1 (Proposition 3.1.8), that the above quotient can be expressed in terms of $\overset{\circ}{\mathrm{IndCoh}}(Y)$ by a *topological operation* using the above-mentioned crystal structure on $\overset{\circ}{\mathrm{IndCoh}}(Y)$. Namely, we have

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \text{D-mod}(Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y)) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y). \tag{0.5}$$

The word 'topological' refers to the fact that we are dealing with D-modules rather than (quasi)-coherent sheaves.

0.3.4. Assume now that in the above situation the scheme Z is quasi-smooth as well. Recall from [AG, Sect. 2.4] that in this case we have a canonically defined map

$$\mathrm{Sing}(f) : Z \underset{Y}{\times} \mathrm{Sing}(Y) \to \mathrm{Sing}(Z),$$

called the *singular codifferential* of f.

Furthermore, recall the category

$$\mathrm{QCoh}(Z)_{\mathrm{conn}/Y} \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

We have:

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{QCoh}(Z)_{\mathrm{conn}/Y} \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

where all the inclusions are, generally speaking, strict.

The key point for us is that the quotient

$$\mathrm{QCoh}(Z)_{\mathrm{conn}/Y}/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

can be described explicitly *in topological terms* using the equivalence (0.5). Namely, in Theorem 3.2.9 we prove:

Theorem 0.3.5. *Under the identification* (0.5)*, the full subcategory*

$$\mathrm{QCoh}(Z)_{\mathrm{conn}/Y}/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

corresponds to

$$\text{D-mod}(\mathbb{P}(\mathrm{Sing}(f)^{-1}(\{0\}))) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y) \subset \text{D-mod}(Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y)) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y).$$

In particular, we have a canonical equivalence:

$$\mathrm{QCoh}(Z)_{\mathrm{conn}\,/Y}/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \tag{0.6}$$
$$\simeq \text{D-mod}(\mathbb{P}(\mathrm{Sing}(f)^{-1}(\{0\}))) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y).$$

0.3.6. We have now set up an abstract framework for handling Conjecture 0.1.6. For simplicity, we work with schemes rather than stacks.

Let $Z_i \overset{f_i}{\to} Y$ be a diagram of quasi-smooth schemes, indexed by some category I. Suppose that the maps f_i are proper. Let $\mathcal{N} \subset \mathrm{Sing}(Y)$ be a fixed conical Zariski-closed subset. For each $i \in I$, we consider the composition

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Z_i)_{\mathrm{conn}\,/Y} \to \mathrm{QCoh}(Z_i)_{\mathrm{conn}\,/Y}.$$

Taken together, these functors yield a functor

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \to \mathrm{Glue}(\mathrm{QCoh}(Z_i)_{\mathrm{conn}\,/Y}, i \in I). \tag{0.7}$$

We want to determine whether (0.7) is fully faithful.

In Theorem 4.4.5 we prove the following sufficient condition.

Theorem 0.3.7. *Suppose the following two conditions hold:*

(1) *The corresponding functor*

$$\mathrm{QCoh}(Y) \to \lim_i \mathrm{QCoh}(Z_{i,\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

is fully faithful.

(2) *the corresponding functor*

$$\text{D-mod}(\mathbb{P}(\mathcal{N})) \to \mathrm{Glue}\left(\text{D-mod}(\mathbb{P}(\mathrm{Sing}(f_i)^{-1}(\{0\}))), i \in I\right) \tag{0.8}$$

is fully faithful.

Then the functor (0.7) *is fully faithful as well.*

Let us note that in the formation of the category $\mathrm{Glue}\left(\text{D-mod}(\mathbb{P}(\mathrm{Sing}(f_i)^{-1}(\{0\}))), i \in I\right)$, the functors

$$\text{D-mod}(\mathbb{P}(\mathrm{Sing}(f_j)^{-1}(\{0\})) \to \text{D-mod}(\mathbb{P}(\mathrm{Sing}(f_i)^{-1}(\{0\}))$$

for an arrow $i \to j$ in I *are not mere pullbacks*, but rather are given by pull-push along the *correspondence*

$$\begin{array}{ccc} Z_i \underset{Z_j}{\times} \mathbb{P}(\mathrm{Sing}(f_j)^{-1}(\{0\}) & \longrightarrow & \mathbb{P}(\mathrm{Sing}(f_i)^{-1}(\{0\}) \\ \big\downarrow & & \\ \mathbb{P}(\mathrm{Sing}(f_j)^{-1}(\{0\}). & & \end{array}$$

0.3.8. Finally, assume that in the above situation, the schemes Z_i are *proper* over Y. In this case, in Corollary 6.3.8 we show that the question of full faithfulness of the functor (0.8) can be reduced to that of *homological contractibility* of certain homotopy types.

Namely, for a k-point $\nu \in \mathcal{N}$ let $W_{i,\nu}$ denote the preimage of ν under the map

$$\mathrm{Sing}(f_i)^{-1}(\{0\}) \hookrightarrow Z_i \underset{Y}{\times} \mathrm{Sing}(Y) \to \mathrm{Sing}(Y).$$

For an arrow $i \to j$ in the category of indices I, the schemes $W_{i,\nu}$ and $W_{j,\nu}$ are related by the correspondence

$$\begin{array}{ccc} Z_i \underset{Z_j}{\times} W_{j,\nu} & \longrightarrow & W_{i,\nu} \\ \downarrow & & \\ W_{j,\nu}. & & \end{array}$$

In Sect. 6.3.3 we show how such a data gives rise to a prestack, denoted $W_{\mathrm{Glued},\nu}$. Namely, $W_{\mathrm{Glued},\nu}$ is the prestack colimit over the category of *strings*

$$i_0 \to i_1 \to \cdots \to i_n, \quad n \in \mathbb{N}, \quad i_j \in I$$

of the diagram of schemes that assigns to a string as above the scheme

$$Z_{i_0} \underset{Z_{i_n}}{\times} W_{i_n,\nu}.$$

We will prove:

Theorem 0.3.9. *The functor* (0.8) *is fully faithful if and only if for every ν not in the zero-section, the prestack $W_{\mathrm{Glued},\nu}$ is* homologically contractible*, i.e., the map*

$$\mathrm{C}_*(W_{\mathrm{Glued},\nu}) \to k$$

is an isomorphism.

Here C_* stands for homology (with coefficients in k). Note that if a prestack $\mathcal{W}$ is the colimit of schemes

$$\mathcal{W} = \operatorname*{colim}_{a \in A} W_a,$$

then its homology can be computed as

$$\mathrm{C}_*(\mathcal{W}) = \operatorname*{colim}_{a \in A} \mathrm{C}_*(W_a).$$

If the ground field k is $\mathbb{C}$, we can assign to $\mathcal{W}$ the homotopy type

$$\mathcal{W}^{\mathrm{top}} := \operatorname*{colim}_{a \in A} W_a^{\mathrm{top}}$$

(here the colimit is taken in the ∞-category of *spaces*, and for a scheme W_a we denote by W_a^{top} the underlying analytic space). In this case we have

$$\mathrm{C}_*(\mathcal{W}) \simeq \mathrm{C}_*(\mathcal{W}^{\mathrm{top}}).$$

So, the homology $\mathrm{C}_*(W_{\mathrm{Glued},\nu})$ appearing in Theorem 0.3.9 is indeed the homology of a canonically defined homotopy type.

0.3.10. The above discussion applies to the case when Y is a quasi-smooth algebraic stack rather than a scheme, and Z_i's are quasi-smooth algebraic stacks proper and schematic over Y.

The upshot is that the question of fully faithfulness of the functor

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \to \mathrm{Glue}(\mathrm{QCoh}(Z_i)_{\mathrm{conn}/Y}, i \in I)$$

is equivalent to that of homological contractibility, as stated in Theorem 0.3.9.

0.4. **The methods: global Springer fibers.**

0.4.1. Recall that our goal is to show that the functor (0.3) is fully faithful. According to Sect. 0.3.10, this follows from homological contractibility of certain homotopy types constructed using the maps

$$\begin{array}{ccc} \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G) & \longrightarrow & \mathrm{Sing}(\mathrm{LocSys}_P) \\ \downarrow & & \\ \mathrm{Sing}(\mathrm{LocSys}_G). & & \end{array}$$

Namely, fix a k-point $\mathrm{Nilp}_{\mathrm{glob}}$. We can think of such a point as a pair (σ, A), where σ is a G-local system on X, and A is a *horizontal* section of the vector bundle $\mathfrak{g}_\sigma$ associated with the adjoint representation.

Then, the corresponding scheme $W_{i,\nu}$ of Sect. 0.3.8 for $i = P$ and $\nu = (\sigma, A)$ is that of reductions of σ to P for which A is contained in the sub-bundle $\mathfrak{u}(P)_\sigma$, where $\mathfrak{u}(P)$ denotes the Lie algebra of the unipotent radical of P. We denote this scheme by

$$\mathrm{Spr}^{\sigma,A}_{P,\mathrm{unip}}.$$

In addition, we consider the schemes

$$\mathrm{Spr}^{\sigma,A}_{P,\mathrm{unip}} \subset \mathrm{Spr}^{\sigma,A}_{P} \subset \mathrm{Spr}^{\sigma}_{P}, \tag{0.9}$$

where Spr^{σ}_P is that of reductions of σ to P, and $\mathrm{Spr}^{\sigma,A}_P$ is the subscheme that corresponds to those reductions for which A is contained in $\mathfrak{p}_\sigma$.

All three of the above schemes can be viewed as global versions of the Springer fiber.

0.4.2. For $P_1 \subset P_2$, the schemes $\mathrm{Spr}^{\sigma,A}_{P_1,\mathrm{unip}}$ and $\mathrm{Spr}^{\sigma,A}_{P_2,\mathrm{unip}}$ are related by the correspondence

$$\begin{array}{ccc} \mathrm{Spr}^{\sigma}_{P_1} \underset{\mathrm{Spr}^{\sigma}_{P_2}}{\times} \mathrm{Spr}^{\sigma,A}_{P_2,\mathrm{unip}} & \longrightarrow & \mathrm{Spr}^{\sigma,A}_{P_1,\mathrm{unip}} \\ \downarrow & & \\ \mathrm{Spr}^{\sigma,A}_{P_2,\mathrm{unip}} & & \end{array}$$

and the colimit described in Sect. 0.3.8 yields a prestack, denoted $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$.

Combining the results of Sect. 0.3, we obtain that Conjecture 0.1.6 follows from the next result (it appears in the paper as Theorem 7.1.8):

Theorem 0.4.3. *For any (σ, A) with a nilpotent A, the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$ is homologically contractible.*

0.4.4. Although Theorem 0.4.3 is a concrete statement, it involves the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$, which is defined by a complicated procedure using correspondences. However, in Sect. 7, we show that Theorem 0.4.3 is equivalent to a statement about simpler objects.

Namely, let $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}$ be the colimit of the diagram of schemes

$$P \mapsto \mathrm{Spr}^{\sigma,A}_{P},$$

taken over the poset of standard *proper* parabolics of G (where $\mathrm{Spr}^{\sigma,A}_P$ is as in (0.9)).

We show, *assuming that Theorem 0.4.3 holds for proper Levi subgroups of G*, that Theorem 0.4.3 is equivalent to the next assertion (it appears in the paper as Theorem 7.2.5):

Theorem 0.4.5. *For any (σ, A) with a* non-zero *nilpotent A, the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}$ is homologically contractible.*

0.4.6. Theorem 0.4.5 is an essentially combinatorial statement that is proved in Sect. 8. The idea of the proof is the following:

By the Jacobson-Morozov Theorem, the section A defines a reduction of σ to a canonically defined parabolic P_0. This reduction gives rise to a stratification of each $\mathrm{Spr}_P^{\sigma,A}$ by elements of the Weyl group that measure the relative position of a given reduction to P with the canonical reduction to P_0.

For each $w \in W$, let

$$\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,<w} \subset \mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,\leq w} \subset \mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A}$$

be the corresponding substacks. Consider also

$$\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,<w} := \mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,\leq w} \underset{\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,<w}}{\sqcup} \mathrm{pt}\,.$$

We prove, by an analysis of the Weyl group combinatorics, that the prestack

$$\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A,<w}$$

is homologically contractible for every w.

This implies Theorem 0.4.5 by induction on the length of w.

0.5. **Contents.** The present paper is naturally divided into three parts.

0.5.1. In Part 1 we discuss the crystal structure on the category of singularities of a quasi-smooth scheme or algebraic stack, and its corollaries.

In Sect. 1 we state the main result of Part I, Theorem 1.4.2, which says that for a quasi-smooth scheme Z, there exists a canonically defined crystal of categories over $\mathbb{P}\,\mathrm{Sing}(Z)$, denoted $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}$, such that the category of singularities of Z, denoted

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z) := \mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z),$$

is recovered as the category of global sections of $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}$.

As was mentioned above, this theorem can be viewed as saying that $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ can be 'localized' onto $\mathbb{P}\,\mathrm{Sing}(Z)$. Due to the 1-affineness property of de Rham prestacks, this theorem can be equivalently phrased as saying that the (symmetric) monoidal category

$$\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Z)) := \mathrm{QCoh}(\mathbb{P}(\mathrm{Sing}(Z))_{\mathrm{dR}})$$

acts on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$.

In Sect. 2 we prove Theorem 1.4.2. Let us emphasize that it is naturally proved in the 'crystal of categories' formulation, rather than in the 'action of the category of D-modules' one.

In Sect. 3 we study the category $\mathrm{IndCoh}(Z)_{\mathrm{conn}\,/Y}$, defined for a morphism $Z \to Y$, and its various subcategories that can be described in terms of the crystal structure.

0.5.2. In Part II of the paper we state our main result, Theorem 4.3.4, and reduce it to the assertion that certain homotopy types are homologically contractible, namely, Theorem 7.1.8.

In Sect. 4 we recall the general paradigm of gluing of DG categories and state Theorem 4.3.4, which says that the Gluing Conjecture holds. In addition, we state Theorem 4.4.5, which says that a certain fully faithfulness condition purely at the level of D-modules implies a fully faithfulness result for ind-coherent sheaves.

It is fair to say that Theorem 4.4.5 contains the *main idea* of the present paper: it allows us to reduce the Gluing Conjecture to the question of homological contractibility.

Sect. 5 is devoted to the proof of Theorem 4.4.5.

In Sect. 6 we reformulate the condition of Theorem 4.4.5 (the pullback functor to the category obtained by gluing certain categories of D-modules is fully faithful) as homological contractibility of certain prestacks.

0.5.3. In Part III of the paper we prove Theorem 7.1.8, which verifies the required homological contractibility condition for the Gluing Conjecture.

In Sect. 7 we introduce global Springer fibers, state Theorem 7.1.8, and show that it is equivalent to a simpler homological contractibility statement (Theorem 7.2.5).

In Sect. 8 we prove Theorem 7.2.5 using an analysis of Weyl group combinatorics and Schubert strata.

Finally, in Sect. 9, we give an alternative proof of a special case Theorem 7.2.5, using the Springer correspondence.

0.6. **Conventions.**

0.6.1. *DG categories and ∞-categories.* This paper uses the language of ∞-categories. For example, the main result, Theorem 4.3.4, concerns the lax limit of ∞-categories. Our conventions regarding ∞-categories follow those of [AG]. In particular, the reader *does not* need to know how the theory of ∞-categories is constructed, but rather how to use it.

The primary object of study in this paper is DG categories (e.g., Theorem 4.3.4 says that a certain functor between DG categories is fully faithful). Again, the conventions pertaining to DG categories follow those of [AG]. Thus, all DG categories are assumed to be presentable, and in particular *cocomplete* (i.e., containing arbitrary direct sums); all functors are assumed *continuous* (i.e., preserving colimits).

0.6.2. We let $\mathrm{DGCat}_{\mathrm{cont}}$ denote the $(\infty,1)$-category of (presentable) DG categories and continuous functors. This $(\infty,1)$-category has a natural symmetric monoidal structure, given by tensor product

$$\mathbf{C}_1, \mathbf{C}_2 \to \mathbf{C}_1 \otimes \mathbf{C}_2.$$

Thus, we can talk about monoidal DG-categories (i.e., algebra objects in $\mathrm{DGCat}_{\mathrm{cont}}$ with respect to the above (symmetric) monoidal structure), and modules over them.

Given a monoidal DG category $\mathbf{O}$, we denote by $\mathbf{O}\text{-}\mathbf{mod}$ the category of $\mathbf{O}$-modules. Thus, $\mathbf{C} \in \mathbf{O}\text{-}\mathbf{mod}$ means that $\mathbf{C}$ is a DG category equipped with an action of $\mathbf{O}$

$$\mathbf{O} \otimes \mathbf{C} \to \mathbf{C}.$$

0.6.3. *Derived algebraic geometry.* This paper concerns quasi-coherent and ind-coherent sheaves on derived stacks. This puts us in the framework of *derived algebraic geometry*.

Our conventions regarding derived algebraic geometry follow those of [AG].

By a prestack we mean an arbitrary contravariant functor form the ∞-category of affine DG schemes that that of ∞-groupoids. (In particular, we say 'prestack' rather than 'DG prestack.') By an 'algebraic stack' we mean a derived algebraic stack. For a prestack $\mathcal{Y}$ there is a canonically defined category $\mathrm{QCoh}(\mathcal{Y})$ of quasi-coherent sheaves on $\mathcal{Y}$.

All DG schemes and prestacks considered in this paper are locally almost of finite type. For such schemes and prestacks, one has the theory of *ind-coherent* sheaves. The key tenets of this theory are recorded in [Ga1]. However, the main construction of this theory, namely that of the !-pullback, does not as yet appear in the published literature. A book-in-progress that contains this, as well as some other fundamental constructions of this theory, is available in the form of [GR2].

The following notation is used throughout the paper: for a prestack $\mathcal{Y}$ (assumed as always to be locally almost of finite type) there is a canonically defined object

$$\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y}),$$

the dualizing sheaf. We have a canonically defined functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \omega_{\mathcal{Y}}.$$

0.6.4. *Sheaves of categories.* In Part I of the paper, we use the notion of *sheaf of categories over a prestack* and some fundamental results about it (such as the notion of 1-affineness, its implications and its criteria). The reader is referred to [Ga2, Sects. 1 and 2] for a summary.

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Part I: Crystals and singular support.

1. The category of singularities as a crystal

Let Z be an affine DG scheme almost of finite type. In this section, we study the singularity category of Z

$$\overset{\circ}{\mathrm{IndCoh}}(Z) := \mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z).$$

The category $\overset{\circ}{\mathrm{IndCoh}}(Z)$ obviously 'lives over' Z, in the sense that its objects can be tensored by quasi-coherent sheaves on Z.

In this section we show that if Z is quasi-smooth, then the category $\overset{\circ}{\mathrm{IndCoh}}(Z)$ has a richer structure. Namely, it 'lives over' the relative de Rham prestack of $\mathrm{Sing}(Z)$, where the latter is the classical scheme measuring how far Z is from being smooth.

1.1. Recollections: singular support.

1.1.1. Let Z be an affine quasi-smooth DG scheme. Consider the DG categories $\mathrm{IndCoh}(Z)$ and $\mathrm{QCoh}(Z)$. Recall that according to [AG, Sect. 4.2.4], there is a canonically defined fully faithful functor

$$\Xi_Z : \mathrm{QCoh}(Z) \hookrightarrow \mathrm{IndCoh}(Z),$$

which admits a (continuous) right adjoint, denoted Ψ_Z.

We identify $\mathrm{QCoh}(Z)$ with the full subcategory $\Xi_Z(\mathrm{QCoh}(Z)) \subset \mathrm{IndCoh}(Z)$ using the functor Ξ_Z.

Remark 1.1.2. Recall there is another canonically defined functor

$$\Upsilon_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \omega_Z,$$

where $\omega_Z \in \mathrm{IndCoh}(Z)$ is the dualizing sheaf. Fortunately, when Z is quasi-smooth, the functors Ξ_Z and Υ_Z differ by tensoring by a line bundle. Hence their essential images in $\mathrm{IndCoh}(Z)$ coincide.

1.1.3. Define the *singularity category* of Z to be the quotient DG category

$$\overset{\circ}{\mathrm{IndCoh}}(Z) := \mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z).$$

Note that $\overset{\circ}{\mathrm{IndCoh}}(Z)$ identifies with the full subcategory [3] $\mathrm{QCoh}(Z)^{\perp} \subset \mathrm{IndCoh}(Z)$ (which equals $\ker(\Psi_Z)$).

Recall also that $\mathrm{IndCoh}(Z)$ is naturally a module category over $\mathrm{QCoh}(Z)$, and both functors Ξ_Z and Ψ_Z are compatible with the $\mathrm{QCoh}(Z)$-actions. Hence, $\overset{\circ}{\mathrm{IndCoh}}(Z)$ also acquires a natural structure of $\mathrm{QCoh}(Z)$-module category.

[3] Here and elsewhere, for a full subcategory $\mathbf{C}' \subset \mathbf{C}$, we denote by $(\mathbf{C}')^{\perp} \subset \mathbf{C}$ its *right* orthogonal, i.e., the full subcategory consisting of objects that receive no non-zero maps from objects of $\mathbf{C}'$.

1.1.4. Recall (see [AG, Sect. 2.3]) that to the DG scheme Z one attaches the classical scheme $\mathrm{Sing}(Z)$ equipped with

- a $\mathbb{G}_m$-action,
- a projection $\mathrm{Sing}(Z) \to Z$,
- a zero section ${}^{\mathrm{cl}}Z \to \mathrm{Sing}(Z)$.

By a slight abuse of notation, we denote the image of the zero section by $\{0\} \subset \mathrm{Sing}(Z)$.

The action of $\mathbb{G}_m$ on $\mathrm{Sing}(Z) - \{0\}$ is free. Put

$$\mathbb{P}\,\mathrm{Sing}(Z) := (\mathrm{Sing}(Z) - \{0\})/\mathbb{G}_m.$$

1.1.5. The main construction of the paper [AG] (namely [AG, Defn. 4.1.4], essentially borrowed from [BIK]) assigns to an object $\mathcal{F} \in \mathrm{IndCoh}(Z)$ a Zariski-closed conical subset

$$\mathrm{SingSupp}(\mathcal{F}) \subset \mathrm{Sing}(Z).$$

Conversely, a Zariski-closed conical subset $\mathcal{N} \subset \mathrm{Sing}(Z)$ yields a full subcategory

$$\mathrm{IndCoh}_{\mathcal{N}}(Z) := \{\mathcal{F} \,|\, \mathrm{SingSupp}(\mathcal{F}) \subset \mathcal{N}\} \subset \mathrm{IndCoh}(Z).$$

The following is [AG, Theorem 4.2.6]:

Theorem 1.1.6. *The full subcategories* $\mathrm{IndCoh}(Z)_{\{0\}}$ *and* $\mathrm{QCoh}(Z)$ *of* $\mathrm{IndCoh}(Z)$ *coincide.*

1.1.7. From Theorem 1.1.6 we obtain that to an object $\mathcal{F} \in \overset{\circ}{\mathrm{IndCoh}}(Z)$ we can assign a Zariski-closed subset

$$\mathbb{P}\,\mathrm{SingSupp}(\mathcal{F}) \subset \mathbb{P}\,\mathrm{Sing}(Z).$$

Conversely, a Zariski-closed conical subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(Z)$ yields a full subcategory

$$\overset{\circ}{\mathrm{IndCoh}}_{\mathcal{N}}(Z) := \{\mathcal{F} \,|\, \mathbb{P}\,\mathrm{SingSupp}(\mathcal{F}) \subset \mathcal{N}\} \subset \overset{\circ}{\mathrm{IndCoh}}(Z).$$

1.2. **Recollections: sheaves of categories.**

1.2.1. Recall the notion of a *quasi-coherent sheaf of categories over a prestack* introduced in [Ga2, Sect. 1.1]. For a prestack $\mathcal{Y}$, a quasi-coherent sheaf of categories $\mathcal{C}$ over $\mathcal{Y}$ consists of the following data:

- A $\mathrm{QCoh}(S)$-module $\mathcal{C}_{S,y} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod}$ for every $(S, y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$;
- An identification of $\mathrm{QCoh}(S')$-modules

$$\mathcal{C}_{S',y'} \simeq \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathcal{C}_{S,y}$$

  for every morphism $S' \xrightarrow{f} S$, where $(S, y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ and $y' = y \circ f$;
- A homotopy-coherent system of compatibilities between the identifications for higher-order compositions.

Denote the category of quasi-coherent sheaves of categories over $\mathcal{Y}$ by $\mathrm{ShvCat}(\mathcal{Y})$.

1.2.2. If $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$, the *category of global sections* of $\mathcal{C}$ is defined as

$$\mathbf{\Gamma}(\mathcal{Y}, \mathcal{C}) := \underset{(S,y) \in \mathrm{PreStk}_{/\mathcal{Y}}}{lim} \mathcal{C}_{S,y}.$$

It is a DG category equipped with a natural action of the (symmetric) monoidal category $\mathrm{QCoh}(\mathcal{Y})$ (see [Ga2, Sect. 1.2]); indeed, $\mathrm{QCoh}(\mathcal{Y})$ acts on each term $\mathcal{C}_{S,y}$.

1.2.3. The category $\mathrm{ShvCat}(\mathcal{Y})$ is naturally enriched over $\mathrm{DGCat}_{\mathrm{cont}}$. Using this structure, we can think of of $\mathbf{\Gamma}(\mathcal{Y},\mathcal{C})$ as the DG category of maps from $\mathrm{QCoh}_{/\mathcal{Y}}$ to $\mathcal{C}$, where $\mathrm{QCoh}_{/\mathcal{Y}}$ is the *unit* sheaf of categories given by

$$(\mathrm{QCoh}_{/\mathcal{Y}})_{S,y} := \mathrm{QCoh}(S) \quad \text{for all } (S,y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}.$$

Note that $\mathbf{\Gamma}(\mathcal{Y},\mathrm{QCoh}_{/\mathcal{Y}}) \simeq \mathrm{QCoh}(\mathcal{Y})$.

1.2.4. Recall (see [Ga2, Sefinition 1.3.7]) that a prestack $\mathcal{Y}$ is said to be *1-affine* if the above functor

$$\mathbf{\Gamma}(\mathcal{Y},-) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$$

is an equivalence of categories.

1.2.5. For future reference, recall the following constructions. Let $g : \mathcal{Z} \to \mathcal{Y}$ be a map of prestacks. In this case, we have a tautologically defined functor

$$\mathbf{cores}_g : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{ShvCat}(\mathcal{Z}),$$

given by restriction: for $(S,z) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Z}}$ we have

$$(\mathbf{cores}_g(\mathcal{C}))_{S,z} := \mathcal{C}_{S,g\circ z}.$$

Note that $\mathbf{cores}_g(\mathrm{QCoh}_{/\mathcal{Y}}) \simeq \mathrm{QCoh}_{/\mathcal{Z}}$.

