

GEOMETRIC CONSTANT TERM FUNCTOR(S)

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ABSTRACT. We study the Eisenstein series and constant term functors in the framework of geometric theory of automorphic functions. Our main result says that for a parabolic $P \subset G$ with Levi quotient M , the $!$ -constant term functor

$$\mathrm{CT}_! : \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_M)$$

is canonically isomorphic to the $*$ -constant term functor

$$\mathrm{CT}_*^- : \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_M),$$

taken with respect to the opposite parabolic P^- .

INTRODUCTION

0.1. Conventions. We fix an algebraically closed field k of characteristic 0. Unless stated otherwise, we say “scheme” instead of “scheme locally of finite type over k .” When we say “stack” we mean an algebraic stack locally of finite type over k such that the automorphism group of any k -point is affine.¹

A morphism of stacks $f : Y_1 \rightarrow Y_2$ is said to be *representable* (resp. *schematic*) if for any scheme S the stack $Y_1 \times_{Y_2} S$ is an algebraic space (resp. scheme).

0.2. Posing the problem.

0.2.1. One of the main tools in studying automorphic functions on an adèle group G is the pair of mutually adjoint operators, called “Eisenstein series” and “constant term” that connect this space to similar spaces for Levi subgroups.

The goal of this paper is to make several basic observations regarding the analogs of these operators in the geometric context.

0.2.2. Let X be a smooth projective connected curve over k , and G a reductive group. Let Bun_G be the moduli stack of principal G -bundles on X . Our geometric analog of the space of automorphic functions is the DG category of (not necessarily holonomic) D-modules on Bun_G , denoted by $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$. We refer the reader to Sect. 0.6.2 for the explanation of what exactly we understand by “DG category of D-modules.” Here we just mention that the homotopy category of this DG category is the derived category of D-modules.

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¹The latter condition is called “locally QCA” in [DrGa1]. It ensures that the theory of D-modules on our stack is reasonable.

Let P be a parabolic in G with Levi quotient M . We have the following fundamental diagram of stacks:

$$(0.1) \quad \begin{array}{ccc} & \text{Bun}_P & \\ \mathfrak{q} \swarrow & & \searrow \mathfrak{p} \\ \text{Bun}_M & & \text{Bun}_G \end{array}$$

Recall that the set of connected components of the stack Bun_M is in bijection with $\pi_1(M)$. For an element $\mu \in \pi_1(M)$ we let Bun_M^μ denote the corresponding connected component; let Bun_P^μ denote the preimage of $\text{Bun}_M^\mu \subset \text{Bun}_M$ under the above map $\mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$; one can show that Bun_P^μ is connected, i.e., is a single connected component of Bun_P .

By a slight abuse of notation we will denote by the same symbols \mathfrak{q} and \mathfrak{p} , respectively, the restrictions of the corresponding maps to $\text{Bun}_P^\mu \subset \text{Bun}_P$. Thus, we obtain a diagram

$$(0.2) \quad \begin{array}{ccc} & \text{Bun}_P^\mu & \\ \mathfrak{q} \swarrow & & \searrow \mathfrak{p} \\ \text{Bun}_M^\mu & & \text{Bun}_G \end{array}$$

0.2.3. Applying pull-push along diagram (0.2) we obtain the functors

$$\text{Eis}_*^\mu : \text{D-mod}(\text{Bun}_M^\mu) \rightarrow \text{D-mod}(\text{Bun}_G), \quad \text{Eis}_*^\mu := \mathfrak{p}_* \circ \mathfrak{q}^!$$

and

$$\text{CT}_*^\mu : \text{D-mod}(\text{Bun}_G) \rightarrow \text{D-mod}(\text{Bun}_M^\mu), \quad \text{CT}_*^\mu := \mathfrak{q}_* \circ \mathfrak{p}^!$$

(here Eis stands for ‘‘Eisenstein’’ and CT stands for ‘‘constant term’’).

We will address the following questions:

Question 0.2.4. *Can one define the functor*

$$\text{Eis}_!^\mu := \mathfrak{p}_! \circ \mathfrak{q}^* : \text{D-mod}(\text{Bun}_M^\mu) \rightarrow \text{D-mod}(\text{Bun}_G),$$

left adjoint to CT_^μ ?*

Question 0.2.5. *Can one define the functor*

$$\text{CT}_!^\mu := \mathfrak{q}_! \circ \mathfrak{p}^* : \text{D-mod}(\text{Bun}_G) \rightarrow \text{D-mod}(\text{Bun}_M^\mu),$$

left adjoint to Eis_^μ ?*

The next few remarks explain why these questions are non-obvious.

Remark 0.2.6. For a morphism $f : Y_1 \rightarrow Y_2$ between schemes or stacks the functors

$$f_* : \text{D-mod}(Y_1) \rightarrow \text{D-mod}(Y_2) \text{ and } f^! : \text{D-mod}(Y_1) \rightarrow \text{D-mod}(Y_2)$$

are well-defined. But their left adjoints

$$f^* : \text{D-mod}(Y_2) \rightarrow \text{D-mod}(Y_1) \text{ and } f_! : \text{D-mod}(Y_2) \rightarrow \text{D-mod}(Y_1)$$

are only *partially* defined (because we work with non-necessarily holonomic D-modules).

Remark 0.2.7. If f is quasi-compact, representable and proper then $f_!$ is always defined and equals f_* . If f is smooth then f^* is always defined and equals $f^!$ up to a cohomological shift.

Remark 0.2.8. If a morphism of stacks $f : Y_1 \rightarrow Y_2$ is *safe* then the functor f_* is “well-behaved” (more precisely, f_* is *continuous*). For more details (including the definitions of safety and continuity), see Sect. 0.6.3.

Remark 0.2.9. In diagram (0.2) the morphism \mathfrak{p} is quasi-compact and representable (and moreover, schematic), but neither smooth nor proper. The morphism \mathfrak{q} is smooth and safe ² (but not representable).

0.3. Statement of the results.

0.3.1. First, we show that the functor $\text{Eis}_!^\mu := \mathfrak{p}_! \circ \mathfrak{q}^*$ is well-defined. (More precisely, since \mathfrak{q} is smooth the functor \mathfrak{q}^* is well-defined, and we will show that $\mathfrak{p}_!$ is well-defined on the essential image of \mathfrak{q}^*).

0.3.2. We now turn to $\text{CT}_!^\mu$. We do eventually show that the functor $\text{CT}_!^\mu$ is well-defined, but this comes as a result of a more precise assertion (see Theorem 0.3.4 below) that describes this functor explicitly.

Here we note that a direct attempt to define $\text{CT}_!^\mu = \mathfrak{q}_! \circ \mathfrak{p}^*$ brings problems different from those in the case of $\text{Eis}_!^\mu$. Namely, it turns out that the second functor, i.e., $\mathfrak{q}_!$, is always well-defined (see Sect. 4.1.6), while the first one, namely \mathfrak{p}^* , is not. However, we will show that their composition is well-defined in a certain sense (see Sect. 4.3.2 for the general paradigm when such compositions are well-defined; we will show that this paradigm is applicable to Bun_G in Sect. 4.7 and Sect. 4.1.6).

0.3.3. Now we are ready to state the main result of this paper, namely, Theorem 1.2.3:

Theorem 0.3.4. *The functor $\text{CT}_!^\mu$ exists and is isomorphic to the functor $\text{CT}_*^{\mu,-}$, where the superscript “ $-$ ” means that instead of P we are considering the opposite parabolic P^- .*

The assertion of this theorem can be viewed as some kind of non-standard functional equation. It does not have an immediate analog in the classical theory of automorphic functions (where one has only one type of pullback operator and one type of push forward).

Theorem 0.3.4 has an implication to the relation between the functors $\text{Eis}_!$ and Eis_* , which will be discussed in a future publication ([Ga]). Here we will only mention that this implication *does* have a manifestation in the classical theory of automorphic functions.

0.4. Relation to the geometric Langlands conjecture. The contents of this subsection play a motivational role and may be skipped by the reader.

0.4.1. In addition to the functors

$$\text{Eis}_*^\mu, \quad \text{CT}_*^\mu, \quad \text{Eis}_!^\mu, \quad \text{CT}_!^\mu$$

one can consider the full Eisenstein and constant term functors:

$$\text{Eis}_* := \bigoplus_{\mu \in \pi_1(M)} \text{Eis}_*^\mu, \quad \text{Eis}_! := \bigoplus_{\mu \in \pi_1(M)} \text{Eis}_!^\mu$$

and

$$\begin{aligned} \text{CT}_* &:= \bigoplus_{\mu \in \pi_1(M)} \text{CT}_*^\mu \simeq \prod_{\mu \in \pi_1(M)} \text{CT}_*^\mu, \\ \text{CT}_! &:= \bigoplus_{\mu \in \pi_1(M)} \text{CT}_!^\mu \simeq \prod_{\mu \in \pi_1(M)} \text{CT}_!^\mu, \end{aligned}$$

²For any surjection of algebraic groups $\phi : H_1 \rightarrow H_2$ the corresponding map $\Phi : \text{Bun}_{H_1} \rightarrow \text{Bun}_{H_2}$ is smooth, which can be seen through the calculation of its differential. If $\ker(\phi)$ is unipotent, then Φ is safe.

(where the sum equals the product because each CT_*^μ and $\mathrm{CT}_!^\mu$ lands in its own component of Bun_M^μ).

Tautologically, the functors $(\mathrm{Eis}_!, \mathrm{CT}_*)$ form an adjoint pair.

Note, however, that the functors $\mathrm{CT}_!$ and Eis_* *do not* form an adjoint pair. Indeed, the right adjoint of $\mathrm{CT}_!$ is given by $\prod_{\mu \in \pi_1(M)} \mathrm{Eis}_*^\mu$ (which is different from $\mathrm{Eis}_* := \bigoplus_{\mu \in \pi_1(M)} \mathrm{Eis}_*^\mu$); in particular, the above right adjoint is not continuous.

Define also

$$\mathrm{CT}_*^- := \bigoplus_{\mu \in \pi_1(M)} \mathrm{CT}_*^{\mu,-}.$$

Of course, Theorem 0.3.4 implies that we have a canonical isomorphism

$$\mathrm{CT}_! \simeq \mathrm{CT}_*^-.$$

0.4.2. It turns out that the functor $\mathrm{Eis}_!$, although less straightforward to define than Eis_* , *plays a more fundamental role from the point of the geometric Langlands conjecture.*

Namely, according to [AG], this conjecture predicts an equivalence of categories

$$\mathbb{L}_G : \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \rightarrow \mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{G}}),$$

where $\mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{G}})$ is a certain modification of the DG category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$ of local systems on X with respect to the Langlands dual group \check{G} .

Now, the equivalence \mathbb{L}_G is expected to be compatible with the corresponding equivalence \mathbb{L}_M for its Levi subgroup M via the diagram

$$(0.3) \quad \begin{array}{ccc} \mathrm{D}\text{-mod}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{G}}) \\ \mathrm{Eis}_! \uparrow & & \uparrow \mathrm{Eis}_{\mathrm{spec}} \\ \mathrm{D}\text{-mod}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{M}}), \end{array}$$

which commutes up to an auto-equivalence of $\mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{M}})$, given by tensoring with a certain canonically defined graded line bundle on $\mathrm{LocSys}_{\check{M}}$.

In the above diagram, $\mathrm{Eis}_{\mathrm{spec}}$ is the *spectral Eisenstein series functor*, defined as pull-push along the diagram

$$(0.4) \quad \begin{array}{ccc} & \mathrm{LocSys}_{\check{M}} & \\ \mathrm{p}_{\mathrm{spec}} \swarrow & & \searrow \mathrm{q}_{\mathrm{spec}} \\ \mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{P}}, \end{array}$$

see [AG, Sect. 12.2] for more details.

The point that we would like to emphasize is that *the commutation in diagram (0.3) takes place for the functor $\mathrm{Eis}_!$ and not for Eis_** (moreover, this cannot be remedied by any auto-equivalence of the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ ³.)

It is possible to explicitly describe the functor

$$\mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{M}}) \rightarrow \mathrm{IndCoh}_{\mathrm{NilP}_{glob}}(\mathrm{LocSys}_{\check{G}})$$

³See, however, Sect. 0.4.4 below.

that corresponds via \mathbb{L}_G and \mathbb{L}_M to

$$\mathrm{Eis}_* : \mathrm{D}\text{-mod}(\mathrm{Bun}_M) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G),$$

but this description is more involved.

0.4.3. The fact that the functor $\mathrm{Eis}_!$ is more fundamental than Eis_* can also be explained as follows.

Recall (see [DrGa2, Sect. 4.3.3]) that in addition to the category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$, there exists another DG category, denoted $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}}$, that can be naturally assigned to the stack Bun_G .

Furthermore, according to [DrGa2, Sect. 4.4.3], there is a canonically defined functor

$$\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, \mathrm{naive}} : \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}} \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G),$$

which is *not* an equivalence, unless G is a torus.

Now, it is not difficult to see that there is a naturally defined functor

$$\mathrm{Eis}_{*, \mathrm{co}} : \mathrm{D}\text{-mod}(\mathrm{Bun}_M)_{\mathrm{co}} \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}}$$

that makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, \mathrm{naive}}} & \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \\ \mathrm{Eis}_{*, \mathrm{co}} \uparrow & & \mathrm{Eis}_* \uparrow \\ \mathrm{D}\text{-mod}(\mathrm{Bun}_M)_{\mathrm{co}} & \xrightarrow{\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_M, \mathrm{naive}}} & \mathrm{D}\text{-mod}(\mathrm{Bun}_M). \end{array}$$

Thus, the functor Eis_* is a coarsening of $\mathrm{Eis}_{*, \mathrm{co}}$, since, informally, the functor $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, \mathrm{naive}}$ “loses more information” than $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_M, \mathrm{naive}}$.

0.4.4. Finally, according to [DrGa2, Sect. 4.4.8], we note that in addition to the functor $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, \mathrm{naive}}$, there is another canonically defined functor

$$\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, !} : \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}} \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G).$$

It will be shown in [Ga] that the functor $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, !}$ *is actually an equivalence of categories*.

It will be also shown that the following diagram commutes:

$$(0.5) \quad \begin{array}{ccc} \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, !}} & \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \\ \mathrm{Eis}_{*, \mathrm{co}} \uparrow & & \mathrm{Eis}_! \uparrow \\ \mathrm{D}\text{-mod}(\mathrm{Bun}_M)_{\mathrm{co}} & \xrightarrow{\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_M, !}} & \mathrm{D}\text{-mod}(\mathrm{Bun}_M). \end{array}$$

So, to summarize, although the naive functor Eis_* plays an inferior role to that of $\mathrm{Eis}_!$, its counterpart

$$\mathrm{Eis}_{*, \mathrm{co}} : \mathrm{D}\text{-mod}(\mathrm{Bun}_M)_{\mathrm{co}} \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G)_{\mathrm{co}}$$

is on par with $\mathrm{Eis}_!$ by virtue of being intertwined by the equivalences $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_G, !}$ and $\mathrm{Ps}\text{-Id}_{\mathrm{Bun}_M, !}$.

In [Ga], it will be also explained how diagram (0.5) expresses the compatibility of the Langlands correspondence functors \mathbb{L}_G (resp., \mathbb{L}_M) with Verdier duality on Bun_G (resp., Bun_M) and Serre duality on $\mathrm{LocSys}_{\check{G}}$ (resp., $\mathrm{LocSys}_{\check{M}}$).

0.5. Method of proof. The proof of Theorem 0.3.4 is a variation on the theme of a theorem of T. Braden on hyperbolic restrictions (see [Br]), recently revisited in [DrGa3]. In fact, we give two proofs, in Sects. 2-3 and Sect. 4, respectively.

0.5.1. This first proof mimics the new proof of Braden's theorem given in [DrGa3], and it directly establishes the $(\mathrm{CT}_*^{\mu,-}, \mathrm{Eis}_*^{\mu})$ -adjunction by specifying the unit and co-unit morphisms.

As in the case of the new proof of Braden's theorem given in [DrGa3], the co-unit morphism is straightforward, and essentially corresponds to the embedding of the big Bruhat cell into $P \backslash G / P^-$.

0.5.2. The unit of the adjunction uses a certain geometric construction, namely, an \mathbb{A}^1 -family of subgroups \tilde{G} of $G \times G$, whose fiber at $1 \in \mathbb{A}^1$ is the diagonal copy of G , and whose fiber at $0 \in \mathbb{A}^1$ is the subgroup

$$P \times_M P^- \subset P \times P^- \subset G \times G.$$

For the definition of the group-subscheme $\tilde{G} \subset \mathbb{A}^1 \times G \times G$, see Sects. 2.3.2, 2.3.3, and 2.4. Here let us just mention that \tilde{G} depends on the choice of a co-character

$$(0.6) \quad \gamma : \mathbb{G}_m \rightarrow M,$$

which lands in the center of M , and which is dominant and regular with respect to P .

0.5.3. The group-scheme \tilde{G} is not new in Lie theory. Namely, it can be recovered from the *Vinberg semi-group* corresponding to G (a.k.a. the enveloping semi-group of G); see Appendix D, where this is explained.

The Lie algebra $\mathrm{Lie}(\tilde{G})$ can be directly recovered from the “wonderful compactification” of G defined in [DCP]:

Let W denote the variety of Lie subalgebras of $\mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g} := \mathrm{Lie}(G)$. Let $\gamma : \mathbb{G}_m \rightarrow M$ denote the co-character (0.6). For $t \in \mathbb{G}_m$ let $\Gamma_t \subset \mathfrak{g} \times \mathfrak{g}$ denote the graph of $\mathrm{Ad}_{\mu(t)} : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$. Since W is projective the map $\mathbb{G}_m \rightarrow W$ defined by $t \mapsto \Gamma_t$ extends to a morphism $\mathbb{A}^1 \rightarrow W$, whose image is contained in the “wonderful compactification.” Thus one gets an \mathbb{A}^1 -family of Lie subalgebras of $\mathfrak{g} \times \mathfrak{g}$. This \mathbb{A}^1 -family is $\mathrm{Lie}(\tilde{G})$ (because \tilde{G} is smooth over \mathbb{A}^1 , see Proposition 2.3.8).

0.5.4. The statement that the functors $(\mathrm{CT}_*^{\mu,-}, \mathrm{Eis}_*^{\mu})$ form an adjoint pair bears a strong resemblance to the Second Adjointness Theorem in the theory of \mathfrak{p} -adic groups.

By a slight abuse of notation, let us temporarily denote by G the set of points of a reductive group over a local non-archimedean field, and consider the corresponding groups

$$M \leftarrow P \hookrightarrow G.$$

We have the usual pair of adjoint functors

$$(r_M^G, i_M^G),$$

where i_M^G is the functor of parabolic induction, and r_M^G is the Jacquet functor.

Now, a theorem of J. Bernstein (unpublished) says that, in addition to being the *right* adjoint of r_M^G , the functor i_M^G is also the *left* adjoint of $r_M^{G,-}$, where the latter is the Jacquet functor with respect to the opposite parabolic P^- .

The proof of this result, given recently in [BKa] (which is different from the original proof of Bernstein), is very close in spirit to our proof of the $(\mathrm{CT}_*^{\mu,-}, \mathrm{Eis}_*^{\mu})$ -adjunction:

One constructs the unit of the adjunction using the big Bruhat cell in $N(P)\backslash G/N(P^-)$.

The co-unit of the adjunction uses (the set of points over our field of) the family of schemes over \mathbb{A}^1

$$(\mathbb{A}^1 \times G \times G)/\tilde{G},$$

acted on by $G \times G$. Note that this family interpolates between G (the fiber at $1 \in \mathbb{A}^1$) and $(G/N(P) \times G/N(P^-))/M$ (the fiber at $0 \in \mathbb{A}^1$).

0.5.5. We now turn to the second proof of Theorem 0.3.4, given in Sect. 4. This proof is obtained by deducing the isomorphism

$$\mathrm{CT}_!^\mu \simeq \mathrm{CT}_*^{\mu,-}$$

from Braden's theorem, which involves *schemes* acted on by \mathbb{G}_m .

The schemes in question are obtained by replacing Bun_G , Bun_P , Bun_{P^-} and Bun_M by their versions when one considers a sufficiently deep level structure at one point of the curve.

0.5.6. Finally, we remark that Theorem 0.3.4 is analogous to the corresponding theorem of Lusztig on *restriction of character sheaves*, see [Gi, Theorem 4.1 (iv)].

In fact, the two statements admit a common generalization when instead of Bun_G we consider the moduli stack of G -bundles with level structure at a finite collection of points of X ; the case of character sheaves on G (resp., the Lie algebra \mathfrak{g}) corresponds to the case of $X = \mathbb{P}^1$ with structure of level 1 at $(0, \infty) \in \mathbb{P}^1$ (resp., structure of level 2 at $\infty \in \mathbb{P}^1$).

Just as Theorem 0.3.4, Lusztig's theorem can be deduced from Braden's theorem. We learned this very simple proof of Lusztig's theorem from folklore and wrote it up in [DrGa3, Sect. 0.2].

0.6. Recollections on D-modules on stacks.

0.6.1. *DG categories.* Our conventions and notation pertaining to DG categories follow those of [DrGa2, Sect. 1].

0.6.2. *The category $\mathrm{D}\text{-mod}(\mathcal{Y})$, where \mathcal{Y} is a stack.* For any stack \mathcal{Y} let $\mathrm{D}\text{-mod}(\mathcal{Y})$ denote the DG category of D-modules on \mathcal{Y} as defined in [DrGa1] (the case of quasi-compact schemes is considered in [DrGa1, Sect. 5.1] and the general case in [DrGa1, Sect. 6.2]).

If $\mathcal{Y} = S$ is a quasi-compact scheme then the homotopy category of the DG category $\mathrm{D}\text{-mod}(S)$ is the usual derived category of D-modules on S , but note that we impose *no boundedness or coherence conditions*⁴.

If \mathcal{Y} is a stack then $\mathrm{D}\text{-mod}(\mathcal{Y})$ is defined to be the (projective) limit of the categories $\mathrm{D}\text{-mod}(S)$ over the indexing category of quasi-compact schemes S mapping smoothly to \mathcal{Y} . Informally, an object of $\mathrm{D}\text{-mod}(\mathcal{Y})$ is a compatible collection of objects of $\mathrm{D}\text{-mod}(S)$ for all quasi-compact schemes S mapping smoothly to \mathcal{Y} .

As is explained in [DrGa1], the DG category $\mathrm{D}\text{-mod}(\mathcal{Y})$ is *cocomplete*, i.e., it has arbitrary colimits.

⁴This is because we want $\mathrm{D}\text{-mod}(S)$ to be cocomplete, see below.

0.6.3. *Direct images and safety.* For any morphism $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ between stacks one has the direct image functor $f_* : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y}_2)$ (see [DrGa1, Sect. 7.4] for the definition of f_* in the case that f is not necessarily quasi-compact schematic).

The functor f_* is not necessarily *continuous*. (Recall that a functor between cocomplete DG categories is said to be continuous if it commutes with (infinite) colimits, or equivalently, with (infinite) direct sums.)

However, according to [DrGa1, Sect. 10.2], f_* is continuous if f is *safe*. By definition, safety means that f is quasi-compact and has the following property: for any $y \in \mathcal{Y}_2(k)$ the neutral connected component of the automorphism group of any k -point of the fiber $(\mathcal{Y}_1)_y$ is unipotent. In particular, any quasi-compact representable morphism is safe.

0.6.4. *Terminological remark.* Following the conventions of higher category theory, we call a morphism between two objects in a DG category an *isomorphism* if and only if it is such in the homotopy category.

0.7. **Organization of the article.** In Sect. 1 we reformulate Theorem 0.3.4 as an existence of an adjunction (see Theorem 1.2.5), and we describe the natural transformation that will turn out to be the co-unit of the adjunction. We also discuss the notion of cuspidal object of $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$; the main point is that two *a priori* different notions of cuspidality coincide.

As already said in 0.5, we give two proofs of Theorem 1.2.5.

The first one is given in Sections 2-3. (In Sect. 2 we construct the natural transformation that will turn out to be the unit, and in Sect. 3 we verify that the natural transformations constructed in Sections 1 and 2 satisfy the properties of unit and counit of an adjunction.)

The second proof of Theorem 1.2.5 is given in Sect. 4.

In Appendix A we prove a technical Theorem 4.3.4, which is used in the second proof of Theorem 1.2.5.

In Appendix B we prove Proposition 1.4.6 that describes support of cuspidal objects of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$.

In Appendix C we prove Proposition 2.4.4, which says that the quotient of $\mathbb{A}^1 \times G \times G$ by the “interpolating” group-subscheme $\tilde{G} \subset \mathbb{A}^1 \times G \times G$ is a quasi-affine scheme.

In Appendix D we describe \tilde{G} and $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ in terms of the Vinberg semigroup of G (a.k.a. enveloping semigroup of G).

