

RECENT PROGRESS IN GEOMETRIC LANGLANDS THEORY

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ABSTRACT. This is the English version of the text of the talk at Séminaire Bourbaki on February 16, 2016.

1. INTRODUCTION

Throughout the talk we fix X to be a smooth connected complete curve and G a reductive group over a ground field k .

When discussing connections with the classical (function-theoretic) Langlands theory, we will assume that $k = \mathbb{F}_q$. When talking about the *categorical* geometric Langlands theory, we will take k to be characteristic zero.

1.1. Some history. What is nowadays known as the *geometric Langlands theory* originated from the ideas of four people: A. Beilinson, P. Deligne, V. Drinfeld and G. Laumon.

1.1.1. The first input was Deligne's observation that one can prove the existence of the grossen-character corresponding to an unramified character of the Galois of a function field using algebro-geometric considerations. The idea is the following.

An (unramified) grossen-character can be thought of as a function on the set (rather, groupoid) of \mathbb{F}_q -points of the Picard stack $\text{Pic}(X)$ of X . We will do the construction in two steps. First, starting from an unramified Galois character σ , we will construct an ℓ -adic sheaf on $\text{Pic}(X)$, denoted \mathcal{F}_σ . Then the sought-for grossen-character will be obtained from \mathcal{F}_σ by Grothendieck's sheaves-functions correspondence, i.e., by taking traces of the Frobenius.

The construction of \mathcal{F}_σ is geometric. Namely, we interpret σ as a 1-dimensional ℓ -adic local system on X , denoted E_σ . To E_σ and $d \geq 0$, we attach the *symmetric power*, denoted $E_\sigma^{(d)}$, which is a 1-dimensional local system on the scheme $X^{(d)}$ parameterizing effective divisors on X of degree d . (The sheaf $E_\sigma^{(d)}$ is a natural thing to do from the number-theoretic point of view: the function attached to it is the function corresponding to σ on the set of effective divisors.)

Now, we consider the Abel-Jacobi map

$$X^{(d)} \rightarrow \text{Pic}(X)$$

and the task is to show that there exists an ℓ -adic sheaf \mathcal{F}_σ on $\text{Pic}(X)$ that pulls back to $E_\sigma^{(d)}$ for each d . It is easy to see that it is enough to prove the existence of \mathcal{F}_σ over the connected components $\text{Pic}^d(X)$ of $\text{Pic}(X)$ for d large (i.e., $d \geq d_0$ for some fixed d_0).

The punchline is that for $d > 2g - 2$ (here g is the genus of X), the map $X^{(d)} \rightarrow \text{Pic}^d(X)$ is a smooth fibration with simply-connected fibers, which guarantees the existence (and uniqueness) of the descent of $E_\sigma^{(d)}$ to the sought-for ℓ -adic sheaf \mathcal{F}_σ^d on $\text{Pic}^d(X)$.

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1.1.2. Then came Drinfeld’s ground-breaking paper [Dr]. In a sense it was the extension of Deligne’s construction to the vastly more complicated case, when instead of grossen-characters we consider unramified automorphic functions for the group GL_2 . Here again, we interpret the (unramified) automorphic space as the groupoid of \mathbb{F}_q -points of the moduli space $\text{Bun}_2 := \text{Bun}_{GL_2}$ classifying rank-2 vector bundles on X . Drinfeld’s idea is to attach to a 2-dimensional Galois representation σ an ℓ -adic sheaf (by which we actually mean an object of the corresponding derived category) \mathcal{F}_σ on Bun_2 , and then obtain the sought-for function by taking the traces of the Frobenius.

The main difference from the commutative case, considered by Deligne (which corresponds to the case of the group $\mathbb{G}_m = GL_1$), is that the construction of \mathcal{F}_σ starting from σ is much more involved. The intermediate player, i.e., $E_\sigma^{(d)}$, is now interpreted as the ℓ -adic sheaf that records the *Whittaker* (a.k.a., Fourier) coefficients of \mathcal{F}_σ . So, our task is to reconstruct an automorphic object from its Fourier coefficients. This is again done via appealing to geometry—ultimately the simply-connectedness of fibers of some map.

1.1.3. After Drinfeld’s paper came one by Laumon, [Lau1], which gave a conjectural extension of Drinfeld’s construction from GL_2 to GL_n . To the best of our knowledge, the title of Laumon’s paper was the first place where the combination of words ‘geometric Langlands’ appeared.

While the stated goal of Drinfeld’s paper was to construct an automorphic *function*, Laumon’s paper had the effect of shifting the goal: people became interested in automorphic *sheaves* (ℓ -adic sheaves on $\text{Bun}_n(X)$) for their own sake.

Following the appearance of Laumon’s paper, it became clear that one should also try to attack $\text{Bun}_G(X)$ for an arbitrary reductive G , even though it was not clear how to do this (because the Whittaker model does not work as nicely outside the case of $G = GL_n$).

1.1.4. The next paradigm shift came in the work of Beilinson and Drinfeld, [BD]. They considered the same $\text{Bun}_G(X)$, but now over a ground field k of characteristic zero, and instead of ℓ -adic sheaves, they proposed to consider D-modules.

In this case, a new method for constructing objects becomes available: by generators and relations. A fancy version of ‘generators and relations’ principle—the localization functor, pioneered in [BB], lies in the core of the manuscript [BD], which produces automorphic D-modules using representations of the Kac-Moody Lie algebra (thought of as the Lie algebra of infinitesimal symmetries of a G -bundle on the formal punctured disk).

1.1.5. In an independent development, in [Lau2], Laumon showed that if we take G to be a torus T , a generalized version of the Fourier-Mukai transform identifies the (derived) category of D-modules on the stack $\text{Bun}_T(X)$ with the (derived) category of quasi-coherent sheaves on the stack $\text{LocSys}_{\check{T}}(X)$ of *de Rham local systems* on X with respect to the Langlands dual torus \check{T} .

I.e., Laumon’s paper extends the pointwise Langlands correspondence (i.e., construction of \mathcal{F}_σ corresponding to a fixed local system σ) to a statement about the universal family of local systems.

1.1.6. Finally, combining Laumon’s equivalence for the torus, and accumulated evidence for the general G , Beilinson and Drinfeld came up with the idea of *categorical geometric Langlands equivalence*.

In its crude form, this should be an equivalence between the (derived) category $D(\mathrm{Bun}_G(X))$ of D-modules on the stack $\mathrm{Bun}_G(X)$ and the (derived) category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}(X)$. Such an equivalence is what Beilinson and Drinfeld called the *best hope*, but they never stated it explicitly, because it is (and was) known that it cannot hold *as-is* beyond the case of a torus (the reason for this will be indicated in Sect. 3.1.2).

1.2. What do we mean by ‘geometric Langlands’ nowadays? There are several meta-problems that comprise what one can call the *geometric Langlands theory*; we shall list some of them below; the order in which they will appear reflects (our perception of) the historical development (and the increasing level of technical complexity) rather than how the complete picture should ultimately look like (e.g., we do think that the quantum case is more fundamental than the usual one).

We will only consider the *categorical* geometric Langlands theory; in particular we will assume that the ground field k is of characteristic zero, and on the automorphic side we will work with D-modules rather than ℓ -adic sheaves.

We should remark that whatever conjectures and meta-conjectures we mention below, they are all theorems when the group G is a torus, thanks to the various generalizations of the Deligne-Fourier-Mukai-Laumon transform.

1.2.1. First, we have the *categorical¹ global unramified* geometric Langlands. This is an attempt to formulate and prove a version of the *best hope* by Beilinson and Drinfeld, mentioned above. I.e., we want a category that is a close cousin (or identical twin) of $D(\mathrm{Bun}_G(X))$ to be equivalent to a category that is a close cousin of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

This is the aspect of the geometric Langlands theory that has been developed the most. It will be discussed in Sects. 2 and 3.

1.2.2. Next there is the *local ramified* geometric Langlands theory. Unlike the global case, in the local version we are interested in an equivalence of *2-categories* (rather than 1-categories, i.e., just categories). For a long time it was not even clear how to formulate our wish (specifically, what 2-category to consider on the Galois side). However, recently, a breakthrough has been achieved in the work of S. Raskin, [Ras]. We will discuss this in Sect. 4.

We should also mention that the tamely ramified case of the local ramified geometric Langlands had been settled by R. Bezrukavnikov in [Bez] even before the general program was formulated.

1.2.3. Next, there is the global ramified Langlands theory. Its tamely ramified case has not been explicitly studied in detail, but the current state of knowledge should allow to bring it to the same status as the unramified case.

The general ramified case is wide-open, and there are formidable technical difficulties that one needs to surmount in order to start investigating it. One of the difficulties is that we do not know whether the category of D-modules on the automorphic side, i.e., the (derived) category of D-modules on the moduli space $\mathrm{Bun}_G(X)^{k,x}$ of G -bundles on X equipped with structure of level $k \geq 1$ at a point x is compactly generated².

¹From now on, we will drop the adjective ‘categorical’, because everything will be categorical.

²If one surveys the literature, in most of the statements that involve an equivalence of two triangulated/DG categories, the categories of question are compactly generated. The reason is that we do not know very well how to compute things outside the compactly generated case.