Slightly abusing the notation, we sometimes write

$$\mathbf{\Gamma}(\mathcal{Z},\mathcal{C}) := \mathbf{\Gamma}(\mathcal{Z},\mathbf{cores}_g(\mathcal{C})) \quad \text{for } \mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}).$$

We sometimes write for $(S,y) \in (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$ and $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y})$

$$\mathbf{\Gamma}(S,\mathcal{C}) := \mathcal{C}_{S,y}.$$

1.2.6. The above functor $\mathbf{cores}_g$ admits a right adjoint, which we denote by

$$\mathbf{coind}_g : \mathrm{ShvCat}(\mathcal{Z}) \to \mathrm{ShvCat}(\mathcal{Y}).$$

It can be explicitly described as follows:

$$(\mathbf{coind}_g(\mathcal{C}))_{S,y} = \mathbf{\Gamma}(S \underset{\mathcal{Y}}{\times} \mathcal{Z},\mathcal{C}) \quad \text{for all } (S,y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}},$$

see [Ga2, Sect. 3.1.3]. Here $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Z})$

By adjunction and using Sect. 1.2.3, we have

$$\mathbf{\Gamma}(\mathcal{Y},\mathbf{coind}_g(\mathcal{C})) \simeq \mathbf{\Gamma}(\mathcal{Z},\mathcal{C}).$$

1.3. **Recollections: the de Rham prestack.**

1.3.1. Recall (see e.g., [GR1, Sect. 1.1.1]) that the *de Rham prestack* $\mathcal{Y}_{\mathrm{dR}}$ of a prestack $\mathcal{Y}$ is defined by

$$\mathrm{Maps}(S,\mathcal{Y}_{\mathrm{dR}}) = \mathrm{Maps}({}^{\mathrm{red}}S,\mathcal{Y}), \quad S \in \mathrm{DGSch}^{\mathrm{aff}}.$$

We have a tautological projection

$$p_{\mathcal{Y},\mathrm{dR}} : \mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}}.$$

For this paper, we only consider $\mathcal{Y}_{\mathrm{dR}}$ for prestacks $\mathcal{Y}$ of locally (almost) finite type[4]. In this case, it is shown in [GR1, Proposition 1.3.3] that $\mathcal{Y}_{\mathrm{dR}}$ is classical and also locally almost of finite type.

[4]The word 'almost' is parenthesized because $\mathcal{Y}_{\mathrm{dR}}$ only depends on the classical prestack underlying $\mathcal{Y}$.

The basic fact concerning $\mathcal{Y}_{\mathrm{dR}}$ is a canonical equivalence of categories

$$\mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}) \simeq \text{D-mod}(\mathcal{Y}),$$

which, properly speaking, must be taken as the definition of the category D-mod($\mathcal{Y}$).

1.3.2. The main object of study in this paper is sheaves of categories over prestacks of the form $\mathcal{Y}_{\mathrm{dR}}$. They can be alternatively called 'crystals of categories over $\mathcal{Y}$.'

Let us now list several useful facts about crystals of categories. The first is the following (see [Ga2, Theorem 2.6.3]):

Proposition 1.3.3. *Let Y be a (DG) scheme of finite type. Then Y_{dR} is 1-affine.*

Remark 1.3.4. We should warn the reader that not all prestacks one encounters in practice are 1-affine. For example, although it is shown in [Ga2, Theorem 2.2.6] that a quasi-compact algebraic stack $\mathcal{Y}$ is 1-affine under some mild technical assumptions, the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ is typically *not* 1-affine (see [Ga2, Proposition 2.6.5]).

1.3.5. For the rest of this subsection we fix a prestack $\mathcal{Y}$ and a closed embedding $i : \mathcal{Z} \to \mathcal{Y}$. Note that by the finite type assumption, the complementary open embedding $j : \overset{\circ}{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ is a quasi-compact morphism.

We have the following assertion (see [Ga2, Sect. 4]):

Proposition 1.3.6. *Consider the maps*

$$\mathcal{Z}_{\mathrm{dR}} \xrightarrow{i_{\mathrm{dR}}} \mathcal{Y}_{\mathrm{dR}} \xleftarrow{j_{\mathrm{dR}}} \overset{\circ}{\mathcal{Y}}_{\mathrm{dR}}.$$

(a) *The functor*

$$\mathbf{coind}_{i_{\mathrm{dR}}} : \mathrm{ShvCat}(\mathcal{Z}_{\mathrm{dR}}) \to \mathrm{ShvCat}(\mathcal{Y}_{\mathrm{dR}})$$

is fully faithful. Its essential image consists of those objects that are annihilated by the functor $\mathbf{cores}_{j_{\mathrm{dR}}}$.

(b) *For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}_{\mathrm{dR}})$, the functor*

$$\mathbf{\Gamma}(\mathcal{Y}_{\mathrm{dR}}, \mathcal{C}) \to \mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C})$$

induces an equivalence

$$\ker\left(\mathbf{\Gamma}(\mathcal{Y}_{\mathrm{dR}}, \mathcal{C}) \to \mathbf{\Gamma}(\overset{\circ}{\mathcal{Y}}_{\mathrm{dR}}, \mathcal{C})\right) \to \mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C}).$$

1.3.7. From now on, we use claim (b) of Proposition 1.3.6 to identify $\mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C})$ and $\ker\left(\mathbf{\Gamma}(\mathcal{Y}_{\mathrm{dR}}, \mathcal{C}) \to \mathbf{\Gamma}(\overset{\circ}{\mathcal{Y}}_{\mathrm{dR}}, \mathcal{C})\right)$. Thus, we consider $\mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C})$ as a full subcategory

$$\mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C}) \subset \mathbf{\Gamma}(\mathcal{Y}_{\mathrm{dR}}, \mathcal{C}).$$

We also have the following (tautological) assertion:

Lemma 1.3.8. *If in the situation of Proposition 1.3.6(b) the prestack $\mathcal{Y}_{\mathrm{dR}}$ is 1-affine, then the full subcategory*

$$\mathbf{\Gamma}(\mathcal{Z}_{\mathrm{dR}}, \mathcal{C}) \subset \mathbf{\Gamma}(\mathcal{Y}_{\mathrm{dR}}, \mathcal{C})$$

consists of objects annihilated by the monoidal ideal

$$\ker\left(\mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathcal{Z}_{\mathrm{dR}})\right) \subset \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}).$$

1.4. **Statement of the result.** Return now to the setup of Sect. 1.1. Thus, Z is an affine quasi-smooth DG scheme. The notion of singular support provides natural assignments

$$\mathcal{F} \in \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z) \;\rightsquigarrow\; \mathbb{P}\,\mathrm{SingSupp}(\mathcal{F}) \subset \mathbb{P}\,\mathrm{Sing}(Z)$$

and

$$\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(Z) \;\rightsquigarrow\; \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(\mathrm{IndCoh}(Z)) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$$

(see Sect. 1.1). The goal of this section is to refine the assignments to a richer structure.

1.4.1. Consider the prestack $(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}$. We will prove:

Theorem-Construction 1.4.2. *There exists a canonically defined object*

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim} \in \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}),$$

equipped with an identification

$$\mathbf{\Gamma}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}) \simeq \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z).$$

This construction has the following properties:

(a) *For a Zariski-closed subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(Z)$, the full subcategory $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(Z) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ coincides with*

$$\mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}) \subset \mathbf{\Gamma}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}).$$

(b) *The action of $\mathrm{QCoh}(Z_{\mathrm{dR}})$ on $\mathbf{\Gamma}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim})$ coming from the (symmetric) monoidal functor*

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \to \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}})$$

and the natural action of the latter on $\mathbf{\Gamma}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim})$ identifies with the action of $\mathrm{QCoh}(Z_{\mathrm{dR}})$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ coming from the (symmetric) monoidal functor

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \to \mathrm{QCoh}(Z),$$

and the action of the latter on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z) \subset \mathrm{IndCoh}(Z)$.

Remark 1.4.3. Note that Theorem 1.4.2 relates the category of singularities

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z) := \mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z)$$

and the projectivization $\mathbb{P}\,\mathrm{Sing}(Z)$ of $\mathrm{Sing}(Z)$. It would be interesting to find a similar structure on $\mathrm{IndCoh}(Z)$ itself.

1.4.4. According to Lemma 1.3.8 and Proposition 1.3.3, Theorem 1.4.2 is equivalent to the following:

Corollary 1.4.5. *The category $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ carries a canonically defined action of the (symmetric) monoidal category $\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}})$ such that:*

(a) *For a Zariski-closed subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(Z)$, the full subcategory $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(Z) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ coincides with the full subactegory of objects annihilated by the monoidal ideal*

$$\ker(\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathcal{N}_{\mathrm{dR}})).$$

(b) *The action of $\mathrm{QCoh}(Z_{\mathrm{dR}})$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ coming from the (symmetric) monoidal functor*

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \to \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}})$$

identifies with the action of $\mathrm{QCoh}(Z_{\mathrm{dR}})$ *on* $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ *coming from the (symmetric) monoidal functor*

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \to \mathrm{QCoh}(Z)$$

and the action of $\mathrm{QCoh}(Z)$ *on* $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z) \subset \mathrm{IndCoh}(Z)$.

1.5. **Upgrade to a relative crystal of categories.** We postpone the proof of Theorem 1.4.2 until Sect. 2. Let us state a slight refinement of the theorem concerning the structure of a *relative* crystal of categories on the category of singularities. This refined structure naturally allows us to extend the theory from the case of an affine DG scheme Z to that of an algebraic stack.

1.5.1. Consider the (classical reduced) scheme $\mathbb{P}\,\mathrm{Sing}(Z)$, and the prestack

$$(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z.$$

Informally, this prestack can be thought of as the 'relative' de Rham stack of $\mathbb{P}\,\mathrm{Sing}(Z)$ over the base Z. Let $(\mathrm{id} \times p_{\mathrm{dR},Z})$ denote the tautological map

$$(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z \to (\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}.$$

Consider the corresponding functor

$$\mathbf{coind}_{(\mathrm{id} \times p_{\mathrm{dR},Z})} : \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z) \to \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}).$$

Proposition-Construction 1.5.2. *There exists a canonically defined object*

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim,\mathrm{rel}} \in \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z),$$

equipped with an identification

$$\mathbf{coind}_{(\mathrm{id} \times p_{\mathrm{dR},Z})}(\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim,\mathrm{rel}}) \simeq \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)^{\sim}.$$

Let us now derive Proposition 1.5.2 from Theorem 1.4.2.

1.5.3. First, we claim:

Lemma 1.5.4. *The prestack* $(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z$ *is 1-affine.*

Proof. We can realize $\mathbb{P}\,\mathrm{Sing}(Z)$ as a closed subscheme of $Z \times \mathbb{P}^n$. Hence, we have a map

$$(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z \to (\mathbb{P}^n)_{\mathrm{dR}} \times Z,$$

which is a closed embedding. Hence, by [Ga2, Corollary 3.2.7], it suffices to show that $(\mathbb{P}^n)_{\mathrm{dR}} \times Z$ is 1-affine. However, the latter follows from [Ga2, Corollary 3.2.8].

□

1.5.5. By Lemma 1.5.4, we obtain that in order to prove Proposition 1.5.2, we need to extend the action of the (symmetric) monoidal category $\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}})$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ to that of the (symmetric) monoidal category

$$\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z).$$

We now claim:

Lemma 1.5.6. *For any map of DG schemes almost of finite type $Z' \to Z$, the functor*

$$\mathrm{QCoh}(Z'_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Z_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z)$$

is an equivalence.

Proof. Follows from [Ga2, Proposition 3.1.9]. □

In particular, we obtain that the symmetric monoidal functor

$$\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Z_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Z) \to \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z)$$

is an equivalence.

Now, the action of $\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z)$ on $\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$ is obtained by combining Lemma 1.5.6 and the compatibility statement Theorem 1.4.2(b).

□(Proposition 1.5.2)

1.6. **Extension to algebraic stacks.**

1.6.1. Let now $\mathcal{Z}$ be a quasi-smooth algebraic stack with an affine diagonal (see [AG, Sect. 8.1.1] for the definition).

Let $\mathrm{Sing}(\mathcal{Z})$ be the corresponding (classical) algebraic stack, constructed in [AG, Sect. 8.1.5], and consider the corresponding stack $\mathbb{P}\,\mathrm{Sing}(\mathcal{Z})$.

1.6.2. Consider the category $\mathrm{IndCoh}(\mathcal{Z})$, the subcategory $\mathrm{QCoh}(\mathcal{Z}) \overset{\Xi_{\mathcal{Z}}}{\hookrightarrow} \mathrm{IndCoh}(\mathcal{Z})$, and the quotient category

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Z}) := \mathrm{IndCoh}(\mathcal{Z})/\mathrm{QCoh}(\mathcal{Z}),$$

which identifies with the full subcategory

$$\mathrm{QCoh}(\mathcal{Z})^{\perp} = \ker(\Psi_{\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{QCoh}(\mathcal{Z})) \subset \mathrm{IndCoh}(\mathcal{Z}).$$

The constructions of Sect. 1.1.7 and [AG, Sect. 8.2] generalize to define for every $\mathcal{F} \in \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Z})$ the Zariski-closed subset

$$\mathbb{P}\,\mathrm{SingSupp}(\mathcal{F}) \subset \mathbb{P}\,\mathrm{Sing}(\mathcal{Z}),$$

and for a Zariski-closed subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(\mathcal{Z})$, the full subcategory

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(\mathcal{Z}) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Z}).$$

1.6.3. We claim:

Proposition 1.6.4. *There exists a canonically defined object*

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Z})^{\sim,\mathrm{rel}} \in \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}),$$

equipped with the following system of identifications:

(a) *For an affine DG scheme Z equipped with a* smooth *map $Z \to \mathcal{Z}$, we have a canonical identification*

$$\mathbf{\Gamma}\left(\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(\mathcal{Z})^{\sim}\right) \simeq \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z),$$

as categories equipped with an action of $\mathrm{QCoh}(Z)$.

(b) *For a* smooth *map $g : Z_1 \to Z_2$ of affine DG schemes smooth over $\mathcal{Z}$, the diagram*

$$\begin{array}{ccc} \mathbf{\Gamma}\left(\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z_2, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim}\right) & \longrightarrow & \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_2) \\ \downarrow & & \downarrow{\scriptstyle g^!} \\ \mathbf{\Gamma}\left(\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z_1, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim}\right) & \longrightarrow & \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_1) \end{array}$$

commutes.

Proof. The proof is completely formal:

Let $(\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{smooth}\,/\mathcal{Z}}$ be the category of affine DG schemes Z equipped with a smooth map to $\mathcal{Z}$. By [Ga2, Theorem 1.5.7], in order to construct $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}$, it is sufficient to construct an assignment

$$Z \in (\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{smooth}\,/\mathcal{Z}} \rightsquigarrow \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}|_Z \in \mathrm{ShvCat}\left(\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z\right)$$

equipped with a coherent system of identifications

$$g : Z_1 \to Z_2 \ \rightsquigarrow \ \mathbf{cores}_{\mathrm{id}\times g}(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}|_{Z_2}) \simeq \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}|_{Z_1}.$$

Given $Z \in (\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{smooth}\,/\mathcal{Z}}$, we set

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}|_Z := \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim,\mathrm{rel}}.$$

Note that

$$\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z \simeq (\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z.$$

It remains to construct an identification

$$\mathbf{cores}_{\mathrm{id}\times g}(\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_2)^{\sim,\mathrm{rel}}) \simeq \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_1)^{\sim,\mathrm{rel}} \tag{1.1}$$

for a morphism $g : Z_1 \to Z_2$ in $(\mathrm{DGSch}^{\mathrm{aff}})_{\mathrm{smooth}\,/\mathcal{Z}}$. Since $(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{Z_{\mathrm{dR}}}{\times} Z$ is 1-affine, an identification (1.1) amounts to an identification

$$\mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Z_2))_{\mathrm{dR}} \underset{(Z_2)_{\mathrm{dR}}}{\times} Z_2, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_2)^{\sim,\mathrm{rel}}\right) \underset{\mathrm{QCoh}(Z_2)}{\otimes} \mathrm{QCoh}(Z_1) \simeq$$
$$\simeq \mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Z_1))_{\mathrm{dR}} \underset{(Z_1)_{\mathrm{dR}}}{\times} Z_1, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_1)^{\sim,\mathrm{rel}}\right)$$

in $\mathrm{QCoh}\left((\mathbb{P}\,\mathrm{Sing}(Z_1))_{\mathrm{dR}} \underset{(Z_1)_{\mathrm{dR}}}{\times} Z_1\right)$-**mod**.

Since

$$\mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Z_i))_{\mathrm{dR}} \underset{(Z_i)_{\mathrm{dR}}}{\times} Z_i, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_i)^{\sim,\mathrm{rel}}\right) \simeq \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_i),$$

it remains to construct an identification

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_2) \underset{\mathrm{QCoh}(Z_2)}{\otimes} \mathrm{QCoh}(Z_1) \simeq \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_1).$$

Such identification is given by the functor $g^!$, see [Ga1, Corollary 7.5.7]. □

1.6.5. We now claim:

Proposition 1.6.6. *There exists a canonical identification*

$$\mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim,\mathrm{rel}}\right) \simeq \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z}).$$

Moreover, for a Zariski-closed subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(\mathcal{Z})$, the full subcategory

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}_{\mathcal{N}}(\mathcal{Z}) \subset \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})$$

equals

$$\mathbf{\Gamma}\left(\mathcal{N}_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}\right) \subset \mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}\right).$$

Proof. Follows by combining Theorem 1.4.2(a) and [AG, Proposition 8.3.4]. □

2. Proof of Theorem 1.4.2

2.1. **Idea of the proof.** Before we give the proof, let us explain informally its main idea.

2.1.1. To specify a sheaf of categories $\mathcal{C}$ over $(\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}))_{\mathrm{dR}}$, we need to assign a category $\mathbf{\Gamma}(S, \mathcal{C})$ to any affine DG scheme S equipped with a map

$$^{\mathrm{red}}S \to \mathbb{P}\,\mathrm{Sing}(\mathcal{Z}).$$

In the case of the sheaf $\mathcal{C} = \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}$, we take $\mathbf{\Gamma}(S, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim})$ to be a certain full subcategory in

$$\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z).$$

2.1.2. Namely, for an object $\mathcal{F} \in \mathrm{QCoh}(S) \otimes \mathrm{IndCoh}(Z)$ we can talk about its singular support, which is a closed subset in $S \times \mathrm{Sing}(Z)$, conical with respect to the $\mathbb{G}_m$-action on the second factor. Note that if $\mathcal{F} \in \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(Z)$, then its singular support is contained in $S \times \{0\}$. Hence, to an object of

$$\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)$$

we can attach its singular support, which is a closed subset of $S \times \mathbb{P}\,\mathrm{Sing}(\mathcal{Z})$.

Now, let

$$\mathbf{\Gamma}(S, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}) \subset \mathrm{QCoh}(S) \otimes \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)$$

be the full subcategory of objects whose singular support is contained (set-theoretically) in the graph of the given map $^{\mathrm{red}}S \to \mathbb{P}\,\mathrm{Sing}(\mathcal{Z})$.

2.1.3. To prove that the above construction works, we need to do two things:

(i) Show that the assignment $S \rightsquigarrow \mathbf{\Gamma}(S, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim})$ is indeed a sheaf of categories. This is not difficult.

(ii) Show that a naturally constructed functor $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z) \to \mathbf{\Gamma}(\mathbb{P}\,\mathrm{Sing}(\mathcal{Z}), \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim})$ is an equivalence. To do so, we will reduce to the case when Z is a global complete intersection and use some explicit analysis.

2.1.4. Rather than giving the proof specifically for $\overset{\circ}{\mathrm{IndCoh}}(Z)$, below we do it in an abstract setting, by isolating the relevant pieces of structure.

Namely, instead of $\mathrm{IndCoh}(Z)$ we have an arbitrary DG category $\mathbf{C}$, and the role of the $\mathbb{E}_2$-algebra of Hochschild cochains (whose action on $\mathrm{IndCoh}(Z)$ gives rise to the notion of singular support), we use an arbitrary $\mathbb{E}_2$-algebra $\mathcal{A}$.

2.2. **Abstract setting for Theorem 1.4.2.**

2.2.1. Let $\mathbf{C}$ be a DG category, equipped with an action of an $\mathbb{E}_2$-algebra $\mathcal{A}$ (see [AG, Sect. 3.5] for what this means). Let A be a commutative finitely generated algebra, graded by even non-negative integers, equipped with a grading-preserving homomorphism

$$A \to H^\bullet(\mathcal{A}) := \underset{n}{\oplus} H^n(\mathcal{A}).$$

According to [AG, Sect. 3.5] (by the construction going back to [BIK]), to any $\mathbf{c} \in \mathbf{C}$ we can attach its support, denoted $\mathrm{supp}_A(\mathbf{c})$, which is a conical Zariski-closed subset of $\mathrm{Spec}(A)$.

Vice versa, to a conical Zariski-closed subset $\mathcal{N} \subset \mathrm{Spec}(A)$ we assign the full subcategory

$$\mathbf{C}_{\mathcal{N}} \subset \mathbf{C},$$

consisting of objects with support in $\mathcal{N}$.

2.2.2. Let A^0 be the degree 0 component of A. The projection $\mathrm{Spec}(A) \to \mathrm{Spec}(A^0)$ admits a canonically defined section $\mathrm{Spec}(A^0) \to \mathrm{Spec}(A)$, because we can identify A^0 with the quotient algebra of A by the ideal $A^{>0}$.

Let $\{0\}$ denote the subset of $\mathrm{Spec}(A)$ equal to the image $\mathrm{Spec}(A^0)$ under the above section. Let $\mathbf{C}_{\{0\}}$ be the corresponding full subcategory of $\mathbf{C}$. Define

$$\overset{\circ}{\mathbf{C}} := \mathbf{C}/\mathbf{C}_{\{0\}}.$$

We can also think of $\overset{\circ}{\mathbf{C}}$ as the kernel of the co-localization functor $\mathbf{C} \to \mathbf{C}_{\{0\}}$, right adjoint to the tautological embedding; this is the same as $(\mathbf{C}_{\{0\}})^\perp \subset \mathbf{C}$.

2.2.3. Consider the scheme $\mathrm{Proj}(A)$. The assignment

$$\mathbf{c} \in \mathbf{C} \ \rightsquigarrow \ \mathrm{supp}_A(\mathbf{c}) \subset \mathrm{Spec}(A)$$

gives rise to an assignment

$$\mathbf{c} \in \overset{\circ}{\mathbf{C}} \ \rightsquigarrow \ \mathbb{P}\mathrm{supp}_A(\mathbf{c}) \subset \mathrm{Proj}(A).$$

Vice versa, to a Zariski-closed subset $\mathcal{N} \subset \mathrm{Proj}(A)$ we assign the full subcategory

$$\overset{\circ}{\mathbf{C}}_{\mathcal{N}} = \{\mathbf{c} \in \overset{\circ}{\mathbf{C}} \,|\, \mathbb{P}\mathrm{supp}_A(\mathbf{c}) \subset \mathcal{N}\} \subset \overset{\circ}{\mathbf{C}}.$$

2.3. **Plan of this section.**

2.3.1. In Sect. 2.4, we attach a certain sheaf of categories $\mathcal{C}_A \in \mathrm{ShvCat}(\mathrm{Proj}(A)_{\mathrm{dR}})$ to the data $(\mathbf{C}, \mathcal{A}, A)$ as above.

In Sect. 2.5, we show that $\mathcal{C}_A$ comes equipped with a functor

$$\overset{\circ}{\mathbf{C}} \to \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A). \tag{2.1}$$

More generally, for a Zariski-closed subset $\mathcal{N} \subset \mathrm{Proj}(A)$, there is a functor

$$\overset{\circ}{\mathbf{C}}_{\mathcal{N}} \to \mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}}, \mathcal{C}_A). \tag{2.2}$$

We then provide additional conditions on the triple $(\mathbf{C}, \mathcal{A}, A)$ (in Sect. 2.5.5) that guarantee that the functor (2.2), and in particular (2.1), is an equivalence. The proof of this claim (Proposition 2.5.6) occupies Sections 2.7–2.9.

2.3.2. In Sect. 2.6 we apply this discussion to

$$\mathbf{C} := \mathrm{IndCoh}(Z),\ \mathcal{A} := \mathrm{HC}(Z),\ A := \Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)}).$$

In the above formula, $\mathrm{HC}(Z)$ is the $\mathbb{E}_2$-algebra of Hochschild cochains on Z, or, which is the same, the $\mathbb{E}_2$-center of the DG category $\mathrm{IndCoh}(Z)$, see [AG, Appendix F].

The resulting sheaf of categories category $\mathcal{C}_A$ is the sought-for

$$\overset{\circ}{\mathrm{IndCoh}}(Z)^{\sim} \in \mathrm{ShvCat}((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}).$$

The equivalence (2.2) proves point (a) of Theorem 1.4.2.

2.3.3. To establish point (b) of Theorem 1.4.2, we study the interaction of the construction

$$(\mathbf{C}, \mathcal{A}, A) \rightsquigarrow \mathcal{C}_A$$

with some pre-existing monoidal actions; this is done in Sect. 2.10.

2.4. **Construction of the sheaf of categories.**

2.4.1. For $S \in \mathrm{DGSch}^{\mathrm{aff}}$ consider the category

$$\mathrm{QCoh}(S) \otimes \mathbf{C}.$$

The action of $\mathcal{A}$ on $\mathbf{C}$ and the action of the $\mathbb{E}_\infty$-algebra $\Gamma(S, \mathcal{O}_S)$, viewed as a $\mathbb{E}_2$-algebra, on $\mathrm{QCoh}(S)$ give rise to the action of the $\mathbb{E}_2$-algebra $\Gamma(S, \mathcal{O}_S) \otimes \mathcal{A}$ on $\mathrm{QCoh}(S) \otimes \mathbf{C}$.

Note that we have a canonical map of commutative algebras

$$A_S := H^0(\Gamma(S, \mathcal{O}_S)) \otimes A \to H^\bullet(\Gamma(S, \mathcal{O}_S) \otimes \mathcal{A}).$$

2.4.2. Consider the corresponding categories $(\mathrm{QCoh}(S)\otimes\mathbf{C})_{\{0\}}\subset\mathrm{QCoh}(S)\otimes\mathbf{C}$ and

$$\mathrm{QCoh}(S)\otimes\mathbf{C}/(\mathrm{QCoh}(S)\otimes\mathbf{C})_{\{0\}}.$$

By [AG, Proposition 3.5.7], we have

$$(\mathrm{QCoh}(S)\otimes\mathbf{C})_{\{0\}}=\mathrm{QCoh}(S)\otimes\mathbf{C}_{\{0\}},$$

as full subcategories in $\mathrm{QCoh}(S)\otimes\mathbf{C}$.

Hence,

$$\mathrm{QCoh}(S)\otimes\mathbf{C}/(\mathrm{QCoh}(S)\otimes\mathbf{C})_{\{0\}}\simeq\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}}.$$

Note that $\mathrm{Proj}(A_S)\simeq S\times\mathrm{Proj}(A)$. Thus, to a Zariski-closed subset $\mathcal{N}'\subset S\times\mathrm{Proj}(A)$, we can attach the full subcategory

$$(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})_{\mathcal{N}'}\subset\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}}.$$

2.4.3. Assume now that S is equipped with a map to $\mathrm{Proj}(A)_{\mathrm{dR}}$, i.e., ${}^{\mathrm{red}}S$ is equipped with a map f to $\mathrm{Proj}(A)$.

Define

$$\mathbf{\Gamma}(S,\mathcal{C}_A):=(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f},$$

where Graph_f is the Zariski-closed *subset* of $S\times\mathrm{Proj}(A)$ equal to the graph of the map f.

2.4.4. For a map $S_1\to S_2$ we have a tautological identification

$$\mathrm{QCoh}(S_1)\underset{\mathrm{QCoh}(S_2)}{\otimes}(\mathrm{QCoh}(S_2)\otimes\overset{\circ}{\mathbf{C}}))\simeq\mathrm{QCoh}(S_1)\otimes\overset{\circ}{\mathbf{C}}.$$

It is easy to see that under this identification we have an inclusion

$$\mathrm{QCoh}(S_1)\underset{\mathrm{QCoh}(S_2)}{\otimes}(\mathrm{QCoh}(S_2)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_{f_2}}\subset(\mathrm{QCoh}(S_1)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_{f_1}},\tag{2.3}$$

where $f_2:{}^{\mathrm{red}}S_2\to\mathrm{Proj}(A)$ and f_1 is the composition of ${}^{\mathrm{red}}S_1\to{}^{\mathrm{red}}S_2$ and f_2.