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1. THE STATEMENT

1.1. The functor Eis_1^μ .

1.1.1. Consider the diagram

$$(1.1) \quad \begin{array}{ccc} & \text{Bun}_P^\mu & \\ \text{q} \swarrow & & \searrow \text{p} \\ \text{Bun}_M^\mu & & \text{Bun}_G \end{array}$$

By Remark 0.2.9, the map q is smooth, so the functor q^* , left adjoint to q_* , is well-defined. We are going to prove:

Proposition 1.1.2. *The partially defined left adjoint $\text{p}_!$ to $\text{p}^!$ is defined on the essential image of the functor q^* .*

Corollary 1.1.3. *The functor $\text{Eis}_!^\mu := \text{p}_! \circ \text{q}^* : \text{D-mod}(\text{Bun}_M^\mu) \rightarrow \text{D-mod}(\text{Bun}_G)$, left adjoint to CT_*^μ , is well-defined.*

The proof of Proposition 1.1.2, given below, is based on some results of [BG]. Let us recall them.

1.1.4. First, the diagram (1.1) was extended in [BG, Sect. 1.2] to a diagram

$$(1.2) \quad \begin{array}{ccc} \text{Bun}_P^\mu & \xrightarrow{r} & \widetilde{\text{Bun}}_P^\mu \\ & \searrow \tilde{\text{q}} & \searrow \tilde{\text{p}} \\ \text{Bun}_M^\mu & & \text{Bun}_G \end{array}$$

so that $\tilde{\text{p}}$ is proper, r is an open embedding, and

$$\text{p} = \tilde{\text{p}} \circ r, \quad \text{q} = \tilde{\text{q}} \circ r.$$

Moreover, the following basic fact was established (see [BG, Theorem 5.1.5]):

The object $r_!(k_{\text{Bun}_P^\mu}) \in \text{D-mod}(\widetilde{\text{Bun}}_P^\mu)$ is universally locally acyclic (ULA) with respect to the map $\tilde{\text{q}}$. Here for a stack \mathcal{Y} , we denote by $k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ the ‘‘constant sheaf’’ D-module on \mathcal{Y} (i.e., the Verdier dual of $\omega_{\mathcal{Y}}$).

1.1.5. Let us recall the definition of the ULA property. First, recall that on $\text{D-mod}(\mathcal{Y})$ there are two tensor products, namely

$$\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2 := \Delta_{\mathcal{Y}}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \quad \text{and} \quad \mathcal{F}_1 \overset{*}{\otimes} \mathcal{F}_2 := \Delta_{\mathcal{Y}}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \quad \mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathcal{Y}),$$

where $\Delta_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is the diagonal morphism. (Note that Δ^* and $\overset{*}{\otimes}$ are only partially defined.)

Suppose now that we have a morphism $\phi : \mathcal{Y} \rightarrow \mathcal{Z}$ with \mathcal{Z} smooth. According to [BG, Sect. 5.1.1], $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is said to be ULA with respect to ϕ if for every $\mathcal{F}' \in \text{D-mod}(\mathcal{Z})$ the following holds: $\mathcal{F} \overset{*}{\otimes} \phi^*(\mathcal{F}')$ is a well-defined object of $\text{D-mod}(\mathcal{Y})$ and a certain canonical morphism

$$\mathcal{F} \overset{*}{\otimes} \phi^*(\mathcal{F}') \rightarrow \mathcal{F} \overset{!}{\otimes} \phi^!(\mathcal{F}') [2 \dim(\mathcal{Z})],$$

defined in [BG, Sect. 5.1.1], is an isomorphism.

1.1.6. *Proof of Proposition 1.1.2.* Since $\mathfrak{p} = \tilde{\mathfrak{p}} \circ r$ and the functor $\tilde{\mathfrak{p}}_! = \tilde{\mathfrak{p}}_*$ is well-defined, it suffices to show that $r_! \circ \mathfrak{q}^*(\mathcal{F})$ is well-defined for every $\mathcal{F} \in \text{D-mod}(\text{Bun}_M^\mu)$.

This follows from the ULA property of $r_!(k_{\text{Bun}_P^\mu})$ because $r_! \circ \mathfrak{q}^*(\mathcal{F}) = r_!(k_{\text{Bun}_P^\mu})^* \otimes \tilde{\mathfrak{q}}^*(\mathcal{F})$. \square

1.2. **The functor $\text{CT}_!^\mu$.** Our next task is to analyze the existence of the left adjoint of the functor Eis_* .

1.2.1. Let P^- be a parabolic opposite to P . We will identify the Levi factors of P and P^- via the embedding of $M \simeq P \cap P^-$ into P and P^- , respectively.

In particular, we have the diagram

$$(1.3) \quad \begin{array}{ccc} & \text{Bun}_{P^-}^\mu & \\ \mathfrak{q}^- \swarrow & & \searrow \mathfrak{p}^- \\ \text{Bun}_M^\mu & & \text{Bun}_G \end{array}$$

Let $\text{Eis}_*^{\mu,-}$, $\text{Eis}_!^{\mu,-}$ and $\text{CT}_*^{\mu,-}$ be the counterparts of Eis_*^μ , $\text{Eis}_!^\mu$ and CT_*^μ , obtained via the diagram (1.3).

1.2.2. We are now ready to formulate our main result:

Theorem 1.2.3. *The functor*

$$\text{CT}_!^\mu : \text{D-mod}(\text{Bun}_G) \rightarrow \text{D-mod}(\text{Bun}_M^\mu),$$

left adjoint to Eis_^μ , exists and is canonically isomorphic to $\text{CT}_*^{\mu,-}$.*

Thus we have a sequence of three functors

$$\text{Eis}_!^{\mu,-}, \quad \text{CT}_*^{\mu,-} = \text{CT}_!^\mu, \quad \text{Eis}_*^\mu$$

in which each neighboring pair forms an adjoint pair of functors.

1.2.4. Theorem 1.2.3 can be tautologically restated as follows:

Theorem 1.2.5. *The functors $(\text{CT}_*^{\mu,-}, \text{Eis}_*^\mu)$ form an adjoint pair.*

In Sect. 2 we will prove Theorem 1.2.5 by essentially repeating the argument from the paper [DrGa3] that gives a new proof of a theorem of T. Braden (see [Br]) on hyperbolic restrictions.

In Sect. 4 we will give a proof of Theorem 1.2.5 by directly deducing it from the above theorem of Braden.

1.3. **Description of the co-unit of the adjunction.**

1.3.1. Let us specify the co-unit of the adjunction for the functors $(CT_*^{\mu,-}, Eis_*^\mu)$ in Theorem 1.2.5, i.e., the natural transformation

$$(1.4) \quad CT_*^{\mu,-} \circ Eis_*^\mu \rightarrow \text{Id}_{\text{D-mod}(\text{Bun}_M^\mu)}.$$

Consider the diagram

$$\begin{array}{ccccc}
 & & \text{Bun}_M^\mu & & \\
 & & \downarrow j & & \\
 & \text{id} & \text{Bun}_P^\mu \times_{\text{Bun}_G} \text{Bun}_{P^-}^\mu & \text{id} & \\
 & \swarrow & \downarrow \begin{array}{l} \text{'p}^- \\ \text{'p} \end{array} & \searrow & \\
 \text{Bun}_M^\mu & & \text{Bun}_P^\mu & & \text{Bun}_{P^-}^\mu \\
 \swarrow q & & \downarrow p & & \downarrow p^- \\
 \text{Bun}_M^\mu & & \text{Bun}_G & & \text{Bun}_M^\mu \\
 \searrow q^- & & & &
 \end{array}$$

where j is the (open) embedding of the big Bruhat cell, i.e., the locus where the reductions of a given G -bundle to P and P^- are mutually transversal.⁵

By base change,

$$CT_*^{\mu,-} \circ Eis_*^\mu = (q^-)_* \circ (p^-)^! \circ p_* \circ q^! \simeq (q^-)_* \circ (\text{'p})_* \circ (\text{'p}^-)^! \circ q^!.$$

Now, the open embedding j gives rise to a natural transformation

$$\text{Id}_{\text{D-mod}(\text{Bun}_{P^-}^\mu \times_{\text{Bun}_G} \text{Bun}_P^\mu)} \rightarrow j_* \circ j^* \simeq j_* \circ j^!,$$

and hence to

$$\begin{aligned}
 CT_*^{\mu,-} \circ Eis_*^\mu &\simeq (q^-)_* \circ (\text{'p})_* \circ (\text{'p}^-)^! \circ q^! \rightarrow \\
 &\rightarrow (q^-)_* \circ (\text{'p})_* \circ j_* \circ j^! \circ (\text{'p}^-)^! \circ q^! \simeq \\
 &\simeq (q^- \circ \text{'p} \circ j)_* \circ (q \circ \text{'p}^- \circ j)^! \simeq \text{id}_* \circ \text{id}^! = \text{Id}_{\text{D-mod}(\text{Bun}_M^\mu)}.
 \end{aligned}$$

This is the sought-for co-unit of the adjunction.

⁵The morphism j is of the type considered in Lemma 2.5.8(ii) below.

1.3.2. Let us observe that Theorem 1.2.3 can be reformulated also as saying that the functors $(\mathrm{Eis}_!^{\mu,-}, \mathrm{CT}_!^\mu)$ form an adjoint pair. The above description of the co-unit for the $(\mathrm{CT}_*^{\mu,-}, \mathrm{Eis}_*^\mu)$ -adjunction gives rise to a description of the unit of the $(\mathrm{Eis}_!^{\mu,-}, \mathrm{CT}_!^\mu)$ -adjunction:

By definition, the functors $\mathrm{Eis}_!^{\mu,-}$ and $\mathrm{CT}_!^\mu$ are the left adjoints of the functors $\mathrm{CT}_*^{\mu,-}$ and Eis_*^μ , respectively. Now, the unit map

$$\mathrm{Id}_{\mathrm{D-mod}(\mathrm{Bun}_M^\mu)} \rightarrow \mathrm{CT}_!^\mu \circ \mathrm{Eis}_!^{\mu,-}$$

is obtained by passing to left adjoints in the map (1.4).

1.4. Cuspidality.

1.4.1. We will say that an object $\mathcal{F} \in \mathrm{D-mod}(\mathrm{Bun}_G)$ is **-cuspidal* if $\mathrm{CT}_*^\mu(\mathcal{F}) = 0$ for all proper parabolics of G .

We will say that an object $\mathcal{F} \in \mathrm{D-mod}(\mathrm{Bun}_G)$ is *!-cuspidal* if $\mathrm{CT}_!^\mu(\mathcal{F}) = 0$ for all proper parabolics of G .

1.4.2. From Theorem 1.2.3 we obtain:

Corollary 1.4.3. *The notions of !- and *-cuspidality coincide.*

Hence, from now on, we will rename the above property as just cuspidality. Let $\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ be the full subcategory of $\mathrm{D-mod}(\mathrm{Bun}_G)$ formed by cuspidal objects. By construction, $\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ is cocomplete (i.e., closed under colimits).

Let us denote by $\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis},!}$ (resp., $\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis},*}$) the full subcategory of $\mathrm{D-mod}(\mathrm{Bun}_G)$ generated by the essential images of the functors $\mathrm{Eis}_!^\mu$ (resp., Eis_*^μ) for all proper parabolics.

Viewing the notion of cuspidality from the *-perspective, we have:

$$\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}} = (\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis},!})^\perp.$$

1.4.4. In addition, we have the inclusion

$$(1.5) \quad {}^\perp(\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis},*}) \subset \mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}},$$

that comes from

$$\begin{aligned} {}^\perp(\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{Eis},*}) &= {}^\perp\left(\bigcup_{P,\mu} \mathrm{Im}(\mathrm{Eis}_*^\mu)\right) \subset \bigcap_{P,\mu} {}^\perp(\mathrm{Im}(\mathrm{Eis}_*^\mu)) = \\ &= \bigcap_{P,\mu} \ker(\mathrm{CT}_!^\mu) = \mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}. \end{aligned}$$

It is not clear whether the inclusion (1.5) is an equality; we conjecture that it is not if X has genus ≥ 2 .

1.4.5. Let us note another important property of cuspidal objects:

Proposition 1.4.6. *There exists an open substack $j : \mathcal{U} \hookrightarrow \mathrm{Bun}_G$ such that*

- (i) *The intersection of \mathcal{U} with each connected component of Bun_G is quasi-compact;*
- (ii) *For any $\mathcal{F} \in \mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$, the canonical maps*

$$j_! \circ j^*(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j_* \circ j^*(\mathcal{F})$$

*are isomorphisms.*⁶

⁶The statement about $j_! \circ j^*(\mathcal{F})$ should be understood as follows: the partially defined functor $j_!$ is defined on $j^*(\mathcal{F})$ and the map $j_! \circ j^*(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism.

The proof is given in Appendix B. It is parallel to the proof of a similar statement in the classical theory of automorphic forms. In fact, Appendix B we describe an explicit open substack \mathcal{U} with the properties required in Proposition 1.4.6.

2. PROOF OF THEOREM 1.2.5: CONSTRUCTING THE UNIT OF THE ADJUNCTION

We will prove Theorem 1.2.5 (and thereby Theorem 1.2.3) by mimicking the new proof of Braden's theorem given in [DrGa3].

The idea of the proof is to define a natural transformation

$$(2.1) \quad \mathrm{Id}_{\mathrm{D}\text{-mod}(\mathrm{Bun}_G)} \rightarrow \mathrm{Eis}_*^{\mu} \circ \mathrm{CT}_*^{\mu, -}$$

and to show that (1.4) and (2.1) define an adjunction datum.

In this section we will construct the natural transformation (2.1). The fact that (1.4) and (2.1) indeed define an adjunction datum will be proved in Sect. 3.

2.1. Digression: functors given by kernels. As was mentioned in Sect. 0.1, all algebraic stacks in this paper are assumed locally QCA.

2.1.1. Let $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ be a *quasi-compact* map. We note that in addition to the usual de Rham direct image functor

$$f_* : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y}_2)$$

(defined as in [DrGa1, Sect. 7.4]), there exists another canonically defined functor

$$f_{\blacktriangle} : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y}_2).$$

The main feature of the functor f_{\blacktriangle} is that, unlike f_* , it is *continuous*. In addition, it has the following properties (all of which fail for f_*):

- (i) The formation $f \rightsquigarrow f_{\blacktriangle}$ is compatible with compositions of morphisms;
- (ii) It satisfies base change (with respect to $!$ -pullbacks);
- (iii) It satisfies the projection formula.

We have a natural transformation

$$(2.2) \quad f_{\blacktriangle} \rightarrow f_*$$

If the morphism f is representable, or, more generally, *safe*, then (2.2) is an isomorphism.

All of the above facts are established in [DrGa1, Sect. 9.3].

Remark 2.1.2. Technically, in [DrGa1, Sect. 9.3], the functor f_{\blacktriangle} was defined when \mathcal{Y}_2 (and, hence, \mathcal{Y}_1) is quasi-compact. However, the base-change property ensures that the definition canonically extends to the case of general quasi-compact morphisms.

2.1.3. Let \mathcal{Y}_1 and \mathcal{Y}_2 be two algebraic stacks. We will say that an object

$$\mathcal{Q} \in \mathrm{D}\text{-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

has a *quasi-compact $!$ -support relative to \mathcal{Y}_2* if the following condition holds:

For every quasi-compact open substack $U_2 \xrightarrow{j_2} \mathcal{Y}_2$ there exists a quasi-compact open substack $U_1 \xrightarrow{j_1} \mathcal{Y}_1$ such that the map

$$(2.3) \quad (\mathrm{id} \times j_2)^*(\mathcal{Q}) \rightarrow (j_1 \times \mathrm{id})_* \circ (j_1 \times \mathrm{id})^* \circ (\mathrm{id} \times j_2)^*(\mathcal{Q}) = (j_1 \times \mathrm{id})_* \circ (j_1 \times j_2)^*(\mathcal{Q})$$

is an isomorphism in $\mathrm{D}\text{-mod}(\mathcal{Y}_1 \times U_2)$.

Let us repeat the same in words: the restriction of \mathcal{Q} to the open substack $\mathcal{Y}_1 \times U_2$ is a $*$ -extension from $U_1 \times U_2$ for some quasi-compact open substack $U_1 \subset \mathcal{Y}_1$.

Note that $(j_1 \times \text{id})_* \simeq (j_1 \times \text{id})_\blacktriangle$ (indeed, the morphism $j_1 \times \text{id}$ is representable because it is an open embedding).

2.1.4. Let $\mathcal{Q} \in \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)$ have a quasi-compact $!$ -support relative to \mathcal{Y}_2 . We claim that such \mathcal{Q} canonically gives rise to a continuous functor

$$(2.4) \quad F_{\mathcal{Q}} : \text{D-mod}(\mathcal{Y}_1) \rightarrow \text{D-mod}(\mathcal{Y}_2), \quad F_{\mathcal{Q}}(\mathcal{F}) := (\text{pr}_2)_\blacktriangle \left(\text{pr}_1^*(\mathcal{F}) \overset{!}{\otimes} \mathcal{Q} \right).$$

Since the morphism pr_2 is not assumed quasi-compact, we have to explain how to understand formula (2.4).

By the definition of $\text{D-mod}(\mathcal{Y}_2)$ (see [DrGa2, Sect. 2.3] for a detailed review), the datum of a functor $\text{D-mod}(\mathcal{Y}_1) \rightarrow \text{D-mod}(\mathcal{Y}_2)$ amounts to a compatible ⁷ family of functors

$$\text{D-mod}(\mathcal{Y}_1) \rightarrow \text{D-mod}(U_2)$$

for quasi-compact open substacks $U_2 \xrightarrow{j_2} \mathcal{Y}_2$.

For a given U_2 , let $U_1 \xrightarrow{j_1} \mathcal{Y}_1$ be as in Sect. 2.1.3, and let

$$\text{pr}_2^{U_1, U_2} : U_1 \times U_2 \rightarrow U_2$$

denote the projection. Then we set

$$(2.5) \quad j_2^*(F_{\mathcal{Q}}(\mathcal{F})) := (\text{pr}_2^{U_1, U_2})_\blacktriangle \circ (j_1 \times j_2)^* \left(\text{pr}_1^*(\mathcal{F}) \overset{!}{\otimes} \mathcal{Q} \right)$$

(here the \blacktriangle -puhsforward is defined because the morphism $\text{pr}_2^{U_1, U_2}$ is quasi-compact).

It is easy to see that the isomorphism (2.3) implies that the right-hand side in (2.5) is independent of the choice of U_1 .

Furthermore, it is easy to see that the base change property for the \blacktriangle -puhsforward for quasi-compact morphisms implies that the functors (2.5) are indeed compatible under the inclusions $U'_2 \subset U''_2$.

2.1.5. As an example, consider a diagram

$$(2.6) \quad \begin{array}{ccc} & \mathcal{Y} & \\ f_1 \swarrow & & \searrow f_2 \\ \mathcal{Y}_1 & & \mathcal{Y}_2, \end{array}$$

where the morphism f_2 is quasi-compact. In this case the morphism

$$(f_1 \times f_2) : \mathcal{Y} \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is quasi-compact as well.

Set

$$\mathcal{Q} := (f_1 \times f_2)_\blacktriangle(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2),$$

where $\omega_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is the dualizing complex.

It is easy to see that \mathcal{Q} has a quasi-compact $!$ -support relative to \mathcal{Y}_2 . Moreover, in this case the functor $F_{\mathcal{Q}}$ identifies canonically with $(f_2)_\blacktriangle \circ f_1^*$.

⁷The compatibility is with respect to $*$ -restrictions for the inclusions $U'_2 \subset U''_2$.

In particular, for $\mathcal{Y}_1 = \mathcal{Y} = \mathcal{Y}_2$ and $f_1 = f_2 = \text{id}$, we obtain that for $\mathcal{Q} := (\Delta_{\mathcal{Y}})_{\blacktriangle}(\omega_{\mathcal{Y}})$, the corresponding functor $F_{\mathcal{Q}}$ identifies canonically with $\text{Id}_{\text{D-mod}(\mathcal{Y})}$.

2.2. Plan of the construction.

2.2.1. Note that since the morphism \mathfrak{p} is quasi-compact and representable (in fact, schematic), we have:

$$\text{Eis}_*^{\mu} = \mathfrak{p}_* \circ \mathfrak{q}^! \simeq \mathfrak{p}_{\blacktriangle} \circ \mathfrak{q}^!$$

Since \mathfrak{q} is *safe*, we have also

$$\text{CT}_*^{\mu} = \mathfrak{q}_* \circ \mathfrak{p}^! \simeq \mathfrak{q}_{\blacktriangle} \circ \mathfrak{p}^!$$

Hence, the functor $\text{Eis}_*^{\mu} \circ \text{CT}_*^{\mu,-}$ is given as pull-push along the following diagram:

$$(2.7) \quad \begin{array}{ccccc} & & \text{Bun}_{P^-}^{\mu} \times_{\text{Bun}_M^{\mu}} \text{Bun}_P^{\mu} & & \\ & & \swarrow \text{'q} & & \searrow \text{'q}^- \\ & \text{Bun}_{P^-}^{\mu} & & & \text{Bun}_P^{\mu} \\ & \swarrow \mathfrak{p}^- & & \swarrow \mathfrak{q}^- & \searrow \mathfrak{q} \\ \text{Bun}_G & & \text{Bun}_M^{\mu} & & \text{Bun}_G \\ & & \swarrow \mathfrak{q} & & \searrow \mathfrak{p} \end{array}$$

Here and in the sequel, by “push” we understand the \blacktriangle -pushforward, so the base change formula applies to any quasi-compact morphism.

2.2.2. Denote

$$\mathcal{Q}_0 := ((\mathfrak{p}^- \circ \text{'q}) \times (\mathfrak{p} \circ \text{'q}^-))_{\blacktriangle} (\omega_{\text{Bun}_{P^-}^{\mu} \times_{\text{Bun}_M^{\mu}} \text{Bun}_P^{\mu}}) \in \text{D-mod}(\text{Bun}_G \times \text{Bun}_G)$$

and

$$\mathcal{Q}_1 := (\Delta_{\text{Bun}_G})_{\blacktriangle} (\omega_{\text{Bun}_G}) \in \text{D-mod}(\text{Bun}_G \times \text{Bun}_G).$$

By Sect. 2.1.5 we have:

Lemma 2.2.3. *The functors $\text{Id}_{\text{D-mod}(\text{Bun}_G)}$ and $\text{Eis}_*^{\mu} \circ \text{CT}_*^{\mu,-}$ are given by $F_{\mathcal{Q}_1}$ and $F_{\mathcal{Q}_0}$, respectively.*

2.2.4. We are going to define the natural transformation (2.1) by constructing a map

$$(2.8) \quad \mathcal{Q}_1 \rightarrow \mathcal{Q}_0$$

in $\text{D-mod}(\text{Bun}_G \times \text{Bun}_G)$. To do this, we will construct an object

$$\mathcal{Q} \in \text{D-mod}(\mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G)$$

such that:

(i) $\iota_0^!(\mathcal{Q}) = \mathcal{Q}_0$ and $\iota_1^!(\mathcal{Q}) = \mathcal{Q}_1$, where $\iota_t : \text{Bun}_G \times \text{Bun}_G \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G$ is the closed embedding corresponding to $t \in \mathbb{A}^1$;

(ii) \mathcal{Q} is \mathbb{G}_m -monodromic with respect to the usual action of \mathbb{G}_m on \mathbb{A}^1 ; by definition, this means that \mathcal{Q} belongs to the full subcategory of $\mathrm{D}\text{-mod}(\mathbb{A}^1 \times \mathrm{Bun}_G \times \mathrm{Bun}_G)$ generated by the essential image of the pullback functor

$$\mathrm{D}\text{-mod}((\mathbb{A}^1/\mathbb{G}_m) \times \mathrm{Bun}_G \times \mathrm{Bun}_G) \rightarrow \mathrm{D}\text{-mod}(\mathbb{A}^1 \times \mathrm{Bun}_G \times \mathrm{Bun}_G).$$

Assuming we have such \mathcal{Q} , we take the map $\mathcal{Q}_1 \rightarrow \mathcal{Q}_0$ of (2.8) to be the *specialization map*

$$\mathrm{Sp}_{\mathcal{Q}} : \iota_1^!(\mathcal{Q}) \rightarrow \iota_0^!(\mathcal{Q}),$$

which is defined by virtue of the above property (ii). The definition of the specialization map is recalled in Sect. 2.2.6 below.

The remaining part of Sect. 2 is devoted to constructing an object \mathcal{Q} with the above properties (i)-(ii). Informally, \mathcal{Q} should “interpolate” between \mathcal{Q}_1 and \mathcal{Q}_0 .

Remark 2.2.5. The construction of \mathcal{Q} is geometric, and except for Sect. 2.6.2 at the very end, we do not use the characteristic 0 assumption on k . The idea is to first construct a group-scheme \tilde{G} over \mathbb{A}^1 , which “interpolates” between the groups G and $P \times P^-$, and then to construct a stack over \mathbb{A}^1 which “interpolates” between Bun_G and the stack

$$\mathrm{Bun}_P^\mu \times_{\mathrm{Bun}_M^\mu} \mathrm{Bun}_{P^-}^\mu$$

from diagram (2.7).