1.2.4. Finally, all of the above three aspects: unramified global, ramified local and ramified global admit *quantum* versions. The quantum parameter in the *quantum geometric Langlands theory* is a non-degenerate W -invariant symmetric bilinear form on the Cartan Lie algebra \mathfrak{h} of \mathfrak{g} , whose inverse is a similar kind of datum for $\check{\mathfrak{g}}$.

At the risk of making a controversial statement, the author/speaker has to admit that he came to regard the quantum version as the ultimate reason of ‘why something like the Langlands theory takes place’, with the usual geometric Langlands being its degeneration (letting the quantum parameter tend to zero), and the classical (i.e., function-theoretic) Langlands theory as some sort of residual phenomenon.

We will not say anything about the quantum case in this talk, but rather refer the reader to [Ga2], where the dream of *quantum geometric Langlands* is discussed. Here we will only mention the following two facts.

One is that in the quantum theory restores the symmetry between G and \check{G} : we no longer have the Galois side, but rather both sides are automorphic, but twisted by the quantum parameter.

The other is that the guiding principle of the quantum theory is that ‘Whittaker is dual to Kac-Moody’, which is striking because ‘Whittaker’ has a classical (i.e., number-theoretic) meaning, while ‘Kac-Moody’ does not.

1.3. Terminology and notation.

1.3.1. The global unramified geometric Langlands conjecture can be formulated as an equivalence of (triangulated) categories. But if one wants to dig a tiny bit deeper into attempts of its proof, one needs to work with ∞ -categories—in this case with (k -linear) DG categories. We refer the reader to [GR1, Sect. 10] for the definition of the latter.

For the reader not familiar with ∞ -categories, we recommend the following approach. On the first pass pretend that there no difference between ∞ -categories and ordinary categories. On the second pass pretend that you already know what ∞ -categories are and stay tuned for the language used when working with them (a survey of the syntax of ∞ -categories can be found in [Lu, Sect. 1], or from a somewhat different perspective, in [GR1, Sect. 1]). On the third pass...learn the theory properly!

1.3.2. Another piece of ‘bad news’ is that when working on the Galois side of the geometric Langlands theory, we cannot stay within the realm of classical algebraic geometry, and one needs to plunge oneself into the world of DAG—derived algebraic geometry. For example, the stack $\text{LocSys}_{\check{G}}(X)$ has a non-trivial derived structure for $G = T$ being a torus. The reader is referred to [GR2] for an introduction to DAG.

In what follows, when we say ‘scheme’ or ‘algebraic stack’, we will tacitly mean the corresponding derived notions.

1.3.3. To a scheme or algebraic stack Y one attaches the DG category $\text{QCoh}(Y)$; we will somewhat abusively refer to it as the (*derived*) *category of quasi-coherent sheaves on*³ Y . We refer the reader to [GR3] for the definition. We note, however, that the definition of $\text{QCoh}(Y)$ is much more general: it makes sense for Y which is an arbitrary *prestack*⁴.

³When Y is a *classical* (as opposed to derived) scheme or a sufficiently nice algebraic stack, this category is the derived category of its heart with respect to a naturally defined t-structure.

⁴But in this more general setting, $\text{QCoh}(Y)$ is not at all the *derived* category of any abelian category, even if Y itself is a classical prestack.

1.3.4. If Y is a scheme/algebraic stack/prestack locally of finite type over k , one can attach to it the DG category $D(Y)$; we will also somewhat abusively refer to it as the *(derived) category of D -modules on*⁵ Y . We refer the reader to [GR4] for the definition.

The good news is that when discussing $D(Y)$, the derived structure on Y plays no role. So, on the automorphic side of the geometric Langlands theory we can stay within the realm of classical algebraic geometry.

1.3.5. Whenever we talk about functors between (derived) categories of sheaves/ D -modules on various spaces (!- and *- direct and inverse images), we will always mean the corresponding derived functors. I.e., abelian categories will not appear unless explicitly stated otherwise.

1.3.6. For the motivational parts of the talk (i.e., analogies with the function-theoretic situation), we will assume the reader's familiarity with the basics of algebraic number theory (adèles, ramification, Frobenius elements, etc.)

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2. HECKE ACTION

The classical Langlands correspondence, and historically also the geometric one, were characterized by relating the spectrum of the action of the Hecke operators (resp., functors) on the automorphic side to a Galois datum. We begin by discussing this aspect of the theory.

2.1. **Hecke action on automorphic functions.** Let \mathcal{K} be the field of rational functions on our curve X ; let \mathbb{A} be the ring of adèles, and $\mathbb{O} \subset \mathbb{A}$ the subring of integral adèles. For a place $x \in X$, we let $\mathcal{O}_x \subset \mathcal{K}_x$ denote the corresponding local ring and local field, respectively.

2.1.1. The automorphic space is by definition the quotient $G(\mathbb{A})/G(\mathcal{K})$. It is acted on by left translations by the group $G(\mathbb{A})$. The unramified automorphic space is the set (but, properly speaking, *groupoid*)

$$G(\mathbb{O}) \backslash G(\mathbb{A}) / G(\mathcal{K}).$$

Our object of study is the space $\text{Autom}(X)$ of unramified \mathbb{Q}_ℓ -valued automorphic functions, i.e., functions on $G(\mathbb{O}) \backslash G(\mathbb{A}) / G(\mathcal{K})$, or, which is the same, the space of $G(\mathbb{O})$ -invariant functions on $G(\mathbb{A}) / G(\mathcal{K})$.

2.1.2. Since the subgroup $G(\mathbb{O}) \subset G(\mathbb{A})$ is not normal, we do not have an action of $G(\mathbb{A})$ on $\text{Autom}(X)$. Instead, the action of $G(\mathbb{A})$ on $G(\mathbb{A}) / G(\mathcal{K})$ induces an action on $\text{Autom}(X)$ of the *spherical Hecke algebra* $\mathbb{H}(G)_X$. By definition, as a vector space, $\mathbb{H}(G)_X$ consists of compactly supported $G(\mathbb{O})$ -biinvariant functions on $G(\mathbb{A})$, and it is endowed with a structure of associative algebra via the operation of *convolution*.

The datum of the action of $\mathbb{H}(G)_X$ is equivalent to that of a family of pairwise commuting actions of the local Hecke algebras $\mathbb{H}(G)_x$ for every place x of X , where each $\mathbb{H}(G)_x$ is the algebra (with respect to convolution) of $G(\mathcal{O}_x)$ -biinvariant compactly supported functions on $G(\mathcal{K}_x)$.

Our interest is to find the spectrum of $\mathbb{H}(G)_X$ (i.e., the joint spectrum of the algebras $\mathbb{H}(G)_x$) acting on $\text{Autom}(X)$.

⁵If Y is a scheme, this *is* the derived category of its heart, but this is no longer true for algebraic stacks.

2.1.3. Fix $x \in X$. The first basic fact about the associative algebra $\mathbb{H}(G)_x$ is that it is actually commutative. But, in addition to this, we can actually describe it very explicitly.

Namely, the classical Satake isomorphism says that $\mathbb{H}(G)_x$ identifies canonically with the algebra of ad-invariant regular functions on the algebraic group \check{G} , where \check{G} is the Langlands dual of G , thought of as an algebraic group over \mathbb{Q}_ℓ .

2.1.4. The Satake isomorphism allows us to give a formulation of *Langlands correspondence*.

Namely, given a unramified representation σ of the Galois group of \mathcal{K} into \check{G} (defined up to conjugation) and a common eigenvector $f \in \text{Autom}(X)$ of the algebras $\mathbb{H}(G)_x$, we shall say that f corresponds to σ , if for every $x \in X$, the character by which $\mathbb{H}(G)_x$ acts on f is given, in terms of the Satake isomorphism, by *evaluation* of functions on \check{G} on the conjugacy class of the image of Frob_x under the map σ . Here Frob_x is the Frobenius at $x \in X$, which is a well-defined conjugacy class in the unramified quotient of the Galois group of \mathcal{K} .

2.1.5. The recent result of V. Lafforgue (known as the Automorphic \Rightarrow Galois direction of Langlands correspondence, see [VLaf]) says that for every eigenvector f (assumed cuspidal) *there exists* a σ to which this f corresponds. In this case of $G = GL_n$ the existence statement can be strengthened to one about uniqueness and surjectivity, due to the work of L. Lafforgue [LLaf].

2.2. Geometric Satake and Hecke eigensheaves. We now pass to considering the Hecke action in the geometric context, i.e., when instead of functions on the automorphic space we consider the appropriately defined derived category $\text{D}(\text{Bun}_G(X))$ of ℓ -adic sheaves/D-modules on the automorphic stack $\text{Bun}_G(X)$.

2.2.1. The initial observation is the *geometric Satake* equivalence of Lusztig-Drinfeld-Ginzburg-Mirković-Vilonen (historical order) that says that for every point $x \in X$ the monoidal category $\text{Rep}(\check{G})$ of algebraic representations of \check{G} acts on the category $\text{D}(\text{Bun}_G(X))$.

Thus, we obtain the Hecke functors

$$H_{V,x} : \text{D}(\text{Bun}_G(X)) \rightarrow \text{D}(\text{Bun}_G(X)), \quad x \in X, V \in \text{Rep}(\check{G}).$$

This is the geometric replacement of the $\mathbb{H}(G)_x$ -action on $\text{Autom}(X)$ combined with the Satake isomorphism of Sect. 2.1.3.

