CHAPTER A.1. BASICS OF 2-CATEGORIES

Contents

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Introduction

0.1. **What are $(\infty,2)$-categories?** There are multiple definitions of $(\infty,2)$-categories. In this Chapter we adopt the one that mimics the approach to $(\infty,1)$-categories via complete Segal spaces. Let us first recall the latter.

0.1.1. Given an $(\infty,1)$-category $\mathbf{C}$, we attach to it the simplicial space, denoted $\mathrm{Seq}_\bullet(\mathbf{C})$, whose space $\mathrm{Seq}_n(\mathbf{C})$ of n-simplices is the space of n-fold compositions in $\mathbf{C}$, i.e.,

$$\mathrm{Seq}_n(\mathbf{C}) = \mathrm{Maps}_{1\text{-Cat}}([n],\mathbf{C}),$$

where $[n]$ is the category $0 \to 1 \to ... \to n$.

It turns out that the functor

$$\mathrm{Seq}_\bullet : 1\text{-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is fully faithful, and one can explicitly (and concisely) describe its essential image: it consists of *complete Segal spaces*, see Sect. 1.2 for the definition.

0.1.2. We *define* $(\infty,2)$-categories by a similar procedure: we let the datum of an $(\infty,2)$-category $\mathbb{S}$ to be a simplicial $(\infty,1)$-*category* $\mathrm{Seq}_\bullet(\mathbb{S})$, subject to conditions analogous to those that single out complete Segal spaces among simplicial spaces.

Thus, we obtain an $(\infty,1)$-category 2-Cat, which is a full subcategory in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$.

0.1.3. The idea of $\mathrm{Seq}_\bullet(\mathbb{S})$ is the following. For $n=0$, the $(\infty,1)$-category $\mathrm{Seq}_0(\mathbb{S})$ is the *space* of objects in $\mathbb{S}$, denoted $\mathbb{S}^{\mathrm{Spc}}$.

For $n=1$, the category $\mathrm{Seq}_1(\mathbb{S})$ has as objects 1-morphisms $\alpha : s_0 \to s_1$. Now, morphisms in $\mathrm{Seq}_1(\mathbb{S})$ are diagrams

(0.1)
$$\begin{tikzcd} s_0 \arrow[r, "\alpha"] \arrow[d] & s_1 \arrow[d, "\alpha'"] \arrow[dl, Rightarrow] \\ s_0' \arrow[r] & s_1', \end{tikzcd}$$

where the vertical arrows are *isomorphisms* (indeed, they must be such, because each column in the above diagram must restrict to a morphism in $\mathrm{Seq}_0(\mathbb{S})$).

Thus, $\mathrm{Seq}_1(\mathbb{S})$ splits as a disjoint union according to $\pi_0(\mathrm{Seq}_0(\mathbb{S})) \times \pi_0(\mathrm{Seq}_0(\mathbb{S}))$. For fixed $(s_0,s_1) \in \mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$, the category

$$\mathrm{Seq}_1(\mathbb{S}) \underset{\mathrm{Seq}_0(\mathbb{S})\times\mathrm{Seq}_0(\mathbb{S})}{\times} \{(s_0,s_1)\}$$

has as morphisms diagrams

$$\begin{tikzcd} s_0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & s_1. \end{tikzcd}$$

0.1.4. We note that this approach is morally close to the 'enriched ideology' (although we do not attempt to pursue the latter): for each $s_0, s_1 \in \mathbb{S}^{\mathrm{Spc}}$ we upgrade the *space*

$$\mathrm{Maps}_{\mathbb{S}}(s_0, s_1)$$

(which records the structure of the $(\infty,1)$-category underlying $\mathbb{S}$) to an $(\infty,1)$-category

$$\mathbf{Maps}_{\mathbb{S}}(s_0, s_1),$$

the latter being $\mathrm{Seq}_1(\mathbb{S}) \underset{\mathrm{Seq}_0(\mathbb{S})\times\mathrm{Seq}_0(\mathbb{S})}{\times} \{(s_0, s_1)\}$. I.e.,

$$\mathrm{Maps}_{\mathbb{S}}(s_0, s_1) = (\mathbf{Maps}_{\mathbb{S}}(s_0, s_1))^{\mathrm{Spc}}.$$

0.1.5. Having defined $(\infty,2)$-categories, we can now explain the main example of one such: the $(\infty,2)$-category of $(\infty,1)$-categories, denoted $\mathbf{1\text{-}Cat}$, so that

$$(\mathbf{1\text{-}Cat})^{\text{1-Cat}} = \text{1-Cat}.$$

Here is its definition: the corresponding $(\infty,1)$-category $\mathrm{Seq}_n(\mathbf{1\text{-}Cat})$ has as objects *Cartesian* fibrations over $[n]^{\mathrm{op}}$.