2.2.6. The specialization map. Let \mathcal{Y} be a stack. Let $\iota_0, \iota_1 : \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ denote the closed embeddings corresponding to $0 \in \mathbb{A}^1$ and $1 \in \mathbb{A}^1$, respectively. Suppose that $\mathcal{K} \in \mathrm{D}\text{-mod}(\mathbb{A}^1 \times \mathcal{Y})$ is \mathbb{G}_m -monodromic with respect to the usual action of \mathbb{G}_m on \mathbb{A}^1 . In this situation we can define the *specialization morphism*

$$(2.9) \quad \mathrm{Sp}_{\mathcal{K}} : \iota_1^!(\mathcal{K}) \rightarrow \iota_0^!(\mathcal{K}).$$

We need the following lemma (which is actually a particular case of Proposition 4.1.5(2)):

Lemma 2.2.7. *Let $\pi : \mathbb{A}^1 \times \mathcal{Y} \rightarrow \mathcal{Y}$ denote the projection. Then the functor $\pi_!$, left adjoint to $\pi^!$, is defined and is canonically isomorphic to $\iota_0^!$ on \mathbb{G}_m -monodromic objects of $\mathrm{D}\text{-mod}(\mathbb{A}^1 \times \mathcal{Y})$.*

Proof. The morphism in one direction

$$\iota_0^! \rightarrow \pi_!$$

(assuming that $\pi_!$ exists) is given by

$$\iota_0^! \simeq \pi_! \circ (\iota_0)_! \circ \iota_0^! \rightarrow \pi_!.$$

To verify the existence of $\pi_!$ and the fact that the above map is an isomorphism, it suffices to do so for the corresponding functors $\mathrm{D}\text{-mod}((\mathbb{A}^1/\mathbb{G}_m) \times \mathcal{Y}) \rightarrow \mathrm{D}\text{-mod}((\mathrm{pt}/\mathbb{G}_m) \times \mathcal{Y})$. Since

$$\mathrm{D}\text{-mod}((\mathbb{A}^1/\mathbb{G}_m) \times \mathcal{Y}) \simeq \mathrm{D}\text{-mod}(\mathbb{A}^1/\mathbb{G}_m) \otimes \mathrm{D}\text{-mod}(\mathcal{Y})$$

and

$$\mathrm{D}\text{-mod}((\mathrm{pt}/\mathbb{G}_m) \times \mathcal{Y}) \simeq \mathrm{D}\text{-mod}(\mathrm{pt}/\mathbb{G}_m) \otimes \mathrm{D}\text{-mod}(\mathcal{Y}),$$

the assertion reduces to the case when $\mathcal{Y} = \mathrm{pt}$, which is straightforward. \square

Now define the specialization morphism (2.9) to be the composition

$$\iota_1^!(\mathcal{K}) \simeq \pi_! \circ (\iota_1)_! \circ \iota_1^!(\mathcal{K}) \rightarrow \pi_!(\mathcal{K}) \simeq \iota_0^!(\mathcal{K})$$

(note that since ι_1 is a closed embedding the functor $(\iota_1)_!$ is well-defined).

Remark 2.2.8. One can show (using only general nonsense) that the above composition is equal to the composition

$$\iota_1^!(\mathcal{K}) \rightarrow \iota_1^! \circ \pi^! \circ \pi_!(\mathcal{K}) \simeq \pi_!(\mathcal{K}) \simeq \iota_0^!(\mathcal{K}).$$

2.3. Recollections from [DrGa3]. In this subsection we will recall some constructions from [DrGa3, Sects. 1-2], to be subsequently applied to the group G .

2.3.1. Attractors and repellers. Let Z be a quasi-compact scheme acted on by \mathbb{G}_m . According to [DrGa3, Proposition 1.3.4 and Corollary 1.5.3(ii)], there exist quasi-compact schemes Z^0 , Z^+ , and Z^- representing the following functors:

$$\text{Maps}(S, Z^0) = \text{Maps}^{\mathbb{G}_m}(S, Z),$$

$$\text{Maps}(S, Z^+) = \text{Maps}^{\mathbb{G}_m}(\mathbb{A}^1 \times S, Z),$$

$$\text{Maps}(S, Z^-) = \text{Maps}^{\mathbb{G}_m}(\mathbb{A}_-^1 \times S, Z),$$

where S is a test affine scheme and \mathbb{A}^1 (resp. \mathbb{A}_-^1) is the affine line equipped with the usual \mathbb{G}_m -action (resp. the \mathbb{G}_m -action opposite to the usual one).

The scheme Z^0 (resp. Z^+ and Z^-) is called the *scheme of \mathbb{G}_m -fixed points* (resp. the *attractor* and *repeller*).

Let $p^+ : Z^+ \rightarrow Z$ and $q^+ : Z^+ \rightarrow Z^0$ denote the maps corresponding to evaluating a \mathbb{G}_m -equivariant morphism $\mathbb{A}^1 \times S \rightarrow Z$ at $1 \in \mathbb{A}^1$ and $0 \in \mathbb{A}^1$, respectively. One defines $p^- : Z^- \rightarrow Z$ and $q^- : Z^- \rightarrow Z^0$ similarly.

Let $i^+ : Z^0 \rightarrow Z^+$ (resp. $i^- : Z^0 \rightarrow Z^-$) denote the morphism induced by the projection $\mathbb{A}^1 \times S \rightarrow S$ (resp. $\mathbb{A}_-^1 \times S \rightarrow S$).

If Z is affine (the case of interest for us) then the existence of Z^0 and Z^\pm (i.e., the representability of the corresponding functors) is very easy to prove. Moreover, in this case Z^0 and Z^\pm are affine, and the morphisms $p^\pm : Z^\pm \rightarrow Z$ are closed embeddings.

2.3.2. The family of hyperbolas. We now consider the following family of curves over \mathbb{A}^1 , denoted by \mathbb{X} : as a scheme, $\mathbb{X} = \mathbb{A}^2$, and the map $\mathbb{X} \rightarrow \mathbb{A}^1$ is $(t_1, t_2) \mapsto t_1 \cdot t_2$. The fibers of this map are hyperbolas; the zero fiber is the coordinate cross, i.e., a degenerate hyperbola.

We let \mathbb{G}_m act on \mathbb{X} hyperbolically:

$$\lambda \cdot (t_1, t_2) = (\lambda \cdot t_1, \lambda^{-1} \cdot t_2).$$

2.3.3. The scheme \tilde{Z} . According to [DrGa3, Theorem 2.4.2], there exists a quasi-compact scheme \tilde{Z} over \mathbb{A}^1 representing the following functor on the category of schemes over \mathbb{A}^1 :

$$\text{Maps}_{\mathbb{A}^1}(S, \tilde{Z}) := \text{Maps}_{\mathbb{A}^1}^{\mathbb{G}_m}(\mathbb{X} \times S, Z).$$

Again, if Z is affine the existence of \tilde{Z} is very easy to prove; moreover, in this case \tilde{Z} is affine. Let us also mention that if Z is affine and smooth (the case of interest for us) then Proposition 2.3.8(ii) below gives a description of \tilde{Z} which some readers may prefer to consider as a definition.

2.3.4. One has a canonical map $\tilde{p} : \tilde{Z} \rightarrow \mathbb{A}^1 \times Z \times Z$. To define it, first note that any section $\sigma : \mathbb{A}^1 \rightarrow \mathbb{X}$ of the morphism $\mathbb{X} \rightarrow \mathbb{A}^1$ defines a map

$$\sigma^* : \text{Maps}_{\mathbb{A}^1}^{\mathbb{G}_m}(\mathbb{X} \times S, Z) \rightarrow \text{Maps}(S, Z)$$

and therefore a morphism $\tilde{Z} \rightarrow Z$. Let $\pi_1 : \tilde{Z} \rightarrow Z$ and $\pi_2 : \tilde{Z} \rightarrow Z$ denote the morphisms corresponding to the sections

$$t \mapsto (1, t) \in \mathbb{X} \quad \text{and} \quad t \mapsto (t, 1) \in \mathbb{X},$$

respectively.

Finally, define

$$(2.10) \quad \tilde{p} : \tilde{Z} \rightarrow \mathbb{A}^1 \times Z \times Z$$

to be the morphism whose first component is the tautological projection $\tilde{Z} \rightarrow \mathbb{A}^1$, and the second and the third components are π_1 and π_2 , respectively.

It is easy to check that if Z is affine then \tilde{p} is a closed embedding (see [DrGa3, Proposition 2.3.6]). So \tilde{p} identifies \tilde{Z} with a closed subscheme of $\mathbb{A}^1 \times Z \times Z$.

2.3.5. Let \tilde{Z}_t denote the preimage of $t \in \mathbb{A}^1$ under the projection $\tilde{Z} \rightarrow \mathbb{A}^1$. Let \tilde{p}_t denote the corresponding map $\tilde{Z}_t \rightarrow Z \times Z$.

By definition, $(\tilde{Z}_1, \tilde{p}_1)$ identifies with (Z, Δ_Z) . For any $t \in \mathbb{A}^1 - \{0\}$, the pair $(\tilde{Z}_t, \tilde{p}_t)$ is the graph of the action of $t \in \mathbb{G}_m$ on Z . Moreover, the morphism \tilde{p} induces an isomorphism

$$(2.11) \quad \mathbb{G}_m \times_{\mathbb{A}^1} \tilde{Z} \xrightarrow{\sim} \Gamma, \quad \Gamma := \{(t, z_1, z_2) \mid t \cdot z_1 = z_2\}.$$

The scheme \tilde{Z}_0 identifies with $Z^+ \times_{Z^0} Z^-$ so that the morphism $\tilde{p}_0 : \tilde{Z}_0 \rightarrow Z \times Z$ identifies with the composition

$$Z^+ \times_{Z^0} Z^- \hookrightarrow Z^+ \times Z^- \xrightarrow{p^+ \times p^-} Z \times Z.$$

The above-mentioned identification comes from the fact that the coordinate cross \mathbb{X}_0 is a union of \mathbb{A}^1 and \mathbb{A}_-^1 glued together along 0, see [DrGa3, Proposition 2.2.9].

2.3.6. *The action of $\mathbb{G}_m \times \mathbb{G}_m$.* In what follows we will need the action of $\mathbb{G}_m \times \mathbb{G}_m$ on \tilde{Z} that corresponds to the following action of $\mathbb{G}_m \times \mathbb{G}_m$ on \mathbb{X} :

$$(\lambda_1, \lambda_2) \cdot (t_1, t_2) = (\lambda_1 \cdot t_1, \lambda_2 \cdot t_2); \quad \lambda_1, \lambda_2 \in \mathbb{G}_m, \quad (t_1, t_2) \in \mathbb{X}.$$

Note that the map \tilde{p} is equivariant with respect to the following action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^1 \times Z \times Z$:

$$(\lambda_1, \lambda_2) \cdot (t, z_1, z_2) = (\lambda_1^{-1} \cdot \lambda_2^{-1} \cdot t, \lambda_1 \cdot z_1, \lambda_2^{-1} \cdot z_2).$$

2.3.7. *Smoothness.* The following facts are proved in [DrGa3, Propositions 2.5.2 and 2.5.4].

Proposition 2.3.8.

- (i) *If Z is smooth then so is the morphism $\tilde{Z} \rightarrow \mathbb{A}^1$.*
- (ii) *If Z is smooth and affine then the morphism $\tilde{p} : \tilde{Z} \rightarrow \mathbb{A}^1 \times Z \times Z$ induces an isomorphism*

$$\tilde{Z} \xrightarrow{\sim} \bar{\Gamma},$$

where Γ is as in formula (2.11) and $\bar{\Gamma}$ is the scheme-theoretic closure of Γ in $\mathbb{A}^1 \times Z \times Z$.

We will need the proposition only in the case that Z is affine. Let us give a proof in this case, which is different from the one in [DrGa3].

Proof. We can assume that Z has pure dimension n . This easily implies that $\tilde{Z}_0 = Z^+ \times_{Z^0} Z^-$ is smooth and of pure dimension n .

Since Z is affine, \tilde{p} is a closed embedding; so we can consider \tilde{Z} as a closed subscheme of $\mathbb{A}^1 \times Z \times Z$. We have $\tilde{Z} \cap (\mathbb{G}_m \times Z \times Z) = \Gamma$, so $\bar{\Gamma} \subset \tilde{Z}$.

Let us show that

$$(2.12) \quad (\bar{\Gamma})_t = \tilde{Z}_t \text{ for all } t \in \mathbb{A}^1$$

(here $(\bar{\Gamma})_t$ is the fiber over t). The only nontrivial case is $t = 0$. Both $(\bar{\Gamma})_0$ and \tilde{Z}_0 have pure dimension n , and \tilde{Z}_0 is smooth. Since $(\bar{\Gamma})_0 \subset \tilde{Z}_0$, it remains to check that $(\bar{\Gamma})_0$ meets each connected component of \tilde{Z}_0 . This follows from the obvious inclusion $\tilde{Z} \supset \mathbb{A}^1 \times \Delta_Z(Z^0)$.

By definition, $\bar{\Gamma}$ is flat over \mathbb{A}^1 . By (2.12), $(\bar{\Gamma})_t$ is smooth for each $t \in \mathbb{A}^1$. So $\bar{\Gamma}$ is smooth over \mathbb{A}^1 . It remains to show that the embedding $\bar{\Gamma} \hookrightarrow \tilde{Z}$ is an isomorphism. By (2.12), it suffices to check that it is an open embedding. This follows from the next lemma. \square

Lemma 2.3.9. *Let Y and Y' be schemes of finite type over a Noetherian scheme S , and let $f : Y' \rightarrow Y$ be an S -morphism. Assume that:*

- (i) Y' is flat over S ,
- (ii) for any $s \in S$ the morphism $Y'_s \rightarrow Y_s$ induced by f is an open embedding.

Then f is an open embedding.

This follows from Grothendieck's "Critère de platitude par fibres" (Corollary 11.3.11 from EGA IV-3). Here is a direct proof (which is much shorter than the proof of "Critère de platitude par fibres"):

Proof. It suffices to show that f is étale at any $y' \in Y'$. Let y and s be the images of y' in Y and S . Let A be the completed local ring of S at s and m its maximal ideal. Let B be the completed local ring of Y at y . Let B' be the completed local ring of Y' at y' . Then B is a finite A -algebra and $B' = B/I$ for some ideal I . The problem is to show that $I = 0$.

By assumption, B' is flat over A and the image of I in B/mB is zero. So $I/mI = 0$. By Nakayama's lemma, this implies that $I = 0$. \square

2.4. The interpolating family of groups.

2.4.1. Fix a co-character

$$(2.13) \quad \gamma : \mathbb{G}_m \rightarrow M$$

mapping to the center of M , which is dominant and regular with respect to P .

We will apply the set-up of Sect. 2.3 when $Z = G$ and the \mathbb{G}_m -action on G is the adjoint action corresponding to the co-character (2.13).

2.4.2. In this situation $Z^+ = P$, $Z^- = P^-$, and $Z^0 = M$.

Recall that the scheme

$$\tilde{G} := \tilde{Z}$$

is a closed subscheme of $\mathbb{A}^1 \times G \times G$.

Since the constructions of Sect. 2.3 are functorial in Z , this subscheme is a group-scheme over \mathbb{A}^1 , which is a closed group-subscheme of the constant group-scheme $\mathbb{A}^1 \times G \times G$. By Proposition 2.3.8, \tilde{G} is smooth over \mathbb{A}^1 .

The fiber \tilde{G}_1 of \tilde{G} over $1 \in \mathbb{A}^1$ is the diagonal copy of G , and the fiber \tilde{G}_0 over $0 \in \mathbb{A}^1$ is $P \times_M P^-$.

2.4.3. Since \tilde{G} is flat over \mathbb{A}^1 the quotient $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ exists as an algebraic space of finite type over \mathbb{A}^1 . In Appendix C we will prove the following statement.

Proposition 2.4.4. *The quotient $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ is a quasi-affine scheme.*

2.4.5. In Appendix D we will show how the group scheme \tilde{G} can be described using the Vinberg semi-group corresponding to G , see Proposition D.6.4. This description immediately implies Proposition 2.4.4 (see Corollary D.6.5). However, the proof of Proposition 2.4.4 given in Appendix C has the advantage of being short and self-contained.

2.5. The interpolation of moduli of bundles.

2.5.1. The functor that assigns to an affine scheme S over \mathbb{A}^1 the groupoid of torsors on $S \times X$ with respect to the group-scheme $(\tilde{G} \times_{\mathbb{A}^1} S) \times X$ is an (a priori, non-algebraic) stack over \mathbb{A}^1 , denoted by $\text{Bun}_{\tilde{G}}$. The morphism

$$\tilde{p} : \tilde{G} \rightarrow \mathbb{A}^1 \times G \times G$$

gives rise to a morphism ⁸

$$(2.14) \quad \tilde{p} : \text{Bun}_{\tilde{G}} \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G.$$

Proposition 2.5.2.

(i) $\text{Bun}_{\tilde{G}}$ is an algebraic stack smooth over \mathbb{A}^1 , with an affine diagonal.

(ii) The map \tilde{p} is of finite type and representable. Moreover, it is schematic.

Proof. Point (ii) follows from Proposition 2.4.4.

Let us prove (i). Statement (ii) implies that the stack $\text{Bun}_{\tilde{G}}$ is algebraic and locally of finite type. The fact that it has an affine diagonal is immediate from the fact that \tilde{G} itself is affine over \mathbb{A}^1 . It remains to check that the morphism $\text{Bun}_{\tilde{G}} \rightarrow \mathbb{A}^1$ is formally smooth. As usual, this follows from the fact that any coherent sheaf on X has a trivial H^2 . \square

2.5.3. Consider the action of $\mathbb{G}_m \times \mathbb{G}_m$ on \tilde{G} defined as in Sect. 2.3.6. The morphism $\tilde{G} \rightarrow \mathbb{A}^1$ is $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant if \mathbb{A}^1 is equipped with the following $\mathbb{G}_m \times \mathbb{G}_m$ -action:

$$(\lambda_1, \lambda_2) \cdot t = \lambda_1^{-1} \cdot \lambda_2^{-1} \cdot t.$$

Moreover, the action of $\mathbb{G}_m \times \mathbb{G}_m$ on \tilde{G} respects the group structure on \tilde{G} . Therefore it induces a $\mathbb{G}_m \times \mathbb{G}_m$ -action on the stack $\text{Bun}_{\tilde{G}}$, which covers the above $\mathbb{G}_m \times \mathbb{G}_m$ -action on \mathbb{A}^1 .

Lemma 2.5.4. *The map $\tilde{p} : \text{Bun}_{\tilde{G}} \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G$ is equivariant with respect to the above $\mathbb{G}_m \times \mathbb{G}_m$ -action on $\text{Bun}_{\tilde{G}}$ and the $\mathbb{G}_m \times \mathbb{G}_m$ -action on $\mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G$ via the \mathbb{A}^1 -factor.*

Proof. This follows from the $\mathbb{G}_m \times \mathbb{G}_m$ -equivariance of the map

$$\tilde{p} : \tilde{G} \rightarrow \mathbb{A}^1 \times G \times G$$

(see Sect. 2.3.6), and the fact that since the $\mathbb{G}_m \times \mathbb{G}_m$ -action on $G \times G$ is inner, the induced action on $\text{Bun}_G \times \text{Bun}_G$ is canonically isomorphic to the trivial one. \square

⁸We warn the reader of the clash of notations: the map $\tilde{p} : \text{Bun}_{\tilde{G}} \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G$ introduced above has nothing to do with the map $\tilde{p} : \widetilde{\text{Bun}}_P \rightarrow \text{Bun}_G$ of Sect. 1.1.4. The symbol \tilde{p} has been chosen in both cases in order to be consistent with both [DrGa3] and [BG]. The two are unlikely to be confused, as the stack $\widetilde{\text{Bun}}_P$ will not appear again in this paper.

Remark 2.5.5. The action on $\text{Bun}_{\tilde{G}}$ of the subgroup

$$(2.15) \quad \{(\lambda_1, \lambda_2) \in \mathbb{G}_m \times \mathbb{G}_m \mid \lambda_1 \cdot \lambda_2 = 1\} \subset \mathbb{G}_m \times \mathbb{G}_m$$

is canonically trivial because its action on \tilde{G} is inner. Moreover, the action of the subgroup (2.15) on the triple $(\text{Bun}_{\tilde{G}}, \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G, \tilde{\rho})$ is canonically trivial.

2.5.6. Let $(\text{Bun}_{\tilde{G}})_t$ denote the fiber of $\text{Bun}_{\tilde{G}}$ over $t \in \mathbb{A}^1$. Let

$$\tilde{\rho}_t : (\text{Bun}_{\tilde{G}})_t \rightarrow \text{Bun}_G \times \text{Bun}_G$$

denote the corresponding map.

By construction, $(\text{Bun}_{\tilde{G}})_1$ identifies with $\text{Bun}_{\tilde{G}_1} = \text{Bun}_G$, and the morphism

$$\tilde{\rho}_1 : (\text{Bun}_{\tilde{G}})_1 \rightarrow \text{Bun}_G \times \text{Bun}_G,$$

identifies with the diagonal

$$\Delta_{\text{Bun}_G} : \text{Bun}_G \rightarrow \text{Bun}_G \times \text{Bun}_G.$$

Similarly, $(\text{Bun}_{\tilde{G}})_0$ identifies with $\text{Bun}_{\tilde{G}_0} \simeq \text{Bun}_{P \times_M P^-}$. We now claim:

Lemma 2.5.7. *The natural map $\text{Bun}_{P \times_M P^-} \rightarrow \text{Bun}_P \times_{\text{Bun}_M} \text{Bun}_{P^-}$ is an isomorphism.*

The above lemma is a particular case of the following one.

Lemma 2.5.8. *Let*

$$H_1 \xrightarrow{f_1} H \xleftarrow{f_2} H_2$$

be a diagram of algebraic groups.

(i) *If $f_1(H_1) \cdot f_2(H_2) = H$ then the natural map $\text{Bun}_{H_1 \times_H H_2} \rightarrow \text{Bun}_{H_1} \times_{\text{Bun}_H} \text{Bun}_{H_2}$ is an isomorphism.*

(ii) *If $f_1(H_1) \cdot f_2(H_2)$ is open in H then the above map is an open embedding.*

Proof. It suffices to show that the morphism of stacks

$$(2.16) \quad \text{pt}/(H_1 \times_H H_2) \rightarrow \text{pt}/H_1 \times_{\text{pt}/H} \text{pt}/H_2$$

is an isomorphism if (i) holds and an open embedding if (ii) holds. To this end, consider the action of $H_1 \times H_2$ on H defined by

$$(h_1, h_2) * h := f_1(h_1) \cdot h \cdot f_2(h_2)^{-1}, \quad h_i \in H_i, h \in H$$

and note that the morphism (2.16) can be obtained from the embedding $f_1(H_1) \cdot f_2(H_2) \hookrightarrow H$ by passing to the quotient with respect to the action of $H_1 \times H_2$. \square

2.6. **The interpolating kernel.**

2.6.1. Let $\text{Bun}_{\tilde{G}}^\mu$ denote the open substack of $\text{Bun}_{\tilde{G}}$ obtained by removing from the special fiber

$$\text{Bun}_{\tilde{G}_0} \simeq \text{Bun}_P \times_{\text{Bun}_M} \text{Bun}_{P^-}$$

the union of the connected components $\text{Bun}_P^{\mu'} \times_{\text{Bun}_M^{\mu'}} \text{Bun}_{P^-}^{\mu'}$ with $\mu' \neq \mu$. By a slight abuse of notation, the restriction of the map (2.14) to $\text{Bun}_{\tilde{G}}^\mu$ will be denoted by the same symbol

$$(2.17) \quad \tilde{\mathfrak{p}} : \text{Bun}_{\tilde{G}}^\mu \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G .$$

By Lemma 2.5.7,

$$(2.18) \quad (\text{Bun}_{\tilde{G}}^\mu)_0 \simeq \text{Bun}_P^\mu \times_{\text{Bun}_M^\mu} \text{Bun}_{P^-}^\mu .$$

On the other hand, the map

$$(2.19) \quad (\text{Bun}_{\tilde{G}}^\mu)_1 \rightarrow (\text{Bun}_{\tilde{G}})_1 \simeq \text{Bun}_G$$

is an isomorphism.

2.6.2. By Proposition 2.5.2(ii), the morphism $\tilde{\mathfrak{p}}$ in (2.17) is quasi-compact, so the functor $\tilde{\mathfrak{p}}_\bullet$ is well-defined.

Remark 2.6.3. By the same Proposition 2.5.2(ii), $\tilde{\mathfrak{p}}$ is representable, and in fact schematic. As was already mentioned in Sect. 2.1.1, this implies that the morphism $\tilde{\mathfrak{p}}_\bullet \rightarrow \tilde{\mathfrak{p}}_*$ is, in fact, an isomorphism.

We define $\mathcal{Q} \in \text{D-mod}(\mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G)$ by

$$\mathcal{Q} := \tilde{\mathfrak{p}}_\bullet(\omega_{\text{Bun}_{\tilde{G}}^\mu}).$$

Let us check that \mathcal{Q} satisfies properties (i)-(ii) from Sect. 2.2.4.

2.6.4. By (2.18) and (2.19) and base change, the objects \mathcal{Q}_0 and \mathcal{Q}_1 from Sect. 2.2.2 identify with the !-restrictions of \mathcal{Q} under the maps

$$\iota_0, \iota_1 : \text{Bun}_G \times \text{Bun}_G \rightarrow \mathbb{A}^1 \times \text{Bun}_G \times \text{Bun}_G$$

corresponding to $0 \in \mathbb{A}^1$ and $1 \in \mathbb{A}^1$, respectively.

This establishes property (i).