However, in geometry one can do much more: one can make the point x move along X . Thus, for every $V \in \text{Rep}(\check{G})$ we obtain the Hecke functor

$$H_V : \text{D}(\text{Bun}_G(X)) \rightarrow \text{D}(\text{Bun}_G(X) \times X).$$

2.2.2. But in fact, one can do even more than that. Let us take a pair of objects $V_1, V_2 \in \text{Rep}(\check{G})$. To them we can canonically attach the functor

$$H_{V_1, V_2} : \text{D}(\text{Bun}_G(X)) \rightarrow \text{D}(\text{Bun}_G(X) \times X \times X),$$

which is the composition

$$\text{D}(\text{Bun}_G(X)) \xrightarrow{H_{V_1}} \text{D}(\text{Bun}_G(X) \times X) \xrightarrow{H_{V_2}} \text{D}(\text{Bun}_G(X) \times X \times X)$$

and *also* the composition

$$\text{D}(\text{Bun}_G(X)) \xrightarrow{H_{V_2}} \text{D}(\text{Bun}_G(X) \times X) \xrightarrow{H_{V_1}} \text{D}(\text{Bun}_G(X) \times X \times X),$$

up to the permutation of the factors in $X \times X$.

A key property of this functor is that its composition with the restriction functor

$$(\mathrm{id}_{\mathrm{Bun}_G(X)} \times \mathrm{diag}_X)^\dagger : \mathrm{D}(\mathrm{Bun}_G(X) \times X \times X) \rightarrow \mathrm{D}(\mathrm{Bun}_G(X) \times X)$$

identifies with the functor $\mathrm{H}_{V_1 \otimes V_2}$.

2.2.3. Let us now say the same but slightly more generally and abstractly. Let I be a non-empty finite set, and let V_I be an I -tuple of objects of $\mathrm{Rep}(\check{G})$

$$(i \in I) \rightsquigarrow V_i \in \mathrm{Rep}(\check{G}).$$

To this datum we attach a functor

$$\mathrm{H}_{V_I} : \mathrm{D}(\mathrm{Bun}_G(X)) \rightarrow \mathrm{D}(\mathrm{Bun}_G(X) \times X^I).$$

When I is a singleton, we recover the functor H_V .

The assignment $I \mapsto \mathrm{H}_{V_I}$ is compatible with the operation of disjoint union of finite sets: for $I = I_1 \sqcup I_2$, the functor H_{V_I} identifies with

$$\mathrm{D}(\mathrm{Bun}_G(X)) \xrightarrow{\mathrm{H}_{V_{I_1}}} \mathrm{D}(\mathrm{Bun}_G(X) \times X^{I_1}) \xrightarrow{\mathrm{H}_{V_{I_2}}} \mathrm{D}(\mathrm{Bun}_G(X) \times X^{I_1} \times X^{I_2}) \simeq \mathrm{D}(\mathrm{Bun}_G(X) \times X^I).$$

2.2.4. Let us now be given a surjection of finite sets $\phi : I \twoheadrightarrow J$. Given $V_I : I \rightarrow \mathrm{Rep}(\check{G})$ we can create $\phi(V_I) =: V_J$ by

$$(2.1) \quad V_j = \bigotimes_{i \in \phi^{-1}(j)} V_i.$$

Let diag_ϕ denote the map $X^J \rightarrow X^I$, corresponding to ϕ . Then the composition

$$\mathrm{D}(\mathrm{Bun}_G(X)) \xrightarrow{\mathrm{H}_{V_I}} \mathrm{D}(\mathrm{Bun}_G(X) \times X^I) \xrightarrow{(\mathrm{id}_{\mathrm{Bun}_G(X)} \times \mathrm{diag}_\phi)^\dagger} \mathrm{D}(\mathrm{Bun}_G(X) \times X^J)$$

identifies with H_{V_J} .

2.2.5. We will now perform one more manipulation. Let \mathcal{M}_I be an object of $\mathrm{D}(X^I)$. We define the *endo-functor* $\mathrm{H}_{V_I, \mathcal{M}_I}$ of $\mathrm{D}(\mathrm{Bun}_G(X))$ to be the composition of H_{V_I} , followed by the functor

$$\mathrm{D}(\mathrm{Bun}_G(X) \times X^I) \rightarrow \mathrm{D}(\mathrm{Bun}_G(X)), \quad \mathcal{F} \mapsto (\mathrm{pr}_{\mathrm{Bun}_G(X)})_! (\mathcal{F} \overset{!}{\otimes} (\mathrm{pr}_{X^I})^! (\mathcal{M}_I)).$$

Here $\mathrm{pr}_{\mathrm{Bun}_G(X)}$ and pr_{X^I} denote the two projections

$$\mathrm{Bun}_G(X) \leftarrow \mathrm{Bun}_G(X) \times X^I \rightarrow X^I,$$

and $\overset{!}{\otimes}$ is the $!$ -tensor product of sheaves/D-modules (the $!$ -pullback of the external tensor product by the diagonal morphism).

2.2.6. For $I = I_1 \sqcup I_2$ and $\mathcal{M} = \mathcal{M}_1 \boxtimes \mathcal{M}_2 \in \mathrm{D}(X^I) \simeq \mathrm{D}(X^{I_1} \times X^{I_2})$, we have

$$\mathrm{H}_{V_I, \mathcal{M}_I} \simeq \mathrm{H}_{V_{I_1}, \mathcal{M}_{I_1}} \circ \mathrm{H}_{V_{I_2}, \mathcal{M}_{I_2}} \simeq \mathrm{H}_{V_{I_2}, \mathcal{M}_{I_2}} \circ \mathrm{H}_{V_{I_1}, \mathcal{M}_{I_1}}.$$

For $\phi : I \twoheadrightarrow J$ and $\mathcal{M}_J \in \mathrm{D}(X^J)$ we have

$$\mathrm{H}_{V_I, (\mathrm{diag}_\phi)_! (\mathcal{M}_J)} \simeq \mathrm{H}_{V_J, \mathcal{M}_J}.$$

2.2.7. Now, the collection of (I, V_I, \mathcal{M}_I) can be glued to a category⁶, by imposing the following family of relations⁷

$$(I, V_I, \mathcal{M}_I) \simeq (J, V_J, \mathcal{M}_J)$$

each time we have

$$\phi : I \twoheadrightarrow J, \quad \mathcal{M}_I = (\text{diag}_\phi)_!(\mathcal{M}_J), \quad V_I = \phi(V_J).$$

We denote this category by $\text{Rep}(\check{G}, \text{Ran}(X))$ (see [Ga3, Sect. 4.2] for another approach to defining $\text{Rep}(\check{G}, \text{Ran}(X))$).

The operation of disjoint union of finite sets induces on $\text{Rep}(\check{G}, \text{Ran}(X))$ a structure of *non-unital* (symmetric) monoidal category.

2.2.8. The upshot of all the preceding discussion is that, ultimately, the geometric Hecke action amounts to the action of the monoidal category $\text{Rep}(\check{G}, \text{Ran}(X))$ on $\text{D}(\text{Bun}_G(X))$.

2.3. The spectral decomposition. In this subsection we will assume that k is of characteristic zero, and we will take $\text{D}(-)$ to mean the derived category of D-modules.

We consider the stack $\text{LocSys}_{\check{G}}(X)$ of de Rham \check{G} -local systems on X . We will explain the counterpart within the geometric Langlands theory of V. Lafforgue's theorem mentioned in Sect. 2.1.5, which heuristically means that the spectrum of the Hecke functors on $\text{D}(\text{Bun}_G(X))$ is contained in $\text{LocSys}_{\check{G}}(X)$.

2.3.1. First we fix a point $x \in X$ and an object $V \in \text{Rep}(\check{G})$. To this data we associate a coherent sheaf (in fact, a vector bundle) on $\text{LocSys}_{\check{G}}(X)$, denoted $\text{Ev}_{V,x}$.

Namely, the fiber of $\text{Ev}_{V,x}$ at a point $\sigma \in \text{LocSys}_{\check{G}}(X)$ is $(V^\sigma)_x$, where V^σ is the local system (=lisse D-module) associated to the \check{G} -representation V and the \check{G} -local system σ , and $(-)_x$ denotes taking the !-fiber at x .

More generally, given a finite set I and V_I as in Sect. 2.2.3, we can associate to this data an object Ev_{V_I} , which is a quasi-coherent sheaf on $\text{LocSys}_{\check{G}}(X) \times X^I$, *equipped with a connection along X^I* .

Hence, given in addition $\mathcal{M}_I \in \text{D}(X^I)$, we can produce

$$(2.2) \quad \text{Ev}_{V_I, \mathcal{M}_I} \in \text{QCoh}(\text{LocSys}_{\check{G}}(X))$$

by taking the de Rham direct image of $\text{Ev}_{V_I} \otimes (\text{pr}_{X^I})^!(\mathcal{M}_I)$ along the projection

$$\text{pr}_{\text{LocSys}_{\check{G}}(X)} : \text{LocSys}_{\check{G}}(X) \times X^I \rightarrow \text{LocSys}_{\check{G}}(X).$$

The assignment

$$(I, V_I, \mathcal{M}_I) \mapsto \text{Ev}_{V_I, \mathcal{M}_I}$$

defines a symmetric monoidal functor

$$(2.3) \quad \text{Ev} : \text{Rep}(\check{G}, \text{Ran}(X)) \rightarrow \text{QCoh}(\text{LocSys}_{\check{G}}(X)).$$

We have the following result:

Proposition 2.3.2 (D.G and J. Lurie, unpublished). *The functor Ev admits a fully faithful right adjoint.*

⁶As always, 'category' means 'DG category'.