We claim:

Lemma 2.4.5. *The inclusion* (2.3) *is an equality.*

Proof. Follows by combining [AG, Proposition 3.5.5 and Lemma 3.3.12]. □

2.4.6. From Lemma 2.4.5 we obtain that the assignment

$$(S,f)\rightsquigarrow(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$$

defines an object of $\mathrm{ShvCat}(\mathrm{Proj}(A)_{\mathrm{dR}})$.

We denote this object by $\mathcal{C}_A$. Thus by definition,

$$\mathbf{\Gamma}(S,\mathcal{C}_A):=(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$$

for any $(S,f)\in(\mathrm{DGSch}^{\mathrm{aff}})_{/\,\mathrm{Proj}(A)_{\mathrm{dR}}}$.

2.5. **A functor to the category of global sections.**

2.5.1. For $(S,f)\in (\mathrm{DGSch}^{\mathrm{aff}})_{/\,\mathrm{Proj}(A)_{\mathrm{dR}}}$, we define a functor

$$\overset{\circ}{\mathbf{C}}\to (\mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}=\mathbf{\Gamma}(S,\mathcal{C}_A) \tag{2.4}$$

as follows.

It is the composition of the tautological functor

$$\overset{\circ}{\mathbf{C}}\to \mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}},\quad \mathbf{c}\mapsto \mathcal{O}_S\otimes \mathbf{c},$$

followed by the co-localization functor

$$\mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}}\to (\mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f},$$

which is right adjoint to the tautological embedding

$$(\mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}\hookrightarrow \mathrm{QCoh}(S)\otimes \overset{\circ}{\mathbf{C}}.$$

2.5.2. The functors (2.4) are clearly compatible under the maps $S_1\to S_2$ in the category $(\mathrm{DGSch}^{\mathrm{aff}})_{/\,\mathrm{Proj}(A)_{\mathrm{dR}}}$.

Hence, they give rise to a functor

$$\overset{\circ}{\mathbf{C}}\to \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}},\mathcal{C}_A). \tag{2.5}$$

2.5.3. Let now $\mathcal{N}\subset \mathrm{Proj}(A)$ be a Zariski-closed subset. Consider the corresponding full subcategory

$$\overset{\circ}{\mathbf{C}}_{\mathcal{N}}\subset \overset{\circ}{\mathbf{C}}.$$

On the other hand, consider $\mathcal{N}_{\mathrm{dR}}\subset \mathrm{Proj}(A)_{\mathrm{dR}}$. By Sect. 1.3.7, the category $\mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}},\mathcal{C}_A)$ is naturally a full subcategory of $\mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}},\mathcal{C}_A)$. The following assertion results from the construction (see Proposition 1.3.6(b)):

Lemma 2.5.4. *The essential image of the subcategory* $\overset{\circ}{\mathbf{C}}_{\mathcal{N}}\subset \overset{\circ}{\mathbf{C}}$ *under the functor* (2.5) *is contained in* $\mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}},\mathcal{C}_A)\subset \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}},\mathcal{C}_A)$.

Thus, from Lemma 2.5.4, for every $\mathcal{N}$ as above, we obtain a functor

$$\overset{\circ}{\mathbf{C}}_{\mathcal{N}}\to \mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}},\mathcal{C}_A). \tag{2.6}$$

2.5.5. *Imposing additional conditions.* We now return to the general setting of Sect. 2.2 and make the following additional assumption on the pair $(\mathcal{A},A)$:

Suppose there exists an $\mathbb{E}_2$-algebra $\mathcal{B}$, equipped with a homomorphism

$$\mathcal{B}\to\mathcal{A},$$

such that:

- $\mathcal{B}$ is equipped with a grading such that $\mathcal{B}^{\mathrm{shift}}$ is classical (here $(-)^{\mathrm{shift}}$ is as in [AG, Sect. 3.6.1]);
- The resulting map $B:=H^\bullet(\mathcal{B})\to H^\bullet(\mathcal{A})$ can be factored as
$$B\to A\to H^\bullet(\mathcal{A}),$$
where $B\to A$ is a surjection modulo nilpotents.

We claim:

Proposition 2.5.6. *Under the above assumptions on the pair* $(\mathcal{A},A)$*, the functor* (2.6) *is an equivalence.*

We prove Proposition 2.5.6 in Sections 2.7–2.9.

2.6. **The case of ind-coherent sheaves.** In this subsection we deduce Theorem 1.4.2 from Proposition 2.5.6.

2.6.1. Let Z be an affine quasi-smooth DG scheme. In the setting of Sect. 2.2 we take

$$\mathbf{C} = \mathrm{IndCoh}(Z), \quad \mathcal{A} := \mathrm{HC}(Z), \ A := \Gamma(\mathrm{Sing}(Z), \mathcal{O}_{\mathrm{Sing}(Z)}).$$

In this case $\mathrm{Proj}(A) = \mathbb{P}\,\mathrm{Sing}(Z)$. The construction of Sect. 2.4 defines a sheaves of categories $\mathcal{C}_A$ over $(\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}$; this is the sought-for $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}$.

2.6.2. The functor (2.5) gives rise to a functor

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z) \to \mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Z))_{\mathrm{dR}}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}\right). \tag{2.7}$$

Furthermore, for a Zariski-closed subset $\mathcal{N} \subset \mathbb{P}\,\mathrm{Sing}(Z)$ we obtain a functor

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}_{\mathcal{N}}(Z) \to \mathbf{\Gamma}\left(\mathcal{N}_{\mathrm{dR}}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z)^{\sim}\right). \tag{2.8}$$

To prove Theorem 1.4.2(a), we need to show that the functor (2.8), and in particular, (2.7) is an equivalence. We do so by reducing to the situation when Proposition 2.5.6 becomes applicable.

2.6.3. First, we notice that the fact that (2.8) is an equivalence can be checked Zariski-locally on Z. Hence, can (and will) assume that Z is a *global derived complete intersection*. This means that Z fits into a Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & V, \end{array} \tag{2.9}$$

where U is smooth, and V is a vector space.

We claim that in this case the additional assumptions of Sect. 2.5.5 are satisfied.

Indeed, for Z fitting into the diagram (2.9), we take $\mathcal{B}$ to be the $\mathbb{E}_2$-algebra

$$\Gamma(U, \mathcal{O}_U) \otimes \mathrm{Sym}(V[-2]),$$

see [AG, Sect. 5.3.2]. The required pieces of structure on $\mathcal{B}$ are described in [AG, Formula (5.9) and Sect. 5.4], respectively.

2.7. **Proof of Proposition 2.5.6, Step 1.** Let $(\mathcal{A}, A)$ and $(\mathcal{B}, B)$ be as in Sect. 2.5.5. Let us prove that (2.5) is an equivalence in the special case $(\mathcal{A}, A) = (\mathcal{B}, B)$.

2.7.1. According to [AG, Sect. 3.6.2], the category $\overset{\circ}{\mathbf{C}}$ has a natural structure of module over $\mathrm{QCoh}(\mathrm{Proj}(B))$.

Let $\mathcal{C}'_B$ denote the object of $\mathrm{ShvCat}(\mathrm{Proj}(B))$ equal to

$$\mathbf{Loc}_{\mathrm{Proj}(B)}(\overset{\circ}{\mathbf{C}}),$$

where $\mathbf{Loc}_{\mathrm{Proj}(B)}$ is the left adjoint functor to

$$\mathbf{\Gamma}(\mathrm{Proj}(B),-):\mathrm{QCoh}(\mathrm{Proj}(B))\text{-}\mathbf{mod}\to\mathrm{ShvCat}(\mathrm{Proj}(B)),$$

see [Ga2, Sect. 1.3.1]. Explicitly, for an affine DG scheme S mapping to $\mathrm{Proj}(B)$, we have

$$\mathbf{\Gamma}(S,\mathcal{C}'_B):=\mathrm{QCoh}(S)\underset{\mathrm{QCoh}(\mathrm{Proj}(B))}{\otimes}\overset{\circ}{\mathbf{C}}.$$

2.7.2. Recall that $p_{\mathrm{dR},\mathrm{Proj}(B)}$ denotes the tautological map $\mathrm{Proj}(B)\to\mathrm{Proj}(B)_{\mathrm{dR}}$. The key observation is provided by the following lemma:

Lemma 2.7.3. *There exists a canonical isomorphism*

$$\mathcal{C}_B\simeq\mathbf{coind}_{p_{\mathrm{dR},\mathrm{Proj}(B)}}(\mathcal{C}'_B)$$

in $\mathrm{ShvCat}(\mathrm{Proj}(B)_{\mathrm{dR}})$*; under this identification, the composed map*

$$\overset{\circ}{\mathbf{C}}\to\mathbf{\Gamma}(\mathrm{Proj}(B),\mathcal{C}'_B)\simeq\mathbf{\Gamma}(\mathrm{Proj}(B)_{\mathrm{dR}},\mathbf{coind}_{p_{\mathrm{dR},\mathrm{Proj}(B)}}(\mathcal{C}'_B))\simeq\mathbf{\Gamma}(\mathrm{Proj}(B)_{\mathrm{dR}},\mathcal{C}_B)$$

identifies with (2.10).

Proof. Fix $S\overset{f}{\longrightarrow}\mathrm{Proj}(B)$, and let $(S\times\mathrm{Proj}(B))^{\wedge}_{\mathrm{Graph}_f}$ be the formal completion of $S\times\mathrm{Proj}(B)$ along the graph of f, i.e.,

$$(S\times\mathrm{Proj}(B))^{\wedge}_{\mathrm{Graph}_f}:=(S\times\mathrm{Proj}(B))\underset{(S\times\mathrm{Proj}(B))_{\mathrm{dR}}}{\times}(\mathrm{Graph}_f)_{\mathrm{dR}}.$$

The sheaf of categories $\mathbf{coind}_{p_{\mathrm{dR},\mathrm{Proj}(B)}}(\mathcal{C}'_B)$ assigns to (S,f) as above the category

$$\mathrm{QCoh}\left((S\times\mathrm{Proj}(B))^{\wedge}_{\mathrm{Graph}_f}\right)\underset{\mathrm{QCoh}(\mathrm{Proj}(B))}{\otimes}\overset{\circ}{\mathbf{C}}.$$

which tautologically identifies with

$$\mathrm{QCoh}\left((S\times\mathrm{Proj}(B))^{\wedge}_{\mathrm{Graph}_f}\right)\underset{\mathrm{QCoh}(S\times\mathrm{Proj}(B))}{\otimes}(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})\simeq$$

$$\simeq\mathrm{QCoh}(S\times\mathrm{Proj}(B))_{\mathrm{Graph}_f}\underset{\mathrm{QCoh}(S\times\mathrm{Proj}(B))}{\otimes}(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}}).$$

Now, the latter category identifies with $(\mathrm{QCoh}(S)\otimes\overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$ by [AG, Corollay 3.6.5]. □

2.7.4. From Lemma 2.7.3, we obtain that in order to prove that (2.5) is an isomorphism, it suffices to show that the map

$$\overset{\circ}{\mathbf{C}}\to\mathbf{\Gamma}(\mathrm{Proj}(B),\mathcal{C}'_B)=\mathbf{\Gamma}(\mathrm{Proj}(B),\mathbf{Loc}_{\mathrm{Proj}(B)}(\overset{\circ}{\mathbf{C}}))$$

is an isomorphism.

However, the latter follows from the fact that $\mathrm{Proj}(B)$ is 1-affine, being a quasi-compact DG scheme, see [Ga2, Theorem 2.1.1].

2.8. **Proof of Proposition 2.5.6, Step 2.** Suppose now that for $(\mathcal{A}, A)$ as in Sect. 2.5.5, the map (2.5) is an equivalence.

2.8.1. Applying the construction of Sects. 2.4-2.5 to $(\mathcal{B}, B)$, we obtain

$$\mathcal{C}_B \in \mathrm{ShvCat}(\mathrm{Proj}(B)),$$

and a functor

$$\overset{\circ}{\mathbf{C}} \to \mathbf{\Gamma}(\mathrm{Proj}(B)_{\mathrm{dR}}, \mathcal{C}_B). \tag{2.10}$$

2.8.2. By assumption, the homomorphism $B \to A$ induces a map $g : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$, which is moreover a closed embedding of the underlying classical reduced schemes.

Consider the corresponding map $g_{\mathrm{dR}} : \mathrm{Proj}(A)_{\mathrm{dR}} \to \mathrm{Proj}(B)_{\mathrm{dR}}$ and the resulting adjoint pair of functors functor

$$\mathbf{cores}_{g_{\mathrm{dR}}} : \mathrm{ShvCat}(\mathrm{Proj}(B)_{\mathrm{dR}}) \rightleftarrows \mathrm{ShvCat}(\mathrm{Proj}(A)_{\mathrm{dR}}) : \mathbf{coind}_{g_{\mathrm{dR}}}.$$

Tautologically, we have:

$$\mathbf{cores}_{g_{\mathrm{dR}}}(\mathcal{C}_B) \simeq \mathcal{C}_A.$$

Moreover, under this identification, the composed map

$$\overset{\circ}{\mathbf{C}} \to \mathbf{\Gamma}(\mathrm{Proj}(B)_{\mathrm{dR}}, \mathcal{C}_B) \to \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathbf{cores}_{g_{\mathrm{dR}}}(\mathcal{C}_B)) \simeq \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A)$$

identifies with (2.5).

By adjunction, we obtain a map in $\mathrm{ShvCat}(\mathrm{Proj}(B)_{\mathrm{dR}})$:

$$\mathcal{C}_B \to \mathbf{coind}_{g_{\mathrm{dR}}}(\mathcal{C}_A). \tag{2.11}$$

We claim:

Lemma 2.8.3. *The map* (2.11) *is an isomorphism.*

Proof. Clearly, $\mathbb{P}\mathrm{supp}_B(\mathbf{c}) \subset \mathrm{Proj}(A) \subset \mathrm{Proj}(B)$ for any $\mathbf{c} \in \overset{\circ}{\mathbf{C}}$. Hence the restriction of $\mathcal{C}_B$ to

$$\mathrm{Proj}(B)_{\mathrm{dR}} - \mathrm{Proj}(A)_{\mathrm{dR}}$$

vanishes. Now the claim follows from Proposition 1.3.6(a). □

2.8.4. As we showed in Step 1 of the proof, the functor (2.10) is an equivalence. Since

$$\mathbf{\Gamma}(\mathrm{Proj}(B)_{\mathrm{dR}}, \mathbf{coind}_{g_{\mathrm{dR}}}(\mathcal{C}_A)) \simeq \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A),$$

Lemma 2.8.3 implies that (2.5), as claimed.

2.9. **Proof of Proposition 2.5.6, Step 3.**

2.9.1. To complete the proof, it remains to show that the functor

$$\overset{\circ}{\mathbf{C}}_{\mathcal{N}} \to \mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}}, \mathcal{C}_A)$$

of Sect. 2.6 is an equivalence.

2.9.2. Let $\mathcal{N}'$ be the conical Zariski-closed subset of $\mathrm{Spec}(A)$ such that $\mathcal{N}' \supset \{0\}$ and $\mathcal{N} = \mathbb{P}(\mathcal{N}')$. Consider the corresponding full subcategory $\mathbf{C}' := \mathbf{C}_{\mathcal{N}'} \subset \mathbf{C}$. We have an equality

$$\overset{\circ}{\mathbf{C}}{}' = \overset{\circ}{\mathbf{C}}{}_{\mathcal{N}}$$

of full subcategories of $\overset{\circ}{\mathbf{C}}$.

Consider the corresponding sheaf of categories $\mathcal{C}'_A$ over $\mathrm{Proj}(A)_{\mathrm{dR}}$. We have a canonical identification

$$\mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}'_A) \simeq \mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}}, \mathcal{C}_A),$$

such that the diagram

$$\begin{array}{ccc} \overset{\circ}{\mathbf{C}}{}_{\mathcal{N}} & \longrightarrow & \mathbf{\Gamma}(\mathcal{N}_{\mathrm{dR}}, \mathcal{C}_A) \\ \sim\uparrow & & \uparrow\sim \\ \overset{\circ}{\mathbf{C}}{}' & \longrightarrow & \mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}'_A) \end{array}$$

commutes.

As shown on Step 2 of the proof (applied to $\mathbf{C}'$), the bottom arrow of this diagram is an equivalence. Hence, the top arrow is an equivalence as well. This completes the proof.

2.10. **Compatibility of monoidal actions.**

2.10.1. We now enhance the setting of Sect. 2.2 to include certain pre-existing monoidal actions.

Suppose $\widetilde{\mathcal{A}}$ is a commutative (i.e., $\mathbb{E}_\infty$) algebra and $\widetilde{\mathcal{A}} \to \mathcal{A}$ is a homomorphism of $\mathbb{E}_2$-algebras. Assume that

- $\widetilde{\mathcal{A}}$ is connective, i.e., $H^n(\widetilde{\mathcal{A}}) = 0$ for $n > 0$;
- We are given a factorization of the homomorphism $H^0(\widetilde{\mathcal{A}}) \to H^0(\mathcal{A})$ as

$$H^0(\widetilde{\mathcal{A}}) \to A^0 \to H^0(\mathcal{A}).$$

The homomorphism $\widetilde{\mathcal{A}} \to \mathcal{A}$ and the action of $\mathcal{A}$ on $\mathbf{C}$ define an action of $\widetilde{\mathcal{A}}$ on $\mathbf{C}$. In particular, the (symmetric) monoidal category $\widetilde{\mathcal{A}}$-mod $= \mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}}))$ acts on $\mathbf{C}$, and hence on $\overset{\circ}{\mathbf{C}}$.

2.10.2. Thus, on the one hand, the category $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ acts on $\overset{\circ}{\mathbf{C}}$ via the (symmetric) monoidal functor

$$\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})) = \widetilde{\mathcal{A}}\text{-mod} \to \mathcal{A}\text{-mod}$$

(where the first arrow coresonding to the tautological projection $\mathrm{Spec}(\widetilde{\mathcal{A}}) \to \mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}$), and the action of $\mathcal{A}$-mod on $\overset{\circ}{\mathbf{C}} \subset \mathbf{C}$.

On the other hand, we have the (symmetric) monoidal functor

$$\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}) \simeq \mathrm{QCoh}(\mathrm{Spec}(H^0(\widetilde{\mathcal{A}}))_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Spec}(A^0)_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Proj}(A)_{\mathrm{dR}}),$$

while $\mathrm{QCoh}(\mathrm{Proj}(A)_{\mathrm{dR}})$ acts on $\mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A)$.

We claim:

Proposition 2.10.3. *The functor* (2.5) *intertwines the above actions of* $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ *on* $\overset{\circ}{\mathbf{C}}$ *and* $\mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A)$*, respectively.*

2.10.4. We apply Proposition 2.10.3 as follows. We take $\widetilde{\mathcal{A}} = \Gamma(Z, \mathcal{O}_Z)$, which is equipped with a canonical map to $\mathrm{HC}(Z)$. The conclusion of Proposition 2.10.3 in this case implies the compatibility statement in Theorem 1.4.2(b).

The rest of this subsection is devoted to the proof of Proposition 2.10.3.

2.10.5. The action of $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ on $\mathbf{\Gamma}(\mathrm{Proj}(A)_{\mathrm{dR}}, \mathcal{C}_A)$ amounts to a compatible family of actions of $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ on the categories

$$\mathbf{\Gamma}(S, \mathcal{C}_A) = (\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f},$$

for $(S, f) \in (\mathrm{DGSch}^{\mathrm{aff}})_{/\,\mathrm{Proj}(A)_{\mathrm{dR}}}$.

For every (S, f), the action in question is obtained as the composition

$$\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}) \simeq \mathrm{QCoh}(\mathrm{Spec}(H^0(\widetilde{\mathcal{A}}))_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Spec}(A^0)_{\mathrm{dR}}) \to$$
$$\to \mathrm{QCoh}(\mathrm{Proj}(A)_{\mathrm{dR}}) \to \mathrm{QCoh}(S) \to \mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}\text{-mod}$$

and the action of $\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}$-mod on $(\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$, obtained from the monoidal functor $\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}\text{-mod} \to \mathrm{QCoh}(S) \otimes \mathcal{A}\text{-mod}$.

We need to show that the functor

$$\overset{\circ}{\mathbf{C}} \to (\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$$

of (2.4) intertwines the above action with the action of $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ on $\overset{\circ}{\mathbf{C}}$, obtained from

$$\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})) = \widetilde{\mathcal{A}}\text{-mod},$$

and the action of $\widetilde{\mathcal{A}}$-mod on $\overset{\circ}{\mathbf{C}} \subset \mathbf{C}$, obtained from the monoidal functor $\widetilde{\mathcal{A}}\text{-mod} \to \mathcal{A}\text{-mod}$.

Tautologically, the functor (2.4) intertwines the action of $\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}})$ on $\overset{\circ}{\mathbf{C}}$ with its action on $(\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$ obtained from the composition

$$\mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}) \to \mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})) = \widetilde{\mathcal{A}}\text{-mod} \overset{\mathcal{O}_S \otimes -}{\longrightarrow} \mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}\text{-mod}$$

and the action of $\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}$-mod on $(\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$ obtained from the monoidal functor $\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}\text{-mod} \to \mathrm{QCoh}(S) \otimes \mathcal{A}\text{-mod}$.

2.10.6. Note, however, that the action of $\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}$-mod on $(\mathrm{QCoh}(S) \otimes \overset{\circ}{\mathbf{C}})_{\mathrm{Graph}_f}$ factors through

$$\mathrm{QCoh}(S) \otimes \widetilde{\mathcal{A}}\text{-mod} = \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(\mathrm{Spec}(\widetilde{\mathcal{A}})) \simeq$$
$$\simeq \mathrm{QCoh}(S \times \mathrm{Spec}(\widetilde{\mathcal{A}})) \to \mathrm{QCoh}((S \times \mathrm{Spec}(\widetilde{\mathcal{A}}))^{\wedge}_{\mathrm{Graph}_{\widetilde{f}}}),$$

where $(S \times \mathrm{Spec}(\widetilde{\mathcal{A}}))^{\wedge}_{\mathrm{Graph}_{\widetilde{f}}}$ is the formal completion of $S \times \mathrm{Spec}(\widetilde{\mathcal{A}})$ along the graph of the composed map, denoted $\widetilde{f}$:

$$^{\mathrm{red}}S \to \mathrm{Proj}(A) \to \mathrm{Spec}(A^0) \to \mathrm{Spec}(H^0(\widetilde{\mathcal{A}})) \to \mathrm{Spec}(\widetilde{\mathcal{A}}).$$

Now, the required assertion follows from the commutativity of the next diagram:

$$\begin{array}{ccccc} (S\times \mathrm{Spec}(\widetilde{\mathcal{A}}))^{\wedge}_{\mathrm{Graph}_{\widetilde{f}}} & \longrightarrow & S\times \mathrm{Spec}(\widetilde{\mathcal{A}}) & \xrightarrow{\mathrm{pr}_2} & \mathrm{Spec}(\widetilde{\mathcal{A}}) \\ \downarrow & & & & \downarrow \\ S\times \mathrm{Spec}(\widetilde{\mathcal{A}}) & \xrightarrow{\mathrm{pr}_1} & S & \xrightarrow{\widetilde{f}} & \mathrm{Spec}(\widetilde{\mathcal{A}})_{\mathrm{dR}}. \end{array}$$

3. Relative crystals

Let $f: Z \to Y$ be a map of DG schemes almost of finite type. We are interested in the category

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

Objects of this category can be viewed as ind-coherent sheaves on Z *equipped with a connection along the fibers of the map* $Z \to Y$. When the map is smooth, the words 'connection along the fibers' can be understood literally. In general, the definition requires the language of de Rham prestacks.

When Z is quasi-smooth, one can use singular support to construct subcategories of $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$. In this section we study the interaction of this construction with the crystal structure on the category of singularities studied in Sect. 1.

3.1. **Relative crystals as a tensor product.** Let $f: Z \to Y$ be a map of DG schemes almost of finite type. Let us describe the category $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$ in terms of $\mathrm{IndCoh}(Y)$.

3.1.1. First, we claim:

Proposition 3.1.2. *The functor*

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y), \tag{3.1}$$

induced by the $\mathrm{QCoh}(Y)$*-linear functor*

$$(f_{\mathrm{dR}} \times \mathrm{id})^{!}: \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

is an equivalence.

3.1.3. *Proof of Proposition 3.1.2, Step 0.* First, it is easy to see that the assertion is Zariski-local with respect to Z. Hence, we can assume that the map f can be factored as

$$Z \overset{i}{\hookrightarrow} Z' \to Y,$$

where i is a closed embedding, and Z' is of the form $W \times Y$.

Let $\overset{\circ}{Z} \overset{j}{\hookrightarrow} Z'$ be the embedding of the complementary open.

3.1.4. *Proof of Proposition 3.1.2, Step 1.* We claim that the assertion of the proposition holds for Z'. Indeed, we have:

$$Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y \simeq W_{\mathrm{dR}} \times Y,$$

and hence

$$\mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \simeq \mathrm{QCoh}(W_{\mathrm{dR}}) \otimes \mathrm{IndCoh}(Y).$$

Similarly,

$$\mathrm{IndCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{IndCoh}(W_{\mathrm{dR}}) \otimes \mathrm{IndCoh}(Y).$$

Now, the functor

$$\mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

identifies with

$$\Upsilon_{W_{\mathrm{dR}}} \otimes \mathrm{Id} : \mathrm{QCoh}(W_{\mathrm{dR}}) \otimes \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(W_{\mathrm{dR}}) \otimes \mathrm{IndCoh}(Y),$$

which is an equivalence by [GR1, Proposition 2.4.4].

3.1.5. *Proof of Proposition 3.1.2, Step 2.* Note that the map $Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y \to Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$ is an isomorphism from $Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$ to its own formal completion inside $Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$.

Hence, we have a localization sequence

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \rightleftarrows \mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \rightleftarrows \mathrm{QCoh}(\overset{\circ}{Z}_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

which gives rise to the localization sequence

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \rightleftarrows \mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \rightleftarrows$$
$$\rightleftarrows \mathrm{QCoh}(\overset{\circ}{Z}_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y).$$

Similarly, we have a localization sequence

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \rightleftarrows \mathrm{IndCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \rightleftarrows \mathrm{IndCoh}(\overset{\circ}{Z}_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

Combined with the fact that

$$\mathrm{QCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Z'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

is an equivalence, this implies that (3.1) is fully faithful.

Applying this to $\overset{\circ}{Z}$, we obtain that the functor (3.1) *for* $\overset{\circ}{Z}$ is conservative. Comparing the localization sequences, this implies that the functor (3.1) is essentially surjective for Z, as required.

□(Proposition 3.1.2)

3.1.6. Assume now that Y is quasi-smooth. Consider the category

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y),$$

appearing on the left-hand side of the equivalence in Proposition 3.1.2. It contains as a full subcategory

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq$$
$$\simeq \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Y) \overset{\mathrm{Id}\otimes\Xi_Y}{\hookrightarrow} \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y).$$

The resulting embedding

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \to \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \simeq \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

differs from the canonical embedding $\Upsilon_{Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y}$ (given by the action of the left-hand side on $\omega_{Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y}$) by tensoring by the pullback of ω_Y. In particular, the two embeddings have the same essential image.