2.6.5. By Lemma 2.5.4 and base change, \mathcal{Q} is naturally $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant, i.e., it is naturally a pullback of an object of

$$\text{D-mod}((\mathbb{A}^1/(\mathbb{G}_m \times \mathbb{G}_m)) \times \text{Bun}_G \times \text{Bun}_G),$$

where the $\mathbb{G}_m \times \mathbb{G}_m$ -action on \mathbb{A}^1 is

$$(\lambda_1, \lambda_2) \cdot t = \lambda_1^{-1} \cdot \lambda_2^{-1} \cdot t.$$

In particular, \mathcal{Q} is \mathbb{G}_m -monodromic with respect to the action of \mathbb{G}_m on \mathbb{A}^1 by dilations.

This establishes property (ii).

3. VERIFICATION OF ADJUNCTION

We have the natural transformation (1.4)

$$\mathrm{CT}_{*}^{\mu,-} \circ \mathrm{Eis}_{*}^{\mu} \rightarrow \mathrm{Id}_{\mathrm{D}\text{-mod}(\mathrm{Bun}_{M}^{\mu})}$$

and the natural transformation (2.1)

$$\mathrm{Id}_{\mathrm{D}\text{-mod}(\mathrm{Bun}_{G})} \rightarrow \mathrm{Eis}_{*}^{\mu} \circ \mathrm{CT}_{*}^{\mu,-},$$

where the latter is defined using the map $\mathcal{Q}_1 \rightarrow \mathcal{Q}_0$ of (2.8).

It remains to show that these two natural transformations satisfy the adjunction properties. I.e., the composition

$$(3.1) \quad \mathrm{CT}_{*}^{\mu,-} \rightarrow \mathrm{CT}_{*}^{\mu,-} \circ \mathrm{Eis}_{*}^{\mu} \circ \mathrm{CT}_{*}^{\mu,-} \rightarrow \mathrm{CT}_{*}^{\mu,-}$$

is the identity endomorphism of $\mathrm{CT}_{*}^{\mu,-}$, and

$$(3.2) \quad \mathrm{Eis}_{*}^{\mu} \rightarrow \mathrm{Eis}_{*}^{\mu} \circ \mathrm{CT}_{*}^{\mu,-} \circ \mathrm{Eis}_{*}^{\mu} \rightarrow \mathrm{Eis}_{*}^{\mu}$$

is the identity endomorphism of Eis_{*}^{μ} .

We will do so for the composition (3.1). The case of (3.2) is similar and will be left to the reader.

The computation of the composition (3.1) repeats *verbatim* the corresponding computation in [DrGa3, Sect. 5]. We include it for the sake of completeness.

3.1. The diagram describing the composed functor.

3.1.1. We will use the notation

$$(3.3) \quad \Phi := \mathrm{CT}_{*}^{\mu,-} \circ \mathrm{Eis}_{*}^{\mu} \circ \mathrm{CT}_{*}^{\mu,-} = ((\mathfrak{p}^{-})_{\blacktriangle} \circ (\mathfrak{q}^{-})^{\dagger}) \circ ((\mathfrak{q}^{+})_{\blacktriangle} \circ (\mathfrak{p}^{+})^{\dagger}) \circ ((\mathfrak{p}^{-})_{\blacktriangle} \circ (\mathfrak{q}^{-})^{\dagger}).$$

By base change, Φ is given by pull-push along the following diagram:

$$(3.4) \quad \begin{array}{ccccc} & & \mathrm{Bun}_{\mathrm{Bun}_{G}}^{\mu} \times \mathrm{Bun}_{\mathrm{Bun}_{M}}^{\mu} \times \mathrm{Bun}_{\mathrm{Bun}_{P^{-}}}^{\mu} & & \\ & & \swarrow & & \searrow \\ & & \mathrm{Bun}_{\mathrm{Bun}_{G}}^{\mu} \times \mathrm{Bun}_{P}^{\mu} & & \mathrm{Bun}_{\mathrm{Bun}_{M}}^{\mu} \times \mathrm{Bun}_{P^{-}}^{\mu} \\ & \swarrow & & \swarrow & \searrow \\ & \mathrm{Bun}_{P^{-}}^{\mu} & & \mathrm{Bun}_{P}^{\mu} & & \mathrm{Bun}_{P^{-}}^{\mu} \\ & \swarrow \scriptstyle \mathfrak{q}^{-} & \searrow \scriptstyle \mathfrak{p}^{-} & \swarrow \scriptstyle \mathfrak{p}^{+} & \searrow \scriptstyle \mathfrak{q}^{+} & \swarrow \scriptstyle \mathfrak{q}^{-} & \searrow \scriptstyle \mathfrak{p}^{-} \\ \mathrm{Bun}_{M}^{\mu} & & \mathrm{Bun}_{G} & & \mathrm{Bun}_{M}^{\mu} & & \mathrm{Bun}_{G} \end{array}$$

3.1.2. Recall the stack $\mathrm{Bun}_{\tilde{G}}^\mu$, see Sect. 2.6.1.

Consider the stack

$$(3.5) \quad \mathrm{Bun}_{P^-, \tilde{G}}^\mu := \mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_{\tilde{G}}^\mu,$$

where the fiber product is formed using the composition

$$\mathrm{Bun}_{\tilde{G}}^\mu \xrightarrow{\tilde{p}} \mathbb{A}^1 \times \mathrm{Bun}_G \times \mathrm{Bun}_G \rightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G \xrightarrow{\mathrm{pr}_1} \mathrm{Bun}_G.$$

Let

$$r : \mathrm{Bun}_{P^-, \tilde{G}}^\mu \rightarrow \mathbb{A}^1 \times \mathrm{Bun}_M^\mu \times \mathrm{Bun}_G$$

denote the composition

$$\mathrm{Bun}_{P^-, \tilde{G}}^\mu = \mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_{\tilde{G}}^\mu \rightarrow \mathbb{A}^1 \times \mathrm{Bun}_{P^-}^\mu \times \mathrm{Bun}_G \xrightarrow{\mathrm{id}_{\mathbb{A}^1} \times \mathrm{q}^- \times \mathrm{id}_{\mathrm{Bun}_G}} \mathbb{A}^1 \times \mathrm{Bun}_M^\mu \times \mathrm{Bun}_G,$$

where the first arrow is obtained by base change from

$$\tilde{p} : \mathrm{Bun}_{\tilde{G}}^\mu \rightarrow \mathbb{A}^1 \times \mathrm{Bun}_G \times \mathrm{Bun}_G.$$

Let $(\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_t$ denote the fiber of $\mathrm{Bun}_{P^-, \tilde{G}}^\mu$ over $t \in \mathbb{A}^1$. Let r_t denote the corresponding map $(\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_t \rightarrow \mathrm{Bun}_M^\mu \times \mathrm{Bun}_G$.

3.1.3. The isomorphism (2.18) defines an isomorphism

$$(\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_0 \simeq \mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_P^\mu \times_{\mathrm{Bun}_M^\mu} \mathrm{Bun}_{P^-}^\mu$$

such that the compositions

$$\mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_P^\mu \times_{\mathrm{Bun}_M^\mu} \mathrm{Bun}_{P^-}^\mu \rightarrow \mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_P^\mu \rightarrow \mathrm{Bun}_{P^-}^\mu \rightarrow \mathrm{Bun}_M^\mu$$

and

$$\mathrm{Bun}_{P^-}^\mu \times_{\mathrm{Bun}_G} \mathrm{Bun}_P^\mu \times_{\mathrm{Bun}_M^\mu} \mathrm{Bun}_{P^-}^\mu \rightarrow \mathrm{Bun}_P^\mu \times_{\mathrm{Bun}_M^\mu} \mathrm{Bun}_{P^-}^\mu \rightarrow \mathrm{Bun}_{P^-}^\mu \rightarrow \mathrm{Bun}_G$$

from diagram (3.4) are equal, respectively, to the compositions

$$(\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_0 \xrightarrow{r} \mathrm{Bun}_M^\mu \times \mathrm{Bun}_G \xrightarrow{\mathrm{pr}_1} \mathrm{Bun}_M^\mu$$

and

$$(\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_0 \xrightarrow{r} \mathrm{Bun}_M^\mu \times \mathrm{Bun}_G \xrightarrow{\mathrm{pr}_2} \mathrm{Bun}_G.$$

Hence, the functor Φ is given by pull-push along the diagram

$$(3.6) \quad \begin{array}{ccc} & (\mathrm{Bun}_{P^-, \tilde{G}}^\mu)_0 & \\ \mathrm{pr}_1 \text{ or} \swarrow & & \searrow \mathrm{pr}_2 \text{ or} \\ \mathrm{Bun}_M^\mu & & \mathrm{Bun}_G. \end{array}$$

3.2. **The natural transformations at the level of kernels.**

3.2.1. Denote

$$\mathcal{S} := r_{\blacktriangle}(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}) \in \text{D-mod}(\mathbb{A}^1 \times \text{Bun}_M^\mu \times \text{Bun}_G).$$

As in Sect. 2.6.5 one shows that

$$\mathcal{S} \in \text{D-mod}(\mathbb{A}^1 \times \text{Bun}_M^\mu \times \text{Bun}_G)^{\mathbb{G}_m\text{-mon}}.$$

Set also

$$\mathcal{S}_0 := (r_0)_{\blacktriangle}(\omega_{(\text{Bun}_{P^-, \tilde{G}}^\mu)_0}) \in \text{D-mod}(\text{Bun}_M^\mu \times \text{Bun}_G)$$

and

$$\mathcal{S}_1 := (r_1)_{\blacktriangle}(\omega_{(\text{Bun}_{P^-, \tilde{G}}^\mu)_1}) \in \text{D-mod}(\text{Bun}_M^\mu \times \text{Bun}_G).$$

3.2.2. By Sects. 2.1.5 and 3.1.3, the functor Φ identifies with $F_{\mathcal{S}_0}$. Let us now describe the natural transformations

$$\Phi \rightarrow (\mathfrak{p}^-)_{\blacktriangle} \circ (\mathfrak{q}^-)^! \text{ and } (\mathfrak{p}^-)_{\blacktriangle} \circ (\mathfrak{q}^-)^! \rightarrow \Phi$$

at the level of kernels.

Set

$$\mathcal{T} := (\mathfrak{q}^- \times \mathfrak{p}^-)_{\blacktriangle}(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}).$$

We have

$$(\mathfrak{p}^-)_{\blacktriangle} \circ (\mathfrak{q}^-)^! \simeq F_{\mathcal{T}}.$$

3.2.3. Recall the open embedding

$$j : \text{Bun}_M^\mu \hookrightarrow \text{Bun}_{P^-, \text{Bun}_G}^\mu \times \text{Bun}_{P^-, \text{Bun}_M}^\mu.$$

Let j_{P^-} denote the open embedding

$$\text{Bun}_{P^-, \text{Bun}_G}^\mu \hookrightarrow \text{Bun}_{P^-, \text{Bun}_G}^\mu \times_{\text{Bun}_G} \text{Bun}_{P^-, \text{Bun}_M}^\mu \times_{\text{Bun}_M} \text{Bun}_{P^-, \text{Bun}_G}^\mu \simeq (\text{Bun}_{P^-, \tilde{G}}^\mu)_0,$$

obtained by base change.

We have

$$(\mathfrak{q}^- \times \mathfrak{p}^-) = r_0 \circ j_{P^-}, \quad \text{Bun}_{P^-, \text{Bun}_G}^\mu \rightarrow \text{Bun}_M^\mu \times \text{Bun}_G.$$

3.2.4. By construction, the natural transformation $\Phi \rightarrow (\mathfrak{p}^-)_{\blacktriangle} \circ (\mathfrak{q}^-)^!$ comes from the map

$$(3.7) \quad \mathcal{S}_0 \rightarrow \mathcal{T}$$

equal to

$$(r_0)_{\blacktriangle}(\omega_{(\text{Bun}_{P^-, \tilde{G}}^\mu)_0}) \rightarrow (r_0)_{\blacktriangle} \circ (j_{P^-})_{\blacktriangle}(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}) \simeq (\mathfrak{q}^- \times \mathfrak{p}^-)_{\blacktriangle}(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}),$$

where the first arrow comes from

$$\omega_{(\text{Bun}_{P^-, \tilde{G}}^\mu)_0} \rightarrow (j_{P^-})_* \circ (j_{P^-})^*(\omega_{(\text{Bun}_{P^-, \tilde{G}}^\mu)_0}) \simeq (j_{P^-})_*(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}) \simeq (j_{P^-})_{\blacktriangle}(\omega_{\text{Bun}_{P^-, \tilde{G}}^\mu}).$$

3.2.5. The identification $(\text{Bun}_{\tilde{G}}^\mu)_1 \simeq \text{Bun}_G$ of (2.19) gives rise to an identification

$$(3.8) \quad (\text{Bun}_{P^-, \tilde{G}}^\mu)_1 \simeq \text{Bun}_{P^-, \tilde{G}}^\mu,$$

so that

$$r_1 = (\mathfrak{q}^- \times \mathfrak{p}^-).$$

Hence, we obtain a tautological identification

$$(3.9) \quad \mathcal{T} \simeq \mathcal{S}_1.$$

3.2.6. Now, the map Sp_g of (2.9) defines a canonical map

$$(3.10) \quad \mathcal{S}_1 \rightarrow \mathcal{S}_0.$$

By the functoriality of the construction of the natural transformation Sp (see, e.g., [DrGa3, Sect. 4.1.5]), the natural transformation $(\mathfrak{p}^-)_{\blacktriangle} \circ (\mathfrak{q}^-)^{\dagger} \rightarrow \Phi$ comes from the map

$$(3.11) \quad \mathcal{T} \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_0,$$

equal to the composition of (3.9) and (3.10).

3.2.7. We obtain that in order to prove that the composition (3.1) is the identity map, it suffices to show that the composed map

$$(3.12) \quad \mathcal{T} \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_0 \rightarrow \mathcal{T}$$

is the identity map on \mathcal{T} .

3.3. Passing to an open substack.

3.3.1. Recall the open embedding

$$j_{P^-} : \mathrm{Bun}_{P^-}^{\mu} \hookrightarrow (\mathrm{Bun}_{P^-, \tilde{G}}^{\mu})_0,$$

introduced in Sect. 3.2.3.

Let

$$\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu} \subset \mathrm{Bun}_{P^-, \tilde{G}}^{\mu}$$

be the open substack obtained by removing the closed substack

$$\left((\mathrm{Bun}_{P^-, \tilde{G}}^{\mu})_0 - j_{P^-}(\mathrm{Bun}_{P^-}^{\mu}) \right) \subset (\mathrm{Bun}_{P^-, \tilde{G}}^{\mu})_0 \subset \mathrm{Bun}_{P^-, \tilde{G}}^{\mu}.$$

Let $(\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu})_t$ denote the fiber of $\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu}$ over $t \in \mathbb{A}^1$.

Note that the isomorphism $\mathrm{Bun}_{P^-}^{\mu} \rightarrow (\mathrm{Bun}_{P^-, \tilde{G}}^{\mu})_1$ of (3.8) defines an isomorphism

$$(3.13) \quad \mathrm{Bun}_{P^-}^{\mu} \rightarrow (\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu})_1.$$

By definition, the open embedding $j_{P^-} : \mathrm{Bun}_{P^-}^{\mu} \hookrightarrow (\mathrm{Bun}_{P^-, \tilde{G}}^{\mu})_0$ defines an *isomorphism*

$$(3.14) \quad \mathrm{Bun}_{P^-}^{\mu} \rightarrow (\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu})_0.$$

3.3.2. Let

$$\mathring{r} : \mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu} \rightarrow \mathbb{A}^1 \times \mathrm{Bun}_M^{\mu} \times \mathrm{Bun}_G \quad \text{and} \quad \mathring{r}_t : \mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu} \rightarrow \mathrm{Bun}_M^{\mu} \times \mathrm{Bun}_G$$

denote the corresponding maps.

Set

$$\mathring{\mathcal{S}} := \mathring{r}_{\blacktriangle}(\omega_{\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu}}),$$

and also

$$\mathring{\mathcal{S}}_0 := (\mathring{r}_0)_{\blacktriangle}(\omega_{(\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu})_0}) \quad \text{and} \quad \mathring{\mathcal{S}}_1 := (\mathring{r}_1)_{\blacktriangle}(\omega_{(\mathring{\mathrm{Bun}}_{P^-, \tilde{G}}^{\mu})_1}).$$

The open embedding $\mathring{\text{Bun}}_{P^-, \tilde{G}}^\mu \hookrightarrow \text{Bun}_{P^-, \tilde{G}}^\mu$ gives rise to the maps

$$\mathcal{S} \rightarrow \mathring{\mathcal{S}}, \quad \mathcal{S}_0 \rightarrow \mathring{\mathcal{S}}_0, \quad \mathcal{S}_1 \rightarrow \mathring{\mathcal{S}}_1.$$

As in Sect. 3.2, we have the natural transformations

$$(3.15) \quad \mathcal{T} \rightarrow \mathring{\mathcal{S}}_1 \rightarrow \mathring{\mathcal{S}}_0 \rightarrow \mathcal{T}.$$

Moreover, the diagram

$$\begin{array}{ccccccc} \mathcal{T} & \longrightarrow & \mathcal{S}_1 & \longrightarrow & \mathcal{S}_0 & \longrightarrow & \mathcal{T} \\ \text{id} \downarrow & & \downarrow & & \downarrow & & \downarrow \text{id} \\ \mathcal{T} & \longrightarrow & \mathring{\mathcal{S}}_1 & \longrightarrow & \mathring{\mathcal{S}}_0 & \longrightarrow & \mathcal{T} \end{array}$$

commutes.

Hence, we obtain that in order to show that the composed map (3.12) is the identity map, it suffices to show that the composed map (3.15) is the identity map.

3.4. Digression: description of fiber products.

3.4.1. Let Z be a quasi-compact scheme equipped with a \mathbb{G}_m -action. Consider the fiber product $Z^- \times_Z \tilde{Z}$, formed using the composition

$$\tilde{Z} \xrightarrow{\tilde{p}} \mathbb{A}^1 \times Z \times Z \rightarrow Z \times Z \xrightarrow{\text{pr}_1} Z.$$

In [DrGa3, Sect. 2.6] we define a canonical morphism

$$(3.16) \quad \mathbb{A}^1 \times Z^- \rightarrow Z^- \times_Z \tilde{Z}$$

of schemes over \mathbb{A}^1 .

3.4.2. If Z is affine (the case of interest for us) the definition from [DrGa3] can be reformulated as follows. Recall that if Z is affine then Z^- is a closed subscheme of Z and \tilde{Z} is a closed subscheme of $\mathbb{A}^1 \times Z \times Z$, so $Z^- \times_Z \tilde{Z}$ is a closed subscheme of $\mathbb{A}^1 \times Z \times Z$.

Now define the map

$$\mathbb{A}^1 \times Z^- \rightarrow Z^- \times_Z \tilde{Z} \subset \mathbb{A}^1 \times Z \times Z$$

by

$$(3.17) \quad (t, z) \mapsto (t, t^{-1} \cdot z, z) \in \mathbb{A}^1 \times Z \times Z.$$

It is easy to see that the image of the map (3.17) indeed belongs to $Z^- \times_Z \tilde{Z}$. Indeed, it suffices to consider the case $t \neq 0$, when this is obvious.

3.4.3. By [DrGa3, Proposition 2.6.3 and Remark 2.6.4], for any Z the morphism (3.16) is an open embedding, and if Z is affine it is an isomorphism.

The latter is very easy to check if Z is affine and smooth: in this case the map (3.16) is a morphism between *smooth* schemes over \mathbb{A}^1 , and it suffices to check that it induces an isomorphism between the fibers over any k -point of \mathbb{A}^1 .

3.4.4. Consider now the situation of Sect. 2.4.1, i.e., $Z = G$ and the \mathbb{G}_m -action on G is the adjoint action corresponding to a co-character $\gamma : \mathbb{G}_m \rightarrow M$ such that $\gamma(\mathbb{G}_m)$ is contained in the center of M and γ is dominant and regular with respect to P .

Then the map (3.16) is a homomorphism

$$(3.18) \quad \mathbb{A}^1 \times P^- \rightarrow P^- \times_G \tilde{G}$$

of group-schemes over \mathbb{A}^1 . Its composition with the embedding $P^- \times_G \tilde{G} \hookrightarrow \mathbb{A}^1 \times G \times G$ is given by

$$(t, g) \mapsto (t, \gamma(t)^{-1} \cdot g \cdot \gamma(t), g).$$

3.4.5. The homomorphism (3.18) induces a maps of moduli of bundles

$$(3.19) \quad \mathbb{A}^1 \times \text{Bun}_{P^-} \rightarrow \text{Bun}_{P^-} \times_{\text{Bun}_G} \text{Bun}_{\tilde{G}}.$$

Lemma 3.4.6. *The map (3.19) is an open embedding.*

This lemma is a counterpart of [DrGa3, Proposition 2.6.3].

Proof. Note that Lemma 2.3.9 remains valid for stacks. So it suffices to show that the map (3.19) induces an open embedding of fibers over any $t \in \mathbb{A}^1$. This is clear if $t \neq 0$, so it remains to consider the case of $t = 0$.

The morphism in question is the map

$$\begin{aligned} \text{Bun}_{P^-} &\rightarrow \text{Bun}_{P^-} \times_{\text{Bun}_G} \text{Bun}_{P \times_M P^-} \simeq \text{Bun}_{P^-} \times_{\text{Bun}_G} (\text{Bun}_P \times_{\text{Bun}_M} \text{Bun}_{P^-}) = \\ &= (\text{Bun}_{P^-} \times_{\text{Bun}_G} \text{Bun}_P) \times_{\text{Bun}_M} \text{Bun}_{P^-} \end{aligned}$$

which equals the map

$$\text{Bun}_{P^-} \simeq \text{Bun}_M \times_{\text{Bun}_M} \text{Bun}_{P^-} \xrightarrow{j \times \text{id}} (\text{Bun}_{P^-} \times_{\text{Bun}_G} \text{Bun}_P) \times_{\text{Bun}_M} \text{Bun}_{P^-}.$$

It is an open embedding because so is the morphism $j : \text{Bun}_M \rightarrow \text{Bun}_{P^-} \times_{\text{Bun}_G} \text{Bun}_P$. \square

Remark 3.4.7. One can also prove Lemma 3.4.6 using a variant of Lemma 2.5.8(ii). In this variant H_1, H_2 , and H are flat group-schemes over some scheme S (e.g., $S = \mathbb{A}^1$) and instead of openness of $f_1(H_1) \cdot f_2(H_2)$ one requires flatness of the morphism $H_1 \times_S H_2 \rightarrow H$ given by $(h_1, h_2) \mapsto f_1(h_1) \cdot f_2(h_2)$.

3.5. The key argument.

3.5.1. By construction, the open embedding (3.19) defines an *isomorphism*

$$\mathbb{A}^1 \times \text{Bun}_{P^-}^\mu \rightarrow \text{Bun}_{P^-, \tilde{G}}^\mu$$

with the following properties:

- The map

$$\mathring{r} : \text{Bun}_{P^-, \tilde{G}}^\mu \rightarrow \mathbb{A}^1 \times \text{Bun}_M^\mu \times \text{Bun}_G$$

identifies with the map

$$\text{id}_{\mathbb{A}^1} \times (\mathfrak{q}^- \times \mathfrak{p}^-).$$

- The isomorphism $\mathrm{Bun}_{P^-}^\mu \rightarrow (\mathrm{Bun}_{P^-, \bar{G}}^\mu)_1$ of (3.13) corresponds to the identity map

$$\mathrm{Bun}_{P^-}^\mu \simeq (\mathbb{A}^1 \times \mathrm{Bun}_{P^-}^\mu) \times_{\mathbb{A}^1} \{1\} \simeq \mathrm{Bun}_{P^-}^\mu .$$

- The isomorphism $\mathrm{Bun}_{P^-}^\mu \rightarrow (\mathrm{Bun}_{P^-, \bar{G}}^\mu)_0$ of (3.14) corresponds to the identity map

$$\mathrm{Bun}_{P^-}^\mu \simeq (\mathbb{A}^1 \times \mathrm{Bun}_{P^-}^\mu) \times_{\mathbb{A}^1} \{0\} \simeq \mathrm{Bun}_{P^-}^\mu .$$

3.5.2. Hence, we obtain that the composition (3.15) identifies with

$$\mathcal{T} \simeq \iota_1^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}) \rightarrow \iota_0^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}) \simeq \mathcal{T},$$

where

$$\iota_1^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}) \rightarrow \iota_0^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T})$$

is the specialization map (2.9) for the object

$$\omega_{\mathbb{A}^1} \boxtimes \mathcal{T} \in \mathrm{D}\text{-mod}(\mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}).$$

However, the latter is readily seen to be the identity map.

4. AN ALTERNATIVE PROOF

In this section we will give an alternative proof of Theorem 1.2.5 by directly deducing it from Braden's theorem ([Br]). It has the advantage of being more elementary than the proof of Theorem 1.2.5 given in Sect. 2, if one accepts Braden's theorem as a black box.

However, the two proofs are closely related, because Braden's theorem itself can be proved by an argument similar to one used in the proof of Theorem 1.2.5 from Sect. 2.

4.1. Contraction principle.

4.1.1. Let \mathcal{Y} be a stack equipped with an action of \mathbb{G}_m . We let

$$\mathrm{D}\text{-mod}(\mathcal{Y})^{\mathbb{G}_m, \mathrm{mon}} \subset \mathrm{D}\text{-mod}(\mathcal{Y})$$

denote the corresponding monodromic subcategory, i.e., the full subcategory generated by the essential image of the pullback functor $\mathrm{D}\text{-mod}(\mathcal{Y}/\mathbb{G}_m) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y})$.