⁷Formally, we take the co-end in DGCat of the following functors from the category of non-empty finite sets and surjections: one takes I to $\text{Rep}(\check{G})^{\otimes I}$ and another to $\text{D}(X^I)$.

In other words, the above proposition says that $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ is a localization (a.k.a., Verdier quotient) of $\mathrm{Rep}(\check{G}, \mathrm{Ran}(X))$ by a full subcategory, which is moreover a monoidal ideal.

2.3.3. According to Proposition 2.3.2, given an action of the monoidal category $\mathrm{Rep}(\check{G}, \mathrm{Ran}(X))$ on some category \mathbf{C} , if this action factors through an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on \mathbf{C} , then it does so uniquely. Moreover, this happens if and only if the objects in

$$\ker(\mathrm{Ev}) \subset \mathrm{Rep}(\check{G}, \mathrm{Ran}(X))$$

act on \mathbf{C} by zero.

2.3.4. We are now ready to state the theorem (this is [Ga3, Theorem 4.5.2]) about the spectral decomposition of $\mathrm{D}(\mathrm{Bun}_G(X))$ along $\mathrm{LocSys}_{\check{G}}(X)$. Recall the action of $\mathrm{Rep}(\check{G}, \mathrm{Ran}(X))$ on $\mathrm{D}(\mathrm{Bun}_G(X))$ from Sect. 2.2.8.

Theorem 2.3.5 (V. Drinfeld, D.G.). *The action of*

$$\ker(\mathrm{Ev}) \subset \mathrm{Rep}(\check{G}, \mathrm{Ran}(X))$$

on $\mathrm{D}(\mathrm{Bun}_G(X))$ is zero.

According to Sect. 2.3.3, from Theorem 2.3.5 we obtain a canonically defined action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on $\mathrm{D}(\mathrm{Bun}_G(X))$, in such a way that the objects $\mathrm{Ev}_{V_I, \mathcal{M}_I}$ (see (2.2)) acts as the endo-functors $\mathrm{H}_{V_I, \mathcal{M}_I}$.

We refer to this action as the ‘spectral decomposition of $\mathrm{D}(\mathrm{Bun}_G(X))$ along $\mathrm{LocSys}_{\check{G}}(X)$ ’.

2.4. **Relation to the ‘vanishing conjecture’ of [FGV].** In the paper [FGV] a certain conjecture was proposed (for which the sheaf-theoretic context can be either ℓ -adic sheaves or D-modules), and it was shown that this conjecture implies the existence of Hecke eigensheaves for GL_n . This conjecture was subsequently proved in [Gal].

In this subsection we will show that in the context of D-modules, the vanishing conjecture from [FGV] is a particular case of Theorem 2.3.5.

2.4.1. Let G be GL_n . We consider the stack $\mathrm{Bun}_n(X) := \mathrm{Bun}_{GL_n}$. For a non-negative integer d , let $\mathrm{Mod}_{n,d}(X)$ be the stack classifying triples

$$(\mathcal{M}, \mathcal{M}', \alpha),$$

where $\mathcal{M}, \mathcal{M}'$ are rank- n vector bundles on X , and α is an injection $\mathcal{M} \hookrightarrow \mathcal{M}'$ as coherent sheaves so that the quotient \mathcal{M}'/\mathcal{M} (which is a priori a torsion sheaf on X) has length d .

We have the projections

$$\mathrm{Bun}_n(X) \xleftarrow{\overleftarrow{h}} \mathrm{Mod}_{n,d}(X) \xrightarrow{\overrightarrow{h}} \mathrm{Bun}_n(X)$$

where $\overleftarrow{h}(\mathcal{M}, \mathcal{M}', \alpha) = \mathcal{M}$ and $\overrightarrow{h}(\mathcal{M}, \mathcal{M}', \alpha) = \mathcal{M}'$.

Let

$$\overset{\circ}{\mathrm{Mod}}_{n,d}(X) \xrightarrow{j} \mathrm{Mod}_{n,d}(X)$$

be the open substack corresponding to the condition that the quotient \mathcal{M}'/\mathcal{M} be *regular semi-simple* (i.e., the direct sum of d sky-scrapers concentrated in distinct points of X).

We have a projection

$$\overset{\circ}{\mathrm{Mod}}_{n,d}(X) \xrightarrow{s} \overset{\circ}{X}^{(d)}$$

that remembers the support of \mathcal{M}'/\mathcal{M} , where $\overset{\circ}{X}^{(d)} \subset X^{(d)}$ is the open subscheme of multiplicity-free divisors.

2.4.2. Let E be a local system on X (of an arbitrary finite rank). We form its symmetric power $E^{(d)}$, which is a local system of rank $\text{rk}(E) \circ d$ when restricted to $\overset{\circ}{X}^{(d)}$. We consider the object

$$\mathfrak{L}_{E,d} \in \text{Mod}_{n,d}(X),$$

(known as Laumon's sheaf) defined as

$$\mathfrak{L}_{E,d} := j_{!*}(s^*(E^{(d)})),$$

where $j_{!*}$ is the operation of Goresky-MacPherson extension (applied to the local system $s^*(E^{(d)})$ on $\overset{\circ}{\text{Mod}}_{n,d}(X)$).

We define the *averaging functor*

$$\text{Av}_{E,d} : \text{D}(\text{Bun}_n(X)) \rightarrow \text{D}(\text{Bun}_n(X)), \quad \text{Av}_{E,d}(\mathcal{F}) := \overrightarrow{h}_!(\overleftarrow{h}^!(\mathcal{F}) \otimes^! \mathfrak{L}_{E,d}).$$

The vanishing conjecture of [FGV]/theorem of [Gal] says:

Theorem 2.4.3. *Suppose that E is irreducible and $\text{rk}(E) > n$ and $d > (2g - 2) \cdot n \cdot \text{rk}(E)$. Then $\text{Av}_{E,d} = 0$.*

2.4.4. Let us specialize again to the case when k is of characteristic 0, and $\text{D}(-)$ is the derived category of D-modules. We claim that in the case, Theorem 2.4.3 is a tiny particular case of Theorem 2.3.5.

Indeed, it is easy to see that the functor $\text{Av}_{E,d}$ is given by the action of a particular object $\mathcal{A}_{E,d} \in \text{Rep}(\check{G}, \text{Ran}(X))$. Moreover,

$$\text{Ev}(\mathcal{A}_{E,d}) \in \text{QCoh}(\text{LocSys}_{\check{G}}(X))$$

is calculated as follows:

We note that for $G = GL_n$, we have $\check{G} = GL_n$ and the fiber of $\text{Ev}(\mathcal{A}_{E,d})$ at $\sigma \in \text{LocSys}_{\check{G}}(X)$ is given by

$$H(X^{(d)}, (E \otimes E_\sigma)^{(d)}),$$

where E_σ is then-dimensional local system corresponding to σ .

Now, in order to deduce Theorem 2.4.3, we notice that the above cohomology identifies with

$$\text{Sym}^d(H(X, E \otimes E_\sigma)),$$

and the latter vanishes for all σ under the conditions on E and d specified in the theorem.

2.4.5. Let us also note that for $k = \mathbb{F}_q$, Theorem 2.4.3 says something quite non-trivial even about the classical Hecke operators acting on $\text{Autom}(X)$. Namely, it says that if f is a joint eigenvector of the Hecke algebras $\mathbb{H}(G)_x$ with characters

$$(\lambda_{x,1}, \dots, \lambda_{x,n}/\text{permutation}),$$

then the Rankin-Selberg L-function

$$L(E, f, t) = \prod_x \frac{1}{1 - t^{\deg_x} \cdot (\lambda_{x,1} + \dots + \lambda_{x,n}) \cdot \text{Tr}(\text{Frob}_x, E_x)}$$

is actually a polynomial of degree $\leq (2g - 2) \cdot n \cdot \text{rk}(E)$.

3. GLOBAL UNRAMIFIED GEOMETRIC LANGLANDS

In this section we let the ground field k be of characteristic zero.

3.1. Why the *best hope* does not work.

3.1.1. The categorical global unramified geometric Langlands theory aims to compare the categories $D(\text{Bun}_G(X))$ and $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$. So far, by Theorem 2.3.5, we have that the monoidal category $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ acts on $D(\text{Bun}_G(X))$. Therefore, the datum of a $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ -linear functor

$$\text{QCoh}(\text{LocSys}_{\check{G}}(X)) \rightarrow D(\text{Bun}_G(X))$$

amounts to a choice of an object in $D(\text{Bun}_G(X))$.

Based on many pieces of evidence, the object in $D(\text{Bun}_G(X))$ that we want to choose for the *global geometric Langlands* equivalence is the ‘first Whittaker coefficient’⁸. Thus, we obtain a functor

$$(3.1) \quad \mathbb{L}_G : \text{QCoh}(\text{LocSys}_{\check{G}}(X)) \rightarrow D(\text{Bun}_G(X)).$$

3.1.2. When $G = T$ is a torus, the above functor \mathbb{L}_G is the generalized Fourier-Mukai transform studied by G. Laumon. In particular, it is an equivalence.