Set

$$\overset{\circ}{\mathrm{IndCoh}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) := \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

We view it as a full subcategory of $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$ by identifying it with

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)^{\perp} \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

In terms of the equivalence of Proposition 3.1.2, we have

$$\overset{\circ}{\mathrm{IndCoh}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) = \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y),$$

as full subcategories of

$$\mathrm{Ind}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y).$$

3.1.7. We now claim:

Proposition 3.1.8. *There exist canonical equivalences*

$$\overset{\circ}{\mathrm{IndCoh}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y) \simeq \tag{3.2}$$

$$\simeq \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \overset{\circ}{\mathrm{IndCoh}}(Y)^{\sim}\right)$$

3.1.9. *Proof of Proposition 3.1.8.* Let us show that we have a canonical isomorphism

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}_Y \simeq \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y$$

for any $\mathbf{C}_Y \in \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)\text{-}\mathbf{mod}$.

First, the fact Y_{dR} is 1-affine implies that

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

is an isomorphism, see Lemma 1.5.6.

Hence,

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathbf{C}_Y \simeq \mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y.$$

Next, we rewrite

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y \simeq$$

$$\simeq \left(\mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})\right) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y.$$

Now, the fact that both Z_{dR} and Y_{dR} are 1-affine implies that the functor

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \to \\ \to \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} (\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) = \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})$$

is an equivalence.

Hence,

$$\mathrm{QCoh}(Z_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y \simeq \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathbf{C}_Y,$$

as desired.

Finally, the fact that

$$\mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \to \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right)$$

is an equivalence follows from the fact that both $\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}$ and $(Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}$ are 1-affine.

□

3.2. **Relative crystals with prescribed singular support.** Let $f : Z \to Y$ be as before. We now assume that Z is quasi-smooth and that f has a perfect relative cotangent complex (this is automatic if Y is also quasi-smooth).

In this subsection we show how conical subvarieties on $\mathrm{Sing}(Z)$ give rise to subcategories of $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$.

3.2.1. The tautological map $p_{\mathrm{dR}/Y,Z} : Z \to Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$ gives rise to the forgetful functor

$$(p_{\mathrm{dR}/Y,Z})^! : \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \to \mathrm{IndCoh}(Z).$$

According to [GR2, Part III.3, Proposition 3.1.2], the functor $(p_{\mathrm{dR}/Y,Z})^!$ is conservative and admits a left adjoint, denoted $(p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*$. Informally, if one views ind-coherent sheaves on $Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$ as (relative) D-modules for the morphism $Z \to Y$, then $(p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*$ is the induction functor from ind-coherent sheaves on Z to relative D-modules.

The composition $((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*)$ acquires a natural structure of a monad acting on $\mathrm{IndCoh}(Z)$. Denote by

$$((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*)\text{-mod}(\mathrm{IndCoh}(Z))$$

the category of modules over this monad. The Barr-Beck-Lurie theorem provides an equivalence

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq ((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*)\text{-mod}(\mathrm{IndCoh}(Z)).$$

(The assumption that Z is quasi-smooth is not required for this equivalence.)

3.2.2. Now fix a conical Zariski-closed subset $\mathcal{N} \subset \mathrm{Sing}(Z)$. Let

$$\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

denote the preimage of

$$\mathrm{IndCoh}_{\mathcal{N}}(Z) \subset \mathrm{IndCoh}(Z)$$

under the functor $(p_{\mathrm{dR}/Y,Z})^!$.

We claim:

Proposition 3.2.3. *The functor* $(p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*$ *sends* $\mathrm{IndCoh}_{\mathcal{N}}(Z)$ *to* $\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$.

Proof. The assertion of the proposition is equivalent to the fact that

$$((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*),$$

viewed as a plain endofunctor of $\mathrm{IndCoh}(Z)$, preserves the full subcategory $\mathrm{IndCoh}_{\mathcal{N}}(Z)$.

Recall that according to [GR2, Part 4.4, Theorem 11.3.2], $((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*)$ admits a filtration whose n-th associated graded is isomorphic to the functor

$$\mathrm{Sym}^n(T(Z/Y)) \overset{!}{\otimes} -, \tag{3.3}$$

where Sym^n is taken in the symmetric monoidal category $(\mathrm{IndCoh}(Z), \overset{!}{\otimes})$, and $T(Z/Y) \in \mathrm{IndCoh}(Z)$ is as in [GR2, Part III.1, Sect. 4.3.8].

Thus, it suffices to show that the functor (3.3) preserves the subcategory $\mathrm{IndCoh}_{\mathcal{N}}(Z)$.

Let $T^*(Z/Y) \in \mathrm{QCoh}(Z)$ be the cotangent complex of Z. The assumption that $T^*(Z/Y)$ be perfect implies that $T(Z/Y) \in \mathrm{IndCoh}(Z)$ is canonically isomorphic to

$$\Upsilon_Z((T^*(Z/Y))^\vee),$$

where $(T^*(Z/Y))^\vee \in \mathrm{QCoh}(Z)$ is the monoidal dual of $T^*(Z/Y)$, and where Υ_Z is as in [GR2, Part II.3, Sect. 3.2.5].

Therefore,

$$\mathrm{Sym}^n(T(Z/Y)) \simeq \Upsilon_Z\left(\mathrm{Sym}^n((T^*(Z/Y))^\vee)\right),$$

where Sym^n is now taken in the symmetric monoidal category $(\mathrm{QCoh}(Z), \otimes)$. Hence, the functor (3.3) is given by

$$\mathrm{Sym}^n((T^*(Z/Y))^\vee) \otimes -,$$

where $\otimes$ denotes the action of $\mathrm{QCoh}(Z)$ on $\mathrm{IndCoh}(Z)$, and therefore preserves $\mathrm{IndCoh}_{\mathcal{N}}(Z)$, see [AG, Lemma 4.2.2]. □

3.2.4. As a corollary of Proposition 3.2.3, we obtain:

Corollary 3.2.5. *There exists a canonical equivalence*

$$\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq ((p_{\mathrm{dR}/Y,Z})^! \circ (p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_*)\text{-mod}(\mathrm{IndCoh}_{\mathcal{N}}(Z)),$$

commuting with the forgetful functors to $\mathrm{IndCoh}_{\mathcal{N}}(Z)$.

3.2.6. Let us now assume that Y is quasi-smooth as well. Let $\mathcal{N} \subset \mathrm{Sing}(Z)$ be a Zariski-closed conical subset that contains the zero-section. We have

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

as follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) & \xrightarrow{\Upsilon_{Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y}} & \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Z) & \xrightarrow{\Upsilon_Z} & \mathrm{IndCoh}(Z). \end{array}$$

Let us denote by

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \tag{3.4}$$

the quotient

$$\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

considered as a full subcategory of

$$\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

3.2.7. Recall now the map

$$\mathrm{Sing}(f) : Z \underset{Y}{\times} \mathrm{Sing}(Y) \to \mathrm{Sing}(Z),$$

see [AG, Sect. 2.4.1]. For $\{0\} \subset \mathcal{N} \subset \mathrm{Sing}(Z)$ as above, consider the closed subset

$$\mathrm{Sing}(f)^{-1}(\mathcal{N}) \subset Z \underset{Y}{\times} \mathrm{Sing}(Y).$$

Consider the corresponding closed subset

$$\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})) \subset Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y).$$

Consider the corresponding full subcategory

$$\mathbf{\Gamma}\left((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right) \subset \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right), \tag{3.5}$$

or, which is the same,

$$\begin{aligned} \mathrm{QCoh}\left((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}}\right) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \subset \\ \subset \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y). \end{aligned} \tag{3.6}$$

3.2.8. Using Proposition 3.1.8, we identify

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y)^{\sim}\right) \tag{3.7}$$

or, equivalently,

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y). \tag{3.8}$$

We claim:

Theorem 3.2.9. *The full subcategory*

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{Ind}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

of (3.4) *corresponds under the identifications* (3.7) *and* (3.8) *to the full subcategory*

$$\mathbf{\Gamma}\left((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y)^{\sim}\right) \subset \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y)^{\sim}\right)$$

from (3.5)*, or, equivalently, to the full subcategory*

$$\mathrm{QCoh}\left((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}}\right) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y) \subset$$
$$\subset \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y).$$

from (3.6).

3.2.10. *An example.* Let us take $\mathcal{N} = \{0\}$. In this case we have the following three full subcategories of $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$. The largest is $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$ itself.

The smallest is

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

The middle category is $\mathrm{IndCoh}_{\{0\}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$, i.e., the preimage of $\mathrm{QCoh}(Z) \subset \mathrm{IndCoh}(Z)$ under the forgetful functor

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \to \mathrm{IndCoh}(Z).$$

In terms of the identification

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) = \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)/\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq$$
$$\simeq \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y).$$

of Proposition 3.1.8, the subcategory

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}_{\{0\}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

corresponds to subscheme

$$\mathbb{P}(\mathrm{Sing}(f)^{-1}(\{0\})) \subset Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y).$$

3.3. **Proof of Theorem 3.2.9, Step 1.** We first show that the assertion of the theorem holds when $f : Z \to Y$ is a closed embedding.

3.3.1. Note that in this case $Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y$ is the formal completion $Y^{\wedge}_{Z}$ of Y along Z. In particular, $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$ identifies with the full subcategory

$$\mathrm{IndCoh}(Y)_Z \subset \mathrm{IndCoh}(Y),$$

consisting of objects that are set-theoretically supported on $Z \subset Y$.

Recall that the categories $\mathrm{IndCoh}(Z)$ and $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$ are related by a pair of adjoint functors

$$(p_{\mathrm{dR}/Y,Z})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(Z) \rightleftarrows \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) : (p_{\mathrm{dR}/Y,Z})^!$$

(the induction functor and the forgetful functor). Under the equivalence

$$\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{IndCoh}(Y)_Z,$$

they are identified with the pair of adjoint functors

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(Z) \rightleftarrows \mathrm{IndCoh}(Y)_Z : f^!|_{\mathrm{IndCoh}(Y)_Z}.$$

Under the identification of (3.1), the full subcategory

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \subset \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y)$$

corresponds to

$$\mathrm{IndCoh}(Y)_Z \cap \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \subset \mathrm{IndCoh}(Y)_Z \simeq \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

Furthermore, the diagram

$$\begin{array}{ccc}
\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) & \xrightarrow[\sim]{\text{Proposition 3.1.8}} & \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right) \\
\downarrow \sim & & \downarrow \\
\mathrm{IndCoh}(Y)_Z \cap \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) & & \mathbf{\Gamma}\left((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right) \\
\downarrow & & \downarrow \sim \\
\mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) & \xrightarrow{\mathrm{Id}} & \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)
\end{array}$$

commutes.

3.3.2. Set

$$\mathcal{M} := \mathrm{Sing}(f)^{-1}(\mathcal{N}) \subset Z \underset{Y}{\times} \mathrm{Sing}(Y) \subset \mathrm{Sing}(Y).$$

Let $\mathbb{P}\mathcal{M}$ denote the corresponding Zariski-closed subset of $\mathbb{P}\,\mathrm{Sing}(Y)$.

Then, by Theorem 1.4.2(a),

$$\mathbf{\Gamma}\left(\mathbb{P}\mathcal{M}_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right) \subset \mathbf{\Gamma}\left((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}, \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)^{\sim}\right) \subset \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$$

identifies with the full subcategory of $\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y)$ equal to

$$\overset{\circ}{\mathrm{Ind}}\mathrm{Coh}_{\mathbb{P}\mathcal{M}}(Y) = \overset{\circ}{\mathrm{Ind}}\mathrm{Coh}(Y) \cap \mathrm{IndCoh}_{\mathcal{M}}(Y).$$

Therefore, in order to establish the assertion of the theorem, it is sufficient to show that

$$\mathrm{IndCoh}_{\mathcal{M}}(Y) = \mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

as subcategories of $\mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{IndCoh}(Y)_Z$.

3.3.3. Thus, we need to show that $\mathrm{IndCoh}_{\mathcal{M}}(Y) \subset \mathrm{IndCoh}(Y)_Z$ equals the preimage of $\mathrm{IndCoh}_{\mathcal{N}}(Z)$ under the functor $f^! : \mathrm{IndCoh}(Y)_Z \to \mathrm{IndCoh}(Z)$.

We note that the inclusion

$$\mathrm{IndCoh}_{\mathcal{M}}(Y) \subset (f^!)^{-1}(\mathrm{IndCoh}_{\mathcal{N}}(Z))$$

follows from [AG, Proposition 7.1.3(a)].

For the opposite inclusion, by Corollary 3.2.5, it suffices to show that the essential image of $\mathrm{IndCoh}_{\mathcal{N}}(Z)$ under the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Y)_Z$$

is contained in $\mathrm{IndCoh}_{\mathcal{M}}(Y)$. However, this follows from [AG, Proposition 7.1.3(b)].

3.4. **Proof of Theorem 3.2.9, Step 2.** We now consider the case of a general morphism $f : Z \to Y$.

3.4.1. It is easy to see that the assertion of the theorem is Zariski-local on Z. Hence, we can assume that the morphism f factors as

$$Z \overset{f'}{\to} Y' \overset{g}{\to} Y,$$

where $Z \to Y'$ is a closed embedding, and g is smooth. Furthermore, we can assume that Y' is isomorphic to $Y \times W$ with W smooth.

By Step 1, we know that the statement of the theorem holds for the morphism $Z \to Y'$.

3.4.2. Consider the (forgetful) functor

$$(\mathrm{id} \times g)^! : \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \to \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y'_{\mathrm{dR}}}{\times} Y'). \tag{3.9}$$

By definition,

$$\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

is the preimage under (3.9) of

$$\mathrm{IndCoh}_{\mathcal{N}}(Z_{\mathrm{dR}} \underset{Y'_{\mathrm{dR}}}{\times} Y') \subset \mathrm{IndCoh}(Z_{\mathrm{dR}} \underset{Y'_{\mathrm{dR}}}{\times} Y').$$

3.4.3. The fact that g is smooth implies that

$$\mathrm{Sing}(g) : Y' \underset{Y}{\times} \mathrm{Sing}(Y) \to \mathrm{Sing}(Y')$$

is an isomorphism. In particular,

$$Z \underset{Y}{\times} \mathrm{Sing}(Y) \simeq Z \underset{Y'}{\times} \mathrm{Sing}(Y').$$

Under this identification, the loci

$$\mathrm{Sing}(f)^{-1}(\mathcal{N}) \subset Z \underset{Y}{\times} \mathrm{Sing}(Y) \text{ and } \mathrm{Sing}(f')^{-1}(\mathcal{N}) \subset Z \underset{Y'}{\times} \mathrm{Sing}(Y')$$

correspond to one another.

Under the identifications of Proposition 3.1.8 for Y and Y', respectively, the pullback functor

$$\mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \to \mathrm{QCoh}(Z_{\mathrm{dR}} \underset{Y'_{\mathrm{dR}}}{\times} Y') \underset{\mathrm{QCoh}(Y')}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y')$$

corresponds to the functor

$$\begin{aligned} (3.10) \quad \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \to \\ \to \mathrm{QCoh}((Z \underset{Y'}{\times} \mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y'), \end{aligned}$$

Hence, we obtain that in order to prove the theorem, it suffices to show that the preimage of

$$\begin{aligned} (3.11) \quad \mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f')^{-1}(\mathcal{N})))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y') \subset \\ \subset \mathrm{QCoh}((Z \underset{Y'}{\times} \mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y'). \end{aligned}$$

under the functor (3.10) equals

$$\begin{aligned} (3.12) \quad \mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \subset \\ \subset \mathrm{QCoh}((Z \underset{Y}{\times} \mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}}) \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y). \end{aligned}$$

I.e., it suffices to show that the functor

$$\begin{aligned} \mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}})^{\perp} \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y) \to \\ \to \mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f')^{-1}(\mathcal{N})))_{\mathrm{dR}})^{\perp} \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y') \end{aligned}$$

is conservative.

3.4.4. Since g is smooth, the functor $g^!$ induces an equivalence

$$\mathrm{QCoh}(Y') \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y').$$

Hence,

$$\mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f')^{-1}(\mathcal{N})))_{\mathrm{dR}})^{\perp} \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y'))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y')$$

is obtained from

$$\mathrm{QCoh}((\mathbb{P}(\mathrm{Sing}(f)^{-1}(\mathcal{N})))_{\mathrm{dR}})^{\perp} \underset{\mathrm{QCoh}((\mathbb{P}\,\mathrm{Sing}(Y))_{\mathrm{dR}})}{\otimes} \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Y)$$

by the procedure

$$- \underset{\mathrm{QCoh}(Y'_{\mathrm{dR}})}{\otimes} \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y).$$

Now, we claim that for any $\mathbf{C} \in \mathrm{QCoh}(Y'_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)\text{-}\mathbf{mod}$, the resulting functor

$$\mathbf{C} \to \mathbf{C} \underset{\mathrm{QCoh}(Y'_{\mathrm{dR}})}{\otimes} \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y') \tag{3.13}$$

is conservative.

To show this, it is enough to prove that the pullback functor

$$\mathrm{QCoh}(Y'_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y') \tag{3.14}$$

admits a left adjoint, which is comatible with the action of $\mathrm{QCoh}(Y'_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y)$, and whose essential image generates $\mathrm{QCoh}(Y'_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y)$ as a DG category.

Indeed, such a left adjoint implies the existence of a left adjoint to (3.13), whose essential image generates generates $\mathbf{C}$.

3.4.5. To establish the required property of (3.14), we use the assumption that $Y' = Y \times W$ with W smooth.

We write

$$\mathrm{QCoh}(Y'_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(W_{\mathrm{dR}})$$

and

$$\mathrm{QCoh}(Y') \simeq \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(W).$$

Thus, our assertion follows from the fact that the forgetful functor

$$\mathrm{QCoh}(W_{\mathrm{dR}}) \to \mathrm{QCoh}(W)$$

does admit a left adjoint with the required properties.

Part II: Gluing.

4. A paradigm for gluing

In this section we formulate the main result of this paper, Theorem 4.3.4.

4.1. Gluing and lax limits: a reminder.

4.1.1. Let I be an index ∞-category, and let

$$i \mapsto \mathbf{C}_i, \quad (\alpha : i \to j) \mapsto (\Phi_\alpha : \mathbf{C}_i \to \mathbf{C}_j).$$

be a functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$.

Let $\mathbf{C}_I$ be the corresponding co-Cartesian fibration over I. The lax limit

$$\underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$$

is the object of $\mathrm{DGCat}_{\mathrm{cont}}$ equal to the category of *all* (i.e., not necessarily co-Cartesian) sections $I \to \mathbf{C}_I$ of the projection $\mathbf{C}_I \to I$.

We have a fully faithful embedding

$$\lim_{i \in I} \mathbf{C}_i \hookrightarrow \underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$$

that corresponds to taking *co-Cartesian* sections.

4.1.2. Objects of $\underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$ can be concretely described as follows: An object of $\underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$ is a collection

$$\mathbf{c}_i \in \mathbf{C}_i \quad \text{for all } i \in I,$$

equipped with a family of morphisms (but not necessarily isomorphisms)

$$\Phi_\alpha(\mathbf{c}_i) \to \mathbf{c}_j \quad \text{for all } \alpha : i \to j,$$

compatible with compositions of α's, and endowed with a homotopy-coherent system of compatibilities for multi-fold compositions.

An object as above belongs to $\lim_{i \in I} \mathbf{C}_i$ if and only if the above maps $\Phi_\alpha(\mathbf{c}_i) \to \mathbf{c}_j$ are all isomorphisms.

4.1.3. Unwinding the definitions, for a given $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$, the datum of a functor

$$\mathsf{F} : \mathbf{D} \to \underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$$

consists of a collection of functors

$$\mathsf{F}_i : \mathbf{D} \to \mathbf{C}_i \quad \text{for all } i \in I$$

equipped with a compatible family of natural transformations (but not necessarily isomorphisms)

$$\Phi_\alpha \circ \mathsf{F}_i \to \mathsf{F}_j \quad \text{for all } \alpha : i \to j.$$

In particular, by taking $\mathbf{D} = \mathrm{Vect}$, we obtain the description of objects of $\underset{i \in I}{\text{lax-lim}}\, \mathbf{C}_i$, given above.

4.1.4. We think of $\underset{i\in I}{\text{lax-lim}}\,\mathbf{C}_i$ as glued from the categories $\mathbf{C}_i$ using the functors Φ_α.

For this reason, we also denote

$$\underset{i\in I}{\text{lax-lim}}\,\mathbf{C}_i =: \mathrm{Glue}(\mathbf{C}_i,\, i\in I).$$

Remark 4.1.5. The category $\mathrm{Glue}(\mathbf{C}_i,\, i\in I)$ can be defined in a more general situation. Namely, we do not need

$$i\mapsto \mathbf{C}_i,\quad I\to \mathrm{DGCat}_{\mathrm{cont}}$$

to be a functor, but only (either left or right) *lax* functor. I.e., we do not need to have an isomorphism between $\Phi_\alpha\circ\Phi_\beta$ and $\Phi_{\alpha\circ\beta}$, but only a morphism in one direction.

We do not need this more general set-up in the present paper.

4.1.6. *Example.* Let Y be a topological space, and let $Y_0 \overset{j}{\hookrightarrow} Y$ be an open subset and $Y_1 \overset{i}{\hookleftarrow} Y$ be the complementary closed. Let I be the category $0\to 1$, and set

$$\mathbf{C}_0=\mathrm{Shv}(Y_0),\ \mathbf{C}_1=\mathrm{Shv}(Y_1),\ \Phi_{0\to 1}=i^!\circ j_!.$$

Then the functor

$$\mathrm{Shv}(Y)\to \mathrm{Glue}(\mathbf{C}_i,\, i\in I),\quad \mathcal{F}\mapsto (j^!(\mathcal{F}), i^!(\mathcal{F}), i^!\circ j_!\circ j^!(\mathcal{F})\to i^!(\mathcal{F}))$$

is an equivalence. The inverse functor sends

$$(\mathcal{F}_0,\mathcal{F}_1, i^!\circ j_!(\mathcal{F}_0)\to\mathcal{F}_1)\mapsto \mathrm{Cone}\left(i_!(\ker(i^!\circ j_!(\mathcal{F}_0)\to\mathcal{F}_1))\to j_!(\mathcal{F}_0)\right).$$

4.1.7. *Example.* Example 4.1.6 can be generalized to arbitrary stratified topological spaces, but this requires taking lax limits over lax functors, as in Remark 4.1.5. Namely, let $Y=\bigcup_{a\in A} Y_a$ be a stratification of a topological space Y indexed by a finite poset A. Thus, the subspaces $Y_a\subset Y$ are disjoint and locally closed, and

$$\overline{Y_a}\subset \bigcup_{a'\geq a} Y_{a'}\subset Y$$

for all $a\in A$. Denote the embedding $Y_a\hookrightarrow Y$ by ι_a.

For every pair $a_1,a_2\in A$ with $a_1\leq a_2$, consider the functor

$$\Phi_{a_1\to a_2}:=\iota_{a_2}^!\circ\iota_{a_1,!}:\mathrm{Shv}(Y_{a_1})\to\mathrm{Shv}(Y_{a_2}).$$

For a triple $a_1,a_2,a_3\in A$ with $a_1\leq a_2\leq a_3$, the adjunction between $\iota_{a_2}^!$ and $\iota_{a_2,!}$ yields a natural transformation

$$(\Phi_{a_2\to a_3}\circ\Phi_{a_1\to a_2})\to\Phi_{a_1\to a_3}.$$

In this way, we obtain a lax functor $I\to\mathrm{DGCat}_{\mathrm{cont}}$ (here I is the category corresponding to the poset A) sending $a\in A$ to the category $\mathrm{Shv}(Y_a)$. Similarly to Example 4.1.6, there is a natural equivalence between the resulting glued category and $\mathrm{Shv}(Y)$.

4.1.8. For every $i_0\in I$ we let ev_{i_0} denote the natural evaluation functor

$$\underset{i\in I}{\text{lax-lim}}\,\mathbf{C}_i\to\mathbf{C}_{i_0}.$$

The functor ev_{i_0} admits a left adjoint, denoted ins_{i_0}. Explicitly, for $\mathbf{c}_{i_0}\in\mathbf{C}_{i_0}$ and $i\in I$, we have

$$\mathrm{ev}_i\circ\mathrm{ins}_{i_0}(\mathbf{c}_{i_0})\simeq\underset{\alpha\in\mathrm{Maps}_I(i_0,i)}{\mathrm{colim}}\Phi_\alpha(\mathbf{c}_{i_0}).$$

Remark 4.1.9. The latter expression for $\mathrm{ev}_i\circ\mathrm{ins}_{i_0}$ is a feature of *lax limits* of DG categories; it is false for usual limits.

4.1.10. *Subcategories.* Let $i \mapsto \mathbf{C}_i$ be as before. Suppose now that for every $i \in I$ we chose a full subcategory

$$\mathbf{C}'_i \subset \mathbf{C}_i.$$

These subcategories define a full subcategory $\mathbf{C}'_I \subset \mathbf{C}_I$.

Assume that the following condition holds: for every $(\alpha : i \to j) \in I$, the functor Φ_α sends $\mathbf{C}'_i$ to $\mathbf{C}'_j$. In this case, the composition

$$\mathbf{C}'_I \hookrightarrow \mathbf{C}_I \to I$$

is a co-Cartesian fibration, and hence gives rise to a functor

$$i \mapsto \mathbf{C}'_i, \quad I \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Consider the corresponding category

$$\underset{i \in I}{\text{lax-lim}}\, \mathbf{C}'_i =: \mathrm{Glue}(\mathbf{C}'_i,\, i \in I).$$

By construction, we have a canonical fully faithful functor

$$\mathrm{Glue}(\mathbf{C}'_i,\, i \in I) \to \mathrm{Glue}(\mathbf{C}_i,\, i \in I), \tag{4.1}$$

that commutes with the evaluation functors ev_{i_0}.

Finally, assume that in the above setting, each of the embeddings $\mathbf{C}'_i \hookrightarrow \mathbf{C}_i$ admits a continuous right adjoint. In this case, it is easy to show that the functor (4.1) also admits a continuous right adjoint.

The resulting right adjoint $\mathrm{Glue}(\mathbf{C}_i,\, i \in I) \to \mathrm{Glue}(\mathbf{C}'_i,\, i \in I)$ also commutes with the evaluation functors ev_{i_0}.

4.2. **Gluing of** IndCoh**.**

4.2.1. Consider the following set-up. Let $\mathcal{Y}$ be an algebraic stack. Let I be an index category, and let

$$i \mapsto \mathcal{Z}_i, \quad (\alpha : i \to j) \mapsto (f_\alpha : \mathcal{Z}_j \to \mathcal{Z}_i).$$

be an I^{op}-diagram of algebraic stacks over $\mathcal{Y}$. We denote by f_i the corresponding morphisms $\mathcal{Z}_i \to \mathcal{Y}$.

We assume that $\mathcal{Y}$ and all $\mathcal{Z}_i$ are quasi-smooth.

4.2.2. We consider

$$i \mapsto \mathrm{IndCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}), \quad (\alpha : i \to j) \mapsto ((f_\alpha)_{\mathrm{dR}} \times \mathrm{id}_{\mathcal{Y}})^!$$

as a functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$.

Let now

$$\mathcal{N}_i \subset \mathrm{Sing}(\mathcal{Z}_i)$$

be conical Zariski-closed subsets. We assume that for every $\alpha : i \to j$ the map

$$\mathrm{Sing}(f_\alpha) : \mathcal{Z}_j \underset{\mathcal{Z}_i}{\times} \mathrm{Sing}(\mathcal{Z}_i) \to \mathrm{Sing}(\mathcal{Z}_j)$$

sends $\mathcal{Z}_j \underset{\mathcal{Z}_i}{\times} \mathcal{N}_i$ to $\mathcal{N}_j$.