Note that if the \mathbb{G}_m -action is trivial then $\mathrm{D}\text{-mod}(\mathcal{Y})^{\mathbb{G}_m, \mathrm{mon}} = \mathrm{D}\text{-mod}(\mathcal{Y})$ (because the morphism $\mathcal{Y} \rightarrow \mathcal{Y}/\mathbb{G}_m$ admits a section).

4.1.2. Let \mathcal{Y} be an algebraic stack equipped with an action of \mathbb{A}^1 , where \mathbb{A}^1 is viewed as a monoid with respect to multiplication. (A concrete example of this situation will be considered in Sect. 4.1.6 below).

Define the stack \mathcal{Y}^0 by

$$\mathrm{Maps}(S, \mathcal{Y}^0) := \mathrm{Maps}^{\mathbb{A}^1}(S, \mathcal{Y}), \quad S \in \mathrm{Sch}^{\mathrm{aff}}.$$

The groupoid $\mathrm{Maps}(S, \mathcal{Y}^0)$ admits also the following description (which was used in [DrGa2, Sect. C.5] as a definition): objects of $\mathrm{Maps}(S, \mathcal{Y}^0)$ are pairs (y, α) , where $y \in \mathrm{Maps}(S, \mathcal{Y})$ and α is an isomorphism $0 \cdot y \xrightarrow{\sim} y$ such that the two isomorphisms

$$0 \cdot (0 \cdot y) = (0 \cdot 0) \cdot y = 0 \cdot y \xrightarrow{\sim \alpha} y \quad \text{and} \quad 0 \cdot (0 \cdot y) \xrightarrow{\sim 0 \cdot \alpha} 0 \cdot y \xrightarrow{\sim \alpha} y$$

$0 \cdot (0 \cdot y) \rightrightarrows y$ are equal to each other. Morphisms in $\mathrm{Maps}(S, \mathcal{Y}^0)$ are defined as morphisms between the y 's that intertwine the data of α .

From the latter description, it is easy to see that \mathcal{Y}^0 is again a (locally QCA) stack.

4.1.3. The forgetful map and the action of $0 \in \mathbb{A}^1$ define a retraction

$$(4.1) \quad \mathcal{Y}^0 \xrightarrow{\iota} \mathcal{Y} \xrightarrow{\mathfrak{q}} \mathcal{Y}^0.$$

Under these circumstances we will say that we are given a *contraction* of \mathcal{Y} onto $\mathcal{Y}^0 \xrightarrow{\iota} \mathcal{Y}$.

Since $\mathfrak{q} \circ \iota = \text{id}_{\mathcal{Y}^0}$, it follows that ι is quasi-compact and representable.

4.1.4. From now on we will assume that the morphism \mathfrak{q} is quasi-compact. In this case the functor

$$\mathfrak{q}_\bullet : \text{D-mod}(\mathcal{Y}) \rightarrow \text{D-mod}(\mathcal{Y}^0)$$

is defined.

We have the following assertion:

Proposition 4.1.5.

(1) *The partially defined left adjoint $i^* : \text{D-mod}(\mathcal{Y}) \rightarrow \text{D-mod}(\mathcal{Y}^0)$ of $i_* \simeq i_\bullet$ is defined on $\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}$, and we have a canonical isomorphism*

$$i^*|_{\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}} \simeq q_\bullet|_{\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}}.$$

More precisely, for each $\mathcal{F} \in \text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}$ the natural map

$$q_\bullet(\mathcal{F}) \rightarrow q_\bullet \circ i_* \circ i^*(\mathcal{F}) \simeq q_\bullet \circ i_\bullet \circ i^*(\mathcal{F}) = (q \circ i)_\bullet \circ i^*(\mathcal{F}) = i^*(\mathcal{F})$$

is an isomorphism.

(2) *The partially defined left adjoint $q! : \text{D-mod}(\mathcal{Y}) \rightarrow \text{D-mod}(\mathcal{Y}^0)$ of $q^!$ is defined on $\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}$, and we have a canonical isomorphism*

$$q!|_{\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}} \simeq i^!|_{\text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}}.$$

More precisely, for each $\mathcal{F} \in \text{D-mod}(\mathcal{Y})^{\mathbb{G}_m, \text{mon}}$ the natural map

$$i^!(\mathcal{F}) \rightarrow i^! \circ q^! \circ q!(\mathcal{F}) = (q \circ i)^! \circ q!(\mathcal{F}) = q!(\mathcal{F})$$

is an isomorphism.

For the proof see [DrGa2, Theorem C.5.3].

4.1.6. *A contraction of Bun_P onto Bun_M .*

Let us take $\mathcal{Y} = \text{Bun}_P$. Let $\gamma : \mathbb{G}_m \rightarrow M$ be a co-character as in (2.13). The resulting adjoint action of \mathbb{G}_m on P extends to an action of \mathbb{A}^1 on P such that $0 \in \mathbb{A}^1$ acts as

$$P \rightarrow M = P \cap P^- \hookrightarrow P.$$

The action of \mathbb{A}^1 on P induces an action of \mathbb{A}^1 on Bun_P .

It is easy to check that the resulting diagram (4.1) identifies with

$$\text{Bun}_M \xrightarrow{\iota} \text{Bun}_P \xrightarrow{\mathfrak{q}} \text{Bun}_M,$$

where the maps ι and \mathfrak{q} come from the above homomorphisms $M \hookrightarrow P$ and $P \rightarrow M$.

Note that the \mathbb{G}_m -action on Bun_P , corresponding to the above \mathbb{A}^1 -action, is canonically isomorphic to the trivial one⁹ (because the \mathbb{G}_m -action on P comes from a homomorphism $\mathbb{G}_m \rightarrow P$ and the conjugation action of P on itself).

⁹This trivialization is *not* compatible with the projection $\mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$, so the \mathbb{G}_m -action on the *fibers* of the morphism $\text{Bun}_P \rightarrow \text{Bun}_M$ is non-trivial!

Therefore, the inclusion

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_P)^{\mathbb{G}_m, \mathrm{mon}} \subset \mathrm{D}\text{-mod}(\mathrm{Bun}_P)$$

is an equality. So by Proposition 4.1.5, the functors

$$\mathfrak{q}_! : \mathrm{D}\text{-mod}(\mathrm{Bun}_P) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_M) \text{ and } \iota^* : \mathrm{D}\text{-mod}(\mathrm{Bun}_P) \rightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_M)$$

left adjoint to $\mathfrak{q}^!$ and $\iota_* \simeq \iota_{\blacktriangle}$ are defined and are canonically isomorphic to $\iota^!$ and $\mathfrak{q}_{\blacktriangle}$, respectively.

Note also that in this example, $\mathfrak{q}_{\blacktriangle} \simeq \mathfrak{q}_*$, because the morphism \mathfrak{q} is safe.

The same is, of course, true for P replaced by P^- , when we replace the above action of \mathbb{G}_m by its inverse.

4.2. Hyperbolic restrictions and Braden's theorem. In this subsection we will recall the statement of Braden's theorem in the set-up and notations of [DrGa3, Sect. 3.1].

4.2.1. The material in this subsection uses the notion of pro-completion $\mathrm{Pro}(\mathbf{C})$ of a DG category \mathbf{C} ; we refer the reader to [DrGa3, Appendix A], where the corresponding definitions are given.

Let $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ be a quasi-compact and safe morphism between stacks. Consider the functors

$$f^! : \mathrm{D}\text{-mod}(\mathcal{Y}_2) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y}_1) \text{ and } f_* \simeq f_{\blacktriangle} : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y}_2),$$

We will regard their partially defined left adjoints as functors

$$f_! : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}_2)) \text{ and } f^* : \mathrm{D}\text{-mod}(\mathcal{Y}_2) \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}_1)).$$

4.2.2. Let $Z, Z^0, Z^{\pm}, p^{\pm}, i^{\pm}$ be as in Sect. 2.3.1.

Consider the corresponding commutative (but not necessarily Cartesian) diagram

$$(4.2) \quad \begin{array}{ccc} Z^0 & \xrightarrow{i^+} & Z^+ \\ i^- \downarrow & & \downarrow p^+ \\ Z^- & \xrightarrow{p^-} & Z. \end{array}$$

We enlarge it to the diagram

$$(4.3) \quad \begin{array}{ccccc} & & Z^0 & & \\ & & \searrow^{i^+} & & \\ & & & & Z^+ \\ & j \searrow & & \xrightarrow{p^-} & \\ & & Z^+ \times_{Z^-} & & \\ & & \downarrow p^+ & & \downarrow p^+ \\ & & Z^- & \xrightarrow{p^-} & Z, \end{array}$$

in which the square is Cartesian, and where (according to [DrGa3, Proposition 1.9.4]) the map

$$j : Z^0 \rightarrow Z^+ \times_{Z^-}$$

is an open embedding.

We consider the categories

$$\mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}}, \mathrm{D}\text{-mod}(Z^+)^{\mathbb{G}_m\text{-mon}}, \mathrm{D}\text{-mod}(Z^-)^{\mathbb{G}_m\text{-mon}}$$

and

$$\mathrm{D}\text{-mod}(Z^0)^{\mathbb{G}_m\text{-mon}} = \mathrm{D}\text{-mod}(Z^0).$$

Consider the functors

$$(p^+)!: \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{D}\text{-mod}(Z^+)^{\mathbb{G}_m\text{-mon}} \text{ and } (i^-)!: \mathrm{D}\text{-mod}(Z^-)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{D}\text{-mod}(Z^0).$$

Consider also the functors

$$(p^-)^*: \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(Z^-)^{\mathbb{G}_m\text{-mon}})$$

and

$$(i^+)^*: \mathrm{D}\text{-mod}(Z^+)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(Z^0)),$$

left adjoint in the sense of [DrGa3, Sect. B.3] to

$$(p^-)_*: \mathrm{D}\text{-mod}(Z^-)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}}$$

and

$$(i^+)_*: \mathrm{D}\text{-mod}(Z^0) \rightarrow \mathrm{D}\text{-mod}(Z^+)^{\mathbb{G}_m\text{-mon}},$$

respectively.

4.2.3. Consider the composed functors

$$(i^+)^* \circ (p^+)! \text{ and } (i^-)! \circ (p^-)^*, \quad \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(Z^0)).$$

They are called the functors of *hyperbolic restriction*.

We note that there is a canonical natural transformation

$$(4.4) \quad (i^+)^* \circ (p^+)! \rightarrow (i^-)! \circ (p^-)^*.$$

Namely, the natural transformation (4.4) is obtained via the $((i^+)^*, (i^+)_*)$ -adjunction from the natural transformation

$$(p^+)! \rightarrow (i^+)_* \circ (i^-)! \circ (p^-)^*,$$

defined in terms of (4.3) as follows:

$$\begin{aligned} (p^+)! &\rightarrow (p^+)! \circ (p^-)_* \circ (p^-)^* \simeq ({}'p^-)_* \circ ({}'p^+)! \circ (p^-)^* \rightarrow \\ &\rightarrow ({}'p^-)_* \circ j_* \circ j^! \circ ({}'p^+)! \circ (p^-)^* \simeq (i^+)_* \circ (i^-)! \circ (p^-)^*, \end{aligned}$$

where the map

$$\mathrm{Id} \rightarrow j_* \circ j^!$$

comes from the $(j^!, j_*)$ -adjunction using the fact that j is an open embedding, and where $(p^+)! \circ (p^-)_* \simeq ({}'p^-)_* \circ ({}'p^+)!$ is the base change isomorphism.

4.2.4. Braden's theorem of [Br] (as stated in [DrGa3, Theroem 3.2.5]) reads:

Theorem 4.2.5. *The functors*

$$(i^+)^* \circ (p^+)! \text{ and } (i^-)! \circ (p^-)^*, \quad \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(Z^0))$$

take values in $\mathrm{D}\text{-mod}(Z^0) \subset \mathrm{Pro}(\mathrm{D}\text{-mod}(Z^0))$ and the map (4.4) is an isomorphism.

4.3. **The setting for stacks.**

are smooth, and ψ^0 surjective. Assume also that the morphism $\psi : Z \rightarrow \mathcal{Y}$ can be given a \mathbb{G}_m -equivariant structure (with respect to the trivial action of \mathbb{G}_m on \mathcal{Y}).

We claim:

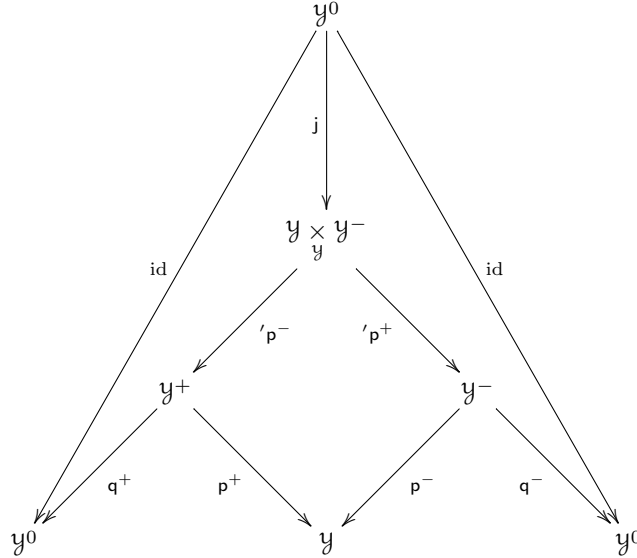
Theorem 4.3.4. *Under the above circumstances the diagram (4.5) is hyperbolic.*

The proof is given in Appendix A.

4.4. Hyperbolicity and adjunction.

4.4.1. Assume that in the situation of diagram (4.5), the morphisms ι^- and ι^+ in (4.5) admit left inverses, denoted \mathfrak{q}^- and \mathfrak{q}^+ , respectively. Assume also that the morphisms \mathfrak{q}^\pm are quasi-compact.

The diagram



gives rise to a natural transformation

$$(4.8) \quad (\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)^\dagger \circ (\mathfrak{p}^+)_* \circ (\mathfrak{q}^+)^\dagger \simeq (\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)^\dagger \circ (\mathfrak{p}^+)_\blacktriangle \circ (\mathfrak{q}^+)^\dagger \rightarrow \mathrm{Id}_{\mathrm{D}\text{-mod}(\mathcal{Y}^0)},$$

which by adjunction gives rise to a natural transformation

$$(4.9) \quad (\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)^\dagger \rightarrow (\mathfrak{q}^+)^\dagger \circ (\mathfrak{p}^+)^*, \quad \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0)).$$

4.4.2. Assume now that the natural transformation

$$(\mathfrak{q}^-)_\blacktriangle \circ (\iota^-)_* \simeq (\mathfrak{q}^-)_\blacktriangle \circ (\iota^-)_\blacktriangle \xrightarrow{\sim} \mathrm{Id}_{\mathrm{D}\text{-mod}(\mathcal{Y}^0)}$$

identifies the functor $(\mathfrak{q}^-)_\blacktriangle$ with the left adjoint $(\iota^-)^*$ of $(\iota^-)_*$.

Assume also that the natural transformation

$$(\iota^+)^\dagger \circ (\mathfrak{q}^+)^\dagger \xrightarrow{\sim} \mathrm{Id}_{\mathrm{D}\text{-mod}(\mathcal{Y}^0)}$$

identifies the functor $(\iota^+)^\dagger$ with the left adjoint $(\mathfrak{q}^+)^\dagger$ of $(\mathfrak{q}^+)^\dagger$.

In particular, both of the above left adjoints take values in

$$\mathrm{D}\text{-mod}(\mathcal{Y}^0) \subset \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0)).$$

(For example, such behavior of the morphisms (q^-, ι^-) and (q^+, ι^+) occurs in the situation described in Sect. 4.1.2.)

In this case diagram chase shows that the following diagram of natural transformations commutes:

$$\begin{array}{ccc} (\iota^-)^* \circ (\mathfrak{p}^-)! & \xrightarrow{(4.6)} & (\iota^+)^! \circ (\mathfrak{p}^+)^* \\ \sim \downarrow & & \downarrow \sim \\ (\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)! & \xrightarrow{(4.9)} & (\mathfrak{q}^+)_! \circ (\mathfrak{p}^+)^*. \end{array}$$

From here we obtain:

Lemma 4.4.3. *The diagram (4.5) is hyperbolic if and only if the natural transformation*

$$((\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)^!) \circ ((\mathfrak{p}^+)_* \circ (\mathfrak{q}^+)^!) \rightarrow \text{Id}_{\text{D-mod}(y^0)}$$

of (4.8) defines the co-unit of an adjunction, making the functor $(\mathfrak{q}^-)_\blacktriangle \circ (\mathfrak{p}^-)^!$ into a left adjoint of $(\mathfrak{p}^+)_* \circ (\mathfrak{q}^+)^!$.

4.4.4. Note that Lemma 4.4.3, combined with Proposition 4.1.5, imply that the assertion of Theorem 1.2.5 (with the co-unit specified in Sect. 1.3.1) is equivalent to the following:

Corollary 4.4.5. *The diagram*

$$\begin{array}{ccc} \text{Bun}_M^\mu & \xrightarrow{\iota} & \text{Bun}_P^\mu \\ \iota^- \downarrow & & \downarrow \mathfrak{p} \\ \text{Bun}_{P^-}^\mu & \xrightarrow{\mathfrak{p}^-} & \text{Bun}_G \end{array}$$

is hyperbolic.

4.5. Proof of Theorem 1.2.5: reduction to the quasi-compact case.

4.5.1. We need to show that for $\mathcal{F}_G \in \text{D-mod}(\text{Bun}_G)$ and $\mathcal{F}_M \in \text{D-mod}(\text{Bun}_M^\mu)$, the map

$$(4.10) \quad \mathcal{H}om_{\text{D-mod}(\text{Bun}_G)}(\mathcal{F}_G, \text{Eis}_*^\mu(\mathcal{F}_M)) \rightarrow \mathcal{H}om_{\text{D-mod}(\text{Bun}_M^\mu)}(\text{CT}_*^{\mu,-}(\mathcal{F}_G), \mathcal{F}_M),$$

induced by (1.4), is an isomorphism.

In this subsection we will show that it is sufficient to prove that (4.10) is an isomorphism for any \mathcal{F}_M of the form $(j_M)_*(\mathcal{F}'_M)$, where

$$U_M \xrightarrow{j_M} \text{Bun}_M^\mu$$

is an open quasi-compact substack and $\mathcal{F}'_M \in \text{D-mod}(U_M)$.

4.5.2. Let open-qc_G denote the poset of open quasi-compact substacks of Bun_G , and let open-qc_M be the corresponding poset for Bun_M^μ .

For every

$$(U_M \xrightarrow{j_M} \text{Bun}_M^\mu) \in \text{open-qc}_M$$

we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{F}_G, \text{Eis}_*^\mu(\mathcal{F}_M)) & \longrightarrow & \mathcal{H}om(\mathcal{F}_G, \text{Eis}_*^\mu \circ (j_M)_* \circ (j_M)^*(\mathcal{F}_M)) \\ \downarrow & & \downarrow \\ \mathcal{H}om(\text{CT}_*^{\mu,-}(\mathcal{F}_G), \mathcal{F}_M) & \longrightarrow & \mathcal{H}om(\text{CT}_*^{\mu,-}(\mathcal{F}_G), (j_M)_* \circ (j_M)^*(\mathcal{F}_M)). \end{array}$$

Hence, it suffices to show that the maps

$$(4.11) \quad \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_M)}(\text{CT}_*^{\mu,-}(\mathcal{F}_G), \mathcal{F}_M) \rightarrow \\ \rightarrow \lim_{\leftarrow U_M \in \text{open-qc}_M} \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_M)}(\text{CT}_*^{\mu,-}(\mathcal{F}_G), (J_M)_* \circ (J_M)^*(\mathcal{F}_M))$$

and

$$(4.12) \quad \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_G)}(\mathcal{F}_G, \text{Eis}_*^\mu(\mathcal{F}_M)) \rightarrow \\ \rightarrow \lim_{\leftarrow U_M \in \text{open-qc}_M} \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_G)}(\mathcal{F}_G, \text{Eis}_*^\mu \circ (J_M)_* \circ (J_M)^*(\mathcal{F}_M))$$

are isomorphisms.

4.5.3. The fact that (4.11) is an isomorphism is evident from the definition of $\mathbf{D}\text{-mod}(\text{Bun}_M)$ as

$$\lim_{\leftarrow U_M \in \text{open-qc}_M} \mathbf{D}\text{-mod}(U_M),$$

since

$$\mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_M)}(\text{CT}_*^{\mu,-}(\mathcal{F}_G), (J_M)_* \circ (J_M)^*(\mathcal{F}_M)) \simeq \\ \simeq \mathcal{H}om_{\mathbf{D}\text{-mod}(U_M)}((J_M)^* \circ \text{CT}_*^{\mu,-}(\mathcal{F}_G), (J_M)^*(\mathcal{F}_M)).$$

Similarly, the map

$$(4.13) \quad \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_G)}(\mathcal{F}_G, \text{Eis}_*^\mu(\mathcal{F}_M)) \rightarrow \\ \rightarrow \lim_{\leftarrow U_G \in \text{open-qc}_G} \mathcal{H}om_{\mathbf{D}\text{-mod}(\text{Bun}_G)}(\mathcal{F}_G, (J_G)_* \circ (J_G)^* \circ \text{Eis}_*^\mu(\mathcal{F}_M))$$

is an isomorphism.

4.5.4. Note now that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{F}_G, \text{Eis}_*^\mu(\mathcal{F}_M)) & \xrightarrow{(4.12)} & \lim_{\leftarrow U_M \in \text{open-qc}_M} \mathcal{H}om(\mathcal{F}_G, \text{Eis}_*^\mu \circ (J_M)_* \circ (J_M)^*(\mathcal{F}_M)) \\ & \searrow (4.13) & \downarrow (4.13) \\ & \lim_{\leftarrow U_G \in \text{open-qc}_G} \mathcal{H}om(\mathcal{F}_G, (J_G)_* \circ (J_G)^* \circ \text{Eis}_*^\mu(\mathcal{F}_M)) & \\ & \searrow & \downarrow \\ & \lim_{\leftarrow U_G, U_M} \mathcal{H}om(\mathcal{F}_G, (J_G)_* \circ (J_G)^* \circ \text{Eis}_*^\mu \circ (J_M)_* \circ (J_M)^*(\mathcal{F}_M)) & \end{array}$$

It remains to show that the map

$$\begin{aligned} \lim_{\leftarrow U_G \in \text{open-qc}_G} \mathcal{H}om(\mathcal{F}_G, (j_G)_* \circ (j_G)^* \circ \text{Eis}_*^\mu(\mathcal{F}_M)) &\rightarrow \\ &\rightarrow \lim_{\leftarrow U_G, U_M} \mathcal{H}om(\mathcal{F}_G, (j_G)_* \circ (j_G)^* \circ \text{Eis}_*^\mu \circ (j_M)_* \circ (j_M)^*(\mathcal{F}_M)) \end{aligned}$$

is an isomorphism.

However, we claim that for any fixed U_G , the corresponding map

$$\begin{aligned} \mathcal{H}om(\mathcal{F}_G, (j_G)_* \circ (j_G)^* \circ \text{Eis}_*^\mu(\mathcal{F}_M)) &\rightarrow \\ &\rightarrow \lim_{\leftarrow U_M \in \text{open-qc}_M} \mathcal{H}om(\mathcal{F}_G, (j_G)_* \circ (j_G)^* \circ \text{Eis}_*^\mu \circ (j_M)_* \circ (j_M)^*(\mathcal{F}_M)) \end{aligned}$$

is already an isomorphism. In fact, the map

$$(j_G)^* \circ \text{Eis}_*^\mu(\mathcal{F}_M) \rightarrow (j_G)^* \circ \text{Eis}_*^\mu \circ (j_M)_* \circ (j_M)^*(\mathcal{F}_M)$$

is an isomorphism, whenever U_M is such that

$$\mathfrak{q}(\mathfrak{p}^{-1}(U_G) \subset U_M,$$

and such U_M are cofinal in open-qc_M .

4.6. Proof of Theorem 1.2.5: reduction to the question of hyperbolicity.

4.6.1. According to Sect. 4.5, it suffices to show that for an open quasi-compact substack

$$U_M \xrightarrow{j_M} \text{Bun}_M^\mu,$$

the natural transformation

$$j_M^* \circ \text{CT}_*^{\mu, -} \circ \text{Eis}_*^\mu \circ (j_M)_* \xrightarrow{(1.4)} j_M^* \circ (j_M)_* \rightarrow \text{Id}_{\text{D-mod}(U_M)}$$

defines the co-unit of an adjunction.