However, the functor \mathbb{L}_G cannot be an equivalence as long G is non-commutative. The reason is that there are objects in $D(\text{Bun}_G(X))$ that are Whittaker-degenerate. For example, the constant D-module on $\text{Bun}_G(X)$ is Whittaker-degenerate.

Remark 3.1.3. Another heuristic piece of evidence for why the category $D(\text{Bun}_G(X))$ cannot be equivalent to $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ comes from the classical theory of automorphic functions:

It is known that automorphic representations are parameterized not by Langlands parameters (i.e., Galois representations) but by *Arthur parameters*, where the latter are conjugacy classes of pairs (σ, A) with σ being a representation of the Galois group of X into \check{G} , and A is a nilpotent element of the Lie algebra of \check{G} that commutes with σ .

3.2. How to make the best hope work? In [AG1] an idea was suggested as to how one can modify the *best hope* to make it work.

This modification consists of tweaking the Galois side, i.e., replacing $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ by some other (but closely related) category, while leaving the automorphic side intact. This tweak happens within homological algebra and has to do with the fact that the category $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ is of infinite cohomological dimension.

3.2.1. In order to explain it we consider the following example. Consider the differential graded algebra $A := k[\epsilon]$, where ϵ is a free generator in degree -1 and its differential is zero (so the differential on all of A is actually zero).

Consider the (derived) category $A\text{-mod}$ of A -modules. Inside we consider the full subcategory

$$A\text{-mod}^{\text{perf}} \subset A\text{-mod}$$

spanned by *perfect complexes*, i.e., those objects that can be obtained by a finite process of taking direct sums, summands and cones from the object $A \in A\text{-mod}$ itself.

We can also consider the full subcategory $A\text{-mod}^{\text{f.g.}} \subset A\text{-mod}$ spanned by objects that have finite-dimensional cohomologies, all of which are finite-dimensional as vector spaces over k .

Since $A \in A\text{-mod}^{\text{f.g.}}$, we have the inclusion

$$A\text{-mod}^{\text{perf}} \subset A\text{-mod}^{\text{f.g.}},$$

⁸It is denoted $\text{Poinc}(\mathcal{W}_{\text{vac}})$ in [Ga3, Sect. 5.7.4].

but it is not an equality. In fact, the Verdier quotient $A\text{-mod}^{\text{f.g.}}/A\text{-mod}^{\text{perf}}$ is equivalent to the category $B\text{-mod}^{\text{f.g.}}$, where $B = k[\eta, \eta^{-1}]$, where η is a free generator in degree 2 and its differential is zero.

Remark 3.2.2. A similar phenomenon in the category of representations of a finite group with torsion coefficients leads to the notion of *Tate cohomology*.

3.2.3. More generally, let V be a finite-dimensional vector space, and consider the DG algebra (with zero differential) $A := \text{Sym}(V[1])$. As above we consider the categories

$$A\text{-mod}^{\text{perf}} \subset A\text{-mod}^{\text{f.g.}}.$$

However, one can now notice that to every conical Zariski-closed subset $\mathcal{N} \subset V$ one can attach a full subcategory

$$A\text{-mod}_{\mathcal{N}}^{\text{f.g.}} \subset A\text{-mod}^{\text{f.g.}},$$

such that:

$$\begin{cases} A\text{-mod}_{\mathcal{N}}^{\text{f.g.}} = A\text{-mod}^{\text{perf}} & \text{if } \mathcal{N} = 0; \\ A\text{-mod}_{\mathcal{N}}^{\text{f.g.}} = A\text{-mod}^{\text{f.g.}} & \text{if } \mathcal{N} = V. \end{cases}$$

3.2.4. Even more generally, let Y be a scheme (or algebraic stack), which is a *locally complete intersection*. To Y one attaches another scheme/stack (see [AG1, Sect. 2.3]), denoted $\text{Sing}(Y)$, whose k -points are pairs (y, ξ) , where y is a k -point of Y , and ξ is an element of the vector space $H^{-1}(T_y^*(Y))$, where $T_y^*(Y)$ is the *derived cotangent space* at y , i.e., the fiber of the cotangent complex⁹ of Y at y .

Let $\text{Coh}(Y)$ be the full subcategory of $\text{QCoh}(Y)$ consisting of objects with finitely many cohomologies, each of which is coherent (i.e., locally finitely generated) as a sheaf on Y . In [AG1, Sect. 4] the following construction is performed: to every conical Zariski-closed subset $\mathcal{N} \subset \text{Sing}(Y)$ one attaches a full subcategory

$$\text{Coh}_{\mathcal{N}}(Y) \subset \text{Coh}(Y).$$

Again, we have:

$$\begin{cases} \text{Coh}_{\mathcal{N}}(Y) = \text{Perf}(Y) & \text{if } \mathcal{N} = \{ \bigcup_{y \in Y} (y, 0) \}; \\ \text{Coh}_{\mathcal{N}}(Y) = \text{Coh}(Y) & \text{if } \mathcal{N} = \text{Sing}(Y), \end{cases}$$

where $\text{Perf}(Y) \subset \text{Coh}(Y)$ is the subcategory of perfect objects (complexes that locally on Y can be represented by a finite complex of free sheaves of finite rank).

3.2.5. The enlargement

$$\text{Perf}(Y) \rightsquigarrow \text{Coh}_{\mathcal{N}}(Y)$$

is exactly the tweak that we will perform on the Galois side of the global unramified geometric Langlands theory. However, there is one point of difference.

For multiple reasons, it is more convenient to work with large (technical term: *cocomplete compactly generated*) categories (such as $\text{QCoh}(Y)$), i.e., categories that admit arbitrary direct sums (and generated by a set of compact objects). The datum of such a category is equivalent to the datum of its full subcategory of compact objects (in the case of $\text{QCoh}(Y)$, its subcategory of compact objects is exactly $\text{Perf}(Y)$), which is a small category. The inverse procedure (recovering a large category from a small one) is called *ind-completion*, see [GR1, Sect. 7.2].

⁹We recall that locally complete intersections are characterized by the property that $H^{-i}(T_y^*(Y)) = 0$ for all y and $i > 1$.

The large subcategory corresponding to $\mathrm{Coh}_{\mathcal{N}}(Y)$ is denoted $\mathrm{IndCoh}_{\mathcal{N}}(Y)$. The inclusion $\mathrm{Perf}(Y) \subset \mathrm{Coh}_{\mathcal{N}}(Y)$ extends to a fully faithful functor

$$\mathrm{QCoh}(Y) \hookrightarrow \mathrm{IndCoh}_{\mathcal{N}}(Y),$$

which admits a right adjoint, given by ind-extending the tautological embedding

$$\mathrm{Coh}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{Coh}(Y) \hookrightarrow \mathrm{QCoh}(Y).$$

3.2.6. For any \mathcal{N} , the category $\mathrm{IndCoh}_{\mathcal{N}}(Y)$ carries a t-structure¹⁰, and the functor

$$\mathrm{IndCoh}_{\mathcal{N}}(Y) \rightarrow \mathrm{QCoh}(Y)$$

(right adjoint to the tautological inclusion) is t-exact. Moreover, the above functor *induces an equivalence of the corresponding bounded below subcategories*

$$\mathrm{IndCoh}_{\mathcal{N}}(Y)^+ \rightarrow \mathrm{QCoh}(Y)^+,$$

see [AG1, Sect. 4.4].

So, the difference between $\mathrm{QCoh}(Y)$ and $\mathrm{IndCoh}_{\mathcal{N}}(Y)$ ‘occurs at $-\infty$ ’. Note that there is no contradiction here: the t-structure on $\mathrm{IndCoh}_{\mathcal{N}}(Y)$ is non-separated, that is, there are non-zero objects all of whose cohomologies vanish.

3.3. **Back to $\mathrm{LocSys}_{\check{G}}(X)$.** The modification of the Galois side of geometric Langlands, proposed in [AG1], is of the form

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}(X)),$$

for a particular conical Zariski-closed subset $\mathcal{N} \subset \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}(X))$.

3.3.1. First, we describe the stack $\mathrm{Sing}(\mathrm{LocSys}_{\check{G}}(X))$. By unwinding the definition of the cotangent complex (see [AG1, Sect. 10.4.6]), we obtain that $\mathrm{Sing}(\mathrm{LocSys}_{\check{G}}(X))$ is the moduli stack of pairs (σ, A) , where $\sigma \in \mathrm{LocSys}_{\check{G}}(X)$ and A is a *horizontal* section of the local system associated with the co-adjoint representation of \check{G} .

Choosing an ad-invariant symmetric bilinear form on $\check{\mathfrak{g}}$, we can think of A as a section of the local system associated with the adjoint representation of \check{G} . We let

$$\mathcal{N} \subset \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}(X))$$

be the *global nilpotent cone*, i.e., the set of those (σ, A) for which A is nilpotent as a section of the local system of Lie algebras $\check{\mathfrak{g}}_{\sigma}$ (equivalently, the value of A in the fiber of $\check{\mathfrak{g}}_{\sigma}$ at some/every point of X should be nilpotent).

3.3.2. Thus, the proposed category on the Galois side of global unramified geometric Langlands is $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}(X))$ for the above choice of \mathcal{N} .

We note that if $G = T$ is a torus, the nilpotent cone in $\check{\mathfrak{g}}$ is zero. So, in this case

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{T}}(X)) = \mathrm{QCoh}(\mathrm{LocSys}_{\check{T}}(X)),$$

i.e., the Galois side is the same as in the original *best hope* (as it should be, because the *best hope* is realized by the Fourier-Mukai transform).