Morphisms in this $(\infty,1)$-category are functors over $[n]^{\mathrm{op}}$ (that do *not necessarily* take Cartesian edges to Cartesian edges) but ones that induce an equivalence over each $i \in [n]$.

0.2. What if did not insist that the vertical arrows be isomorphisms?

0.2.1. Having defined $(\infty,2)$-categories, it is natural to ask the following question. Let us attach to an $(\infty,2)$-category $\mathbb{S}$ another simplicial category (denote it by $\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S})$), as follows:

We let $\mathrm{Seq}^{\mathrm{ext}}_0(\mathbb{S})$ be the $(\infty,1)$-*category* $\mathbb{S}^{\text{1-Cat}}$ underlying $\mathbb{S}$ (i.e., $\mathbb{S}^{\text{1-Cat}}$ is obtained from $\mathbb{S}$ by removing non-invertible 2-morphisms). Recall, by contrast, that $\mathrm{Seq}_0(\mathbb{S})$ was the space $\mathbb{S}^{\mathrm{Spc}}$.

The category $\mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$ will still have as objects 1-morphisms $s_0 \to s_1$. But morphisms in $\mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$ will be diagrams (0.1), where we allow *arbitrary* 1-morphisms along the vertical edges (i.e., we no longer require that these 1-morphisms be isomorphisms).

0.2.2. The above construction indeed defines a functor

$$\mathrm{Seq}^{\mathrm{ext}}_\bullet : \text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}},$$

and this functor also happens to be fully faithful (this is one of the results that are left unproved in this book).

Thus, the $(\infty,1)$-category 2-Cat admits two *different* realizations as a full subcategory in $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$.

0.2.3. Of course, it is nice to know that the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ is fully faithful. But do we actually need this in order to develop the theory?

The answer is 'yes', and that is mainly for the following reason: we will use the $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ realization of 2-Cat in order to talk about adjunctions.

In more detail, for any $(\infty,2)$-category $\mathbb{T}$, it makes sense to ask whether a given 1-morphism $t_0 \to t_1$ admits a left or right adjoint. Now, let

$$F : \mathbb{S} \to \mathbb{T}$$

be a functor, such that for every 1-morphism $s_0 \overset{\alpha}{\to} s_1$, the corresponding 1-morphism

$$F(s_0) \xrightarrow{F(\alpha)} F(s_1)$$

admits a left (resp., right) adjoint.

Then it is natural to expect that in this case, we will be able to canonically construct a functor

$$F^L : \mathbb{S}^{1\&2\text{-op}} \to \mathbb{T} \text{ or } F^R : \mathbb{S}^{1\&2\text{-op}} \to \mathbb{T}$$

(here $\mathbb{S}^{1\&2\text{-op}}$ is the $(\infty,2)$-category obtained from $\mathbb{S}$ by inverting 1- and 2-morphisms), which is the same as F at the level of objects, and at the level of 1-morphisms replaces each $F(\alpha)$ by its left (resp., right) adjoint.

Such a construction is indeed possible, and the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ will be the main tool for carrying it out.

0.2.4. Finally, one can ask the following question: if $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ is so good, why do we not use that instead of $\mathrm{Seq}_\bullet$ in the definition of $(\infty,2)$-categories?

The answer is that we need the $\mathrm{Seq}_\bullet$-realization in order to define the notion of *lax functor* between $(\infty,2)$-categories, see Sect. 0.3.2 below.

So, to summarize, we need both realizations $\mathrm{Seq}_\bullet$ and $\mathrm{Seq}^{\mathrm{ext}}_\bullet$.

0.3. **What else is done in this chapter?**

0.3.1. In Sect. 1 we recall the realization of 1-Cat via complete Segal spaces, and in Sect. 2 we introduce $(\infty,2)$-categories according to the recipe explained above.

Skipping Sect. 3 for a second, in Sect. 4 we explain the approach to $(\infty,2)$-categories via the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$, and in Sect. 5 we describe its essential image (rather, that of its variant $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}$).

In Sect. 6 we upgrade the $(\infty,1)$-category 2-Cat to an $(\infty,2)$-category $\mathbf{2\text{-}Cat}$.