Consider the corresponding full subcategories

$$\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}) \subset \mathrm{IndCoh}((\mathcal{Z}_j)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}).$$

According to [AG, Lemma 8.4.2], the above condition on f_α implies that the functor

$$((f_\alpha)_{\mathrm{dR}} \times \mathrm{id}_{\mathcal{Y}})^!$$

sends $\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y})$ to $\mathrm{IndCoh}_{\mathcal{N}_j}((\mathcal{Z}_j)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y})$.

4.2.3. We consider the corresponding pair of adjoint functors

$$\mathrm{Glue}(\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i \in I) \rightleftarrows \mathrm{Glue}(\mathrm{IndCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i \in I). \tag{4.2}$$

The functors $((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!$ define a functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \lim_{i \in I} \mathrm{IndCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}).$$

Thus, for a given conical Zariski-closed subset $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$ we obtain the functor

$$\begin{aligned} \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{IndCoh}(\mathcal{Y}) \to \lim_{i \in I} \mathrm{IndCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}) \hookrightarrow \\ \hookrightarrow \mathrm{Glue}(\mathrm{IndCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i \in I) \to \mathrm{Glue}(\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i \in I), \end{aligned} \tag{4.3}$$

where the last arrow is the right adjoint from (4.2). This functor is our main object of interest.

Remark 4.2.4. Note that the image of (4.3) is usually *not* contained in the full subcategory

$$\lim_{i \in I} \mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}) \subset \mathrm{Glue}(\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i \in I).$$

4.3. **The setting for the main theorem.**

4.3.1. We now consider a particular case of the above situation. Let G be a reductive group.

We let I^{op} be the category corresponding to the poset $\mathrm{Par}(G)$ of standard parabolics in G (i.e., the set of subsets of vertices of the Dynkin diagram of G).

Given a curve X, we let $\mathcal{Y} := \mathrm{LocSys}_G$ be the algebraic stack of G-local systems on X. We consider the functor

$$P \in \mathrm{Par}(G) \;\mapsto\; \mathcal{Z}_P := \mathrm{LocSys}_P\,.$$

We take

$$\mathcal{N} := \mathrm{Nilp}_{\mathrm{glob}} \subset \mathrm{Sing}(\mathrm{LocSys}_G)$$

to be the global nilpotent cone, see [AG, Sect. 11.1.1]. See also Sect. 7.1.3 for an explicit description of $\mathrm{Nilp}_{\mathrm{glob}}$.

For every $P \in \mathrm{Par}(G)$, we take $\mathcal{N}_P \subset \mathrm{Sing}(\mathrm{LocSys}_P)$ to be the zero-section $\{0\}$.

4.3.2. The following conjecture was made by us (it was recorded as [Ga3, Conjecture 9.3.7]):

Conjecture 4.3.3. *The functor*

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_G) \to \mathrm{Glue}(\mathrm{IndCoh}_{\{0\}}((\mathrm{LocSys}_P)_{\mathrm{dR}} \underset{(\mathrm{LocSys}_G)_{\mathrm{dR}}}{\times} \mathrm{LocSys}_G),\, P \in \mathrm{Par}(G)^{\mathrm{op}})$$

of (4.3) *is fully faithful.*

The main result of this paper is:

Theorem 4.3.4. *Conjecture 4.3.3 holds.*

The rest of this paper is devoted to the proof of this theorem.

4.4. **Gluing for D-modules.** In this subsection we formulate another gluing situation, in the context of D-modules. We then state a result that says that (under certain circumstances) the full faithfulness of the functor (4.3) is equivalent to the full faithfulness of a certain functor in the context of D-modules.

4.4.1. In what follows, for a prestack $\mathcal{Y}$ locally almost of finite type we consider the category D-mod($\mathcal{Y}$). By definition,

$$\text{D-mod}(\mathcal{Y}) := \text{QCoh}(\mathcal{Y}_{\text{dR}}),$$

and thus can be viewed as a symmetric monoidal category.

Recall also that according to [GR1, Proposition 2.4.4], the functor $\Upsilon_{\mathcal{Y}_{\text{dR}}}$ defines an equivalence

$$\text{QCoh}(\mathcal{Y}_{\text{dR}}) \to \text{IndCoh}(\mathcal{Y}_{\text{dR}}).$$

For a morphism $g : \mathcal{Y}_1 \to \mathcal{Y}_2$ we denote by $g^{\text{dR},!}$ the corresponding pullback functor

$$\text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

By definition, $g^{\text{dR},!}$ identifies with either of the vertical arrows in the following diagram:

$$\begin{array}{ccc} \text{QCoh}((\mathcal{Y}_1)_{\text{dR}}) & \xrightarrow{\Upsilon_{(\mathcal{Y}_1)_{\text{dR}}}} & \text{IndCoh}((\mathcal{Y}_1)_{\text{dR}}) \\ {\scriptstyle (g_{\text{dR}})^*}\uparrow & & \uparrow{\scriptstyle (g_{\text{dR}})^!} \\ \text{QCoh}((\mathcal{Y}_2)_{\text{dR}}) & \xrightarrow{\Upsilon_{(\mathcal{Y}_2)_{\text{dR}}}} & \text{IndCoh}((\mathcal{Y}_2)_{\text{dR}}). \end{array}$$

If g is schematic and quasi-compact, we denote by $g_{\text{dR},*}$ the corresponding direct image functor

$$\text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2).$$

4.4.2. Let $\mathcal{Y}'$ be a prestack locally almost of finite type. Let I be again an index category, and let

$$i \mapsto \mathcal{Z}'_i, \quad (\alpha : i \to j) \mapsto (f'_\alpha : \mathcal{Z}'_j \to \mathcal{Z}'_i).$$

be an I^{op}-diagram of algebraic stacks over $\mathcal{Y}$. We denote by f'_i the corresponding morphisms $\mathcal{Z}'_i \to \mathcal{Y}'$.

We consider

$$i \mapsto \text{D-mod}(\mathcal{Z}'_i), \quad (\alpha : i \to j) \mapsto (f'_\alpha)^{\text{dR},!}$$

as a functor $I \to \text{DGCat}_{\text{cont}}$.

Let now

$$\mathcal{M}_i \subset \mathcal{Z}'_i$$

be Zariski-closed subsets. We assume that for every $\alpha : i \to j$ we have

$$(f'_\alpha)^{-1}(\mathcal{M}_i) \subset \mathcal{M}_j.$$

Let $\mathcal{M}$ be a Zariski-closed subset of $\mathcal{Y}'$.

4.4.3. We consider the corresponding pair of adjoint functors

$$\mathrm{Glue}(\text{D-mod}(\mathcal{M}_i),\, i\in I) \rightleftarrows \mathrm{Glue}(\text{D-mod}(\mathcal{Z}'_i),\, i\in I). \tag{4.4}$$

The functors $(f'_i)^{\mathrm{dR},!}$ define a functor

$$\text{D-mod}(\mathcal{Y}') \to \lim_{i\in I} \text{D-mod}(\mathcal{Z}'_i).$$

Consider the composition

$$\begin{aligned}\text{D-mod}(\mathcal{M}) \hookrightarrow \text{D-mod}(\mathcal{Y}') \to \lim_{i\in I} \text{D-mod}(\mathcal{Z}'_i) \hookrightarrow \\ \hookrightarrow \mathrm{Glue}(\text{D-mod}(\mathcal{Z}'_i),\, i\in I) \to \mathrm{Glue}(\text{D-mod}(\mathcal{M}_i),\, i\in I),\end{aligned} \tag{4.5}$$

where the last arrow is the right adjoint from (4.4).

4.4.4. Consider again the setting of Sect. 4.2. Put

$$\mathcal{Y}' = \mathbb{P}\,\mathrm{Sing}(\mathcal{Y}) \qquad \mathcal{Z}'_i = \mathcal{Z}_i \underset{\mathcal{Y}}{\times} \mathbb{P}\,\mathrm{Sing}(\mathcal{Y})$$

$$\mathcal{M} = \mathbb{P}(\mathcal{N}) \qquad \mathcal{M}_i = \mathbb{P}\left(\mathrm{Sing}(f_i)^{-1}(\mathcal{N}_i)\right) \subset \mathcal{Z}'_i.$$

In Sect. 5 we prove:

Theorem 4.4.5. *Assume that the maps $f_i : \mathcal{Z}_i \to \mathcal{Y}$ are schematic and proper. Assume also that the following conditions hold:*

(1) *For every index i, we have $\{0\} \subset \mathcal{N}_i$ and*

$$\mathrm{Sing}(f_i)^{-1}(\mathcal{N}_i) \subset \mathcal{Z}_i \underset{\mathcal{Y}}{\times} \mathcal{N}.$$

(2) *The functor*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{i\in I} \mathrm{QCoh}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y})$$

is fully faithful;

(3) *The functor*

$$\text{D-mod}(\mathcal{M}) \to \mathrm{Glue}\left(\text{D-mod}\left(\mathcal{M}_i\right),\, i\in I\right)$$

of (4.5) *is fully faithful.*

Then the functor

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Glue}(\mathrm{IndCoh}_{\mathcal{N}_i}((\mathcal{Z}_i)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}),\, i\in I)$$

of (4.3) *is fully faithful.*

Remark 4.4.6. In Sect. 6.3 we express condition (3) in Theorem 4.4.5 in more concrete terms: it amounts to acyclicity of certain explicit objects of Vect, or, equivalently, to *homological contractibility* of certain homotopy types.

Thus, Theorem 4.4.5 claims that a certain full faithfulness assertion for IndCoh is essentially of topological nature. The proof of Theorem 4.4.5 is based on Theorem 3.2.9 from Part I of the paper.

Remark 4.4.7. With a little extra work, one can show that Theorem 4.4.5 holds without the condition that

$$\mathrm{Sing}(f_i)^{-1}(\mathcal{N}_i) \subset \mathcal{Z}_i \underset{\mathcal{Y}}{\times} \mathcal{N}.$$

4.4.8. We will apply Theorem 4.4.5 to deduce Theorem 4.3.4. We take $I = \mathrm{Par}(G)^{\mathrm{op}}$ and $\mathcal{Y}, \mathcal{Z}_i, \mathcal{N}, \mathcal{N}_i$ as in Sect. 4.3.1.

Note that condition (1) of Theorem 4.4.5 is trivially satisfied. Condition (2) is satisfied because the category $\mathrm{Par}(G)^{\mathrm{op}}$ has an initial object (the improper parabolic $P = G$), so

$$\lim_{P \in \mathrm{Par}(G)^{\mathrm{op}}} \mathrm{QCoh}((\mathrm{LocSys}_P)_{\mathrm{dR}} \underset{(\mathrm{LocSys}_G)_{\mathrm{dR}}}{\times} \mathrm{LocSys}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}_G).$$

Thus, Theorem 4.3.4 follows from Theorem 4.4.5, combined with the following result:

Theorem 4.4.9. *The functor*

$$\text{D-mod}\left(\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})\right) \to \mathrm{Glue}\left(\text{D-mod}\left(\mathbb{P}(\mathcal{M}_P)\right),\ P \in \mathrm{Par}(G)^{\mathrm{op}}\right)$$

is fully faithful, where

$$\mathcal{M}_P \subset \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G)$$

is the preimage of $\{0\} \subset \mathrm{Sing}(\mathrm{LocSys}_P)$ *under the map*

$$\mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G) \to \mathrm{Sing}(\mathrm{LocSys}_P).$$

We prove Theorem 4.4.9 in Part III of the paper.

5. Proof of Theorem 4.4.5

5.1. A criterion for fully faithfulness.

5.1.1. Let $(\mathbf{C}_i, \Phi_\alpha)$ be as in Sect. 4.1.1. Let $\mathbf{C}'_i \subset \mathbf{C}_i$ be full subcategories such that

$$\Phi_\alpha(\mathbf{C}'_i) \subset \mathbf{C}'_j, \quad (\alpha : i \to j) \in I.$$

Set $\overset{\circ}{\mathbf{C}}_i := (\mathbf{C}'_i)^{\perp} \subset \mathbf{C}_i$. Assume that

$$\Phi_\alpha(\overset{\circ}{\mathbf{C}}_i) \subset \overset{\circ}{\mathbf{C}}_j, \quad (\alpha : i \to j) \in I.$$

Denote

$$\mathbf{C} := \mathrm{Glue}(\mathbf{C}_i,\ i \in I), \quad \mathbf{C}' := \mathrm{Glue}(\mathbf{C}'_i,\ i \in I), \quad \overset{\circ}{\mathbf{C}} := \mathrm{Glue}(\overset{\circ}{\mathbf{C}}_i,\ i \in I).$$

Thus, we have a pair of full subcategories

$$\mathbf{C}' \hookrightarrow \mathbf{C} \hookleftarrow \overset{\circ}{\mathbf{C}}.$$

We have an inclusion

$$\overset{\circ}{\mathbf{C}} \subset (\mathbf{C}')^{\perp},$$

which, in general, is not an equality.

5.1.2. Let now

$$\mathsf{F}_i : \mathbf{D} \to \mathbf{C}_i$$

be a family of functors as in Sect. 4.1.3.

Let $\mathbf{D}' \subset \mathbf{D}$ be a full subcategory, and set

$$\overset{\circ}{\mathbf{D}} := (\mathbf{D}')^{\perp} \subset \mathbf{D}.$$

We assume that for every $i \in I$, the functor F_i satisifies:

$$\mathsf{F}_i(\mathbf{D}') \subset \mathbf{C}'_i, \quad \mathsf{F}_i(\overset{\circ}{\mathbf{D}}) \subset \overset{\circ}{\mathbf{C}}_i.$$

These conditions imply that F restricts to well-defined functors

$$\mathsf{F}' : \mathbf{D}' \to \mathbf{C}' \text{ and } \overset{\circ}{\mathsf{F}} : \overset{\circ}{\mathbf{D}} \to \overset{\circ}{\mathbf{C}}.$$

We claim:

Proposition 5.1.3. *Assume that:*

(a) *Each of the functors F_i admits a left adjoint, denoted F_i^L, and*

$$\mathsf{F}_i^L(\overset{\circ}{\mathbf{C}}_i) \subset \overset{\circ}{\mathbf{D}} \quad \textit{for all } i \in I.$$

(b) *The functors F' and $\overset{\circ}{\mathsf{F}}$ are both fully faithful.*

Then F is also fully faithful.

5.1.4. The rest of this subsection is devoted to the proof of Proposition 5.1.3, which is rather formal.

It is easy to see that the assumption that the functors F_i each admits a left adjoint implies that the functor $\mathsf{F} : \mathbf{D} \to \mathbf{C}$ admits a left adjoint[5] (denoted F^L), which satisfies

$$\mathsf{F}^L \circ \mathrm{ins}_i \simeq \mathsf{F}_i^L \quad \text{for all } i \in I,$$

where ins_i is as in Sect. 4.1.8.

We will need the following:

Lemma 5.1.5. *If*

$$\mathsf{F}_i^L(\overset{\circ}{\mathbf{C}}_i) \subset \overset{\circ}{\mathbf{D}} \quad \textit{for all } i \in I,$$

then the diagram

$$\begin{array}{ccc} \mathbf{C} & \longleftarrow & \overset{\circ}{\mathbf{C}} \\ {\scriptstyle \mathsf{F}^L}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{\mathsf{F}}{}^L} \\ \mathbf{D} & \longleftarrow & \overset{\circ}{\mathbf{D}} \end{array}$$

commutes.

[5] In Sect. 6.2 we give a more explicit description of the functor F^L.

Proof. It is enough to show that for every $i \in I$ the diagram

$$\begin{array}{ccc} & & \overset{\circ}{\mathbf{C}}_i \\ & & \downarrow{\scriptstyle \mathrm{ins}_i} \\ \mathbf{C} & \longleftarrow & \overset{\circ}{\mathbf{C}} \\ {\scriptstyle \mathsf{F}^L}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{\mathsf{F}}{}^L} \\ \mathbf{D} & \longleftarrow & \overset{\circ}{\mathbf{D}} \end{array}$$

commutes. Note, however, that the diagram

$$\begin{array}{ccc} \mathbf{C}_i & \longleftarrow & \overset{\circ}{\mathbf{C}}_i \\ {\scriptstyle \mathrm{ins}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{ins}_i} \\ \mathbf{C} & \longleftarrow & \overset{\circ}{\mathbf{C}} \end{array}$$

commutes. Hence, it is enough to establish the commutativity of

$$\begin{array}{ccc} \mathbf{C}_i & \longleftarrow & \overset{\circ}{\mathbf{C}}_i \\ {\scriptstyle \mathsf{F}^L \circ \mathrm{ins}_i}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{\mathsf{F}}{}^L \circ \mathrm{ins}_i} \\ \mathbf{D} & \longleftarrow & \overset{\circ}{\mathbf{D}}. \end{array}$$

However, the latter diagram identifies with

$$\begin{array}{ccc} \mathbf{C}_i & \longleftarrow & \overset{\circ}{\mathbf{C}}_i \\ {\scriptstyle \mathsf{F}_i^L}\downarrow & & \downarrow{\scriptstyle \overset{\circ}{\mathsf{F}}{}_i^L} \\ \mathbf{D} & \longleftarrow & \overset{\circ}{\mathbf{D}}, \end{array}$$

and the commutativity follows from the assumption.

□

5.1.6. *Proof of Proposition 5.1.3.* It suffices to check that for $\mathbf{d}' \in \mathbf{D}'$ and $\overset{\circ}{\mathbf{d}} \in \overset{\circ}{\mathbf{D}}$, the map

$$\mathrm{Hom}_{\mathbf{D}}(\overset{\circ}{\mathbf{d}}, \mathbf{d}') \to \mathrm{Hom}_{\mathbf{C}}(\overset{\circ}{\mathsf{F}}(\overset{\circ}{\mathbf{d}}), \mathsf{F}'(\mathbf{d}')) \tag{5.1}$$

is an isomorphism.

Using Lemma 5.1.5, we can identify (5.1) with the map

$$\mathrm{Hom}_{\mathbf{D}}(\overset{\circ}{\mathbf{d}}, \mathbf{d}') \to \mathrm{Hom}_{\mathbf{D}}(\overset{\circ}{\mathsf{F}}{}^L \circ \overset{\circ}{\mathsf{F}}(\overset{\circ}{\mathbf{d}}), \mathbf{d}'),$$

which comes from the co-unit of the adjunction

$$\overset{\circ}{\mathsf{F}}{}^L \circ \overset{\circ}{\mathsf{F}}(\overset{\circ}{\mathbf{d}}) \to \overset{\circ}{\mathbf{d}}.$$

Since, the latter is an an isomorphism ($\overset{\circ}{\mathsf{F}}$ was assumed fully faithful), the assertion of the proposition follows.

□

5.2. **Proof of Theorem 4.4.5, Step 0.**

5.2.1. It is easy to see by descent that the property of the functor (4.3) to be fully faithful is local in the smooth topology on $\mathcal{Y}$. The same is true for conditions (2) and (3) in Theorem 4.4.5.

Hence, we can assume that $\mathcal{Y} =: Y$ and $\mathcal{Z}_i =: Z_i$ are DG schemes.

5.2.2. We prove Theorem 4.4.5 by applying Proposition 5.1.3. We take

$$\mathbf{D} = \mathrm{IndCoh}_{\mathcal{N}}(Y), \quad \mathbf{D}' = \mathrm{QCoh}(Y), \quad \overset{\circ}{\mathbf{D}} = \overset{\circ}{\mathrm{IndCoh}}(Y) \cap \mathrm{IndCoh}_{\mathcal{N}}(Y).$$

We take

$$\mathbf{C}_i := \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

Recall the identification

$$\mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y)$$

of Proposition 3.1.2.

We take

$$\mathbf{C}'_i = \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)) = \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Y) \subset$$
$$\subset \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y) \simeq \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

We have:

$$\mathbf{C}'_i \subset \mathrm{IndCoh}_{\{0\}}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) = \mathbf{C}_i.$$

Thus,

$$\overset{\circ}{\mathbf{C}}_i = \left(\mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y)\right) \cap \left(\mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)\right).$$

The functors F_i are the compositions

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{IndCoh}(Y) \overset{((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!}{\longrightarrow} \tag{5.2}$$
$$\to \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \to \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y),$$

where the last arrow is the right adjoint to the embedding

$$\mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \hookrightarrow \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y).$$

It is clear that the above functor sends $\mathbf{D}' = \mathrm{QCoh}(Y)$ to $\mathbf{C}'_i = \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y))$.

In Steps 1 and 2 below we will verify that the above data satisfies the conditions of Proposition 5.1.3.

5.3. **Proof of Theorem 4.4.5, Step 1.** In this subsection we will show the following:

(i) The above functor $\mathsf{F}_i : \mathbf{D} \to \mathbf{C}_i$

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \to \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

admits a left adjoint.

(ii) The left adjoint in (i) sends $\mathbf{C}'_i$ to $\mathbf{D}'$, i.e.,

$$\mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

to

$$\mathrm{QCoh}(Y) \subset \mathrm{IndCoh}_{\mathcal{N}}(Y).$$

Note that (ii) is equivalent to the fact that F_i sends $\overset{\circ}{\mathbf{D}}$ to $\overset{\circ}{\mathbf{C}}_i$.

(iii) The left adjoint in (i) sends $\overset{\circ}{\mathbf{C}}_i$ to $\overset{\circ}{\mathbf{D}}$.

5.3.1. First, we claim that the functor

$$\mathrm{IndCoh}(Y) \overset{((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!}{\longrightarrow} \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

admits a left adjoint[6]. Indeed, we rewrite

$$\mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y).$$

So, it is enough to show that the functor

$$((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

admits a left adjoint (it automatically commutes with the action of $\mathrm{QCoh}(Y)$, because $\mathrm{QCoh}(Y)$ is rigid as a monoidal category).

We write

$$\mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \simeq \mathrm{QCoh}((Z_i)_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(Y),$$

see Lemma 1.5.6.

So, it is enough to show that the functor

$$(f_i)^*_{\mathrm{dR}} : \mathrm{QCoh}(Y_{\mathrm{dR}}) \to \mathrm{QCoh}((Z_i)_{\mathrm{dR}})$$

admits a left adjoint, which commutes with the action of $\mathrm{QCoh}(Y_{\mathrm{dR}})$.

We interpret the latter functor as

$$f_i^{\mathrm{dR},!} : \text{D-mod}(Y) \to \text{D-mod}(Z_i).$$

Since f_i is proper, the left adjoint in question is the functor

$$(f_i)_{\mathrm{dR},*} : \text{D-mod}(Z_i) \to \text{D-mod}(Y).$$

The commutativity with the action of $\mathrm{QCoh}(Y_{\mathrm{dR}}) = \text{D-mod}(Y)$ is given by the projection formula for $(f_i)_{\mathrm{dR},*}$.

[6]More conceptually, the left adjoint in question exists because the map $(f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)$ is *inf-schematic and nil-proper*, see [GR2, Part III.3, Proposition 3.2.4].

5.3.2. Now, the left adjoint to the functor (5.2) is given by the composition

$$\mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \hookrightarrow \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \overset{(((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!)^L}{\longrightarrow} \mathrm{IndCoh}(Y).$$

We claim that the essential image of the above functor belongs to $\mathrm{IndCoh}_{\mathcal{N}}(Y)$. Indeed, by Corollary 3.2.5, it suffices to show that the composition

$$\mathrm{IndCoh}_{\mathcal{N}_i}(Z_i) \to \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \hookrightarrow \\ \to \mathrm{IndCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \overset{(((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!)^L}{\longrightarrow} \mathrm{IndCoh}(Y)$$

maps to $\mathrm{IndCoh}_{\mathcal{N}}(Y)$. However, the latter functor identifies with

$$\mathrm{IndCoh}_{\mathcal{N}_i}(Z_i) \hookrightarrow \mathrm{IndCoh}(Z_i) \overset{(f_i)^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(Y),$$

and the desired containment follows from condition (1) in Theorem 4.4.5 and [AG, Proposition 7.1.3(b)].

5.3.3. The fact that the left adjoint to (5.2) sends

$$\mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \subset \mathrm{IndCoh}_{\mathcal{N}_i}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y)$$

to

$$\mathrm{QCoh}(Y) \subset \mathrm{IndCoh}_{\mathcal{N}}(Y)$$

follows from the construction.

5.3.4. The fact that the left adjoint to (5.2) sends $\overset{\circ}{\mathbf{C}}_i$ to $\overset{\circ}{\mathrm{IndCoh}}(Y)$ follows from the fact that the functor left adjoint to $((f_i)_{\mathrm{dR}} \times \mathrm{id}_Y)^!$ sends

$$\mathrm{QCoh}((Z_i)_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y)$$

to $\overset{\circ}{\mathrm{IndCoh}}(Y)$, which follows from the description of this left adjoint in Sect. 5.3.1.

5.4. **Proof of Theorem 4.4.5, Step 2.** In order to apply Proposition 5.1.3, we need to show that the functors $\mathrm{QCoh}(Y) \to \mathbf{C}'$ and $\overset{\circ}{\mathrm{IndCoh}}(Y) \cap \mathrm{IndCoh}_{\mathcal{N}}(Y) \to \overset{\circ}{\mathbf{C}}$ are both fully faithful.

5.4.1. The fact that $\mathrm{QCoh}(Y) \to \mathbf{C}'$ is fully faithful is given by condition (2) in Theorem 4.4.5.

5.4.2. It remains to show that the functor

$$\overset{\circ}{\mathrm{IndCoh}}(Y) \cap \mathrm{IndCoh}_{\mathcal{N}}(Y) \to \overset{\circ}{\mathbf{C}} \tag{5.3}$$

is fully faithful.

We are now going to use the results from Part I of the paper. Namely, according to Theorem 3.2.9, the functor

$$I \to \mathrm{DGCat}_{\mathrm{cont}}, \quad i \mapsto \overset{\circ}{\mathbf{C}}_i$$

identifies with the functor

$$i \mapsto \text{D-mod}\left(\mathbb{P}\left(\mathrm{Sing}(f_i)^{-1}(\mathcal{N}_i)\right)\right) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y).$$

Similarly, by Theorem 1.4.2,

$$\overset{\circ}{\mathrm{IndCoh}}(Y) \cap \mathrm{IndCoh}_{\mathcal{N}}(Y) \simeq \text{D-mod}(\mathbb{P}\mathcal{N}) \underset{\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))}{\otimes} \overset{\circ}{\mathrm{IndCoh}}(Y).$$

5.4.3. We have the following general assertion:

Lemma 5.4.4. *Suppose that in the setting of Sect. 4.1.1, the functor* $I \to \mathrm{DGCat}_{\mathrm{cont}}$

$$i \mapsto \mathbf{C}_i, \quad (\alpha : i \to j) \mapsto \Phi_\alpha$$

upgrades to a functor $I \to \mathbf{O}$-mod, *where* $\mathbf{O}$ *is a monoidal DG category. Then for a right* $\mathbf{O}$*-module category* $\widetilde{\mathbf{C}}$*, the functor*

$$\widetilde{\mathbf{C}} \underset{\mathbf{O}}{\otimes} \mathrm{Glue}(\mathbf{C}_i,\, i \in I) \to \mathrm{Glue}(\widetilde{\mathbf{C}} \underset{\mathbf{O}}{\otimes} \mathbf{C}_i,\, i \in I)$$

is an equivalence.