However, this modification is nontrivial as soon as G is non-commutative. The *most singular* point of $\mathrm{LocSys}_{\check{G}}(X)$ is one corresponding to the *trivial local system*. Around this point, the difference between $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}(X))$ and the initial $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ is the largest.

¹⁰This is one of the reasons to work with the large category $\mathrm{IndCoh}_{\mathcal{N}}(Y)$ as opposed to the small category $\mathrm{Coh}_{\mathcal{N}}(Y)$.

Consider now the open substack

$$\mathrm{LocSys}_{\tilde{G}}^{\mathrm{irred}}(X) \subset \mathrm{LocSys}_{\tilde{G}}(X)$$

consisting of irreducible local systems. It is easy to see that the corresponding inclusion

$$\mathrm{QCoh}(\mathrm{LocSys}_{\tilde{G}}^{\mathrm{irred}}(X)) \subset \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\tilde{G}}^{\mathrm{irred}}(X))$$

is an equality. So, the modification does not affect the irreducible locus¹¹.

3.3.3. Thus, the proposed version of the global unramified geometric Langlands equivalence reads as follows:

Conjecture 3.3.4. *There is a canonically defined equivalence of categories*

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\tilde{G}}(X)) \rightarrow \mathrm{D}(\mathrm{Bun}_G(X)).$$

The above statement of Conjecture 3.3.4 is too loose. The paper [Ga3] lists a list of compatibility requirements that fix \mathbb{L}_G is uniquely.

Remark 3.3.5. One can view Conjecture 3.3.4 as restoring the Arthur parameters (as opposed to just Langlands parameters) that were missing in the original *best hope*. Indeed, they appear as obstructions to *temperedness*, i.e., as obstructions for an object of $\mathrm{D}(\mathrm{Bun}_G(X))$ to be in the essential image of $\mathrm{QCoh}(\mathrm{LocSys}_{\tilde{G}}(X))$.

Remark 3.3.6. Recall that in Sect. 3.2.6 we said that the difference between the categories $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\tilde{G}}(X))$ and $\mathrm{QCoh}(\mathrm{LocSys}_{\tilde{G}}(X))$ ‘occurs at $-\infty$ ’ with respect to their respective t-structures. On the other hand, the failure of the functor (3.1) to be an equivalence happens already at the level of the corresponding bounded categories: indeed, recall that the constant D-module on $\mathrm{Bun}_G(X)$ is not in the image of $\mathrm{QCoh}(\mathrm{LocSys}_{\tilde{G}}(X))$. This ‘contradiction’ is explained by the fact that the functor \mathbb{L}_G in Conjecture 3.3.4 is of infinite cohomological amplitude.

3.4. **What is known?** An outline of how one might go about proving Conjecture 3.3.4 was proposed in [Ga3]. Here will summarize the main ideas of the proposed proof and comment on its status. It consists of the following steps.

3.4.1. On the automorphic side one constructs a category, denoted $\mathrm{Whit}_{G,G}^{\mathrm{ext}}(X)$ (called the *extended Whittaker category*, see [Ga3, Sect. 8.2]) and a functor of Whittaker expansion

$$(3.2) \quad \mathrm{coeff}_{G,G}^{\mathrm{ext}} : \mathrm{D}(\mathrm{Bun}_G(X)) \rightarrow \mathrm{Whit}_{G,G}^{\mathrm{ext}}(X).$$

In [Ga3, Conjecture 8.2.9] it is conjectured that the functor $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$ is fully faithful, and this conjecture is proved in [Ber] for $G = GL_n$.

3.4.2. The category $\mathrm{Whit}_{G,G}^{\mathrm{ext}}(X)$ can be thought of as fibered over the space of characters \mathfrak{ch} of $N(\mathbb{A})$ trivial on $N(K)$, where for each $\chi \in \mathfrak{ch}$ we consider the category of D-modules on $G(\mathbb{O}) \backslash G(\mathbb{A})$ that transform according to χ with respect to the action of $N(\mathbb{A})$.

The space \mathfrak{ch} splits according to the pattern of how degenerate the character is, i.e., it is a union of locally closed subspaces \mathfrak{ch}^P , where P runs over the poset of standard parabolics of G . For each P , we obtain the corresponding full subcategory $\mathrm{Whit}_{G,P}(X)$.

For example, for $P = G$, we obtain the the usual Whittaker category, denoted $\mathrm{Whit}_{G,G}(X)$, see [Ga3, Sect. 5].

For $P = B$, the category $\mathrm{Whit}_{G,B}(X)$ is the *principal series category* of [Ga3, Sect. 6].

¹¹This fact may arouse suspicions in the validity of the proposed form of the geometric Langlands conjecture for groups other than GL_n .

3.4.3. On the Galois side one constructs a category, denoted $\text{Glue}(\check{G})_{\text{spec}}$, and a functor

$$(3.3) \quad \text{IndCoh}_{\mathcal{N}}(\text{LocSys}_{\check{G}}(X)) \rightarrow \text{Glue}(\check{G})_{\text{spec}}.$$

In [AG2] it is shown that (3.3) is fully faithful.

3.4.4. The category $\text{Glue}(\check{G})_{\text{spec}}$ is glued from the categories

$$\text{QCoh}_{\text{conn}/\text{LocSys}_{\check{G}}(X)}(\text{LocSys}_{\check{P}}(X)),$$

where \check{P} runs through the poset of standard parabolics of \check{G} .

In the above formula, $\text{QCoh}_{\text{conn}/\text{LocSys}_{\check{G}}(X)}(\text{LocSys}_{\check{P}}(X))$ is the (derived) category of quasi-coherent sheaves on the stack $\text{LocSys}_{\check{P}}(X)$, endowed with a *connection* along the fibers of the map $\text{LocSys}_{\check{P}}(X) \rightarrow \text{LocSys}_{\check{G}}(X)$. These notions need to be understood in the sense of derived algebraic geometry, see [Ga3, Sect. 6.5].

For example, for $P = G$, we have

$$\text{QCoh}_{\text{conn}/\text{LocSys}_{\check{G}}(X)}(\text{LocSys}_{\check{P}}(X)) = \text{QCoh}(\text{LocSys}_{\check{G}}(X)),$$

i.e., this is the usual (unmodified) derived category of quasi-coherent sheaves on $\text{LocSys}_{\check{G}}(X)$.

3.4.5. Assuming Conjecture 3.3.4 for *proper Levi subgroups* of G , and certain auxiliary results, one constructs a *fully faithful* functor

$$(3.4) \quad \text{Glue}(\check{G})_{\text{spec}} \rightarrow \text{Whit}_{G,G}^{\text{ext}}(X).$$

The construction of (3.4) with the required properties is complete for $G = GL_2$, and it is a question of time before it becomes available for any G .

3.4.6. The functor (3.4) is glued from the fully faithful functors

$$\text{QCoh}_{\text{conn}/\text{LocSys}_{\check{G}}(X)}(\text{LocSys}_{\check{P}}(X)) \rightarrow \text{Whit}_{G,P}(X).$$

For example, for $P = G$, the corresponding functor

$$\text{QCoh}(\text{LocSys}_{\check{G}}(X)) \rightarrow \text{Whit}_G(X)$$

is the composition of the functor

$$\text{QCoh}(\text{LocSys}_{\check{G}}(X)) \rightarrow \text{Rep}(\check{G}, \text{Ran}(X)),$$

right adjoint to the functor Ev of (2.3) followed by the Casselman-Shalika equivalence

$$\text{Rep}(\check{G}, \text{Ran}(X)) \simeq \text{Whit}_G(X).$$

Here, $\text{Whit}_G(X)$ is a slight modification of $\text{Whit}_{G,G}(X)$ that has to do with the center of G , see [Ga3, Sect. 5.6.7].

3.4.7. Let us now assume that the functor (3.2) is fully faithful (which we know for GL_n), and that the functor (3.4) with the required properties exists. Let us see how this helps to prove Conjecture 3.3.4.

Consider the composition of the functors (3.3) and (3.4), which is a fully faithful functor

$$(3.5) \quad \text{IndCoh}_{\mathcal{N}}(\text{LocSys}_{\check{G}}(X)) \rightarrow \text{Whit}_{G,G}^{\text{ext}}(X).$$

In [Ga3, Sect. 10], one exhibits a collection of objects $\mathcal{F}_\alpha \in \text{D}(\text{Bun}_G(X))$ and a collection of objects $\mathcal{M}_\alpha \in \text{IndCoh}_{\mathcal{N}}(\text{LocSys}_{\check{G}}(X))$ such that for every α , the image of \mathcal{F}_α under (3.2) is isomorphic to the image of \mathcal{M}_α under (3.5).

In addition, it is shown that the objects \mathcal{F}_α generate $\text{D}(\text{Bun}_G(X))$. And is conjectured (and established for $G = GL_n$) that the objects \mathcal{M}_α generate $\text{IndCoh}_{\mathcal{N}}(\text{LocSys}_{\check{G}}(X))$. This implies that the essential images of (3.2) and (3.5) in $\text{Whit}_{G,G}^{\text{ext}}(X)$, being generated by the same collection of objects, coincide, thus providing the sought-for equivalence \mathbb{L}_G .