4.6.2. Let $U_G \subset \text{Bun}_G$ be a quasi-compact open substack such that

$$\mathfrak{p}(\mathfrak{q}^{-1}(U_M)) \subset U_G \text{ and } \mathfrak{p}^-((\mathfrak{q}^-)^{-1}(U_M)) \subset U_G.$$

Consider the corresponding diagram

$$\begin{array}{ccc} U_M & \begin{array}{c} \xleftarrow{\mathfrak{q}_U} \\ \xrightarrow{\iota_U} \end{array} & U_M \times_{\text{Bun}_M} \text{Bun}_P \\ & \begin{array}{c} \searrow \mathfrak{q}_U^- \\ \swarrow \iota_U^- \end{array} & \downarrow \mathfrak{p}_U \\ U_M \times_{\text{Bun}_M} \text{Bun}_P^- & \xrightarrow{\mathfrak{p}_U^-} & U_G, \end{array}$$

It suffices to show that the functor

$$(\mathfrak{q}_U)! \circ \mathfrak{p}_U^* : \mathbf{D}\text{-mod}(U_G) \rightarrow \mathbf{Pro}(\mathbf{D}\text{-mod}(U_M))$$

takes values in $\mathbf{D}\text{-mod}(U_M)$, and that the natural transformation

$$(4.14) \quad (\mathfrak{q}_U^-)_* \circ (\mathfrak{p}_U^-)! \rightarrow (\mathfrak{q}_U)! \circ (\mathfrak{p}_U)^*$$

of (4.9) is an isomorphism.

4.6.3. The contraction of \mathbf{Bun}_P (resp., \mathbf{Bun}_{P^-}) onto \mathbf{Bun}_M (see Sect. 4.1.6) and Proposition 4.1.5 imply that the morphisms

$$\mathfrak{q}_U : U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_P \xrightarrow{\cong} U_M : \iota_U$$

and

$$\mathfrak{q}_U^- : U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_{P^-} \xrightarrow{\cong} U_M : \iota_U^-$$

satisfy the assumptions of Sect. 4.4.2.

Hence, applying Lemma 4.4.3, we obtain that it is enough to show that the diagram

$$(4.15) \quad \begin{array}{ccc} U_M & \xrightarrow{\iota_U} & U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_P \\ \iota_U^- \downarrow & & \downarrow \mathfrak{p}_U \\ U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_{P^-} & \xrightarrow{\mathfrak{p}_U^-} & U_G \end{array}$$

is hyperbolic in the sense of Sect. 4.3.2.

4.7. Proof of Theorem 1.2.5: verification of hyperbolicity.

4.7.1. Fix any point $x \in X$. For $H = G, P, P^-$ or M , let $\mathbf{Bun}_H^{n,x}$ be the stack classifying G -bundles with a structure of level n at x . It is known that for n large enough, the open substacks

$$\begin{aligned} U_G \times_{\mathbf{Bun}_G} \mathbf{Bun}_G^{n,x} &\subset \mathbf{Bun}_G^{n,x} \\ U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_M^{n,x} &\subset \mathbf{Bun}_M^{n,x}, \\ U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_P^{n,x} &\simeq (U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_P) \times_{\mathbf{Bun}_P} \mathbf{Bun}_P^{n,x} \end{aligned}$$

and

$$U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_{P^-}^{n,x} \simeq (U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_{P^-}) \times_{\mathbf{Bun}_{P^-}} \mathbf{Bun}_{P^-}^{n,x}$$

are actually quasi-compact separated schemes.

By making n even larger, we can assume that the maps in the diagram

$$(4.16) \quad \begin{array}{ccc} U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_M^{n,x} & \longrightarrow & U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_P^{n,x} \\ \downarrow & & \downarrow \\ U_M \times_{\mathbf{Bun}_M} \mathbf{Bun}_{P^-}^{n,x} & \longrightarrow & U_G \times_{\mathbf{Bun}_G} \mathbf{Bun}_G^{n,x} \end{array}$$

are locally closed embeddings.

Combining, we obtain the following diagram:

$$(4.17) \quad \begin{array}{ccccc} U_M \times_{\text{Bun}_M} \text{Bun}_M^{n,x} & \longrightarrow & U_M & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & U_M \times_{\text{Bun}_M} \text{Bun}_P^{n,x} & \longrightarrow & U_M \times_{\text{Bun}_M} \text{Bun}_P & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ U_M \times_{\text{Bun}_M} \text{Bun}_P^{n,x} & \longrightarrow & U_M \times_{\text{Bun}_M} \text{Bun}_{P^-} & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ & U_G \times_{\text{Bun}_G} \text{Bun}_G^{n,x} & \longrightarrow & U_G & \end{array}$$

4.7.2. Let $\gamma : \mathbb{G}_m \rightarrow M$ be a co-character as in (2.13), and consider the corresponding homomorphism

$$\mathbb{G}_m \rightarrow H.$$

The adjoint action of \mathbb{G}_m on H defines a \mathbb{G}_m -action on each $\text{Bun}_H^{n,x}$ so that the forgetful maps

$$\text{Bun}_H^{n,x} \rightarrow \text{Bun}_H$$

are \mathbb{G}_m -equivariant. In particular, the scheme

$$Z := U_G \times_{\text{Bun}_G} \text{Bun}_G^{n,x}$$

acquires a \mathbb{G}_m -action, which is *locally linear*, i.e., Z can be covered by affine subschemes that are preserved by the \mathbb{G}_m -action.

Consider the corresponding diagram of (4.2):

$$(4.18) \quad \begin{array}{ccc} Z^0 & \longrightarrow & Z^+ \\ \downarrow & & \downarrow \\ Z^- & \longrightarrow & Z \end{array}$$

for Z as above.

By Theorem 4.3.4, in order to show that the diagram (4.15) is hyperbolic, it is enough to show that the digram (4.16) admits a map to the diagram (4.18), which is an open embedding on a Zariski neighborhood of $U_M \times_{\text{Bun}_M} \text{Bun}_M^{n,x}$. (Indeed, in this case the diagram (4.16) will identify with the diagram (4.2) for an open \mathbb{G}_m -invariant subscheme of Z .)

4.7.3. We construct the corresponding maps

$$(4.19) \quad U_M \times_{\text{Bun}_M} \text{Bun}_M^{n,x} \rightarrow Z^0,$$

$$(4.20) \quad U_M \times_{\text{Bun}_M} \text{Bun}_P^{n,x} \rightarrow Z^+,$$

and

$$(4.21) \quad U_M \times_{\text{Bun}_M} \text{Bun}_{P^-}^{n,x} \rightarrow Z^0$$

as follows.

The map (4.19) is evident, since the adjoint action of \mathbb{G}_m on M is trivial.

The map (4.20) (resp., (4.21)) is given by the \mathbb{A}^1 -action on $\text{Bun}_P^{n \cdot x}$ (resp., $\text{Bun}_P^{n \cdot x}$) that contracts it onto $\text{Bun}_M^{n \cdot x}$ as in Sect. 4.1.6.

4.7.4. The map (4.19) (resp., (4.20), (4.21)) is a locally closed embedding, because its composition with the map $Z^0 \rightarrow Z$ (resp., $Z^+ \rightarrow Z$, $Z^- \rightarrow Z$) is.

It remains to show that the maps (4.19), (4.20) and (4.21) are open embeddings on a Zariski neighborhood of $U_M \times_{\text{Bun}_M} \text{Bun}_M^{n \cdot x}$.

Since the left-hand sides are smooth, it is enough to show that maps in question give rise to isomorphisms of tangent spaces at every k -point of $U_M \times_{\text{Bun}_M} \text{Bun}_M^{n \cdot x}$.

Let z be a k -point of Z , where Z is as in Sect. 4.2. The \mathbb{G}_m -action on Z induces a linear action of \mathbb{G}_m on the tangent space $T_z(Z)$. It is easy to see that the subspaces

$$T_z(Z^0), T_z(Z^+) \text{ and } T_z(Z^-)$$

identify, respectively, with the subspace of zero, non-negative and non-positive characters of \mathbb{G}_m on $T_z(Z)$.

Let $Z := U_G \times_{\text{Bun}_G} \text{Bun}_G^{n \cdot x}$, and let z be a geometric point of $U_M \times_{\text{Bun}_M} \text{Bun}_M^{n \cdot x}$, corresponding to an M -bundle \mathcal{F}_M .

The tangent space $T_z(U_G \times_{\text{Bun}_G} \text{Bun}_G^{n \cdot x})$ identifies with $H^1(X, \mathfrak{g}_{\mathcal{F}_M}(-n \cdot x))$, while the tangent spaces

$$T_z(U_M \times_{\text{Bun}_M} \text{Bun}_M^{n \cdot x}), T_z(U_M \times_{\text{Bun}_M} \text{Bun}_P^{n \cdot x}) \text{ and } T_z(U_M \times_{\text{Bun}_M} \text{Bun}_P^{n \cdot x})$$

identify with

$$H^1(X, \mathfrak{m}_{\mathcal{F}_M}(-n \cdot x)), H^1(X, \mathfrak{p}_{\mathcal{F}_M}(-n \cdot x)) \text{ and } H^1(X, \mathfrak{p}_{\mathcal{F}_M}^-(n \cdot x)),$$

respectively.

This makes the required assertion manifest.

APPENDIX A. HYPERBOLICITY FOR STACKS: PROOF OF THEOREM 4.3.4

With no restriction of generality, we can assume that \mathcal{Y} is quasi-compact (and hence QCA). Indeed, otherwise, replace it by the image of the map ψ .

A.1. **Step 1.** In this subsection we will show that the functors

$$(A.1) \quad (\iota^-)^* \circ (\mathfrak{p}^-)^! \text{ and } (\iota^+)^* \circ (\mathfrak{p}^+)^!$$

take values in $\text{D-mod}(\mathcal{Y}^0) \subset \text{Pro}(\text{D-mod}(\mathcal{Y}^0))$. By symmetry, it is sufficient to treat the case of the former functor.

A.1.1. Note that for $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})$, the object

$$(\psi^0)^* \circ (\iota^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F}),$$

is, isomorphic to $(i^-)^* \circ (\psi^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F})$ and the latter is, up to a cohomological shift, isomorphic to $(i^-)^* \circ (p^-)^! \circ \psi^*(\mathcal{F})$, since the morphisms ψ and ψ^- are both smooth.

Since $\psi^*(\mathcal{F}) \in \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}}$, from Theorem 4.2.5, we obtain that

$$(i^-)^* \circ (p^-)^! \circ \psi^*(\mathcal{F}) \in \mathrm{D}\text{-mod}(Z^0).$$

Hence, we obtain that

$$(\psi^0)^* \circ (\iota^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F}) \in \mathrm{D}\text{-mod}(Z^0).$$

Now, the fact that $(\iota^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$ follows from the next general assertion:

Lemma A.1.2. *Let $\mathcal{Z} \xrightarrow{g} \mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$ be maps between quasi-compact stacks with g smooth and surjective. Then for $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y}_2)$, the object $f^*(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Y}_1)$ if and only if $g^* \circ f^*(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Z})$.*

A.1.3. *Proof of Lemma A.1.2.* The “only” if direction is obvious. For the “if” direction, with no restriction of generality, we can assume that \mathcal{Z} is a quasi-compact scheme.

Let \mathcal{Z}^\bullet denote the Čech nerve of the map g , and let $g^\bullet : \mathcal{Z}^\bullet \rightarrow \mathcal{Y}_1$ denote the corresponding maps. By assumption $(g^\bullet)^* \circ f^*(\mathcal{F})$ is a well-defined an object of $\mathrm{Tot}(\mathrm{D}\text{-mod}(\mathcal{Z}^\bullet))$. Since g is smooth and surjective, the functor

$$(A.2) \quad (g^\bullet)^* : \mathrm{D}\text{-mod}(\mathcal{Y}_1) \rightarrow \mathrm{Tot}(\mathrm{D}\text{-mod}(\mathcal{Z}^\bullet))$$

is an equivalence. Let $\mathcal{F}' \in \mathrm{D}\text{-mod}(\mathcal{Y}_1)$ denote the object corresponding to $(g^\bullet)^* \circ f^*(\mathcal{F})$ under the equivalence (A.2). We claim that $\mathcal{F}' \simeq f^*(\mathcal{F})$.

It suffices to show that for $\mathcal{F}_1 \in \mathrm{D}\text{-mod}(\mathcal{Y}_1)$, we have a canonical isomorphism

$$\mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Y}_1)}(\mathcal{F}', \mathcal{F}_1) \simeq \mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Y}_2)}(\mathcal{F}, f_*(\mathcal{F}_1)).$$

We have:

$$\mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Y}_1)}(\mathcal{F}', \mathcal{F}_1) \simeq \mathrm{Tot}(\mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Z}^\bullet)}((g^\bullet)^* \circ f^*(\mathcal{F}), (g^\bullet)^*(\mathcal{F}_1))),$$

which can be rewritten by adjunction as

$$\begin{aligned} \mathrm{Tot}(\mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Y}_2)}(\mathcal{F}, (f \circ g^\bullet)_* \circ (g^\bullet)^*(\mathcal{F}_1))) &\simeq \\ &\simeq \mathcal{H}om_{\mathrm{D}\text{-mod}(\mathcal{Y}_2)}(\mathcal{F}, \mathrm{Tot}((f \circ g^\bullet)_* \circ (g^\bullet)^*(\mathcal{F}_1))). \end{aligned}$$

Hence, it suffices to show that the canonical map

$$f_*(\mathcal{F}_1) \rightarrow \mathrm{Tot}((f \circ g^\bullet)_* \circ (g^\bullet)^*(\mathcal{F}_1))$$

is an isomorphism. However, this is given by [DrGa1, Lemma 7.5.2]. □

A.2. **Step 2.** In this subsection we will show that the functors

$$(A.3) \quad (\iota^-)^! \circ (\mathfrak{p}^-)^* \text{ and } (\iota^+)^! \circ (\mathfrak{p}^+)^*$$

take values in $\mathrm{D}\text{-mod}(\mathcal{Y}^0) \subset \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0))$. Again, by symmetry, it is sufficient to do this for $(\iota^-)^! \circ (\mathfrak{p}^-)^*$.

A.2.1. Note that the functor $(\iota^-)^\dagger$ admits a (possibly non-continuous) *right* adjoint, denoted $((\iota^-)^\dagger)^R$. Hence, our assertion is that the functor $(\mathfrak{p}^-)_* \circ ((\iota^-)^\dagger)^R$ admits a *left* adjoint, which takes values in $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$, rather than in $\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0))$.

Since \mathcal{Y} was assumed QCA, the category $\mathrm{D}\text{-mod}(\mathcal{Y})$ is compactly generated. Hence, it is enough to show that the value of $(\iota^-)^\dagger \circ (\mathfrak{p}^-)^*$ on $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})^c$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$.

A.2.2. Recall from [GR, Sect. 4.2.4] that the category $\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0))$ inherits a t-structure from $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$. We claim:

Lemma A.2.3. *For $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})^c$, the object $(\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F}) \in \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0))$ is cohomologically bounded, i.e., belongs to $\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Y}^0))^{\geq -n, \leq n}$ for some n .*

Proof. The fact that $(\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F})$ is bounded above follows by combining the following three facts: (i) \mathcal{F} is bounded above; (ii) the functor $(\iota^-)^\dagger$ has a bounded cohomological amplitude; (iii) the functor \mathfrak{p}^-_* , right adjoint to $(\mathfrak{p}^-)^*$, is left t-exact.

Let us show that $(\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F})$ is bounded below. Consider the object $\mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \in \mathrm{D}\text{-mod}(\mathcal{Y})^c$, where $\mathbb{D}_{\mathcal{Y}}$ denotes the Verdier duality functor on \mathcal{Y} (see [DrGa1, Sect. 7.3.4 and Corollary 8.4.2]).

The functor $(\mathfrak{p}^-)^\dagger$ has a bounded cohomological amplitude, hence $(\mathfrak{p}^-)^\dagger \circ \mathbb{D}_{\mathcal{Y}}(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Y}^-)^{\leq m}$ for some m . Hence, we can write $(\mathfrak{p}^-)^\dagger \circ \mathbb{D}_{\mathcal{Y}}(\mathcal{F}) \in \mathrm{D}\text{-mod}(\mathcal{Y}^-)$ as a filtered colimit

$$\lim_{\substack{\longrightarrow \\ i}} \mathcal{F}_i^-,$$

where $\mathcal{F}_i^- \in \mathrm{D}\text{-mod}(\mathcal{Y}^-)^{\leq m}$. In this case

$$(\mathfrak{p}^-)^*(\mathcal{F}) \simeq \varprojlim_i \mathbb{D}_{\mathcal{Y}^-}(\mathcal{F}_i^-).$$

However, the functor $\mathbb{D}_{\mathcal{Y}^-}$ has a bounded cohomological amplitude, so all $\mathbb{D}_{\mathcal{Y}^-}(\mathcal{F}_i^-)$ belong to $\mathrm{D}\text{-mod}(\mathcal{Y}^-)^{\geq -l}$ for some l . Now,

$$(\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F}) \simeq \varprojlim_i (\iota^-)^\dagger \circ \mathbb{D}_{\mathcal{Y}^-}(\mathcal{F}_i^-),$$

and the boundedness below follows from the fact that the functor $(\iota^-)^\dagger$ has a bounded cohomological amplitude. □

A.2.4. Let us now observe that for any $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})$, the object $(\psi^0)^\dagger \circ (\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(Z^0)$. Indeed, we rewrite

$$(\psi^0)^\dagger \circ (\iota^-)^\dagger \circ (\mathfrak{p}^-)^* \simeq (\iota^0)^\dagger \circ (\psi^-)^\dagger \circ (\mathfrak{p}^-)^*,$$

and the latter is isomorphic, up to a cohomological shift to

$$(i^-)^\dagger \circ (\psi^-)^* \circ (\mathfrak{p}^-)^* \simeq (i^-)^\dagger \circ (\mathfrak{p}^-)^* \circ \psi^*.$$

However, $\psi^*(\mathcal{F}) \in \mathrm{D}\text{-mod}(Z)^{\mathbb{G}_m\text{-mon}}$, and hence

$$(i^-)^\dagger \circ (\mathfrak{p}^-)^* \circ \psi^*(\mathcal{F}) \in \mathrm{D}\text{-mod}(Z^0)$$

by Theorem 4.2.5.

Since ψ^0 is smooth (and so $(\psi^0)^\dagger$ is isomorphic to $(\psi^0)^*$ up to a cohomological shift), we obtain that the object $(\psi^0)^* \circ (\iota^-)^\dagger \circ (\mathfrak{p}^-)^*(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(Z^0)$

A.2.5. Now, the fact that $(\iota^-)^! \circ (\mathfrak{p}^-)^*(\mathcal{F})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$ for $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})^c$ follows from Lemma A.2.3 and the next observation:

Lemma A.2.6. *Let $g : \mathcal{Z} \rightarrow \mathcal{W}$ be a smooth surjective map between quasi-compact stacks. Let $\tilde{\mathcal{F}}$ be a cohomologically bounded object of $\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{W}))$. Then $\tilde{\mathcal{F}} \in \mathrm{D}\text{-mod}(\mathcal{W})$ if and only if $g^*(\tilde{\mathcal{F}})$ belongs to $\mathrm{D}\text{-mod}(\mathcal{Z}) \subset \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{Z}))$.*

Proof. The “only if” direction is obvious. For the “if” direction we proceed as follows.

Let $g^\bullet : \mathcal{Z}^\bullet \rightarrow \mathcal{W}$ be the Čech nerve of $g : \mathcal{Z} \rightarrow \mathcal{W}$. By assumption, $(g^\bullet)^*(\tilde{\mathcal{F}})$ gives rise to a well-defined object of

$$\mathrm{Tot}(\mathrm{D}\text{-mod}(\mathcal{Z}^\bullet)).$$

Applying the inverse of the equivalence

$$(A.4) \quad (g^\bullet)^* : \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{Tot}(\mathrm{D}\text{-mod}(\mathcal{Z}^\bullet)),$$

from $(g^\bullet)^*(\tilde{\mathcal{F}})$ we obtain a well-defined object of $\mathrm{D}\text{-mod}(\mathcal{W})$ that we denote by \mathcal{F} . We claim that $\tilde{\mathcal{F}} \simeq \mathcal{F}$.

By construction, the object $\mathcal{F} \in \mathrm{D}\text{-mod}(\mathcal{Y})$ is cohomologically bounded. Assume that $\tilde{\mathcal{F}}$ belongs to $\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{W}))^{\geq -n, \leq n}$. By [GR, Sect. 4.2.4], we have a canonical equivalence

$$\mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{W}))^{\geq -n, \leq n} \simeq \mathrm{Pro}(\mathrm{D}\text{-mod}(\mathcal{W})^{\geq -n, \leq n}).$$

Hence, in order to establish an isomorphism $\mathcal{F} \simeq \tilde{\mathcal{F}}$, it is sufficient to construct a functorial isomorphism

$$\mathcal{H}om(\tilde{\mathcal{F}}, \mathcal{F}') \simeq \mathcal{H}om(\mathcal{F}, \mathcal{F}')$$

for $\mathcal{F}' \in \mathrm{D}\text{-mod}(\mathcal{W})$ which is also cohomologically bounded.

By construction,

$$\mathcal{H}om(\tilde{\mathcal{F}}, \mathcal{F}') \simeq \mathrm{Tot} \left(\mathcal{H}om \left((g^\bullet)^*(\tilde{\mathcal{F}}), (g^\bullet)^*(\mathcal{F}') \right) \right).$$

Now, the fact that $\tilde{\mathcal{F}}$ is bounded above and that \mathcal{F}' is bounded below implies that

$$\mathrm{Tot} \left(\mathcal{H}om \left((g^\bullet)^*(\tilde{\mathcal{F}}), (g^\bullet)^*(\mathcal{F}') \right) \right) \simeq \mathcal{H}om \left(\tilde{\mathcal{F}}, \mathrm{Tot} \left((g^\bullet)_* \circ (g^\bullet)^*(\mathcal{F}') \right) \right).$$

Now,

$$\mathcal{F}' \rightarrow \mathrm{Tot} \left((g^\bullet)_* \circ (g^\bullet)^*(\mathcal{F}') \right)$$

is an isomorphism since (A.4) is an equivalence. □

A.3. Step 3. Having established that both sides of (4.6) belong to $\mathrm{D}\text{-mod}(\mathcal{Y}^0)$, we are finally ready to show that the map (4.6), is an isomorphism.

Since the functor

$$(\psi^0)^! : \mathrm{D}\text{-mod}(\mathcal{Y}^0) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Z}^0)$$

is conservative, it suffices to show that the natural transformation

$$(A.5) \quad (\psi^0)^* \circ (\iota^-)^* \circ (\mathfrak{p}^-)^* \rightarrow (\psi^0)^* \circ (\iota^+)^! \circ (\mathfrak{p}^+)^*,$$

induced by (4.6), is an isomorphism.

A.3.1. Consider the map $\beta^- : Z^- \rightarrow \mathcal{Y}^- \times_{\mathcal{Y}} Z$. Denote

$$\mathcal{K}^- := (\beta^-)^*(\omega_{\mathcal{Y}^- \times_{\mathcal{Y}} Z}) \in \text{D-mod}(Z^-).$$

The object \mathcal{K}^- is isomorphic to ω_{Z^-} up to a cohomological shift, since both Z^- and $\mathcal{Y}^- \times_{\mathcal{Y}} Z$ are smooth over \mathcal{Y}^- .

Note that for $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ there exists a canonical isomorphism

$$(A.6) \quad (\psi^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F}) \simeq \mathcal{K}^- \overset{!}{\otimes} (\mathfrak{p}^-)^! \circ \psi^*(\mathcal{F}).$$

A.3.2. Consider now the map $\beta^0 : Z^0 \rightarrow \mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+$. Denote

$$\mathcal{K}^0 := (\beta^+)^*(\omega_{\mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+}) \in \text{D-mod}(Z^0).$$

The object \mathcal{K}^0 is isomorphic to ω_{Z^0} up to a cohomological shift, since both Z^0 and $\mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+$ are smooth over \mathcal{Y}^0 .

Note that for $\mathcal{F}^+ \in \text{D-mod}(\mathcal{Y}^+)$ there exists a canonical isomorphism

$$(A.7) \quad (\psi^0)^* \circ (\iota^+)^!(\mathcal{F}^+) \simeq \mathcal{K}^0 \overset{!}{\otimes} (i^+)^! \circ (\psi^+)^*(\mathcal{F}^+).$$

A.3.3. Note now that in the diagram

$$\begin{array}{ccccc}
 Z^0 & & & & \\
 & \searrow^{\beta^+} & & & \\
 & & Z^- \times_{\mathcal{Y}^- \times_{\mathcal{Y}} Z} (\mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+) & \xrightarrow{\quad} & \mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+ \\
 & \searrow^{i^-} & \downarrow & & \downarrow \mathfrak{p}^- \times i^+ \\
 & & Z^- & \xrightarrow{\beta^-} & \mathcal{Y}^- \times_{\mathcal{Y}} Z
 \end{array}$$

the map

$$Z^0 \rightarrow (\mathcal{Y}^0 \times_{\mathcal{Y}^+} Z^+) \times_{\mathcal{Y}^- \times_{\mathcal{Y}} Z} Z^-$$

is an open embedding. Hence, we obtain a canonical isomorphism

$$(A.8) \quad \mathcal{K}^0 \rightarrow (i^-)^!(\mathcal{K}^-).$$

A.3.4. Now, diagram chase shows that for $\mathcal{F} \in \mathbf{D}\text{-mod}(\mathcal{Y})$ the following diagram commutes

$$\begin{array}{ccc}
 (\psi^0)^* \circ (\iota^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F}) & \xrightarrow{(A.5)} & (\psi^0)^* \circ (\iota^+)^! \circ (\mathfrak{p}^+)^*(\mathcal{F}) \\
 \sim \downarrow & & \downarrow (A.7) \\
 (i^-)^* \circ (\psi^-)^* \circ (\mathfrak{p}^-)^!(\mathcal{F}) & & \mathcal{K}^0 \overset{!}{\otimes} (\iota^+)^! \circ (\psi^+)^* \circ (\mathfrak{p}^+)^*(\mathcal{F}) \\
 (A.6) \downarrow & & \downarrow \sim \\
 (i^-)^* \left(\mathcal{K}^- \overset{!}{\otimes} (p^-)^! \circ \psi^*(\mathcal{F}) \right) & & \mathcal{K}^0 \overset{!}{\otimes} (\iota^+)^! \circ (p^+)^* \circ \psi^*(\mathcal{F}) \\
 \sim \downarrow & & \downarrow (A.8) \\
 (i^-)^! (\mathcal{K}^-) \overset{!}{\otimes} (i^-)^* \circ (p^-)^! \circ \psi^*(\mathcal{F}) & \xrightarrow{(4.4)} & (i^-)^! (\mathcal{K}^-) \overset{!}{\otimes} (i^+)^! \circ (p^+)^* \circ \psi^*(\mathcal{F}).
 \end{array}$$

Hence, we obtain that (A.5) is an isomorphism, as desired.