Proof. Follows from Sect. 4.1.8. □

Applying Lemma 5.4.4, we obtain that the functor (5.3) identifies with the functor obtained from

$$\mathsf{F}_{\text{D-mod}} : \text{D-mod}(\mathbb{P}(\mathcal{N})) \to \mathrm{Glue}\left(\text{D-mod}\left(\mathbb{P}\left(\mathrm{Sing}(f_i)^{-1}(\mathcal{N}_i)\right)\right),\, i \in I\right) \tag{5.4}$$

by tensoring over $\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))$ with $\overset{\circ}{\mathrm{IndCoh}}(Y)$.

5.4.5. The functor $\mathsf{F}_{\text{D-mod}}$ admits a left adjoint that commutes with the monoidal action of $\text{D-mod}(\mathbb{P}\,\mathrm{Sing}(Y))$ (by the same argument as in Lemma 5.1.5); denote it by $\mathsf{F}^L_{\text{D-mod}}$. Hence, the functor (5.3) also admits a left adjoint that can be identified with

$$\mathsf{F}^L_{\text{D-mod}} \otimes \mathrm{Id}_{\overset{\circ}{\mathrm{IndCoh}}(Y)}.$$

We need to show that the co-unit of the adjunction

$$(\mathsf{F}_{\text{D-mod}} \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)})^L \circ (\mathsf{F}_{\text{D-mod}} \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)}) \simeq$$
$$\simeq (\mathsf{F}^L_{\text{D-mod}} \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)}) \circ (\mathsf{F}_{\text{D-mod}} \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)}) \simeq (\mathsf{F}^L_{\text{D-mod}} \circ \mathsf{F}_{\text{D-mod}}) \otimes \mathrm{Id}_{\mathrm{IndCoh}(Y)}) \to \mathrm{Id}$$

is an isomorphism.

For that, it is enough to know that $\mathsf{F}^L_{\text{D-mod}} \circ \mathsf{F}_{\text{D-mod}} \to \mathrm{Id}$ is an isomorphism, i.e., that $\mathsf{F}_{\text{D-mod}}$ is fully faithful.

However, the latter is given by condition (3) in Theorem 4.4.5.

6. Gluing for D-modules and homological contractibility

For the rest of the paper we work within the usual (as opposed to derived) algebraic geometry. The reason for this is that for a derived scheme Y, the map ${}^{\mathrm{cl}}Y \to Y$ gives rise to an isomorphism $({}^{\mathrm{cl}}Y)_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ (here ${}^{\mathrm{cl}}Y$ denotes the classical scheme underlying Y), so the pullback functor $\text{D-mod}(Y) \to \text{D-mod}({}^{\mathrm{cl}}Y)$ is an equivalence.

From now on, our goal is to prove Theorem 4.4.9; that is, we need to verify condition (3) in Theorem 4.4.5 in a particular situation. Condition (3) may appear somewhat obscure. In this section, we restate it in more concrete terms as *homological contractibility* of certain homotopy types.

6.1. **D-modules on prestacks.** In this subsection we consider a simplified version of the set-up of Sect. 4.4, namely, one where $\mathcal{M}_i = \mathcal{Z}'_i$, and where instead of the glued category we consider the actual (strict) limit.

Strictly speaking, some of this material is not necessary for the sequel; it is included for completeness and in order to familiarize the reader with the objects involved. For a more comprehensive review of the theory, the reader is referred to [Ga4, Sects. 1 and 7.4].

For the duration of the paper, we let $\mathrm{Sch}^{\mathrm{aff}}$ denote the category of (classical) affine schemes of finite type.

6.1.1. Recall that for a prestack $\mathcal{Z}$, the DG category $\text{D-mod}(\mathcal{Z})$ is defined to be

$$\lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Z}})^{\mathrm{op}}} \text{D-mod}(S),$$

where the limit is formed using !-pullbacks as transition functors.

If $\mathcal{Z}$ is written as a colimit over I^{op} (where I is an index ∞-category) as

$$\mathcal{Z} \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} Z_i, \tag{6.1}$$

where $Z_i \in \mathrm{Sch}$, then the functor

$$I^{\mathrm{op}} \to \mathrm{Sch}_{/\mathcal{Z}}, \quad i \mapsto Z_i$$

is cofinal, and so the restriction map

$$\text{D-mod}(\mathcal{Z}) \to \lim_{i \in I} \text{D-mod}(Z_i).$$

is an equivalence.

6.1.2. A prestack $\mathcal{Z}$ is said to be a *pseudo-scheme* if it admits a presentation (6.1)

$$\mathcal{Z} \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} Z_i,$$

where $Z_i \in \mathrm{Sch}$, and the transition maps $Z_i \to Z_j$ are proper.

In this case, by [DrGa, Proposition 1.7.5], the evaluation functors

$$\mathrm{ev}_i : \text{D-mod}(\mathcal{Z}) \to \text{D-mod}(Z_i)$$

admit left adjoints (to be denoted ins_i), and the resulting functor

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} \text{D-mod}(Z_i) \to \text{D-mod}(\mathcal{Z}), \tag{6.2}$$

is an equivalence. In the formation of the above colimit, for an arrow $i_1 \xrightarrow{\alpha} i_2$ in I and the corresponding map $Z_{i_2} \xrightarrow{f_\alpha} Z_{i_1}$, the functor

$$\text{D-mod}(Z_{i_2}) \to \text{D-mod}(Z_{i_2})$$

is $(f_\alpha)_{\mathrm{dR},!} = (f_\alpha)_{\mathrm{dR},*}$. The functors $\text{D-mod}(Z_i) \to \text{D-mod}(\mathcal{Z})$ in (6.2) are ins_i.

6.1.3. Suppose now that $\mathcal{Z}$ is a prestack over a scheme Y that admits a presentation (6.1) where all Z_i are *proper* over Y. We then say that $\mathcal{Z}$ is *pseudo-proper* over Y.

Let f (reps. f_i) denote the map $\mathcal{Z} \to Y$ (reps. $Z_i \to Y$). Consider the pullback functor

$$f^{\mathrm{dR},!} : \text{D-mod}(Y) \to \text{D-mod}(\mathcal{Z}).$$

By Sect. 6.1.2, the functor $f^{\mathrm{dR},!}$ admits a left adjoint, to be denoted $f_{\mathrm{dR},!}$, which is given in terms of the equivalence (6.2) by the compatible family of functors

$$(f_i)_{\mathrm{dR},!} = (f_i)_{\mathrm{dR},*} : \text{D-mod}(Z_i) \to \text{D-mod}(Y).$$

That is, $f_{\mathrm{dR},!} \circ \mathrm{ins}_i \simeq (f_i)_{\mathrm{dR},!}$.

The properness assumption on the f_i's implies the following base-change property: for a morphism of schemes $Y' \xrightarrow{g} Y$ and the corresponding Cartesian square

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{g_Z} & \mathcal{Z} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array} \tag{6.3}$$

the canonical map

$$(f')_{\mathrm{dR},!} \circ (g_Z)^{\mathrm{dR},!} \to g^{\mathrm{dR},!} \circ f_{\mathrm{dR},!}, \tag{6.4}$$

arising by adjunction from the isomorphism $(g_Z)^{\mathrm{dR},!} \circ f^{\mathrm{dR},!} \simeq (f')^{\mathrm{dR},!} \circ g^{\mathrm{dR},!}$, is an isomorphism.

6.1.4. We say that a prestack $\mathcal{Z}$ over a scheme Y is *homologically contractible* over Y if the pullback functor

$$(f)^{\mathrm{dR},!} : \text{D-mod}(Y) \to \text{D-mod}(\mathcal{Z})$$

is fully faithful.

Since $\mathcal{Z}$ is pseudo-proper over Y, the functor $f_{\mathrm{dR},!}$ admits a right adjoint $f^{\mathrm{dR},!}$. Hence, $\mathcal{Z}$ is homologically contractible over Y if and only if the co-unit of the adjuntion

$$f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!} \to \mathrm{Id}_{\text{D-mod}(Y)}$$

is an isomorphism.

The endofunctor $f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}$ of D-mod(Y) can be described explicitly as

$$f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}(\mathcal{F}) = \operatorname*{colim}_{i \in I^{\mathrm{op}}} (f_i)_{\mathrm{dR},!} \circ (f_i)^{\mathrm{dR},!}(\mathcal{F}). \tag{6.5}$$

Therefore, $\mathcal{Z}$ is homologically contractible over $\mathcal{Z}$ if and only if the natual map

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} (f_i)_{\mathrm{dR},*} \circ (f_i)^{\mathrm{dR},!}(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \text{D-mod}(Y)$.

6.1.5. Assume for a moment that $Y = \mathrm{pt}$, so that D-mod$(Y) = \mathrm{Vect}$. Then the endofunctor $f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}$ of Vect is given by tensor product with the object

$$f_{\mathrm{dR},!}(\omega_{\mathcal{Z}}) = f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}(k).$$

(As a side remark, $f_{\mathrm{dR},!}(\omega_{\mathcal{Z}})$ is defined even if $\mathcal{Z}$ is not pseudo-proper; this is due to the fact that the value of $\omega_{\mathcal{Z}}$ on any $S \in \mathrm{Sch}_{/\mathcal{Z}}$ is ω_S, which is holonomic.)

Put

$$f_{\mathrm{dR},!}(\omega_{\mathcal{Z}}) =: \mathrm{C}_*(\mathcal{Z}) \in \mathrm{Vect}\,.$$

We call $\mathrm{C}_*(\mathcal{Z})$ the *homology* of $\mathcal{Z}$.

Remark 6.1.6. When $k = \mathbb{C}$, we can attach to $\mathcal{Z}$ a homotopy type $\mathcal{Z}^{\mathrm{top}}$ given by

$$\mathcal{Z}^{\mathrm{top}} := \underset{S \in \mathrm{Sch}_{/\mathcal{Z}}}{\mathrm{colim}}\, S^{\mathrm{top}}.$$

Here $S \mapsto S^{\mathrm{top}}$ is the functor sending a scheme to the underlying analytic space, and the colimit is taken in the ∞-category of spaces. The Riemann-Hilbert correspondence yields a canonical isomorphism

$$\mathrm{C}_*(\mathcal{Z}) \simeq \mathrm{C}_*(\mathcal{Z}^{\mathrm{top}}, k),$$

where the right-hand side is the homology of the homotopy type $\mathcal{Z}^{\mathrm{top}}$.

We now claim:

Lemma 6.1.7. *Let $\mathcal{Z}$ be pseudo-proper over Y. Then the following conditions are equivalent:*

(i) *The prestack $\mathcal{Z}$ is homologically contractible over Y;*

(ii) *The prestack $\mathcal{Z}$ is* universally *homologically contractible over Y: for any morphism of schemes $Y' \to Y$, the fiber product $\mathcal{Z}' := \mathcal{Z} \underset{Y}{\times} Y$ is homologically contractible over Y';*

(iii) *The map*

$$f_{\mathrm{dR},!}(\omega_{\mathcal{Z}}) \simeq f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}(\omega_Y) \to \omega_Y$$

is an isomorphism;

(iv) *For every field extension $k' \supset k$ and every k'-point y of Y, the prestack $\mathcal{Z}_y = \mathrm{Spec}(k') \underset{Y}{\times} \mathcal{Z}$ is homologically trivial over* $\mathrm{Spec}(k')$*;*

(v) *For every field extension $k' \supset k$ and every k'-point y of Y, the k'-prestack $\mathcal{Z}_y$ has trivial homology: the natural map*

$$\mathrm{C}_*(\mathcal{Z}_y) \to k'$$

is an isomorphism.

Proof. We have (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) for tautological reasons. The implication (iii) $\Rightarrow$ (v) follows from base change (Sect. 6.1.3). The equivalence (iv) $\Leftrightarrow$ (v) follows from Sect. 6.1.5. Let us prove that (iv) $\Rightarrow$ (ii).

Note first that if for a scheme Y and $\mathcal{F} \in \text{D-mod}(Y)$, we have $\mathcal{F} = 0$ if and only if for every field extension $k' \supset k$ and every k'-point y of Y, the !-fiber of

$$k' \underset{k}{\otimes} \mathcal{F} \in \text{D-mod}(k' \underset{k}{\otimes} Y)$$

at y is zero. Using the fact that the formation of $\mathcal{F} \mapsto f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}(\mathcal{F})$ commutes with field extensions (which follows, for instance, from the description of $f_{\mathrm{dR},!} \circ f^{\mathrm{dR},!}$ as (6.5)), and the base-change isomorphism (6.4), we obtain that (ii) is equivalent to each $\mathcal{Z}_y$ being homologically contractible, as claimed.

□

6.2. **Explicit description of the left adjoint: a digression.** Consider the general set-up of Sect. 4.1.3. Thus, we have an index category I and an I-diagram of categories

$$i \mapsto \mathbf{C}_i, \quad (i \xrightarrow{\alpha} j) \mapsto (\mathbf{C}_i \xrightarrow{\Phi_\alpha} \mathbf{C}_j).$$

Let F be the functor

$$\mathbf{D} \to \mathrm{Glue}(\mathbf{C}_i,\, i \in I)$$

given by a lax-compatible family of functors $\mathsf{F}_i : \mathbf{D} \to \mathbf{C}_i$. Let us assume that each of the functors F_i admits a left adjoint, which we denote F_i^L.

Let us give an explicit formula for the left adjoint of the functor

$$\mathsf{F} : \mathbf{D} \to \mathrm{Glue}(\mathbf{C}_i,\, i \in I).$$

6.2.1. Consider the category $\mathrm{String}(I)$, whose objects are strings of objects of I:

$$(i_0 \to i_1 \to \cdots \to i_n), \tag{6.6}$$

and whose morphisms are induced by order preserving maps $[m] \to [n]$. In other words, $\mathrm{String}(I)$ is the co-Cartesian fibration in groupoids over $\mathbf{\Delta}^{\mathrm{op}}$ corresponding to the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \infty\text{-Grpd}$$

given by the nerve of I.

6.2.2. There exists a canonically defined functor

$$\mathsf{F}^L_{\mathrm{String}} : \mathrm{Glue}(\mathbf{C}_i,\, i \in I) \to \mathrm{Funct}(\mathrm{String}(I), \mathbf{D}).$$

Namely, given an object

$$\{i \mapsto \mathbf{c}_i,\ (i \xrightarrow{\alpha} j) \mapsto (\Phi_\alpha(\mathbf{c}_i) \to \mathbf{c}_j)\} \in \mathrm{Glue}(\mathbf{C}_i,\, i \in I),$$

the functor $\mathsf{F}^L_{\mathrm{String}}$ sends it to the functor $\mathrm{String}(I) \to \mathbf{D}$ that sends (6.6) to

$$\mathsf{F}^L_{i_n}(\Phi_{i_0 \to i_n}(\mathbf{c}_{i_0})).$$

6.2.3. Consider the composition functor

$$\mathrm{Glue}(\mathbf{C}_i,\, i \in I) \xrightarrow{\mathsf{F}^L_{\mathrm{String}}} \mathrm{Funct}(\mathrm{String}(I), \mathbf{D}) \xrightarrow{\mathrm{colim}} \mathbf{D}, \tag{6.7}$$

where the right arrow is the functor of colimit along $\mathrm{String}(I)$. We claim:

Proposition 6.2.4. *The functor* (6.7) *is the left adjoint of the functor* F.

Proof. We can factor F as a composition

$$\mathbf{D} \to \mathrm{Glue}(\mathbf{D},\, i \in I) \to \mathrm{Glue}(\mathbf{C}_i,\, i \in I),$$

where $\mathrm{Glue}(\mathbf{D},\, i \in I)$ is formed using the constant functor

$$I \to \mathrm{DGCat}_{\mathrm{cont}}, \quad i \mapsto \mathbf{D}. \tag{6.8}$$

This reduces the statement of the proposition to the case when $\mathbf{C}_i = \mathbf{D}$, as in (6.8). We then identify

$$\mathrm{Glue}(\mathbf{D},\, i \in I) \simeq \mathrm{Funct}(I, \mathbf{D}),$$

and the assertion becomes equivalent to the usual expression of colimits along I via its nerve. □

6.3. **Full faithfulness as homological contractibility.** We return to the situation of Sect. 4.4 (with a slightly simplified notation). Let $\mathcal{Y} = Y \in \mathrm{Sch}$ be a base scheme and

$$i \mapsto (Z_i \xrightarrow{f_i} Y), \quad (i \xrightarrow{\alpha} j) \mapsto (Z_j \xrightarrow{f_\alpha} Z_i)$$

an I^{op}-diagram of schemes over it. Let $\mathcal{M}_i \subset Z_i$ be closed subschemes such that for every $i \xrightarrow{\alpha} j$ we have

$$(f_\alpha)^{-1}(\mathcal{M}_i) \subset \mathcal{M}_j,$$

i.e., we have the diagrams

$$\begin{array}{ccc} (f_\alpha)^{-1}(\mathcal{M}_i) & \longrightarrow & \mathcal{M}_j \\ {\scriptstyle f_\alpha}\big\downarrow & & \\ \mathcal{M}_i. & & \end{array}$$

6.3.1. We consider the category

$$\mathrm{Glue}(\text{D-mod}(\mathcal{M}_i),\, i \in I).$$

For every i, we let $\mathsf{F}_i : \text{D-mod}(Y) \to \text{D-mod}(\mathcal{M}_i)$ be the functor

$$\text{D-mod}(Y) \xrightarrow{f_i^{\mathrm{dR},!}} \text{D-mod}(Z_i) \to \text{D-mod}(\mathcal{M}_i),$$

where the second arrow is the !-pullback along the embedding $\mathcal{M}_i \hookrightarrow Z_i$.

The functors F_i give rise to a functor

$$\mathsf{F} : \text{D-mod}(Y) \to \mathrm{Glue}(\text{D-mod}(\mathcal{M}_i),\, i \in I),$$

and we want to give an explicit criterion for full faithfulness of F.

6.3.2. Let us assume that all Z_i are proper over Y.

In this situation, each of the functors F_i admits a left adjoint, and we find ourselves in the setting of Sect. 6.2. Hence the functor F admits a left adjoint given by (6.7).

Denote the left adjoint of F by

$$\mathsf{F}^L : \mathrm{Glue}(\text{D-mod}(\mathcal{M}_i),\, i \in I) \to \text{D-mod}(Y).$$

The functor F is fully faithful if and only if the co-unit of the adjunction

$$\mathsf{F}^L \circ F \to \mathrm{Id}_{\text{D-mod}(Y)}$$

is an isomorphism.

Let us describe the functor $\mathsf{F}^L \circ F$ explicitly.

6.3.3. Consider the following prestack over Y, denoted $\mathcal{M}_{\mathrm{Glued}}$:

The prestack is the colimit over the category $\mathrm{String}(I^{\mathrm{op}})$ of the functor

$$\mathrm{String}(I^{\mathrm{op}}) \to \mathrm{PreStk}, \quad (i_0 \to i_1 \to \cdots \to i_n) \mapsto \mathcal{Z}_{i_0} \underset{\mathcal{Z}_{i_n}}{\times} \mathcal{M}_{i_n}.$$

(Note that the categories $\mathrm{String}(I^{\mathrm{op}})$ and $\mathrm{String}(I)$ are naturally equivalent.) Denote by f_{Glued} the natural map

$$\mathcal{M}_{\mathrm{Glued}} \to Y.$$

Note that $\mathcal{M}_{\mathrm{Glued}}$ is by definition pseudo-proper over Y. By the results of Sect. 6.1.3, the functor $(f_{\mathrm{Glued}})^{\mathrm{dR},!}$ admits a left adjoint

$$(f_{\mathrm{Glued}})_{\mathrm{dR},!} : \text{D-mod}(\mathcal{M}_{\mathrm{Glued}}) \to \text{D-mod}(Y).$$

6.3.4. From Proposition 6.2.4 we obtain:

Corollary 6.3.5. *There is a canonical isomorphism of endofunctors of* $\mathrm{D\text{-}mod}(Y)$ over $\mathrm{Id}_{\mathrm{D\text{-}mod}(Y)}$*:*

$$\mathsf{F}^L \circ \mathsf{F} \simeq (f_{\mathrm{Glued}})_{\mathrm{dR},!} \circ (f_{\mathrm{Glued}})^{\mathrm{dR},!}.$$

Hence, we obtain:

Corollary 6.3.6. *The functor* F *is fully faithful if and only if the map* f_{Glued} *is* homologically contractible*, that is, if the functor* $(f_{\mathrm{Glued}})^{\mathrm{dR},!}$ *is fully faithful.*

6.3.7. Let $k' \supset k$ be a field extension and let y be a k'-point of Y. Let $\mathcal{M}_{\mathrm{Glued},y}$ be the fiber of $\mathcal{M}_{\mathrm{Glued}}$ over y, that is,

$$\mathcal{M}_{\mathrm{Glued},y} = \mathrm{Spec}(k') \underset{y,\mathcal{Y}}{\times} \mathcal{M}_{\mathrm{Glued}}.$$

Explicitly,

$$\mathcal{M}_{\mathrm{Glued},y} \simeq \underset{(i_0 \to i_1 \to \cdots \to i_n) \in \mathrm{String}(I^{\mathrm{op}})}{\mathrm{colim}} \mathrm{Spec}(k') \underset{y,\mathcal{Y}}{\times} \left(\mathcal{Z}_{i_0} \underset{\mathcal{Z}_{i_n}}{\times} \mathcal{M}_{i_n} \right).$$

Combining Corollary 6.3.6 and Lemma 6.1.7, we obtain:

Corollary 6.3.8. *The functor*

$$\mathsf{F} : \mathrm{D\text{-}mod}(\mathcal{Y}) \to \mathrm{Glue}(\mathrm{D\text{-}mod}(\mathcal{M}_i),\, i \in I)$$

is fully faithful if and only if for every field extension $k' \supset k$ *and every* k'*-point* y *of* Y*, the prestack* $\mathcal{M}_{\mathrm{Glued},y}$ *is homologically contractible over* k'*; that is, the map*

$$\mathrm{C}_*(\mathcal{M}_{\mathrm{Glued},y}) \to k'$$

is an isomorphism.

Part III: Springer fibers.

7. Reduction to a homological contractibility statement

The goal of the remainder of the paper is to prove Theorem 4.4.9 and thereby finish the proof of Theorem 4.3.4. Recall that we work in the framework of usual (non-derived) algebraic geometry, which suffices for the study of D-modules. In other words, all (DG) schemes/stacks are replaced by the corresponding classical subschemes/substacks.

7.1. What do we need to show?

7.1.1. Recall the statement of Theorem 4.4.9. For any $P \in \mathrm{Par}(G)$, we consider the stack of P-local systems LocSys_P. When $P = G$, we take the global nilpotent cone

$$\mathrm{Nilp}_{\mathrm{glob}} \subset \mathrm{Sing}(\mathrm{LocSys}_G).$$

For every $P \in \mathrm{Par}(G)$, we put

$$\mathcal{Z}_P := \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G),$$

and let

$$\mathcal{M}_P \subset \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G)$$

be the preimage of $\{0\} \subset \mathrm{Sing}(\mathrm{LocSys}_P)$ under the singular codifferential map

$$\mathcal{Z}_P = \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G) \to \mathrm{Sing}(\mathrm{LocSys}_P).$$

Theorem 4.4.9 is the statement that the natural functor

$$\text{D-mod}\left(\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})\right) \to \mathrm{Glue}\left(\text{D-mod}\left(\mathbb{P}(\mathcal{M}_P)\right),\ P \in \mathrm{Par}(G)^{\mathrm{op}}\right)$$

is fully faithful.

7.1.2. According to Corollary 6.3.8, Theorem 4.4.9 is equivalent to homological contractibility of the following prestacks. Let $k' \supset k$ be a field extension, and let y be a k'-point of $\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})$. Construct the prestack $\mathcal{M}_{\mathrm{Glued},y}$ as follows.

Consider the category $\mathrm{String}(\mathrm{Par}(G))$. By definition, its objects are chains of standard parabolic subgroups

$$(P_0 \subset P_1 \subset \cdots \subset P_n) \quad (n \geq 0, P_i \in \mathrm{Par}(G)),$$

and morphisms are induced by order-preserving maps $[m] \to [n]$. Now consider the functor $\mathrm{String}(\mathrm{Par}(G)) \to \mathrm{Sch}$ given by

$$(P_0 \subset P_1 \subset \cdots \subset P_n) \mapsto \mathrm{Spec}(k') \underset{y,\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})}{\times} (\mathcal{Z}_{P_0} \underset{\mathcal{Z}_{P_n}}{\times} \mathbb{P}(\mathcal{M}_{P_n})),$$

and put

$$\mathcal{M}_{\mathrm{Glued},y} = \underset{(P_0 \subset P_1 \subset \cdots \subset P_n) \in \mathrm{Strings}(\mathrm{Par}(G))}{\mathrm{colim}} \mathrm{Spec}(k') \underset{y,\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})}{\times} (\mathcal{Z}_{P_0} \underset{\mathcal{Z}_{P_n}}{\times} \mathbb{P}(\mathcal{M}_{P_n})).$$

Theorem 4.4.9 is equivalent to homological contractibility of prestacks $\mathcal{M}_{\mathrm{Glued},y}$ for every k' and y. Without loss of generality, we can replace k with its extension k'. Therefore, we need to verify that $\mathcal{M}_{\mathrm{Glued},y}$ is homologically contractible for every k-point y of $\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})$.

7.1.3. Let us now restate the above condition in explicit terms. First, recall the description of k-points of the algebraic stack $\mathrm{Sing}(\mathrm{LocSys}_G)$ and of the substack

$$\mathrm{Nilp}_{\mathrm{glob}} \subset \mathrm{Sing}(\mathrm{LocSys}_G),$$

see [AG, Sect. 11.1].

Namely, this groupoid of k-points $\mathrm{Sing}(\mathrm{LocSys}_G)(k)$ consists of pairs (σ, A), where σ is a G-local system on X, and A is a horizontal section of the vector bundle $\mathfrak{g}^*_\sigma$ associated with the co-adjoint representation. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a G-invariant bilinear form, and thus think of A as a horizontal section of $\mathfrak{g}_\sigma$.

The sub-groupoid of k-points $\mathrm{Nilp}_{\mathrm{glob}}(k)$ corresponds to pairs (σ, A) with *nilpotent* A.

7.1.4. Given a k-point (σ, A) of $\mathrm{Nilp}_{\mathrm{glob}}$ and a standard parabolic $P \in \mathrm{Par}(G)$, we define schemes

$$\mathrm{Spr}^{\sigma,A}_{P,\mathrm{unip}} \subset \mathrm{Spr}^{\sigma,A}_P \subset \mathrm{Spr}^\sigma_P$$

as follows.

Spr^σ_P is the scheme of reductions of σ (as a local system) to the parabolic P, and $\mathrm{Spr}^{\sigma,A}_{P,\mathrm{unip}}$ and $\mathrm{Spr}^{\sigma,A}_P$ are its subschemes corresponding to the condition that A be a section of

$$\mathfrak{u}(P)_\sigma \subset \mathfrak{g}_\sigma \text{ or } \mathfrak{p}_\sigma \subset \mathfrak{g}_\sigma,$$

respectively, where $\mathfrak{u}(P)$ denotes the Lie algebra of the unipotent radical $U(P)$ of P.

7.1.5. For fixed $\sigma \in \mathrm{LocSys}_G(k)$ as above, the diagram

$$P \rightsquigarrow \mathrm{Spr}^\sigma_P$$

identifies with the diagram of schemes

$$P \rightsquigarrow \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \{\sigma\}.$$

For fixed $(\sigma, A) \in \mathrm{Nilp}_{\mathrm{glob}}(k)$, the diagram

$$P \rightsquigarrow \mathrm{Spr}^{\sigma,A}_{P,\mathrm{unip}}$$

identifies with the diagram of schemes

$$P \rightsquigarrow \mathcal{M}_P \underset{\mathrm{Nilp}_{\mathrm{glob}}}{\times} \{(\sigma, A)\},$$

where

$$\mathcal{M}_P \subset \mathrm{LocSys}_P \underset{\mathrm{LocSys}_G}{\times} \mathrm{Sing}(\mathrm{LocSys}_G)$$

is as in Theorem 4.4.9.