Remark 3.4.8. While all the preceding steps in the proposed proof of Conjecture 3.3.4 were geometric in nature (i.e., used the standard sheaf-theoretic functors on the categories D-modules when working on the automorphic side) and had clear counterparts in the classical theory of automorphic functions, the construction of the objects \mathcal{F}_α (resp., \mathcal{M}_α) is different in nature, and is based on the ideas from [BD]:

On the automorphic side, the objects \mathcal{F}_α are obtained by the *localization functor* from modules over the Kac-Moody algebra at the critical level. On the Galois side, the objects \mathcal{M}_α are obtained by taking direct images along a map to $\text{LocSys}_{\check{G}}(X)$ from the scheme classifying \check{G} -opers on X (see [Ga3, Sect. 10]).

4. LOCAL GEOMETRIC LANGLANDS

4.1. What is the object of study on the representation-theoretic side?

4.1.1. Recall that in the classical global Langlands theory, the object of study on the representation-theoretic (=automorphic) side is the space of functions on the quotient $G(\mathbb{A})/G(K)$, viewed as a representation of the group $G(\mathbb{A})$. In the unramified case, the corresponding object of study is the space of functions on $G(\mathbb{O}) \backslash G(\mathbb{A})/G(K)$, viewed as a module over the Hecke algebra.

By contrast, the object of study on the representation-theoretic side in the classical *local* theory is the *category* of representations of the group $G(\mathcal{K})$, where \mathcal{K} is a local field. So, by going from global to local we raise the level by one in the hierarchy

$$\text{Elements of a Set} \rightarrow \text{Objects of a Category} \rightarrow \text{Objects in a 2-Category}.$$

In the global geometric Langlands theory, in the unramified case, the object of study on the representation-theoretic (=automorphic) side was the *category* $\text{D}(\text{Bun}_G(X))$, viewed as acted on by the Hecke functors.

Hence, by the above analogy, on the representation-theoretic in the local geometric theory, the object of study should be a certain 2-category, attached to the group G and the local field $\mathcal{K} = k((t))$.

We stipulate that the 2-category in question is that of *categories equipped with an action of $G(\mathcal{K})$* . We will now explain what we mean by this.

4.1.2. First, when we say ‘category’ in the above context, we mean a k -linear *DG category*¹², defined, e.g., as in [GR1, Sect. 10]. The important fact is that the totality of such categories and k -linear functors between them¹³ forms an $(\infty, 2)$ -category, denoted DGCat , equipped with a symmetric monoidal structure, called the *Lurie tensor product*.

Second, when we write $G(\mathcal{K})$ we mean the group ind-scheme, defined as a functor on the category of affine schemes by

$$\mathrm{Hom}(\mathrm{Spec}(A), G(\mathcal{K})) := \mathrm{Hom}(\mathrm{Spec}(A((t))), G).$$

We now have to define the notion of action¹⁴ of $G(\mathcal{K})$ on a category. The corresponding general notion has been developed in [Ga4].

However, one can give also the following explicit definition: according to [Ber], we have a well-defined category $\mathrm{D}(G(\mathcal{K}))$ of D-modules on $G(\mathcal{K})$; the group structure on G defines on $\mathrm{D}(G(\mathcal{K}))$ a monoidal structure. I.e., $\mathrm{D}(G(\mathcal{K}))$ acquires a structure of *associative algebra* in the monoidal category DGCat .

We define the notion of category equipped with an action of $G(\mathcal{K})$ to be a module over $\mathrm{D}(G(\mathcal{K}))$ in DGCat . The totality of such has a structure of $(\infty, 2)$ -category (see [GR1, Sect. 8.3]); we denote it by $G(\mathcal{K})\text{-mod}$.

4.1.3. Here are some examples of objects of $G(\mathcal{K})\text{-mod}$.

(i) The first example is $\mathbf{C} := \mathrm{D}(G(\mathcal{K}))$, equipped with an action on itself by left multiplication.

(ii) For any subgroup $H \subset G(\mathcal{K})$, we can take $\mathbf{C} := \mathrm{D}(G(\mathcal{K})/H)$.

As particular cases of the above example (ii), we can take $H = G(\mathcal{O})$ or $H = I$, the latter being the Iwahori subgroup. The resulting categories are the categories of D-modules on the affine Grassmannian Gr_G and the affine flag scheme Fl_G , respectively.

(iii) We can take $\mathbf{C} = \widehat{\mathfrak{g}}_\kappa\text{-mod}$, i.e., the category of representations of the Kac-Moody algebra for any *integral* level κ (see [FG, Sect. 23] for the definition); here the action of $G(\mathcal{K})$ comes from its adjoint action on $\widehat{\mathfrak{g}}_\kappa$.

(iv) We consider the stack $\mathrm{Bun}_G(X)^{\mathrm{level}_x}$, classifying principal G -bundles on the curve X equipped with a full level structure at a point x (i.e., a trivialization over the formal neighborhood of x). We take $\mathbf{C} := \mathrm{D}(\mathrm{Bun}_G(X)^{\mathrm{level}_x})$. This is the object of $G(\mathcal{K})\text{-mod}$, corresponding to the global geometric Langlands theory with ramification allowed at x .

4.2. The object of study on the Galois side.

4.2.1. Recall that in the global unramified geometric theory, the object of study on the Galois side was (a modification of) the derived category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}(X)$ that classifies \check{G} -local systems on the curve X .

Based on the analogy with the local classical theory, the object of study on the Galois side in the *local geometric theory* should be a certain 2-category attached to the space $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ of \check{G} -local systems on the *punctured formal disc* $\overset{\circ}{\mathcal{D}}$.

¹²All our DG categories are assumed *cocomplete*.

¹³All our functors are assumed *continuous*, i.e., preserving infinite direct sums

¹⁴On the geometric/automorphic side of Langlands, when we talk about actions of groups on categories, we mean *strong* actions.

In what follows we will explain what we mean by the space $\mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$, and what the resulting 2-category (in fact, $(\infty, 2)$ -category) is.

4.2.2. Recall also that in the global case, the category that one obtains from $\mathrm{LocSys}_{\check{G}}(X)$ in the most tautological way, i.e., $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$, did not quite match the automorphic side—we needed to introduce a correction that had to do with the difference between perfect complexes and coherent ones.

The $(\infty, 2)$ -category $\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}}))$ that we will define below is the counterpart of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$. It will be responsible for the tempered part of $G(\mathcal{K})$ -**mod**.

The extension of $\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}}))$ that takes into account all local Arthur parameters (as opposed to just Langlands parameters) has been recently proposed by D. Arinkin. But we will not explicitly discuss it in this talk.

4.3. **Sheaves of categories.** In order to talk about sheaves of categories, we need to place ourselves in the context of *derived algebraic geometry*. Thus, in what follows, when we say ‘affine scheme’ we shall mean a derived affine scheme over k . By definition, the category of such is the opposite of the category of *connective*¹⁵ commutative DG algebras over k , see [GR2, Sect. 1.1].

4.3.1. For an affine scheme S , we consider the (symmetric) monoidal category $\mathrm{QCoh}(S)$. This is a (commutative) algebra object in the symmetric monoidal category DGCat .

We let $\mathrm{ShvCat}(S)$ denote the $(\infty, 2)$ -category $\mathrm{QCoh}(S)$ -**mod**, i.e., that of $\mathrm{QCoh}(S)$ -modules in the (symmetric) monoidal $(\infty, 2)$ -category DGCat .

The assignment $S \mapsto \mathrm{ShvCat}(S)$ is a functor from $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ to the ∞ -category of $(\infty, 2)$ -categories.

4.3.2. Let now \mathcal{Y} be an arbitrary *prestack*, i.e., a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \mathrm{Spc},$$

where Spc is the ∞ -category of *spaces*¹⁶.

We define the $(\infty, 2)$ -category $\mathrm{ShvCat}(\mathcal{Y})$ to be

$$\lim_{S \xrightarrow{\mathcal{Y}} \mathcal{Y}} \mathrm{ShvCat}(S),$$

where the index category is $((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}$ and the limit is taken in the ∞ -category of $(\infty, 2)$ -categories.

4.3.3. In other words, informally, an object of $\mathrm{ShvCat}(\mathcal{Y})$ is an assignment for every

$$(4.1) \quad (S \xrightarrow{\mathcal{Y}} \mathcal{Y}) \rightsquigarrow \mathbf{C}_{S, \mathcal{Y}} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

and for every

$$((S_1, y_1), (S_2, y_2), S_1 \xrightarrow{f} S_2, y_2 \circ f \sim y_1) \rightsquigarrow \mathrm{QCoh}(S_1) \otimes_{\mathrm{QCoh}(S_2)} \mathbf{C}_{S_2, y_2} \simeq \mathbf{C}_{S_1, y_1}.$$

This assignment must be endowed with a *homotopy-coherent system of compatibilities* for compositions.

We call objects of $\mathrm{ShvCat}(\mathcal{Y})$ ‘sheaves of categories over \mathcal{Y} ’.

¹⁵Connective=concentrated in non-positive cohomological degrees.

¹⁶Space= ∞ -groupoid.