APPENDIX B. THE SUPPORT OF CUSPIDAL OBJECTS OF $\mathbf{D}\text{-mod}(\mathbf{Bun}_G)$

The goal of this Appendix is to prove Proposition 1.4.6, i.e., to construct an open substack $\mathcal{U} \subset \mathbf{Bun}_G$ having quasi-compact intersection with each connected component of \mathbf{Bun}_G and such that the $*$ -support and $!$ -support of each object of $\mathbf{D}\text{-mod}(\mathbf{Bun}_G)_{\text{cusp}}$ are contained in \mathcal{U} .

B.1. Definition of \mathcal{U} .

B.1.1. Let Λ_G denote the coweight lattice of G . Set $\Lambda_G^{\mathbb{Q}} := \Lambda_G \otimes \mathbb{Q}$ and let $\Lambda_G^{+, \mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ be the dominant cone.

We will use the Harder-Narasimhan-Shatz stratification of \mathbf{Bun}_G (see [DrGa2, Sect. 7.4]).

Its strata are labeled by elements $\lambda \in \Lambda_G^{+, \mathbb{Q}}$ and denoted by $\mathbf{Bun}_G^{(\lambda)}$. The set of those λ for which $\mathbf{Bun}_G^{(\lambda)} \neq \emptyset$ is discrete in $\Lambda_G \otimes \mathbb{R}$.

B.1.2. Set

$$\mathcal{U} := \bigcup_{\lambda \in \Sigma} \mathbf{Bun}_G^{(\lambda)},$$

where Σ is the set of all $\lambda \in \Lambda_G^{+, \mathbb{Q}}$ such that the image of λ in $\Lambda_{G_{\text{adj}}}^{\mathbb{Q}}$ is $\leq (2g-2)\rho_G$. Here g is the genus of the curve X , $\rho_G \in \Lambda_{G_{\text{adj}}}^{\mathbb{Q}}$ is the half-sum of positive coroots, and the ordering \leq is the usual one (i.e., $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple coroots with non-negative coefficients).

Lemma B.1.3. *The union of strata \mathcal{U} is an open substack of \mathbf{Bun}_G . Its intersection with each connected component of \mathbf{Bun}_G is quasi-compact.*

Proof. Use the standard properties of the Harder-Narasimhan-Shatz stratification (see [DrGa2, Lemma 7.4.9]) and the following obvious fact: for any $\lambda_0 \in \Lambda_G^{\mathbb{Q}}$, the union of $\mathbf{Bun}_G^{(\lambda)}$ for $\lambda \in (\lambda_0 + \Lambda_{[G, G]}^{\mathbb{Q}}) \cap \Lambda_G^{+, \mathbb{Q}}$ is both open and closed. \square

B.2. Statement of the result.

B.2.1. Let $j : \mathcal{U} \hookrightarrow \text{Bun}_G$ denote the embedding. In Sect. B.4 we will prove the following

Proposition B.2.2. *For any $\mathcal{F} \in \text{D-mod}(\text{Bun}_G)_{\text{cusp}}$, the canonical maps*

$$j_! \circ j^*(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j_* \circ j^*(\mathcal{F})$$

are isomorphisms.

The statement about $j_! \circ j^*(\mathcal{F})$ will be understood as follows: the partially defined¹⁰ functor $j_!$ is defined on $j^*(\mathcal{F})$ and the map $j_! \circ j^*(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism.

Remark B.2.3. If G is a torus then $\Sigma = \Lambda_G^{+, \mathbb{Q}} = \Lambda_G^{\mathbb{Q}}$, $\mathcal{U} = \text{Bun}_G$, and $\text{D-mod}(\text{Bun}_G)_{\text{cusp}} = \text{D-mod}(\text{Bun}_G)$. So the proposition holds tautologically.

Remark B.2.4. If $g = 0$ and G is not a torus then $\Sigma = \emptyset$ and therefore $\mathcal{U} = \emptyset$. In this case the proposition says that $\text{D-mod}(\text{Bun}_G)_{\text{cusp}} = 0$.

Remark B.2.5. Let $g = 0$ and $\mathcal{F} \in \text{D-mod}(\text{Bun}_G)$. If the map $\mathcal{F} \rightarrow j_* \circ j^*(\mathcal{F})$ is an isomorphism then so is the map $j_! \circ j^*(\mathcal{F}) \rightarrow \mathcal{F}$. This follows from the fact that the open substack $\mathcal{U} \subset \text{Bun}_G$ equals either \emptyset or Bun_G (see Remarks B.2.3-B.2.4).

B.3. The key lemma.

B.3.1. Recall that for every parabolic $P \subset G$ we denote by

$$\mathfrak{p} : \text{Bun}_P \rightarrow \text{Bun}_G \quad \text{and} \quad \mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$$

the corresponding morphisms (as usual, M is the Levi quotient of P).

Lemma B.3.2. *Let Σ be as in Sect. B.1. Let $\lambda \in \Lambda_G^{+, \mathbb{Q}}$, $\lambda \notin \Sigma$. Then there exists a proper parabolic $P \subset G$ and a locally closed substack $V \subset \text{Bun}_M$ such that:*

- (1) *The fiber of $\mathfrak{q} : \text{Bun}_P \rightarrow \text{Bun}_M$ over any geometric point of V has only one isomorphism class of geometric points;*
- (2) *$\mathfrak{q}^{-1}(V) \subset \mathfrak{p}^{-1}(\text{Bun}_G^{(\lambda)})$;*
- (3) *The morphism $\mathfrak{p}|_{\mathfrak{q}^{-1}(V)} : \mathfrak{q}^{-1}(V) \rightarrow \text{Bun}_G^{(\lambda)}$ is an isomorphism if $g > 0$ and is surjective if $g = 0$.*

Remark B.3.3. Let $N(P)$ denote the unipotent radical of P . For any $\mathcal{P} \in \text{Bun}_M$ let $N(P)_{\mathcal{P}}$ denote the group-scheme $N(P)_{\mathcal{P}}$ over X (here we regard M as acting on $N(P)$ via the embedding $M \hookrightarrow P$). Then the fiber of \mathfrak{q} over $\mathcal{P} \in \text{Bun}_M$ is the stack of $N(P)_{\mathcal{P}}$ -torsors on X . Condition (1) in Lemma B.3.2 says that if $\mathcal{P} \in V$ then all such torsors are trivial.

Proof. Let us first assume that $g = 0$, i.e., $X = \mathbb{P}^1$. We can also assume that G is not a torus (otherwise $\Sigma = \Lambda_G^{+, \mathbb{Q}}$ and there is nothing to prove). Take $P = B$ to be a Borel subgroup (so $M = T$ is the Cartan). Define $V \subset \text{Bun}_T$ to be the substack of T -bundles of degree λ . Property (1) follows from the fact that $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$ for $n \geq 0$. Property (3) holds because any G -bundle on \mathbb{P}^1 admits a reduction to a maximal torus of G (see [BN] and references therein). Property (2) is clear.

Let us now assume that X has genus $g > 0$. The simple roots of G will be denoted by $\check{\alpha}_i$. Define P and M by the following condition: the simple roots of M are those $\check{\alpha}_i$'s for which $\langle \lambda, \check{\alpha}_i \rangle \leq 2g - 2$. We have $P \neq G$ (indeed, let $\bar{\lambda} \in \Lambda_{G_{\text{adj}}}^{+, \mathbb{Q}}$ be the image of λ , then the equality $P = G$ would imply that $(2g - 2)\rho_G - \bar{\lambda}$ is dominant and therefore $\bar{\lambda} \leq (2g - 2)\rho_G$, contrary to the assumption that $\lambda \notin \Sigma$).

¹⁰In fact, the embedding $j : \mathcal{U} \hookrightarrow \text{Bun}_G$ has the remarkable property that the functor $j_!$ is defined on all of $\text{D-mod}(\mathcal{U})$, see [DrGa2, Theorem 9.1.2]. We will not use this fact.

Note that $\Lambda_G^{+, \mathbb{Q}} \subset \Lambda_M^{+, \mathbb{Q}}$, so we can consider the Harder-Narasimhan-Shatz stratum $\text{Bun}_M^{(\lambda)}$. Set $V := \text{Bun}_M^{(\lambda)}$. Then V has the required properties. Properties (2)-(3) follow from the definition of the Harder-Narasimhan-Shatz strata (see [DrGa2, Theorem 7.4.3]) and the assumption $2g - 2 \geq 0$. Remark B.3.3 shows that to prove property (1) it suffices to check the equality

$$H^1(X, \mathfrak{n}_{\mathcal{P}}) = 0 \quad \text{for all } \mathcal{P} \in \text{Bun}_M^{(\lambda)},$$

where $\mathfrak{n} := \text{Lie}(N)$. This equality holds by [DrGa2, Proposition 10.1.3] and the characteristic 0 assumption on k . \square

B.3.4. In the situation of Lemma B.3.2 consider the corresponding morphism

$$\mathfrak{q} : \mathfrak{q}^{-1}(V) \rightarrow V.$$

Lemma B.3.5.

- (i) *Each fiber of the morphism $\mathfrak{q} : \mathfrak{q}^{-1}(V) \rightarrow V$ is a classifying space of a unipotent group.*
- (ii) *The functors*

$$\mathfrak{q}_* : \text{D-mod}(\mathfrak{q}^{-1}(V)) \rightarrow \text{D-mod}(V) \quad \text{and} \quad \mathfrak{q}! : \text{D-mod}(\mathfrak{q}^{-1}(V)) \rightarrow \text{D-mod}(V)$$

are equivalences (that differ by a cohomological shift), with the inverses provided by

$$\mathfrak{q}^* : \text{D-mod}(V) \rightarrow \text{D-mod}(\mathfrak{q}^{-1}(V)) \quad \text{and} \quad \mathfrak{q}^! : \text{D-mod}(V) \rightarrow \text{D-mod}(\mathfrak{q}^{-1}(V)),$$

respectively.

Proof. Statement (i) follows from the interpretation of the fibers of \mathfrak{q} given in Remark B.3.3. Statement (ii) follows from (i) because \mathfrak{q} is smooth. \square

B.4. Proof of Proposition B.2.2.

B.4.1. Let Σ be as in Sect. B.1. Let $\lambda \in \Lambda_G^{+, \mathbb{Q}}$, $\lambda \notin \Sigma$. Let $i_\lambda : \text{Bun}_G^{(\lambda)} \hookrightarrow \text{Bun}_G$ be the corresponding locally closed embedding. We need to show that for any cuspidal $\mathcal{F} \in \text{D-mod}(\text{Bun}_G)$ one has

$$i_\lambda^!(\mathcal{F}) = 0, \quad i_\lambda^*(\mathcal{F}) = 0.$$

Let P and V be as in Lemma B.3.2. Consider the corresponding morphisms

$$\mathfrak{p}|_{\mathfrak{q}^{-1}(V)} : \mathfrak{q}^{-1}(V) \rightarrow \text{Bun}_G^{(\lambda)} \quad \text{and} \quad \mathfrak{q}|_{\mathfrak{q}^{-1}(V)} : \mathfrak{q}^{-1}(V) \rightarrow V.$$

B.4.2. Let us first prove that $i_\lambda^!(\mathcal{F}) = 0$. By cuspidality, $\mathfrak{q}_*(\mathfrak{p}^!(\mathcal{F})) = 0$, so

$$(\mathfrak{q}|_{\mathfrak{q}^{-1}(V)})_* \circ (\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})^! \circ i_\lambda^!(\mathcal{F}) = 0.$$

By Lemma B.3.5(ii), this means that $(\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})^! \circ i_\lambda^!(\mathcal{F}) = 0$. Since $\mathfrak{p}|_{\mathfrak{q}^{-1}(V)} : \mathfrak{q}^{-1}(V) \rightarrow \text{Bun}_G^{(\lambda)}$ is surjective, the functor $(\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})^!$ is conservative by [DrGa1, Lemma 5.1.6]. So $i_\lambda^!(\mathcal{F}) = 0$.

B.4.3. Let us now prove that $\iota_\lambda^*(\mathcal{F}) = 0$. By Remark B.2.5, we can assume that $g > 0$, so the map $\mathfrak{p}|_{\mathfrak{q}^{-1}(V)} : \mathfrak{q}^{-1}(V) \rightarrow \mathrm{Bun}_G^{(\lambda)}$ is an isomorphism (see Lemma B.3.2).

We have $\mathrm{CT}_!^\lambda(\mathcal{F}) = 0$, where $\mathrm{CT}_!^\lambda$ is the constant term functor with respect to P . So the adjunction from Theorem 1.2.3 implies that $\mathrm{Hom}(\mathcal{F}, \mathrm{Eis}_*^\lambda(\mathcal{E})) = 0$ for all $\mathcal{E} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_M^\lambda)$.

Taking \mathcal{E} to be the $*$ -pushforward of an object $\mathcal{E}' \in \mathrm{D}\text{-mod}(V)$ we have

$$\mathrm{Eis}_*(\mathcal{E}) \simeq (\iota_\lambda)_* \circ (\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})_* \circ (\mathfrak{q}|_{\mathfrak{q}^{-1}(V)})^!(\mathcal{E}'),$$

and hence

$$\mathrm{Hom}(\mathcal{F}, (\iota_\lambda)_* \circ (\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})_* \circ (\mathfrak{q}|_{\mathfrak{q}^{-1}(V)})^!(\mathcal{E}')) = 0.$$

Since $(\mathfrak{p}|_{\mathfrak{q}^{-1}(V)})_*$ and $(\mathfrak{q}|_{\mathfrak{q}^{-1}(V)})^!$ are equivalences, we obtain

$$\mathrm{Hom}(\mathcal{F}, (\iota_\lambda)_*(\mathcal{F}')) = 0, \quad \forall \mathcal{F}' \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G^{(\lambda)}),$$

i.e., $\iota_\lambda^*(\mathcal{F}) = 0$. □

APPENDIX C. PROOF OF PROPOSITION 2.4.4

In this Appendix (just as in Proposition 2.4.4 itself) the characteristic of the ground field k can be arbitrary.

C.1. Reduction of Proposition 2.4.4 to Lemma C.1.3.

C.1.1. We have to prove that the quotient $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ is a quasi-affine scheme. We know that this quotient exists as an algebraic space (because \tilde{G} is flat over \mathbb{A}^1). This space is separated because \tilde{G} is closed in $\mathbb{A}^1 \times G \times G$.

C.1.2. Recall that a separated quasi-finite morphism is quasi-affine (see [LM, Theorem A.2] or [Kn, Ch. II, Theorem 6.15]). So it remains to construct a quasi-finite morphism from $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ to an affine scheme Y .

Thus it remains to prove the following lemma:

Lemma C.1.3. *There exists an affine scheme Y equipped with an action of $G \times G$ and a morphism $f : \mathbb{A}^1 \rightarrow Y$ such that for any $t \in \mathbb{A}^1(k)$ the stabilizer of $f(t)$ contains \tilde{G}_t as a subgroup of finite index.*

C.2. Plan of the proof of Lemma C.1.3.

C.2.1. For each dominant weight $\tilde{\lambda}$ of G we will construct a finite-dimensional $(G \times G)$ -module $R_{\tilde{\lambda}}$, equipped with a polynomial map $f_{\tilde{\lambda}} : \mathbb{A}^1 \rightarrow R_{\tilde{\lambda}}$.

Let $\mathrm{Stab}_t^{\tilde{\lambda}} \subset G \times G$ denote the stabilizer of $f_{\tilde{\lambda}}(t)$. We will show that

$$(C.1) \quad \left(\bigcap_{\tilde{\lambda}} \mathrm{Stab}_t^{\tilde{\lambda}} \right)_{\mathrm{red}} = \tilde{G}_t.$$

C.2.2. This will immediately imply Lemma C.1.3: indeed, one can take

$$Y = \prod_{\check{\lambda}} R_{\check{\lambda}}, \quad f = (f_{\check{\lambda}}),$$

where $\check{\lambda}$ runs through the set of all dominant weights of G (or if you prefer, a sufficiently large finite collection of weights).

Recall that \tilde{G}_t is reduced by Proposition 2.3.8(i), so (C.1) is equivalent to the inclusions

$$(C.2) \quad \left(\bigcap_{\check{\lambda}} \text{Stab}_t^{\check{\lambda}} \right)_{\text{red}} \subset \tilde{G}_t \subset \bigcap_{\check{\lambda}} \text{Stab}_t^{\check{\lambda}}.$$

Thus it remains to construct $R_{\check{\lambda}}$ and $f_{\check{\lambda}}$, and to prove (C.2).

C.3. Construction of $R_{\check{\lambda}}$ and $f_{\check{\lambda}}(t)$.

C.3.1. *Definition of $R_{\check{\lambda}}$.* Let $V_{\check{\lambda}}$ denote the irreducible G -module with highest weight $\check{\lambda}$. Set $R_{\check{\lambda}} := \text{End}_k(V_{\check{\lambda}})$. We have a homomorphism of monoids

$$\rho_{\check{\lambda}} : G \rightarrow \text{End}_k(V_{\check{\lambda}}).$$

The action of $G \times G$ on $R_{\check{\lambda}} := \text{End}_k(V_{\check{\lambda}})$ is defined as follows: to $(g_1, g_2) \in G \times G$ we associate the operator

$$\text{End}_k(V_{\check{\lambda}}) \rightarrow \text{End}_k(V_{\check{\lambda}}), \quad A \mapsto \rho_{\check{\lambda}}(g_1) \cdot A \cdot \rho_{\check{\lambda}}(g_2)^{-1}.$$

C.3.2. *Definition of $f_{\check{\lambda}}$.* Recall that to define \tilde{G} we fixed in Sect. 2.13 a co-character

$$\gamma : \mathbb{G}_m \rightarrow M$$

mapping to the center of M , which is dominant and regular with respect to P . Let us also fix a maximal torus $T \subset G$ contained in M and a Borel $B \subset G$ such that $T \subset B \subset P$.

Now, for $t \in \mathbb{G}_m$ we set

$$(C.3) \quad f_{\check{\lambda}}(t) := \check{\lambda}(\gamma(t)) \cdot \rho_{\check{\lambda}}(\gamma(t)^{-1}).$$

Thus we get a homomorphism of algebraic groups $\mathbb{G}_m \rightarrow \text{Aut}_k(V_{\check{\lambda}})$. It extends to a homomorphism of algebraic monoids

$$f_{\check{\lambda}} : \mathbb{A}^1 \rightarrow \text{End}_k(V_{\check{\lambda}})$$

because γ is dominant and all the weights of T in $V_{\check{\lambda}}$ are $\leq \check{\lambda}$.

C.3.3. *The operator $f_{\check{\lambda}}(0)$.* Set $S_{\check{\lambda}} := f_{\check{\lambda}}(0)$. Then $S_{\check{\lambda}}$ is a T -equivariant idempotent operator $V_{\check{\lambda}} \rightarrow V_{\check{\lambda}}$. The subspaces

$$V'_{\check{\lambda}} := \text{Im}(S_{\check{\lambda}}), \quad V''_{\check{\lambda}} := \text{Ker}(S_{\check{\lambda}})$$

can be described in terms of the decomposition

$$V_{\check{\lambda}} = \bigoplus_{\check{\mu}} V_{\check{\lambda}, \check{\mu}}$$

with respect to the weights of T . Namely,

$$V'_{\check{\lambda}} = \bigoplus_{\check{\mu} \in A} V_{\check{\lambda}, \check{\mu}}, \quad V''_{\check{\lambda}} = \bigoplus_{\check{\mu} \notin A} V_{\check{\lambda}, \check{\mu}},$$

where A is the set of weights $\check{\mu}$ such that $\check{\lambda} - \check{\mu}$ is in the root lattice of M (which is a sublattice in that of G).

C.4. The stabilizers $\text{Stab}_t^{\check{\lambda}}$.

C.4.1. Let $\text{Stab}_t^\lambda \subset G \times G$ denote the stabilizer of $f_\lambda(t) \in \text{End}_k(V_\lambda)$ with respect to the action of $G \times G$ defined in Sect. C.3.1. Explicitly,

$$(C.4) \quad \text{Stab}_t^\lambda = \{(g_1, g_2) \in G \times G \mid \rho_\lambda(g_1) \cdot f_\lambda(t) = f_\lambda(t) \cdot \rho_\lambda(g_2)\}.$$

Our goal is to prove the inclusions (C.2).

C.4.2. *The case $t \neq 0$.* Let $H_\lambda \subset G$ denote the kernel of $\rho_\lambda : G \rightarrow \text{Aut}_k(V_\lambda)$.

Lemma C.4.3. *If $t \neq 0$ then $\text{Stab}_t^\lambda = \tilde{G}_t \cdot (H_\lambda \times H_\lambda)$ and*

$$\bigcap_{\lambda} \text{Stab}_t^\lambda = \tilde{G}_t \cdot \left(\bigcap_{\lambda} (H_\lambda \times H_\lambda) \right).$$

Proof. Recall that $\tilde{G}_t = \{(g_1, g_2) \in G \times G \mid g_2 = \gamma(t) \cdot g_1 \cdot \gamma(t)^{-1}\}$. On the other hand, by (C.3) and (C.4), we have

$$\text{Stab}_t^\lambda = \{(g_1, g_2) \in G \times G \mid \rho_\lambda(g_2) = \rho_\lambda(\gamma(t)) \cdot \rho_\lambda(g_1) \cdot \rho_\lambda(\gamma(t)^{-1})\}.$$

The lemma follows. \square

The inclusions (C.2) for $t \neq 0$ follow from the above lemma and the next one.

Lemma C.4.4. *The group $(\bigcap_{\lambda} H_\lambda)_{\text{red}}$ is trivial.*

Proof. $(\bigcap_{\lambda} H_\lambda)_{\text{red}}$ is a reduced normal subgroup of G , which has trivial intersection with T .

The lemma follows. \square

Remark C.4.5. If k has characteristic 2 and $G = PGL(2)$ then the group scheme $\bigcap_{\lambda} H_\lambda$ has order 4. In fact, it equals the kernel of the homomorphism $g : PGL(2) \rightarrow SL(2)$ such that the composition $SL(2) \rightarrow PGL(2) \xrightarrow{g} SL(2)$ is the Frobenius endomorphism of $SL(2)$.

C.4.6. *The case $t = 0$.* Now let us study Stab_0^λ (i.e., the stabilizer Stab_t^λ for $t = 0$). By (C.4), we have

$$(C.5) \quad \text{Stab}_0^\lambda = \{(g_1, g_2) \in G \times G \mid \rho_\lambda(g_1) \cdot S_\lambda = S_\lambda \cdot \rho_\lambda(g_2)\},$$

where $S_\lambda := f_\lambda(0)$.

Note that the subspace $V'_\lambda \subset V_\lambda$ is M -stable. So we have a homomorphism $M \rightarrow \text{Aut}_k(V'_\lambda)$ and therefore homomorphisms

$$P \twoheadrightarrow M \rightarrow \text{Aut}_k(V'_\lambda), \quad P^- \twoheadrightarrow M \rightarrow \text{Aut}_k(V'_\lambda).$$

Lemma C.4.7. *The intersection $\text{Stab}_0^\lambda \cap (P \times P^-)$ is equal to the fiber product of P and P^- over $\text{Aut}_k(V'_\lambda)$.*

Proof. Follows from (C.5) and the description of S_λ given in Sect. C.3.3. \square

Recall that $\tilde{G}_0 = P \times_M P^-$. The above lemma implies that

$$\left(\bigcap_{\lambda} \text{Stab}_0^\lambda \right) \cap (P \times P^-) = P \times_{M/H} P^-,$$

where

$$H := \bigcap_{\check{\lambda}} \text{Ker}(M \rightarrow \text{Aut}_k(V'_\lambda)).$$

Similarly to Lemma C.4.4, the group H_{red} is trivial. Therefore, to prove (C.2) for $t = 0$ it remains to show that

$$\left(\bigcap_{\check{\lambda}} \text{Stab}_0^{\check{\lambda}} \right)_{\text{red}} \subset P \times P^-.$$

This follows from the next lemma.