7.1.6. Note that

$$P \rightsquigarrow \mathrm{Spr}_P^{\sigma,A}$$

is a diagram in the usual sense: for any pair of standard parabolics $P_1 \subset P_2$, there is a morphism between the corresponding schemes

$$\mathrm{Spr}_{P_1}^{\sigma,A} \to \mathrm{Spr}_{P_2}^{\sigma,A}.$$

On the other hand, in the diagram

$$P \rightsquigarrow \mathrm{Spr}_{P,\mathrm{unip}}^{\sigma,A},$$

the schemes

$$\mathrm{Spr}_{P_1,\mathrm{unip}}^{\sigma,A} \quad \text{and} \quad \mathrm{Spr}_{P_2,\mathrm{unip}}^{\sigma,A} \qquad (P_1 \subset P_2)$$

are connected by a correspondence:

$$\begin{array}{ccc} \mathrm{Spr}_{P_1}^{\sigma,A} \underset{\mathrm{Spr}_{P_2}^{\sigma,A}}{\times} \mathrm{Spr}_{P_2,\mathrm{unip}}^{\sigma,A} & \longrightarrow & \mathrm{Spr}_{P_1,\mathrm{unip}}^{\sigma,A} \\ \downarrow & & \\ \mathrm{Spr}_{P_2,\mathrm{unip}}^{\sigma,A}. & & \end{array}$$

7.1.7. Explicitly, in the inclusion

$$\mathrm{Spr}_{P_1}^{\sigma} \underset{\mathrm{Spr}_{P_2}^{\sigma}}{\times} \mathrm{Spr}_{P_2,\mathrm{unip}}^{\sigma,A} \subset \mathrm{Spr}_{P_1,\mathrm{unip}}^{\sigma,A},$$

the left-hand side (resp. the right-hand side) parametrizes reductions of the local system σ to the parabolic P_1 such that A is a section of

$$\mathfrak{u}(P_2)_\sigma \subset \mathfrak{g}_\sigma \qquad (\text{resp. of } \mathfrak{u}(P_1)_\sigma \subset \mathfrak{g}_\sigma).$$

Let us now form the prestack

$$\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A} := \underset{(P_0 \subset P_1 \subset \cdots \subset P_n) \in \mathrm{Strings}(\mathrm{Par}(G))}{\mathrm{colim}} \mathrm{Spr}_{P_0}^{\sigma} \underset{\mathrm{Spr}_{P_n}^{\sigma}}{\times} \mathrm{Spr}_{P_n,\mathrm{unip}}^{\sigma,A}.$$

Provided $A \neq 0$, a pair $(\sigma, A) \in (\sigma, A) \in \mathrm{Nilp}_{\mathrm{glob}}(k)$ projects to a k-point y of $\mathbb{P}(\mathrm{Nilp}_{\mathrm{glob}})$, and $\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A}$ identifies with the prestack $\mathcal{M}_{\mathrm{Glued},y}$. We therefore see that Theorem 4.4.9 is implied by the following assertion:

Theorem 7.1.8. *Let (σ, A) be a k-point of $\mathrm{Nilp}_{\mathrm{glob}}$. Then the prestack $\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A}$ is homologically contractible, that is, the trace map*

$$\mathrm{C}_*(\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A}) \to k$$

is an isomorphism.

The rest of the paper is devoted to the proof of Theorem 7.1.8.

Remark 7.1.9. Note that Theorem 7.1.8 claims, in particular, that for any such (σ, A), $\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A}$ is non-empty; this amounts to checking that $\mathrm{Spr}_{P,\mathrm{unip}}^{\sigma,A} \neq \emptyset$ for some $P \in \mathrm{Par}(G)$. This easily follows from the Jacobson-Morozov Theorem, see Sect. 8.3.1.

Remark 7.1.10. Note that in Theorem 7.1.8 we allow $A = 0$. The case $A = 0$ is not needed to deduce Theorem 4.4.9, but it is used in the inductive step in the proof of Theorem 7.1.8. Note, however, that the case $A = 0$ in Theorem 7.1.8 is reasonably easy:

It is not hard to check (see Remark 7.2.2 below) that for $A = 0$, the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$ identifies with

$$\mathrm{Spr}^{\sigma}_{\mathrm{Glued}} := \underset{P \in \mathrm{Par}(G)}{\mathrm{colim}} \, \mathrm{Spr}^{\sigma}_{P}.$$

Now, the category $\mathrm{Par}(G)$ has a final object (with $P = G$), and $\mathrm{Spr}^{\sigma}_{G} = \mathrm{pt}$. From here, $\mathrm{Spr}^{\sigma}_{\mathrm{Glued}} \simeq \mathrm{pt}$.

7.2. **Reduction to another contractibility statement.** One difficulty with Theorem 7.1.8 is due to a rather complicated colimit used to define the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$. We shall now replace Theorem 7.1.8 by an equivalent statement, namely Theorem 7.2.5, which is simpler from the combinatorial point of view.

7.2.1. Denote by $\mathrm{Par}'(G) \subset \mathrm{Par}(G)$ the subset of *proper* parabolics; thus

$$\mathrm{Par}(G) = \mathrm{Par}'(G) \sqcup \{G\}.$$

Consider the assignment

$$P \rightsquigarrow \mathrm{Spr}^{\sigma,A}_{P}$$

as a functor

$$\mathrm{Par}'(G) \to \{\mathrm{Schemes}\}.$$

Set

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}} := \underset{P \in \mathrm{Par}'(G)}{\mathrm{colim}} \, \mathrm{Spr}^{\sigma,A}_{P}.$$

Remark 7.2.2. The stack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}$ is also equal to the (more complicated) colimit over $\mathrm{String}(\mathrm{Par}'(G))$ of the functor

$$(P_0 \subset P_1 \subset \cdots \subset P_n) \mapsto \mathrm{Spr}^{\sigma,A}_{P_0}.$$

7.2.3. In Sects. 7.3 and 7.4, we prove:

Proposition 7.2.4. *Assume the validity of Theorem 7.1.8 for all* proper *Levi subgroups of G. Then for $A \neq 0$ there exists a naturally defined isomorphism*

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}) \simeq \mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}).$$

Assuming Proposition 7.2.4, we obtain that Theorem 7.1.8 is equivalent to the following:

Theorem 7.2.5. *Let (σ, A) be a k-point of* $\mathrm{Nilp}_{\mathrm{glob}}$ *with $A \neq 0$. Then the prestack* $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}$ *is homologically contractible.*

We prove Theorem 7.2.5 in Sect. 8. In Sect. 9 we give an alternative proof of Theorem 7.2.5 in the special case when σ is the trivial local system.

7.2.6. Both Theorems 7.1.8 and 7.2.5 have topological counterparts. Let us sketch these counterparts in case the reader finds their statement more transparent; they are *not* logically necessary for the proof.

Let A be a nilpotent element of $\mathfrak{g}$, but instead of a local system σ fix a family $\{g_\alpha\}$ of elements in G that centralize A.

For every $P \in \mathrm{Par}(G)$ consider the corresponding partial flag variety Fl_P; we think of it as the scheme classifying parabolics P' in the conjugacy class of P. Let

$$\mathrm{Spr}_{P,\mathrm{unip}}^{\{g_\alpha\},A} \subset \mathrm{Spr}_P^{\{g_\alpha\},A} \subset \mathrm{Spr}_P^{\{g_\alpha\}}$$

be the closed subschemes of Fl_P that correspond to $P' \in \mathrm{Fl}_P$ that satisfy the conditions

$$(g_\alpha \in P', A \in \mathfrak{u}(P')),\ (g_\alpha \in P', A \in \mathfrak{p}'),\ (g_\alpha \in P'),$$

respectively.

We can form the prestacks $\mathrm{Spr}_{\mathrm{Glued,unip}}^{\sigma,A}$ and $\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A}$, and the assertions parallel to Theorems 7.1.8 and 7.2.5 hold in this context as well. We leave it to the reader to verify that the argument of this paper can be used to prove these topological counterparts of the theorems.

Note that when $k = \mathbb{C}$, Theorems 7.1.8 and 7.2.5, as stated above, follow from their topological counterparts via the Riemann-Hilbert correspondence.

Namely, fix a base point $x \in X$, and trivialize the fiber of σ at x. Then the monodromy of σ gives a homomorphism $\pi_1(X,x) \to G$, and we take $\{g_\alpha\}$ to be the images in G of some set of generators of $\pi_1(X,x)$. Then the analytic spaces corresponding to the schemes $\mathrm{Spr}_{P,\mathrm{unip}}^{\sigma,A}$ and $\mathrm{Spr}_{P,\mathrm{unip}}^{\{g_\alpha\},A}$ (resp., $\mathrm{Spr}_P^{\sigma,A}$ and $\mathrm{Spr}_P^{\{g_\alpha\},A}$) are canonically identified.

7.2.7. Let us consider some examples of Theorem 7.2.5.

First, we consider the case of $G = SL_2$, in which case Theorem 4.3.4 is already non-obvious. But all of its complexity is contained in the reduction of Theorem 4.3.4 to Theorem 7.2.5, as the latter is quite easy:

For G of rank 1, the poset $\mathrm{Par}'(G)$ consists of one element, namely, $P = B$. Since $A \neq 0$, the scheme $\mathrm{Spr}_B^{\sigma,A}$ is a 'fat point': it is a nilpotent thickening of a point. Hence, it is homologically contractible.

7.2.8. Consider now the case of $G = SL_3$. We distinguish two cases: (a) when A is a regular nilpotent; (b) when A is a sub-regular nilpotent.

In case (a), for all three parabolics, the corresponding schemes $\mathrm{Spr}_P^{\sigma,A}$ are again fat points. So, the contractibility follows from the fact that the poset $\mathrm{Par}'(G)$

$$P_1 \supset B \subset P_2$$

is contractible as a category (it has an initial object, namely B).

Case (b) is more interesting. The scheme $\mathrm{Spr}_B^{\sigma,A}$ has the shape

$$Z_1 \underset{\mathrm{pt}}{\sqcup} Z_2,$$

i.e., its obtained by joining certain subschemes Z_1 and Z_2 along a common point. (To see this, use the topological version described in Sect. 7.2.6, first with $\{g_\alpha\}$ being trivial, and then deduce the general case.)

The projection

$$\mathrm{Spr}_B^{\sigma,A} \to \mathrm{Spr}_{P_1}^{\sigma,A}$$

maps Z_1 isomorphically onto its image, and it collapses Z_2 onto the image of $\mathrm{pt} = Z_1 \cap Z_2$.

Similarly, the projection

$$\mathrm{Spr}_B^{\sigma,A} \to \mathrm{Spr}_{P_1}^{\sigma,A}$$

maps Z_2 isomorphically onto its image, and it collapses Z_1 onto the image of $\mathrm{pt} = Z_1 \cap Z_2$.

This description makes the statement of Theorem 7.2.5 manifest.

7.3. **Proof of Proposition 7.2.4, Step 1.**

7.3.1. Recall that the prestack $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}$ is the the following colimit over the index category $\mathrm{Strings}(\mathrm{Par}(G))$ of chains of standard parabolic subgroups

$$(P_0 \subset P_1 \subset \cdots \subset P_n)$$

and morphisms are given by order-preserving maps $[m] \to [n]$.

To each $(P_0 \subset P_1 \subset \cdots \subset P_n) \in \mathrm{Strings}(\mathrm{Par}(G))$ we attach the scheme

$$\mathrm{Spr}^{\sigma}_{P_0} \underset{\mathrm{Spr}^{\sigma}_{P_n}}{\times} \mathrm{Spr}^{\sigma,A}_{P_n,\mathrm{unip}}\,.$$

In the above diagram, the P_i's are all standard parabolics. It is possible that $P_n = G$, but in this case $\mathrm{Spr}^{\sigma,A}_{P_n,\mathrm{unip}}$ is empty, because $A \neq 0$. Thus we can work with chains of proper standard parabolic subgroups

$$(P_0, \ldots, P_n) \in \mathrm{String}(\mathrm{Par}'(G)).$$

Let $I_1 := \mathrm{String}(\mathrm{Par}'(G))$ be the index category of chains of proper standard parabolic subgroups. Denote by $\mathsf{F}_1 : I_1 \to \mathrm{Sch}$ the functor

$$(P_0, \ldots, P_n) \mapsto \mathrm{Spr}^{\sigma}_{P_0} \underset{\mathrm{Spr}^{\sigma}_{P_n}}{\times} \mathrm{Spr}^{\sigma,A}_{P_n,\mathrm{unip}}\,.$$

Thus

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}} \simeq \operatorname*{colim}_{i \in I_1} \mathsf{F}_1(i).$$

7.3.2. We recall that by definition,

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}} = \operatorname*{colim}_{i \in I_2} \mathsf{F}_2(i),$$

where we put $I_2 := \mathrm{Par}'(G)$ and

$$\mathsf{F}_2 : I_2 \to \mathrm{Sch} : P \mapsto \mathrm{Spr}^{\sigma,A}_P\,.$$

7.3.3. Consider now the category I whose objects are collections

$$(P_0 \subset \cdots \subset P_n \subset P) \quad (n \geq 0; P_0, \ldots, P_n, P \in \mathrm{Par}'(G)). \tag{7.1}$$

A morphism

$$(P_0^1 \subset \cdots \subset P_{n^1}^1 \subset P^1) \to (P_0^2 \subset \cdots \subset P_{n^2}^2 \subset P^2)$$

is specified by an order-preserving map $[n^2] \to [n^1]$ and an inclusion $P^1 \subset P^2$.

Define a functor $F : I \to \mathrm{Sch}$ by

$$F : (P_0 \subset \cdots \subset P_n \subset P) \mapsto \mathrm{Spr}^{\sigma,A}_{P_n,\mathrm{unip}} \underset{\mathrm{Spr}^{\sigma}_{P_n}}{\times} \mathrm{Spr}^{\sigma}_{P_0},$$

and put

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed}} := \operatorname*{colim}_{i \in I} \mathsf{F}(i).$$

7.3.4. We have canonical forgetful functors

$$I_1 \xleftarrow{\phi_1} I \xrightarrow{\phi_2} I_2.$$

By construction $\mathsf{F} \simeq \mathsf{F}_1 \circ \phi_1$. This isomorphism gives rise to a map

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed}} \to \mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,unip}}\,. \tag{7.2}$$

We claim that the map (7.2) is an isomorphism of prestacks. Indeed, this follows from the fact that the functor ϕ_1 is a co-Cartesian fibration with contractible fibers (each fiber has an initial object, namely $P = P_n$).

Thus, to prove Proposition 7.2.4, we need to construct a homological equivalence between prestacks $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed}}$ and $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}$.

7.3.5. Note now that we have a canonically defined natural transformation

$$\mathsf{F} \to \mathsf{F}_2 \circ \phi_2. \tag{7.3}$$

Indeed, for any $(P_0 \subset \cdots \subset P_n \subset P) \in I$, we have a natural map

$$\mathsf{F}(P_0 \subset \cdots \subset P_n \subset P) = \mathrm{Spr}^{\sigma,A}_{P_n,\mathrm{unip}} \underset{\mathrm{Spr}^{\sigma}_{P_n}}{\times} \mathrm{Spr}^{\sigma}_{P_0} \hookrightarrow \mathrm{Spr}^{\sigma}_{P_0} \to$$
$$\mathrm{Spr}^{\sigma}_{P} = \mathsf{F}_2(P) = \mathsf{F}_2 \circ \phi_2(P_0 \subset \cdots \subset P_n \subset P).$$

Hence, we obtain a map of prestacks

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed}} \to \mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}\,. \tag{7.4}$$

Let us prove that the map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed}}) \to \mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}), \tag{7.5}$$

induced by (7.4) is an isomorphism.

7.3.6. Let

$$\mathsf{F}'_2 : I_2 \to \mathrm{Sch}$$

denote the left Kan extension of the functor F along ϕ_2. By adjunction, the natural transformation (7.3) gives rise to a natural transformation

$$\mathsf{F}'_2 \to \mathsf{F}_2.$$

Composing with the functor

$$\mathrm{C}_* : \mathrm{Sch} \to \mathrm{Vect},$$

we obtain a natural transformation

$$\mathrm{C}_* \circ \mathsf{F}'_2 \to \mathrm{C}_* \circ \mathsf{F}_2 \tag{7.6}$$

of functors $I_2 \to \mathrm{Vect}$.

The map (7.5) is obtained from (7.6) by taking colimits over I_2. Thus, in order to prove that (7.5) is an isomorphism, it suffices to show that the map (7.6) is an isomorphism of functors $I_2 \to \mathrm{Vect}$.

The latter will be done in Step 2, using Theorem 7.1.8 for proper Levi subgroups of G (including the case $A = 0$).

7.4. **Proof of Proposition 7.2.4, Step 2.**

7.4.1. Note that the functor ϕ_2 is also a co-Cartesian fibration. Hence, the value of $\mathrm{C}_* \circ \mathsf{F}'_2$ on an object $P \in \mathrm{Par}'(G) = I_2$ is computed as the colimit of the functor $\mathrm{C}_* \circ \mathsf{F}_2$ over the fiber of ϕ_2 over P. I.e., it is the homology of the prestack equal to the colimit of the restriction of F to the above fiber. Denote this prestack by $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P}$.

Note that we have a tautologically defined map

$$f : \mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P} \to \mathrm{Spr}^{\sigma,A}_{P}.$$

We need to show that the above map f induces an isomorphism on homology. It suffices to check that the trace map

$$f_{\mathrm{dR},!}(\omega_{\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P}}) \to \omega_{\mathrm{Spr}^{\sigma,A}_{P}} \tag{7.7}$$

is an isomorphism in $\text{D-mod}(\mathrm{Spr}^{\sigma,A}_{P})$.

7.4.2. The fact that (7.7) is an isomorphism can be checked at the level of !-fibers at k-points of $\mathrm{Spr}^{\sigma,A}_{P}$.

Fix a point $\sigma_P \in \mathrm{Spr}^{\sigma,A}_{P}(k)$. Thus, σ_P is a reduction of σ to P that is compatible with A. Let M be the Levi quotient of P, and let (σ_M, A_M) be the resulting k-point of $\mathrm{Nilp}_{\mathrm{glob}}$ for the group M. (Note that A_M may be zero.)

Note that $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P}$ is a colimit of schemes each of which is proper, and in particular, maps properly to $\mathrm{Spr}^{\sigma,A}_{P}$. Hence, by proper base change, the !-fiber of $f_{\mathrm{dR},!}(\omega_{\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P}})$ at σ_P is isomorphic to the homology of the fiber of $\mathrm{Spr}_{\mathrm{Glued,mixed},P}$ over σ_P; denote this fiber by $\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P,\sigma_P}$.

7.4.3. Thus, we have to show that the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P,\sigma_P}) \to k$$

is an isomorphism.

However, we notice that there is a canonical isomorphism

$$\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued,mixed},P,\sigma_P} \simeq \mathrm{Spr}^{\sigma_M,A_M}_{\mathrm{Glued,unip}},$$

(the latter prestack taken for the reductive group M).

Hence, the required assertion follows from Theorem 7.1.8, applied to M.

8. Schubert stratification

The goal of this section is to prove Theorem 7.2.5.

8.1. **Conventions regarding roots.**

8.1.1. Recall that we fixed a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\Lambda := \mathrm{Hom}(T, \mathbb{G}_m)$ be the character lattice of T; it is a free abelian group, which we write additively. The standard parabolics $P \subset G$ are the parabolic subgroups containing B.

Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras of T, B, and G respectively. For every $\alpha \in \Lambda$, we denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the corresponding root subspace; in particular, $\mathfrak{g}_0 = \mathfrak{t}$. Let

$$\mathsf{R} = \{\alpha \in \Lambda - \{0\} : \mathfrak{g}_\alpha \neq 0\}$$

be the set of roots. Denote by $\mathsf{S} \subset \mathsf{R}^+ \subset \mathsf{R}$ the subsets of simple and positive roots with respect to B. Thus,

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathsf{R}^+} \mathfrak{g}_\alpha.$$

We identify S and the set of the vertices of the Dynkin diagram of G.

8.1.2. We think of $\mathrm{Par}(G)$ as the poset of subsets $\mathsf{J} \subset \mathsf{S}$ (ordered by inclusion) via $\mathsf{J} \mapsto P_\mathsf{J}$. Explicitly, given $\mathsf{J} \in \mathrm{Par}(G)$, the Lie subalgebras

$$\begin{aligned}
\mathfrak{p}_\mathsf{J} &:= \bigoplus \{\mathfrak{g}_\alpha : \alpha \in \mathsf{R}^+ \cup \mathrm{Span}(\mathsf{J})\} \\
\mathfrak{m}_\mathsf{J} &:= \bigoplus \{\mathfrak{g}_\alpha : \alpha \in \mathrm{Span}(\mathsf{J})\} \\
\mathfrak{u}(P_\mathsf{J}) &:= \bigoplus \{\mathfrak{g}_\alpha : \alpha \in \mathsf{R}^+ - \mathrm{Span}(\mathsf{J})\}
\end{aligned}$$

correspond to P_J, the standard Levi subgroup $M_\mathsf{J} \subset P_\mathsf{J}$, and the unipotent radical $U(P_\mathsf{J})$ of P_J, respectively. We denote by

$$\mathsf{R}_\mathsf{J} := \mathsf{R} \cap \mathrm{Span}(\mathsf{J})$$

the set of roots of M_J, so that $\mathsf{J} \subset \mathsf{R}_\mathsf{J}$ is the set of simple roots.

8.1.3. Let $N(T) \subset G$ be the normalizer of T. The Weyl group $\mathsf{W} = N(T)/T$ acts on Λ preserving R. For any $\mathsf{J} \in \mathrm{Par}(G)$, denote by $\mathsf{W}_\mathsf{J} \subset \mathsf{W}$ the subgroup generated by the reflections around the roots in J. Thus, W_J is the Weyl group of M_J.

8.1.4. Given $\mathsf{J} \in \mathrm{Par}(G)$, we denote by

$$\mathrm{Fl}_\mathsf{J} = \{P' \subset G : P' \text{ is conjugate to } P_\mathsf{J}\}$$

the flag variety of parabolic subgroups of type J. We have a natural isomorphism $\mathrm{Fl}_\mathsf{J} = G/P_\mathsf{J}$. If $\mathsf{J} = \emptyset$, then $P_\mathsf{J} = B$, and we write simply

$$\mathrm{Fl} = \mathrm{Fl}_\mathsf{J} = G/B \qquad (\mathsf{J} = \emptyset)$$

for the complete flag variety.

Whenever $\tilde{\mathsf{J}} \subset \mathsf{J}$ in $\mathrm{Par}(G)$, we have a natural morphism

$$f = f_{\tilde{\mathsf{J}},\mathsf{J}} : \mathrm{Fl}_{\tilde{\mathsf{J}}} \to \mathrm{Fl}_\mathsf{J}\,.$$

8.1.5. Given two Borel subgroups $B', B'' \subset G$, we denote their relative position by $w(B', B'') \in \mathsf{W}$. Explicitly, for $B'' = B$ being the fixed Borel, the equality $w = w(B', B)$ means that

$$B' = \mathrm{Ad}_g(B), \quad g \in BwB.$$

We then expand w to arbitrary pairs $(B', B'') \in \mathrm{Fl} \times \mathrm{Fl}$ by G-invariance.

More generally, suppose $\mathsf{J}_0, \mathsf{J} \in \mathrm{Par}(G)$. The relative position of two parabolic subgroups $P' \in \mathrm{Fl}_{\mathsf{J}_0}$, $P'' \in \mathrm{Fl}_{\mathsf{J}}$ is given by the double coset

$$\{w(B', B'') \in \mathsf{W} : B' \subset P' \text{ and } B'' \subset P'' \text{ are Borel subgroups}\} \in \mathsf{W}_{\mathsf{J}_0} \backslash \mathsf{W} / \mathsf{W}_{\mathsf{J}}.$$

This double coset contains a unique minimal element with respect to the Bruhat order on W; we denote it by $w(P', P'') \in \mathsf{W}$. The condition that $w \in \mathsf{W}$ is minimal in its double coset $\mathsf{W}_{\mathsf{J}_0} w \mathsf{W}_{\mathsf{J}}$ is equivalent to the condition

$$w(\mathsf{J}) \subset \mathsf{R}^+ \qquad \text{and} \qquad w^{-1}(\mathsf{J}_0) \subset \mathsf{R}^+.$$

8.2. **Some Weyl group combinatorics.** In this subsection we fix $\mathsf{J}_0 \in \mathrm{Par}(G)$ and the corresponding standard parabolic subgroup $P_0 := P_{\mathsf{J}_0}$.

8.2.1. Put

$$\mathsf{W}' := \{w \in \mathsf{W} : w^{-1}(\mathsf{J}_0) \subset \mathsf{R}^+\} = \{w \in \mathsf{W} : w \text{ is minimal in } \mathsf{W}_{\mathsf{J}_0} w\}. \tag{8.1}$$

There is a unique maximal element $w_0' \in \mathsf{W}'$; it is characterised by the property that

$$w_0'(\mathsf{R}^+) \cap \mathsf{R}^+ = \mathsf{R}_{\mathsf{J}_0} \cap \mathsf{R}^+.$$

Explicitly, w_0' is the minimal element of the coset $\mathsf{W}_{\mathsf{J}_0} w_0$, where $w_0 \in \mathsf{W}$ is the longest element; also, $w_0' w_0 \in \mathsf{W}_{\mathsf{J}_0}$ is the longest element of the Coxeter group $\mathsf{W}_{\mathsf{J}_0}$.

8.2.2. Fix $w \in \mathsf{W}$, and consider the partition $\mathsf{S} = \mathsf{S}_w^0 \cup \mathsf{S}_w^+ \cup \mathsf{S}_w^-$ given by

$$\begin{aligned} \mathsf{S}_w^0 &:= \mathsf{S} \cap w^{-1}(\mathsf{R}_{\mathsf{J}_0}) \\ \mathsf{S}_w^+ &:= \mathsf{S} \cap w^{-1}(\mathsf{R}^+ \setminus \mathsf{R}_{\mathsf{J}_0}) \\ \mathsf{S}_w^- &:= \mathsf{S} \cap w^{-1}(-\mathsf{R}^+ \setminus \mathsf{R}_{\mathsf{J}_0}). \end{aligned}$$

(For simplicity, the dependence of this partition on J_0 is suppressed in the notation.) The following properties of this partition are clear.

Lemma 8.2.3. *Suppose $w \in \mathsf{W}$. Then*

(1) $\mathsf{S}_w^- = \emptyset$ *if and only if* $w \in \mathsf{W}_{\mathsf{J}_0}$, *and*
(2) $\mathsf{S}_w^+ = \emptyset$ *if and only if* $w \in \mathsf{W}_{\mathsf{J}_0} w_0$. □

Corollary 8.2.4. *Suppose $w \in \mathsf{W}'$. Then*

(1) $\mathsf{S}_w^- = \emptyset$ *if and only if* $w = e$, *and*
(2) $\mathsf{S}_w^+ = \emptyset$ *if and only if* $w = w_0'$. □

8.2.5. Let now P' be another parabolic subgroup (not necessarily a standard one). Consider $w(P_0, P') \in \mathsf{W}$. Clearly, $w(P_0, P') \in \mathsf{W}'$. We need the following easy observation.