4.3.4. The most basic example of a sheaf of categories is $\mathrm{QCoh}/_{\mathcal{Y}}$ (not to be confused with the *category* $\mathrm{QCoh}(\mathcal{Y})$, discussed below). Namely, in terms of the assignment (4.1), the object $\mathrm{QCoh}/_{\mathcal{Y}}$ assigns to

$$(S, y) \rightsquigarrow \mathrm{QCoh}(S) \in \mathrm{QCoh}(S)\text{-}\mathbf{mod}.$$

4.3.5. For \mathcal{Y} as above we can consider the (symmetric) monoidal category

$$(4.2) \quad \mathrm{QCoh}(\mathcal{Y}) := \lim_{S \xrightarrow{\mathcal{Y}} \mathcal{Y}} \mathrm{QCoh}(S).$$

This is what one calls the *(derived) category of quasi-coherent sheaves on a prestack*. (In the case of $\mathcal{Y} = \mathrm{LocSys}_{\mathcal{G}}(X)$, this is the category $\mathrm{QCoh}(\mathrm{LocSys}_{\mathcal{G}}(X))$ considered in the previous sections.)

In other words, informally, an object of $\mathrm{QCoh}(\mathcal{Y})$ is an assignment for every

$$(4.3) \quad (S \xrightarrow{\mathcal{Y}} \mathcal{Y}) \rightsquigarrow \mathcal{F}_{S,y} \in \mathrm{QCoh}(S),$$

and for every

$$((S_1, y_1), (S_2, y_2), S_1 \xrightarrow{f} S_2, y_2 \circ f \sim y_1) \rightsquigarrow f^*(\mathcal{F}_{S_2, y_2}) \simeq \mathcal{F}_{S_1, y_1}.$$

This assignment must be endowed with a *homotopy-coherent system of compatibilities* for compositions.

4.3.6. Here is another candidate for what we might call a ‘sheaf of categories’: we can consider the $(\infty, 2)$ -category

$$\mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}.$$

Note that if \mathcal{Y} is (representable by) an affine scheme S , we have a tautological equivalence

$$\mathrm{QCoh}(S)\text{-}\mathbf{mod} \simeq \mathrm{ShvCat}(S).$$

4.3.7. For a general prestack \mathcal{Y} , the above two $(\infty, 2)$ -categories are related by a pair of adjoint functors

$$(4.4) \quad \mathbf{Loc} : \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \rightleftarrows \mathrm{ShvCat}(\mathcal{Y}) : \mathbf{\Gamma}.$$

In terms of the assignment (4.1), the functor $\mathbf{\Gamma}$ sends a sheaf of categories to

$$\lim_{S \xrightarrow{\mathcal{Y}} \mathcal{Y}} \mathbf{C}_{S,y} \in \mathrm{DGCat},$$

which is equipped with a natural action of (4.2).

The functor \mathbf{Loc} sends $\mathbf{C} \in \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$ to

$$(S, y) \rightsquigarrow \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(\mathcal{Y})} \mathbf{C}.$$

We shall say that a prestack \mathcal{Y} is *1-affine* if the functors (4.4) are mutually inverse equivalences.

Tautologically, every affine scheme is 1-affine.

4.3.8. Here are some examples of prestacks that are (or are not) 1-affine (these examples are taken from [Ga4, Sect. 2]):

- (i) Any quasi-compact, quasi-separated algebraic space (in particular, scheme) is 1-affine.
- (ii) Any quasi-compact algebraic stack of finite type with an affine diagonal is 1-affine.
- (iii) For a non-trivial connected algebraic group G , the quotient $\mathrm{pt}/G(\mathcal{O})$ is *not* 1-affine. (This is not in contradiction with example (ii), because the finite-type condition is violated.)
- (iv) The ind-scheme $\mathbb{A}^\infty = \mathrm{colim}_i \mathbb{A}^i$ is *not* 1-affine.

In general, one can say that infinite-dimensionality is typically an obstruction to 1-affineness.

4.4. The space of local systems on the formal punctured disc.

4.4.1. We now introduce the space $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$, which is the main geometric player on the Galois side of the local geometric Langlands.

We start with the space of $\check{\mathfrak{g}}$ -valued connection forms on $\overset{\circ}{\mathcal{D}}$, i.e., $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$. This is an ind-scheme (of infinite type). A choice of a uniformizer in \mathcal{K} identifies $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ with $\check{\mathfrak{g}}(\mathcal{K})$.

Now, the group $G(\mathcal{K})$ acts on $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ by *gauge transformations*.

We define $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ to be the *prestack quotient* $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}/G(\mathcal{K})$.

4.4.2. As was mentioned in Sect. 4.2.2, the $(\infty, 2)$ -category

$$\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$$

plays the same role vis-à-vis $G(\mathcal{K})$ -**mod** as $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ did vis-à-vis $D(\mathrm{Bun}_G)$.

In particular, we expect that

- (i) The $(\infty, 2)$ -category $\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ (which is equipped with a natural (symmetric) monoidal structure) acts on $G(\mathcal{K})$ -**mod**;
- (ii) $\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ is equivalent to the full subcategory of $G(\mathcal{K})$ -**mod**, consisting of tempered objects.

4.4.3. The following key technical result of [Ras] allows to take the program proposed above off the ground ¹⁷:

Theorem 4.4.4 (S. Raskin).

- (a) *The prestack $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ is 1-affine.*
- (b) *The category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}}))$ is compactly generated.*

Remark 4.4.5. Let us point out that the validity of Theorem 4.4.4(a) is a really nice surprise.

Indeed, $\mathrm{LocSys}_{\check{G}}(\overset{\circ}{\mathcal{D}})$ is obtained by quotienting $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ by $G(\mathcal{K})$. Now, the ind-scheme $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ is *not* 1-affine (because it contains Example(iv) from Sect. 4.3.8). Furthermore, quotienting by groups of infinite type also usually destroys 1-affineness (see Example(iii) from Sect. 4.3.8). Thus, for example, if instead of the *gauge* action of $G(\mathcal{K})$ on $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ we considered the adjoint action, the resulting quotient would *not* be 1-affine.

¹⁷After this talk was delivered, S. Raskin found a gap in his proof, and so far we only know that the functor **Loc** is fully faithful. Hopefully, he will fix the proof in the near future.

However, $\text{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ still manages to be 1-affine: the infinite-dimensional ind-direction in $\check{\mathfrak{g}} \otimes \omega_{\mathcal{K}}$ that prevents it from being 1-affine gets ‘eaten up’ by the ind-direction in $G(\mathcal{K})$, and the action of $G(\mathcal{O})$ is ‘free modulo something finite-dimensional’, i.e., the example Example(iii) from Sect. 4.3.8 does not occur. So, $\text{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ does not have infinite-dimensional features that prevent it from being 1-affine, and yet it is *not* locally of finite type (except if G is a torus).

REFERENCES

- [AG1] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, *Selecta Math. N.S.* **21** (2015), 1–199.
- [AG2] D. Arinkin and D. Gaitsgory, *The category of singularities as a crystal and global Springer fibers*, arXiv: 1412.4394.
- [BB] A. Beilinson and J. Bernstein, *Localisation de g -modules*, *C.R. Acad. Sci. Paris Ser. I Math.* **292** (1981), 15–18.
- [BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin’s integrable system and Hecke eigensheaves*, available at <http://math.uchicago.edu/~mitya/langlands.html>.
- [Ber] D. Beraldo, *On the extended Whittaker category*, arXiv: 1411.7982.
- [Bez] R. Bezrukavnikov, *On two geometric realizations of the affine Hecke algebra*, arXiv:1209.0403.
- [Dr] V. Drinfeld, *Two-dimensional l -adic representations of the fundamental group of a curve over a finite field and automorphic forms on $GL(2)$* , *Amer. Jour. of Math.* **105** (1983), 85–114.
- [FG] E. Frenkel and D. Gaitsgory, *D -modules on the affine flag variety and representations of affine Kac-Moody algebras*, *J. Represent. Theory* **13** (2009), 470–608.
- [FGV] E. Frenkel, D. Gaitsgory and K. Vilonen, *On the geometric Langlands conjecture*, *J. Amer. Math. Soc.* **15** (2002), 367–417.
- [Ga1] D. Gaitsgory, *On a vanishing conjecture appearing in the geometric Langlands correspondence*, *Ann. of Math. (2)* **160**, no. 2 (2004), 617–682.
- [Ga2] D. Gaitsgory, *Quantum Langlands correspondence*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [Ga3] D. Gaitsgory, *Outline of the proof of the geometric Langlands conjecture for $GL(2)$* , *Astérisque* **370** (2015), 1–112.
- [Ga4] D. Gaitsgory, *Sheaves of categories and the notion of 1-affineness*, *Contemporary Mathematics* **643** (2015), 1–99.
- [GR1] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, Part I.1*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GR2] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, Part I.2*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GR3] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, Part I.3*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GR4] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, Part III.4*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [LLaf] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, *Invent. Math.* **147** (1) (2002), 1–241.
- [VLaf] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, arXiv:1209.5352.
- [Lau1] G. Laumon, *Correspondence de Langlands géométrique pour les corps de fonctions*, *Duke Math. Jour.* **54** (1987), 309–359.
- [Lau2] G. Laumon, *Transformation de Fourier généralisée*, arXiv:alg-geom/9603004.
- [Lu] J. Lurie, *Higher Topos Theory*, Princeton Univ. Press (2009).
- [MV] I. Mirkovic and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, *Ann. of Math.* **166** (2007), 95–143.
- [Ras] S. Raskin, *On the notion of spectral decomposition in local geometric Langlands*, arXiv:1511.01378