Lemma C.4.8. *If $\check{\lambda}$ is strictly dominant then $(\text{Stab}_0^{\check{\lambda}})_{\text{red}} \subset P \times P^-$.*

Proof. Set $V''_\lambda := \text{Ker}(S_{\check{\lambda}})$. It is clear that $\text{Stab}_0^{\check{\lambda}} \subset K \times K^-$, where

$$K := \{g \in G \mid \rho_{\check{\lambda}}(g)(V'_\lambda) = V'_\lambda\}, \quad K^- := \{g \in G \mid \rho_{\check{\lambda}}(g)(V''_\lambda) = V''_\lambda\}.$$

So it remains to show that $K_{\text{red}} = P$, $K_{\text{red}}^- = P^-$. Let us prove that $K_{\text{red}} = P$ (the proof of the other equality is similar).

Clearly $P \subset K$, so K_{red} is a parabolic containing P . Thus if $K_{\text{red}} \neq P$ then K_{red} contains the subgroup $SL(2)$ corresponding to some simple root $\check{\alpha}_i$ of G which is not a root of M . Then the weight $\check{\lambda} - \langle \check{\lambda}, \alpha_i \rangle \cdot \check{\alpha}_i$ should occur in the weight decomposition of V'_λ with respect to T . By the definition of V'_λ , this means that $\langle \check{\lambda}, \alpha_i \rangle = 0$, which is contrary to the assumption that $\check{\lambda}$ is strictly dominant. \square

Thus we have proved Lemma C.1.3 and Proposition 2.4.4.

Remark C.4.9. It is not hard to show that if $\text{char } k \neq 2$ or if the group $[G, G]$ is simply connected then the equality (C.1) can be replaced by the stronger equality

$$(C.6) \quad \bigcap_{\check{\lambda}} \text{Stab}_t^{\check{\lambda}} = \tilde{G}_t.$$

On the other hand, (C.6) does not hold if $\text{char } k = 2$ and $G = PGL(2)$ (this follows from Remark C.4.5).

Remark C.4.10. In Sect. C.3.1 we defined the $(G \times G)$ -module $R_{\check{\lambda}}$ to be equal to $\text{End}_k(V_{\check{\lambda}})$, where $V_{\check{\lambda}}$ is the *irreducible* G -module with highest weight $\check{\lambda}$. The reader may prefer to define $R_{\check{\lambda}}$ by

$$(C.7) \quad R_{\check{\lambda}} := \text{End}_k(\Delta_{\check{\lambda}}) \times \text{End}_k(\nabla_{\check{\lambda}}),$$

where $\Delta_{\check{\lambda}}$ (resp. $\nabla_{\check{\lambda}}$) is obtained from the 1-dimensional B -module corresponding to $\check{\lambda}$ by applying the functor

$$\{B\text{-modules}\} \rightarrow \{G\text{-modules}\}$$

left adjoint (resp. right adjoint) to the restriction functor $\{G\text{-modules}\} \rightarrow \{B\text{-modules}\}$. It is not hard to show that if one defines $R_{\check{\lambda}}$ by (C.7) then the equality (C.6) holds for *any* reductive G and *any* characteristic of k .

APPENDIX D. RELATION TO VINBERG'S SEMI-GROUP

In Sect. D.6 we will explain how the group-scheme \tilde{G} of Sect. 2.4 can be obtained from the Vinberg semi-group corresponding to G (a.k.a. enveloping semigroup of G).

Before doing this, we give a brief review of the standard material on the Vinberg semi-group¹¹, which is contained in [Vi, Re, Ri1, Ri3, Bo]. For the general theory of reductive monoids, see [Pu, Re, Vi] and [Ti, Ch. 5, Sect. 27].

In this Appendix (as in Sect. 2.4) the characteristic of the ground field k can be arbitrary.

D.1. The group G_{enh} .

D.1.1. Let G be a reductive group. Let T denote its *abstract* Cartan. Let $\check{\Lambda}_G = \check{\Lambda}_T$ denote the weight lattice of G (=the lattice of characters of T). It is equipped with the usual partial order relation, denoted by \leq .

D.1.2. Let $Z(G)$ denote the center of G . Consider the group

$$G_{\text{enh}} := (G \times T)/Z(G),$$

where $Z(G)$ maps to $G \times T$ anti-diagonally. Note that $Z(G_{\text{enh}}) = T$.

Set $T_{\text{adj}} := T/Z(G)$. We have a canonical homomorphism of algebraic groups

$$G_{\text{enh}} := (G \times T)/Z(G) \rightarrow T/Z(G) = T_{\text{adj}}.$$

D.2. Vinberg's semi-group: definition.

D.2.1. The Vinberg semi-group of G , denoted by $\overline{G_{\text{enh}}}$, is an affine algebraic monoid containing G_{enh} as its group of invertible elements. Such a monoid is uniquely determined by the full monoidal subcategory

$$\text{Rep}(\overline{G_{\text{enh}}}) \subset \text{Rep}(G_{\text{enh}})$$

consisting of those representations for which the action of G_{enh} extends to that of $\overline{G_{\text{enh}}}$. The coordinate ring of $\overline{G_{\text{enh}}}$ is reconstructed from $\text{Rep}(\overline{G_{\text{enh}}})$ as the set of matrix elements of all representations from $\text{Rep}(\overline{G_{\text{enh}}})$.

The category $\text{Rep}(\overline{G_{\text{enh}}})$ is the following. Note that an object $V \in \text{Rep}(G_{\text{enh}})$ canonically splits as

$$V \simeq \bigoplus_{\check{\mu} \in \check{\Lambda}_T} V_{\check{\mu}}$$

according to the action of $T = Z(G_{\text{enh}})$. Let us also note that by the definition of G_{enh} , the central character of the action of $G \hookrightarrow G_{\text{enh}}$ on each $V_{\check{\mu}}$ equals $\check{\mu}|_{Z(G)}$. Now $\text{Rep}(\overline{G_{\text{enh}}})$ is defined to consist of all $V \in \text{Rep}(G_{\text{enh}})$ with the following property: for every $\check{\mu}$, the *weights* of $V_{\check{\mu}}$, regarded as a G -representation, are $\leq \check{\mu}$.

Note that the set of weights of a given G -representation is invariantly defined, i.e., independent of the choice of the Cartan or Borel subgroups.

D.2.2. L. Renner proved that the affine scheme $\overline{G_{\text{enh}}}$ is of finite type and normal (this follows from [Re, Thm 5.4 (a)]). In the characteristic zero case this is also proved in [Vi].

¹¹It was defined and studied by E. Vinberg [Vi] in characteristic 0 and then by A.Rittatore in arbitrary characteristic, see [Ri1, Ri2, Ri3, Ri4].

D.2.3. The Tannakian formalism provides the following description of the S -points of $\overline{G_{\text{enh}}}$ for any k -scheme S : an element of the monoid $\text{Maps}(S, \overline{G_{\text{enh}}})$ is a collection of assignments

$$(D.1) \quad V \in \text{Rep}(\overline{G_{\text{enh}}}) \rightsquigarrow g_V \in \text{End}_{\mathcal{O}_S}(V \otimes \mathcal{O}_S),$$

compatible with maps $V_1 \rightarrow V_2$ in $\text{Rep}(\overline{G_{\text{enh}}})$ and such that

$$g_{V_1 \otimes V_2} = g_{V_1} \otimes g_{V_2}.$$

D.3. **The homomorphism $\bar{\pi} : \overline{G_{\text{enh}}} \rightarrow \overline{T_{\text{adj}}}$.**

D.3.1. *The algebraic monoid $\overline{T_{\text{adj}}}$.* As before, set $T_{\text{adj}} := T/Z(G)$. Let I denote the set of vertices of the Dynkin diagram of G . The simple root corresponding to $i \in I$ is a homomorphism $T_{\text{adj}} \rightarrow \mathbb{G}_m$. These homomorphisms define a canonical isomorphism

$$T_{\text{adj}} \xrightarrow{\sim} \prod_{i \in I} \mathbb{G}_m.$$

The inverse isomorphism is given by the fundamental coweights $\mathbb{G}_m \rightarrow T_{\text{adj}}$.

Let $\overline{T_{\text{adj}}}$ denote the algebraic monoid $\prod_{i \in I} \mathbb{A}^1$. The open embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ induces a canonical open embedding

$$T_{\text{adj}} \hookrightarrow \overline{T_{\text{adj}}}.$$

D.3.2. *The homomorphism $\bar{\pi}$.* Define a homomorphism of algebraic groups $\pi : G_{\text{enh}} \rightarrow T_{\text{adj}}$ to be the composition

$$G_{\text{enh}} := (G \times T)/Z(G) \rightarrow T/Z(G) = T_{\text{adj}}.$$

The functor $\text{Rep}(T_{\text{adj}}) \rightarrow \text{Rep}(G_{\text{enh}})$ corresponding to π maps $\text{Rep}(\overline{T_{\text{adj}}})$ to $\text{Rep}(\overline{G_{\text{enh}}})$. So π extends to a homomorphism of algebraic monoids

$$(D.2) \quad \bar{\pi} : \overline{G_{\text{enh}}} \rightarrow \overline{T_{\text{adj}}}.$$

Note that

$$\bar{\pi}^{-1}(T_{\text{adj}}) = G_{\text{enh}}.$$

This follows from the fact that for every $V \in \text{Rep}(G_{\text{enh}})$ there exists a character $\chi : T_{\text{adj}} \rightarrow \mathbb{G}_m$ such that $V \otimes (\chi \circ \pi) \in \text{Rep}(\overline{G_{\text{enh}}})$ (here $\chi \circ \pi$ is considered as a 1-dimensional representation of G_{enh}).

D.4. **The non-degenerate locus of Vinberg's semi-group.** In this subsection we recall some facts whose proofs can be found in [Vi, Sect. 8] in the characteristic zero case and in [Ri1, Ri3] in general (there are also some hints in [Bo, Sect. 1.1]).

D.4.1. Recall that $G_{\text{enh}} := (G \times T)/Z(G)$, so a weight of G_{enh} is a pair

$$(\check{\lambda}_1, \check{\lambda}_2) \in \check{\Lambda}_G \times \check{\Lambda}_T$$

such that $\check{\lambda}_1 - \check{\lambda}_2$ belongs to the root lattice. For each dominant $\check{\lambda} \in \Lambda_G$ let $V_{\check{\lambda}, \check{\lambda}} \in \text{Rep}(\overline{G_{\text{enh}}})$ denote the irreducible representation with highest weight $(\check{\lambda}, \check{\lambda})$.

Given a point $g \in G_{\text{enh}}(k)$, let A_g denote the set of dominant weights $\check{\lambda} \in \Lambda_G$ such that the action of g on $V_{\check{\lambda}, \check{\lambda}}$ is *non-zero*.

The following fact is well-known:

Lemma D.4.2. *The following properties of $g \in G_{\text{enh}}$ are equivalent:*

- (i) A_g is the set of all dominant weights of G ;
- (ii) A_g is not contained in any wall of the dominant cone of $\check{\Lambda}_G \otimes \mathbb{Q}$.

It is clear that the set of elements of $G_{\text{enh}}(k)$ with property (ii) corresponds to an open subscheme of $\overline{G_{\text{enh}}}$; we denote it by $\overline{G_{\text{enh}}}^{\circ}$.

Definition D.4.3. The subscheme $\overline{G_{\text{enh}}}^{\circ} \subset \overline{G_{\text{enh}}}$ is called the *non-degenerate locus* of $\overline{G_{\text{enh}}}$.

D.4.4. An element $(g_1, g_2) \in G_{\text{enh}} \times G_{\text{enh}}$ defines a regular map $\overline{G_{\text{enh}}} \rightarrow \overline{G_{\text{enh}}}$ given by

$$x \mapsto g_1 \cdot x \cdot g_2^{-1}.$$

Thus $G_{\text{enh}} \times G_{\text{enh}}$ acts on the variety $\overline{G_{\text{enh}}}$. The open subscheme $\overline{G_{\text{enh}}}^{\circ} \subset \overline{G_{\text{enh}}}$ is clearly preserved under this action.

D.4.5. It is known that the scheme $\overline{G_{\text{enh}}}^{\circ}$ is smooth and moreover, the restriction of the morphism $\overline{\pi} : \overline{G_{\text{enh}}} \rightarrow \overline{T_{\text{adj}}}$ to $\overline{G_{\text{enh}}}^{\circ}$ is smooth.

It is also known that the action of $G \times G$ on $\overline{G_{\text{enh}}}^{\circ}$ is *transitive relative to $\overline{T_{\text{adj}}}$* , in the sense that the action map

$$(D.3) \quad (G \times G) \times \overline{G_{\text{enh}}}^{\circ} \rightarrow \overline{G_{\text{enh}}}^{\circ} \times_{\overline{T_{\text{adj}}}} \overline{G_{\text{enh}}}^{\circ}$$

is surjective and flat. Moreover, it is known that the morphism (D.3) is smooth.

D.4.6. Consider $(G \times G) \times \overline{G_{\text{enh}}}^{\circ}$ as a group-scheme over $\overline{G_{\text{enh}}}^{\circ}$. Define a closed group-subscheme $\text{Stab}_{G \times G} \subset (G \times G) \times \overline{G_{\text{enh}}}^{\circ}$ to be the pre-image of the diagonal $\overline{G_{\text{enh}}}^{\circ} \hookrightarrow \overline{G_{\text{enh}}}^{\circ} \times_{\overline{T_{\text{adj}}}} \overline{G_{\text{enh}}}^{\circ}$ with

respect to the map (D.3). The fiber of $\text{Stab}_{G \times G}$ over $x \in \overline{G_{\text{enh}}}^{\circ}(k)$ is the stabilizer of x in $G \times G$; it will be denoted by $\text{Stab}_{G \times G}(x)$.

It follows from Sect. D.4.5 that the scheme $\text{Stab}_{G \times G}$ is *smooth* over $\overline{G_{\text{enh}}}^{\circ}$.

Remark D.4.7. The torus $T = Z(G_{\text{enh}})$ acts on $\overline{G_{\text{enh}}}$ by translations. It is easy to show that the resulting action of T on $\overline{G_{\text{enh}}}^{\circ}$ is free.¹² It is also easy to show that the quotient $\overline{G_{\text{enh}}}^{\circ}/T$ depends only on $G_{\text{adj}} := G/Z(G)$.

It is well known that this quotient is projective and smooth; this is the *wonderful compactification*¹³ of $G_{\text{adj}} := G/Z(G)$.

It is clear that the subscheme $\text{Stab}_{G \times G}(x) \subset G \times G$ corresponding to $x \in \overline{G_{\text{enh}}}^{\circ}(k)$ depends only on the image of x in $\overline{G_{\text{enh}}}^{\circ}/T$.

¹²Let us note that for most groups G (e.g., for $G = SL(3)$) the set of points of $\overline{G_{\text{enh}}}$ with trivial stabilizer in T is *larger* than $\overline{G_{\text{enh}}}^{\circ}$.

¹³In characteristic 0 the wonderful compactification was defined in [DCP] (one of the equivalent definitions from [DCP] is as follows: consider $\text{diag}(\mathfrak{g}) \subset \mathfrak{g} \times \mathfrak{g}$ as a point of the Grassmannian of $\mathfrak{g} \times \mathfrak{g}$, then take the closure of its $(G \times G)$ -orbit.) Without the characteristic zero assumption, the wonderful compactification was constructed in [Str] and then, in a different way, in [BKu]. An axiomatic definition of the wonderful compactification and a brief survey can be found in [BP, Sect. 1].

D.5. **The section** $\bar{\mathfrak{s}} : \overline{T_{\text{adj}}} \rightarrow \overline{G_{\text{enh}}}$. Now let us choose a Cartan *subgroup* $T_{\text{sub}} \subset G$ and a Borel $B \supset T_{\text{sub}}$. We will construct a section for the map $\bar{\pi} : \overline{G_{\text{enh}}} \rightarrow \overline{T_{\text{adj}}}$, which depends on these choices.

D.5.1. Let $\varphi : T \xrightarrow{\sim} T_{\text{sub}}$ denote the isomorphism corresponding to the chosen Borel $B \supset T_{\text{sub}}$. Define

$$\mathfrak{s} : T_{\text{adj}} = T/Z(G) \rightarrow (T_{\text{sub}} \times T)/Z(G) \hookrightarrow (G \times T)/Z(G) = G_{\text{enh}}$$

to be the homomorphism induced by the map

$$T \rightarrow T_{\text{sub}} \times T, \quad \tau \mapsto (\varphi(\tau)^{-1}, \tau).$$

The composition $T_{\text{adj}} \xrightarrow{\mathfrak{s}} G_{\text{enh}} \xrightarrow{\bar{\pi}} T_{\text{adj}}$ equals the identity.

Lemma D.5.2.

(i) *The homomorphism* $\mathfrak{s} : T_{\text{adj}} \rightarrow G_{\text{enh}}$ *extends to a homomorphism of algebraic monoids*

$$\bar{\mathfrak{s}} : \overline{T_{\text{adj}}} \rightarrow \overline{G_{\text{enh}}}.$$

(ii) $\bar{\mathfrak{s}}(\overline{T_{\text{adj}}}) \subset \overset{\circ}{\overline{G_{\text{enh}}}}$.

Proof. (i) It suffices to show that the functor $\text{Rep}(G_{\text{enh}}) \rightarrow \text{Rep}(T_{\text{adj}})$ corresponding to the homomorphism $\mathfrak{s} : T_{\text{adj}} \rightarrow G_{\text{enh}}$ maps $\text{Rep}(\overline{G_{\text{enh}}})$ to $\text{Rep}(\overline{T_{\text{adj}}})$.

Set

$$T_{\text{enh}} := (T \times T)/Z(G) \xrightarrow{\varphi \times \text{id}} (T_{\text{sub}} \times T)/Z(G) \subset G_{\text{enh}}.$$

The weights of T_{enh} are pairs $(\check{\lambda}_1, \check{\lambda}_2) \in \check{\Lambda}_T \times \check{\Lambda}_T$, such that $\lambda_1|_{Z(G)} = \lambda_2|_{Z(G)}$.

Let $V \in \text{Rep}(\overline{G_{\text{enh}}})$. Then each weight $(\check{\lambda}_1, \check{\lambda}_2)$ of T_{enh} on V satisfies $\lambda_1 \leq \check{\lambda}_2$ (by the definition of $\overline{G_{\text{enh}}}$).

By the definition of \mathfrak{s} , the weights of T_{adj} on V have the form $\check{\lambda}_2 - \check{\lambda}_1$, where $\check{\lambda}_1$ and $\check{\lambda}_2$ are as above. So they are ≥ 0 . Therefore the action of T_{adj} on V extends to an action of $\overline{T_{\text{adj}}}$.

(ii) If $V = V_{\check{\lambda}, \check{\lambda}}$ then one of the weights of $\overline{T_{\text{adj}}}$ in V equals $\check{\lambda} - \check{\lambda} = 0$, so $\overline{T_{\text{adj}}}$ acts on this subspace by the identity endomorphism. \square

It is clear that the map $\bar{\mathfrak{s}} : \overline{T_{\text{adj}}} \rightarrow \overset{\circ}{\overline{G_{\text{enh}}}} \subset \overline{G_{\text{enh}}}$ defined in the lemma is a section for the projection $\bar{\pi} : \overline{G_{\text{enh}}} \rightarrow \overline{T_{\text{adj}}}$. This section depends on the choice of T_{sub} and B .

D.5.3. Let $\text{Stab}_{G \times G}(\bar{\mathfrak{s}})$ denote the $\bar{\mathfrak{s}}$ -pullback of the subscheme $\text{Stab}_{G \times G} \subset \overset{\circ}{\overline{G_{\text{enh}}}} \times (G \times G)$ defined in Sect. D.4.6. Clearly $\text{Stab}_{G \times G}(\bar{\mathfrak{s}})$ is a closed group-subscheme of $\overline{T_{\text{adj}}} \times (G \times G)$ viewed as a group-scheme over $\overline{T_{\text{adj}}}$.

Corollary D.5.4. *$\text{Stab}_{G \times G}(\bar{\mathfrak{s}})$ is smooth over $\overline{T_{\text{adj}}}$. The quotient*

$$(D.4) \quad (\overline{T_{\text{adj}}} \times (G \times G)) / \text{Stab}_{G \times G}(\bar{\mathfrak{s}})$$

exists and is isomorphic to $\overset{\circ}{\overline{G_{\text{enh}}}}$ as a scheme over $\overline{T_{\text{adj}}}$ equipped with an action of $G \times G$ and a section. In particular, the quotient is a quasi-affine scheme of finite type.

Proof. Follows from Sects. D.4.5-D.4.6. \square

D.6. **Vinberg's semigroup and the "interpolating" group-scheme \tilde{G} .**

D.6.1. Let $P, P^- \subset G$ be parabolics opposite to each other and $M := P \cap P^-$. Let

$$\gamma : \mathbb{G}_m \rightarrow Z(M)$$

be a co-character dominant and regular with respect to P . Using these data we defined in Sect. 2.4.2 a closed group-subscheme $\tilde{G} \subset \mathbb{A}^1 \times G \times G$. On the other hand, in Sect. D.4.6 we defined a closed group-subscheme $\text{Stab}_{G \times G} \subset (G \times G) \times \overline{G_{\text{enh}}} = \overline{G_{\text{enh}}} \times (G \times G)$. We will show that \tilde{G} can, in fact, be obtained from $\text{Stab}_{G \times G}$ by base change with respect to a certain map $\mathbb{A}^1 \rightarrow \overline{G_{\text{enh}}}$ (see Proposition D.6.4 below).

D.6.2. Choose a Cartan subgroup $T_{\text{sub}} \subset G$ contained in M ; then $T_{\text{sub}} \supset Z(M)$. Choose a Borel $B \subset G$ so that $T_{\text{sub}} \subset B \subset P$. Then T_{sub} identifies with T .

The composition

$$\mathbb{G}_m \xrightarrow{\gamma} Z(M) \hookrightarrow T_{\text{sub}} \xrightarrow{\sim} T \rightarrow T_{\text{adj}}$$

is a *dominant* co-character $\mathbb{G}_m \rightarrow T_{\text{adj}}$, so it extends to a homomorphism of algebraic monoids

$$(D.5) \quad \mathbb{A}^1 \rightarrow \overline{T_{\text{adj}}}.$$

Composing (D.5) with the map $\bar{\mathfrak{s}} : \overline{T_{\text{adj}}} \rightarrow \overline{G_{\text{enh}}}$ defined in Lemma D.5.2, we get a map

$$\gamma' : \mathbb{A}^1 \rightarrow \overline{G_{\text{enh}}}.$$

D.6.3. Let $\text{Stab}_{G \times G}(\gamma')$ denote the γ' -pullback of the subscheme $\text{Stab}_{G \times G} \subset \overline{G_{\text{enh}}} \times (G \times G)$ defined in Sect. D.4.6. Clearly $\text{Stab}_{G \times G}(\gamma')$ is a closed group-subscheme of $\mathbb{A}^1 \times (G \times G)$ viewed as a group-scheme over \mathbb{A}^1 .

Proposition D.6.4. *The group-subscheme $\tilde{G} \subset \mathbb{A}^1 \times (G \times G)$ defined in Sect. 2.4.2 is equal to $\text{Stab}_{G \times G}(\gamma')$.*

Proof. Both $\text{Stab}_{G \times G}(\gamma')$ and \tilde{G} are closed group-subschemas of $\mathbb{A}^1 \times (G \times G)$. Both are flat (and in fact, smooth) over \mathbb{A}^1 : for \tilde{G} this was proved in Proposition 2.3.8(i) and for $\text{Stab}_{G \times G}(\gamma')$ this follows from smoothness of $\text{Stab}_{G \times G}$ over $\overline{G_{\text{enh}}}$ (see Sect. D.4.6). So it suffices to check that the two group-subschemas have the same fiber over any $t \in \mathbb{G}_m$.

By formula (2.11), the subscheme $\tilde{G}_t \subset G \times G$ is defined by the equation

$$(D.6) \quad \gamma(t) \cdot g_1 \cdot \gamma(t)^{-1} = g_2, \quad g_1, g_2 \in G.$$

On the other hand, the fiber of $\text{Stab}_{G \times G}(\gamma')$ over $t \in \mathbb{G}_m$ is the stabilizer of $\gamma'(t) \in \overline{G_{\text{enh}}}$ with respect to the action of $(G \times G)$ on $\overline{G_{\text{enh}}}$ defined in Sect. D.4.4; so this fiber is defined by the equation

$$(D.7) \quad g_1 \cdot \gamma'(t) \cdot g_2^{-1} = \gamma'(t), \quad g_1, g_2 \in G.$$

By the definition of \mathfrak{s} from Sect. D.5.1, $\gamma'(t)$ differs from $\gamma(t)^{-1}$ only by an element of $Z(G_{\text{enh}})$. So (D.6) and (D.7) are equivalent. \square

Corollary D.6.5. *The quotient $(\mathbb{A}^1 \times G \times G)/\tilde{G}$ exists and is a quasi-affine scheme of finite type over \mathbb{A}^1 .*

Proof. By Proposition D.6.4, we can rewrite $(\mathbb{A}^1 \times G \times G)/\widetilde{G}$ as

$$(D.8) \quad (\mathbb{A}^1 \times G \times G)/\text{Stab}_{G \times G}(\gamma').$$

By Corollary D.5.4, the quotient (D.4) is a quasi-affine scheme of finite type over $\overline{T}_{\text{adj}}$. The quotient (D.8) is obtained from (D.4) by base change with respect to the map (D.5), so it is also quasi-affine. \square

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