Lemma 8.2.6. *Let $U(P_0) \subset P_0$ be the unipotent radical, and let $\mathfrak{p}'$ and $\mathfrak{u}(P_0)$ be the Lie algebras of P' and $U(P_0)$, respectively. Then $w(P_0, P') = w_0'$ if and only if $\mathfrak{p}' \cap \mathfrak{u}(P_0) = \{e\}$.*

Proof. Follows from Corollary 8.2.4(2). □

8.2.7. Let us now fix $\mathsf{J} \in \mathrm{Par}(G)$, and consider the flag variety Fl_{J}. For $w \in \mathsf{W}$, define the *Schubert stratum* in Fl_{J} as follows:

$$\mathrm{Fl}_{\mathsf{J}}^{w} := \{P' \in \mathrm{Fl}_{\mathsf{J}} : w(P_0, P') = w\} \subset \mathrm{Fl}_{\mathsf{J}}\,.$$

Also, put

$$\mathrm{Fl}_{\mathsf{J}}^{\leq w} := \{P' \in \mathrm{Fl}_{\mathsf{J}} : w(P_0, P') \leq w\} \subset \mathrm{Fl}_{\mathsf{J}}$$

and

$$\mathrm{Fl}_{\mathsf{J}}^{< w} := \{P' \in \mathrm{Fl}_{\mathsf{J}} : w(P_0, P') < w\} \subset \mathrm{Fl}_{\mathsf{J}}\,.$$

(One again, we omit the parabolic subgroup P_0 from the notation.)

Remark 8.2.8. We emphasize that in the definition of $\mathrm{Fl}_{\mathsf{J}}^{w}$, the equality $w(P_0, P') = w$ takes place in W and *not* in $W_{\mathsf{J}_0}\backslash W/W_{\mathsf{J}}$ (and similarly for $\mathrm{Fl}_{\mathsf{J}}^{\leq w}$ and $\mathrm{Fl}_{\mathsf{J}}^{< w}$).

Hence, if $w \notin \mathsf{W}'$, then $\mathrm{Fl}_{\mathsf{J}}^{w} = \emptyset$ and $\mathrm{Fl}_{\mathsf{J}}^{\leq w} = \mathrm{Fl}_{\mathsf{J}}^{< w}$. Also, $\mathrm{Fl}_{\mathsf{J}}^{\leq w'_0} = \mathrm{Fl}_{\mathsf{J}}$.

8.2.9. Suppose $\tilde{\mathsf{J}} \subset \mathsf{J}$ in $\mathrm{Par}(G)$. Consider the natural map $f : \mathrm{Fl}_{\tilde{\mathsf{J}}} \to \mathrm{Fl}_{\mathsf{J}}$. Clearly,

$$f(\mathrm{Fl}_{\tilde{\mathsf{J}}}^{\leq w}) \subset \mathrm{Fl}_{\mathsf{J}}^{\leq w} \quad \text{and} \quad f(\mathrm{Fl}_{\tilde{\mathsf{J}}}^{< w}) \subset \mathrm{Fl}_{\mathsf{J}}^{< w};$$

however, it is not true in general that $f(\mathrm{Fl}_{\tilde{\mathsf{J}}}^{w}) \subset \mathrm{Fl}_{\mathsf{J}}^{w}$.

Lemma 8.2.10. *Fix $w \in \mathsf{W}$ (and recall that $\mathsf{J} \in \mathrm{Par}(G)$ is also fixed).*

(1) *If $\mathsf{J} \cap \mathsf{S}_w^- \neq \emptyset$, then $\mathrm{Fl}_{\mathsf{J}}^{w} = \emptyset$.*

(2) *Put $\tilde{\mathsf{J}} = \mathsf{J} \setminus \mathsf{S}_w^+$. Then the map $f : \mathrm{Fl}_{\tilde{\mathsf{J}}} \to \mathrm{Fl}_{\mathsf{J}}$ induces an isomorphism $\mathrm{Fl}_{\tilde{\mathsf{J}}}^{w} \simeq \mathrm{Fl}_{\mathsf{J}}^{w}$.*

Proof. (1) Indeed, if $\mathsf{J} \cap \mathsf{S}_w^- \neq \emptyset$, then $w(\mathsf{J}) \not\subset \mathsf{R}^+$ and w is not the minimal element of $\mathsf{W}_{\mathsf{J}_0} w \mathsf{W}_{\mathsf{J}}$.

(2) The inverse map sends $P' \in \mathrm{Fl}_{\mathsf{J}}$ to the parabolic subgroup $(P' \cap P_0)U(P') \subset P'$, where $U(P') \subset P'$ is the unipotent radical. □

8.3. **Proof of Theorem 7.2.5: setting up the induction.**

8.3.1. Recall that in Theorem 7.2.5 we fix a G-local system σ and a *non-zero* horizontal section A of $\mathfrak{g}_\sigma$.

By the Jacobson-Morozov Theorem, A determines a canonical reduction of σ to a standard parabolic subgroup, which we denote P_0. Moreover, A belongs to the nilradical of this reduction, in the sense that A lies in $\mathfrak{u}(P_0)_\sigma \subset \mathfrak{g}_\sigma$. (Here we abuse the notation slightly by writing σ for the reduction to P_0.) Equivalently, the reduction corresponds to a point of $\mathrm{Spr}_{P_0,\mathrm{unip}}^{\sigma,A}$. In particular, since $A \neq 0$, we have $\mathfrak{u}(P_0) \neq 0$ and hence $P_0 \neq G$.

Remark 8.3.2. For most of the argument, we only need to know that σ is reduced to a proper parabolic. The fact that A belongs to the nilradical of the reduction is used only in Sect. 8.5.6.

8.3.3. Set $P_0 = P_{\mathsf{J}_0}$; that is, J_0 is the type of the standard parabolic P_0. Let us use the formalism of Sect. 8.2 for this choice of J_0.

Each of the schemes $\mathrm{Spr}_P^{\sigma,A}$ comprising $\mathrm{Spr}_{\mathrm{Glued}}^{\sigma,A}$ acquires a stratification by the set W', where W' is given by (8.1); denote the corresponding subschemes by

$$\mathrm{Spr}_P^{\sigma,A,<w} \subset \mathrm{Spr}_P^{\sigma,A,\leq w} \supset \mathrm{Spr}_P^{\sigma,A,w}\,.$$

Explicitly, the stratification is determined by the relative position the reduction of σ to P (corresponding to a point of $\mathrm{Spr}_P^{\sigma,A}$) and the fixed reduction of σ to P_0.

Consider the corresponding prestacks

$$\mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}} = \underset{P\in \mathrm{Par}'(G)}{\mathrm{colim}}\, \mathrm{Spr}^{\sigma,A,<w}_P$$
$$\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} = \underset{P\in \mathrm{Par}'(G)}{\mathrm{colim}}\, \mathrm{Spr}^{\sigma,A,\leq w}_P .$$

(Note that the schemes $\mathrm{Spr}^{\sigma,A,w}_P$ do not form a diagram indexed by $P \in \mathrm{Par}'(G)$.)

Consider also the quotients

$$\mathrm{Spr}^{\sigma,A,\leq w}_P \,/\, \mathrm{Spr}^{\sigma,A,<w}_P := \mathrm{Spr}^{\sigma,A,\leq w}_P \underset{\mathrm{Spr}^{\sigma,A,<w}_P}{\sqcup} \mathrm{pt}$$

and

$$\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} \,/\, \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}} := \mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} \underset{\mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}}{\sqcup} \mathrm{pt},$$

the latter being the same as

$$\underset{P\in \mathrm{Par}'(G)}{\mathrm{colim}}\, \mathrm{Spr}^{\sigma,A,\leq w}_P \,/\, \mathrm{Spr}^{\sigma,A,<w}_P,$$

since the category $\mathrm{Par}'(G)$ is contractible (having an initial object).

In what follows we also use the notation

$$\mathrm{Spr}^{\sigma,A,<w}_{\mathsf{J}} := \mathrm{Spr}^{\sigma,A,<w}_P \ \text{ for } P = P_{\mathsf{J}},$$

etc.

8.3.4. We need to show that the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A}_{\mathrm{Glued}}) \to k$$

is an isomorphism.

We will prove that for every $w \in \mathsf{W}'$, the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}}) \to k \tag{8.2}$$

is an isomorphism. (That is, $\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}}$ is a homologically contractible k-prestack.) Applying this to $w = w'_0$, we obtain the desired result.

8.3.5. The proof that (8.2) is an isomorphism uses the following two statements, proved in Sections 8.4 and 8.5, respectively:

Case $w = 1$**:** the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A,1}_{\mathrm{Glued}}) \to k$$

is an isomorphism;

Case $w \neq 1$**:** For any $1 \neq w \in \mathsf{W}'$, the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} \,/\, \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}) \to k$$

is an isomorphism.

Let us show how the combination of these two statements implies that (8.2) is an isomorphism. This will be completely formal.

We argue by induction on the poset W'. The base of the induction is the statement in Case $w = 1$. Let us now perform the induction step, so take $w \neq 1$.

We have a push-out square of prestacks

$$\begin{array}{ccc} \mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} & \longrightarrow & \mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} \,/\, \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}} \\ \uparrow & & \uparrow \\ \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}} & \longrightarrow & \mathrm{pt}\,. \end{array}$$

and hence a cofiber square in Vect:

$$\begin{array}{ccc} \mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}}) & \longrightarrow & \mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w}_{\mathrm{Glued}} \,/\, \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}) \\ \uparrow & & \uparrow \\ \mathrm{C}_*(\mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}) & \longrightarrow & \mathrm{C}_*(\mathrm{pt}) \simeq k. \end{array}$$

Taking into account the statement in Case $w \neq 1$, it suffices to show that the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}) \to k$$

is an isomorphism. This is done below.

8.3.6. Consider the prestack

$$\operatorname*{colim}_{w_1<w} \mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}\,.$$

We have an isomorphism

$$\operatorname*{colim}_{w_1<w} \mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}} \to \mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}},$$

and hence an isomorphism

$$\mathrm{C}_*\left(\operatorname*{colim}_{w_1<w} \mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}\right) \to \mathrm{C}_*(\mathrm{Spr}^{\sigma,A,<w}_{\mathrm{Glued}}).$$

Hence, it remains to show that the trace map

$$\mathrm{C}_*\left(\operatorname*{colim}_{w_1<w} \mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}\right) \to k$$

is an isomorphism. We have

$$\mathrm{C}_*\left(\operatorname*{colim}_{w_1<w} \mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}\right) \simeq \operatorname*{colim}_{w_1<w} \mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}).$$

Now, by the induction hypothesis, for every $w_1 < w$, the trace map

$$\mathrm{C}_*(\mathrm{Spr}^{\sigma,A,\leq w_1}_{\mathrm{Glued}}) \to k$$

is an isomorphism. Hence, the assertion follows from the fact that the index category, i.e., w_1 with $w_1 < w$, is contractible (it contains an initial element $w_1 = 1$).

8.4. **Verifying Case $w = 1$.**

8.4.1. Let us show that the prestack $\mathrm{Spr}^{\sigma,A,1}_{\mathrm{Glued}}$ itself is isomorphic to pt. By definition,

$$\mathrm{Spr}^{\sigma,A,1}_{\mathrm{Glued}} = \underset{\mathsf{J}\in \mathrm{Par}'(G)}{\mathrm{colim}}\, \mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}}.$$

Let M_0 denote the Levi quotient of P_0. Let $\mathrm{Par}(M_0)$ be the poset of all standard parabolics of M_0. We identify $\mathrm{Par}(M_0)$ with the poset of all subsets of J_0 (including J_0 itself).

The inclusion

$$\mathrm{Par}(M_0) \hookrightarrow \mathrm{Par}'(G) \tag{8.3}$$

admits a right adjoint, given by

$$\mathsf{J} \mapsto \mathsf{J} \cap \mathsf{J}_0.$$

Note now that for any $\mathsf{J} \subset \mathsf{S}$, the map

$$\mathrm{Fl}^1_{\mathsf{J}\cap\mathsf{J}_0} \to \mathrm{Fl}^1_{\mathsf{J}}$$

is an isomorphism. Indeed, this is a special case of Lemma 8.2.10(2). Therefore, the map

$$\mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}\cap\mathsf{J}_0} \to \mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}}$$

is an isomorphism as well.

8.4.2. We have the following general assertion:

Let I be an index category, and $I' \overset{\phi}{\hookrightarrow} I$ a full subcategory such that the inclusion ϕ admits a right adjoint, which we denote by ψ.

Let $\mathsf{F} : I \to \mathbf{D}$ be a functor with values in some ∞-category $\mathbf{D}$. Assume that for every $i \in I$, the co-unit of the adjunction

$$\phi \circ \psi(i) \to i$$

induces an isomorphism

$$\mathsf{F} \circ \phi \circ \psi(i) \to \mathsf{F}(i).$$

Lemma 8.4.3. *Under the above circumstances, the canonical map*

$$\underset{i'\in I'}{\mathrm{colim}}\, \mathsf{F} \circ \phi \to \underset{i\in I}{\mathrm{colim}}\, \mathsf{F}$$

is an isomorphism. □

8.4.4. Applying Lemma 8.4.3 to (8.3) and the functor

$$\mathsf{J} \rightsquigarrow \mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}},$$

we see that $\mathrm{Spr}^{\sigma,A,1}_{\mathrm{Glued}}$ is isomorphic to the prestack

$$\underset{\mathsf{J}\subset\mathsf{J}_0}{\mathrm{colim}}\, \mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}}. \tag{8.4}$$

Now, the index category of subsets of J_0 has a final object (namely, $\mathsf{J} = \mathsf{J}_0$), and $\mathrm{Spr}^{\sigma,A,1}_{\mathsf{J}_0} = \mathrm{pt}$. Hence, the colimit in (8.4) is isomorphic to pt.

8.5. **Verifying Case $w \neq 1$.**

8.5.1. We need to show that for $w \neq 1$, the trace map

$$\operatorname*{colim}_{P \in \mathrm{Par}'(G)} \mathrm{C}_*(\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w}) \to k \tag{8.5}$$

is an isomorphism. Consider the case of $w \neq w_0'$ first.

Put

$$\mathrm{Par}'_w(G) := \{\mathsf{J} \in \mathrm{Par}'(G) \,|\, \mathsf{J} \subset \mathsf{S}^0_w \cup \mathsf{S}^-_w\} \subset \mathrm{Par}'(G).$$

Recall (see Sect. 8.2.2) that $\mathsf{J} \subset \mathsf{S}^0_w \cup \mathsf{S}^-_w$ means that for every simple root $\alpha \in \mathsf{J}$, $w(\alpha)$ is either negative, or a root of R_0.

8.5.2. We claim that the inclusion $\mathrm{Par}'_w(G) \hookrightarrow \mathrm{Par}'(G)$ satisfies the conditions of Lemma 8.4.3 for the functor

$$\mathsf{J} \mapsto \mathrm{C}_*(\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w}).$$

Indeed, note that the inclusion $\mathrm{Par}'_w(G) \hookrightarrow \mathrm{Par}'(G)$ admits a right adjoint given by

$$\mathsf{J} \mapsto \tilde{\mathsf{J}} := \mathsf{J} \setminus \mathsf{S}^+_w.$$

Now, we claim that for J and $\tilde{\mathsf{J}}$ as above, the map

$$\mathrm{Spr}_{\tilde{\mathsf{J}}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\tilde{\mathsf{J}}}^{\sigma,A,<w} \to \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w}$$

induces an isomorphism on homology.

This follows by Lemma 8.2.10(2) from the following general assertion:

Lemma 8.5.3. *Let $f : Y_1 \to Y_2$ be a proper map between schemes. Let $Y_i' \subset Y_i$ for $i = 1, 2$ be closed subschemes such that $f(Y_1') \subset Y_2'$, and f induces an isomorphism $Y_1 \setminus Y_1' \to Y_2 \setminus Y_2'$. Then the induced map*

$$\mathrm{C}_*(Y_1 \underset{Y_1'}{\sqcup} \mathrm{pt}) \to \mathrm{C}_*(Y_2 \underset{Y_2'}{\sqcup} \mathrm{pt})$$

is an isomorphism.

Proof. It is enough to show that the map

$$\mathrm{Cone}\left(\mathrm{C}_*(Y_1') \to \mathrm{C}_*(Y_1)\right) \to \mathrm{Cone}\left(\mathrm{C}_*(Y_2') \to \mathrm{C}_*(Y_2)\right),$$

defined by f, is an isomorphism.

Let ι_i (resp. j_i) denote the closed embedding $Y_i' \hookrightarrow Y_i$ (resp. the open embedding $(Y_i \setminus Y_i') \hookrightarrow Y_i$). From the excision exact triangle

$$(\iota_i)_{\mathrm{dR},*}(\omega_{Y_i'}) \to \omega_{Y_i} \to (j_i)_{\mathrm{dR},*}(\omega_{Y_i \setminus Y_i'})$$

we obtain an isomorphism

$$\mathrm{Cone}\left(\mathrm{C}_*(Y_i') \to \mathrm{C}_*(Y_i)\right) \simeq (p_{Y_i})_{\mathrm{dR},!}((j_i)_{\mathrm{dR},*}(\omega_{Y_i \setminus Y_i'})),$$

where $p_{Y_i} : Y_i \to \mathrm{pt}$ is the projection to the point.

Now, the fact that f is proper and the assumption of the lemma imply that

$$f_{\mathrm{dR},!}((j_1)_{\mathrm{dR},*}(\omega_{Y_1 \setminus Y_1'})) \simeq (j_2)_{\mathrm{dR},*}(\omega_{Y_2 \setminus Y_2'}),$$

implying the desired isomorphism.

□

Remark 8.5.4. The above argument involves the excision exact triangle. For this reason, it does not imply that the prestack $\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w} / \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w}$ itself is isomorphic to pt (and we do not know whether this is true).

8.5.5. Thus, by Lemma 8.4.3, the colimit in (8.5) is isomorphic to the colimit

$$\underset{P\in \mathrm{Par}'_w(G)}{\mathrm{colim}}\ \mathrm{C}_*(\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w}\,/\,\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w}),$$

and it suffices to show that the trace map from the latter to k is an isomorphism. Let us show that the prestack

$$\underset{P\in \mathrm{Par}'_w(G)}{\mathrm{colim}}\ \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w}\,/\,\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w} \tag{8.6}$$

itself is isomorphic to pt.

By the assumption that $w\neq w'_0$ and Corollary 8.2.4, the poset $\mathrm{Par}'_w(G)$ contains a maximal element, namely, $\mathsf{J}=\mathsf{S}^-_w\cup\mathsf{S}^0_w$. Hence, the colimit (8.6) is isomorphic to

$$\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,\leq w}\,/\,\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,<w}\,.$$

Now, by the assumption that $w\neq 1$ and Lemma 8.2.10(1), we have

$$\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,\leq w}=\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,<w},$$

and so

$$\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,\leq w}\,/\,\mathrm{Spr}_{\mathsf{S}^-_w\cup\mathsf{S}^0_w}^{\sigma,A,<w}\simeq \mathrm{pt}\,.$$

8.5.6. Finally, we consider the case of $w=w'_0$. We claim that in this case the prestack

$$\underset{P\in \mathrm{Par}'(G)}{\mathrm{colim}}\ \mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w'_0}\,/\,\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w'_0}$$

is isomorphic to pt. In fact, we claim that for every J, we have

$$\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,w'_0}=\emptyset,$$

and so

$$\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,\leq w'_0}\,/\,\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,<w'_0}\simeq \mathrm{pt}\,.$$

Indeed, the fact that $\mathrm{Spr}_{\mathsf{J}}^{\sigma,A,w'_0}$ is empty follows from Lemma 8.2.6 and the fact that A is a horizontal section of $\mathfrak{u}(P_0)_\sigma$, while $A\neq 0$ by assumption.

9. A proof via the Grothendieck-Springer correspondence

In this section we give an alternative proof of Theorem 7.2.5 in the special case of the trivial local system σ.

9.1. **Making the nilpotent vary.**

9.1.1. As was mentioned above, in this section the local system is trivial. Hence, we can think of A as a nilpotent element of the Lie algebra $\mathfrak{g}$, and $\mathrm{Spr}_P^{\sigma,A}$ is thus the usual parabolic Springer fiber

$$\mathrm{Spr}_P^A:=\{P'\in \mathrm{Fl}_P\mid A\in\mathfrak{p}'\}.$$

9.1.2. For an element $P \in \mathrm{Par}(G)$, let

$$\widetilde{\mathfrak{g}}_P := \{(x, P') \in \mathfrak{g} \times \mathrm{Fl}_P \mid x \in \mathfrak{p}'\} \subset \mathfrak{g} \times \mathrm{Fl}_P$$

be the *parabolic Grothendieck-Springer* variety. Denote by π_P the tautological projection $\widetilde{\mathfrak{g}}_P \to \mathfrak{g}$, and put

$$\mathcal{S}_P := (\pi_P)_{\mathrm{dR},!}(\omega_{\mathrm{Fl}_P}) \in \text{D-mod}(\mathfrak{g}).$$

The assignment

$$P \rightsquigarrow \mathcal{S}_P$$

is a functor $\mathrm{Par}(G) \to \text{D-mod}(\mathfrak{g})$. Consider the colimit

$$\mathcal{S}_{\mathrm{Glued}} := \operatorname*{colim}_{P \in \mathrm{Par}'(G)} \mathcal{S}_P \in \text{D-mod}(\mathfrak{g}).$$

9.1.3. Let $\mathrm{Nilp}_{\mathfrak{g}} \overset{i}{\hookrightarrow} \mathfrak{g}$ be the subvariety of nilpotent elements. Consider the object

$$i^!(\mathcal{S}_{\mathrm{Glued}}) \in \text{D-mod}(\mathrm{Nilp}_{\mathfrak{g}}).$$

By construction, the assertion of Theorem 7.2.5 is equivalent to the following:

Proposition 9.1.4. *The trace map*

$$i^!(\mathcal{S}_{\mathrm{Glued}}) \to \omega_{\mathrm{Nilp}_{\mathfrak{g}}}$$

is an isomorphism away from $0 \in \mathrm{Nilp}_{\mathfrak{g}}$.

9.2. **Interpretaion via the Springer theory.** In this subsection we recall some basic facts about the Springer theory.

9.2.1. Put

$$\mathcal{S} := \mathcal{S}_B.$$

It is well known that $\mathcal{S}[-\dim(\mathfrak{g})]$ lies in the heart of the t-structure (note that the usual t-structure for D-modules corresponds to the perverse t-structure under the Riemann-Hilbert correspondence), and that it carries a canonically defined action of W.

Here are some well-known facts regarding $\mathcal{S}$:

Lemma 9.2.2.

(a) *The trace map* $\mathcal{S} \to \omega_{\mathfrak{g}}$ *induces an isomorphism* $\mathrm{coinv}(\mathsf{W}, \mathcal{S}) \to \omega_{\mathfrak{g}}$. *Here* $\mathrm{coinv}(\mathsf{W}, \mathcal{S})$ *is the* D*-module of coinvariants of the action of* W *on* $\mathcal{S}$.

(b) *Let* $\text{anti-inv}(\mathsf{W}, \mathcal{S})$ *be the sign isotopic component in* $\mathcal{S}$. *Then the* $!$*-restriction of* $\text{anti-inv}(\mathsf{W}, \mathcal{S})$ *to* $\mathrm{Nilp}_{\mathfrak{g}}$ *vanishes outside of* $0 \in \mathrm{Nilp}_{\mathfrak{g}}$.

(c) *For a parabolic* $P = P_{\mathsf{J}}$, *we have* $\mathcal{S}_P \simeq \mathrm{coinv}(\mathsf{W}_{\mathsf{J}}, \mathcal{S})$, *and for* $\mathsf{J}_1 \subset \mathsf{J}_2$ *the natural map* $\mathcal{S}_{P_1} \to \mathcal{S}_{P_2}$ *is induced by the inclusion* $\mathsf{W}_{\mathsf{J}_1} \subset \mathsf{W}_{\mathsf{J}_2}$.

9.2.3. In view of the above lemma, Proposition 9.1.4 follows from the next more precise result:

Proposition 9.2.4. *There exists a canonical isomorphism in* $\text{D-mod}(\mathrm{Nilp}_{\mathfrak{g}})$*:*

$$\mathcal{S}_{\mathrm{Glued}} \simeq \mathrm{coinv}(\mathsf{W}, \mathcal{S}) \oplus \text{anti-inv}(\mathsf{W}, \mathcal{S})[\mathrm{rk}(\mathfrak{g}) - 1].$$

9.3. **Proof of Proposition 9.2.4.**

9.3.1. In view of Lemma 9.2.2(c), the object $\mathcal{S}_{\mathrm{Glued}}$ has the form

$$\mathsf{M} \underset{k[\mathsf{W}]}{\otimes} \mathcal{S},$$

for $\mathsf{M} \in \mathrm{Rep}(\mathsf{W})$ equal to

$$\underset{\mathsf{J} \in \mathrm{Par}'(G)}{\mathrm{colim}}\, k[\mathsf{W}] \underset{k[\mathsf{W}_\mathsf{J}]}{\otimes} k.$$

Thus, it remains to show that

$$\mathsf{M} \simeq k \oplus \mathrm{sign}[\mathrm{rk}(\mathfrak{g}) - 1], \tag{9.1}$$

viewed as representations of W.

9.3.2. Instead of proving the isomorphism (9.1) directly, let us provide a more elegant geometric argument.

Consider the diagram of finite sets equipped with an action of W:

$$\mathsf{J} \mapsto \mathsf{W}/\mathsf{W}_\mathsf{J}.$$

Consider the homotopy type

$$\mathsf{W}_{\mathrm{Glued}} := \underset{\mathsf{J} \in \mathrm{Par}'(G)}{\mathrm{colim}}\, \mathsf{W}/\mathsf{W}_\mathsf{J}.$$

We have:

$$\mathsf{M} = \mathrm{C}_*(\mathsf{W}_{\mathrm{Glued}}).$$

9.3.3. We claim that the geometric realization of $\mathsf{W}_{\mathrm{Glued}}$ is W-equivariantly homotopy equivalent to a $(\mathrm{rk}(\mathfrak{g}) - 1)$-dimensional sphere in the Euclidean space

$$\mathfrak{t}_\mathbb{R} := \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R}.$$

Indeed, fix a generic point $\gamma \in \mathfrak{t}_\mathbb{R}$, and let B_γ be the convex hull of the orbit $\mathsf{W}\gamma$. For each $j = 0, \ldots, \mathrm{rk}(\mathfrak{g})$, the j-faces of the polytope B_γ are indexed by the union

$$\coprod_{|\mathsf{J}|=j} \mathsf{W}/\mathsf{W}_\mathsf{J}.$$

From this, we obtain a W-equivariant homotopy equivalence between B_γ and the geometric realization of

$$\underset{\mathsf{J} \in \mathrm{Par}(G)}{\mathrm{colim}}\, \mathsf{W}/\mathsf{W}_\mathsf{J},$$

and also between the boundary $\partial(\mathsf{B}_\gamma)$ (which is homeomorphic to a sphere) and the geometric realization of $\mathsf{W}_{\mathrm{Glued}}$.

The sign representation $\mathsf{W} \to \{\pm 1\}$ identifies with the action of W on the torsor of orientations of $\mathfrak{t}_\mathbb{R}$, and hence also on the torsor of orientations of $\partial(\mathsf{B}_\gamma)$.

This implies the desired formula for $\mathrm{C}_*(\mathsf{W}_{\mathrm{Glued}})$.

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