0.3.2. Let us now return to Sect. 3. In this section we introduce the notion of *right-lax* functor between $(\infty,2)$-categories. Morally, a right-lax functor

$$F : \mathbb{S} \dashrightarrow \mathbb{T}$$

is the same as a functor, with the difference that it only respects composition *up to a not necessarily invertible* 2-morphism.

I.e., for a string

$$s_0 \xrightarrow{\alpha} s_1 \xrightarrow{\beta} s_2$$

in $\mathbb{S}$, we are supposed to be given a 2-morphism in $\mathbb{T}$

$$F(\beta) \circ F(\alpha) \to F(\beta \circ \alpha).$$

0.3.3. Formally, the definition is given as follows. Let $\mathbb{S}^{\oint}$ and $\mathbb{T}^{\oint}$ be the coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, corresponding to the functors

$$\mathrm{Seq}_\bullet(\mathbb{S}), \mathrm{Seq}_\bullet(\mathbb{T}) : \mathbf{\Delta}^{\mathrm{op}} \to \text{1-Cat},$$

respectively.

A genuine (i.e., strict) functor $\mathbb{S} \to \mathbb{T}$ is the same as a functor $\mathbb{S}^{\oint} \to \mathbb{T}^{\oint}$ over $\mathbf{\Delta}^{\mathrm{op}}$ that takes coCartesian edges to coCartesian edges.

A right-lax functor $\mathbb{S} \dashrightarrow \mathbb{T}$ is, by definition, a functor $\mathbb{S}^{\oint} \to \mathbb{T}^{\oint}$ over $\mathbf{\Delta}^{\mathrm{op}}$ that takes coCartesian edges *that lie over idle arrows in* $\mathbf{\Delta}^{\mathrm{op}}$ to coCartesian edges (we refer the reader to Sect. 3.1.2 where the notion of idle arrow in $\mathbf{\Delta}^{\mathrm{op}}$ is defined).

0.3.4. The notion of right-lax functor allows us to introduce the notion of *Gray product* of $(\infty,2)$-categories. Given, $\mathbb{S},\mathbb{T}\in 2\text{-Cat}$, their Gray product, denoted

$$\mathbb{S}\circledast\mathbb{T}$$

is a $(\infty,2)$-category, equipped with a *right-lax* functor

$$\mathbb{S}\times\mathbb{T}\dashrightarrow\mathbb{S}\circledast\mathbb{T},$$

universal with respect to the following property:

For a pair of 1-morphisms

$$s_0\overset{\phi}{\to}s_1 \text{ and } t_0\overset{\psi}{\to}t_1,$$

the diagram

$$\begin{tikzcd}
(s_0,t_0) \arrow[r, "(\mathrm{id},\psi)"] \arrow[d, "(\phi,\mathrm{id})"'] & (s_0,t_1) \arrow[d, "(\phi,\mathrm{id})"] \\
(s_1,t_0) \arrow[r, "(\mathrm{id},\psi)"] & (s_1,t_1)
\end{tikzcd}$$

in $\mathbb{S}\circledast\mathbb{T}$ no longer commutes, but only does so up to a *non-invertible* 2-morphism

$$\begin{tikzcd}
(s_0,t_0) \arrow[r] \arrow[d] & (s_0,t_1) \arrow[d] \arrow[dl, Rightarrow] \\
(s_1,t_0) \arrow[r] & (s_1,t_1).
\end{tikzcd}$$

0.3.5. The Gray product produces something non-trivial even if $\mathbb{S}=\mathbf{I}$ and $\mathbb{T}=\mathbf{J}$ are $(\infty,1)$-categories. Consider the simplest example of $\mathbf{I}=\mathbf{J}=[1]$. In this case, the $(\infty,2)$-category

$$[1]\circledast[1]=:[1,1]$$

can be depicted as

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[dl, Rightarrow] \\
\bullet \arrow[r] & \bullet.
\end{tikzcd}$$

0.3.6. The notion of Gray product allows to introduce the notion of *right-lax natural transformation* between functors. In general, for $\mathbb{S},\mathbb{T}\in 2\text{-Cat}$, we introduce the $(\infty,2)$-category

$$\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}$$

(of genuine functors, but where we allow right-lax natural transformations) by

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{X},\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}})=\mathrm{Maps}_{2\text{-Cat}}(\mathbb{X}\circledast\mathbb{S},\mathbb{T}).$$

The $(\infty,2)$-category contains as a 1-full subcategory the usual $(\infty,2)$-category of functors, denoted $\mathrm{Funct}(\mathbb{S},\mathbb{T})$, and defined by

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{X},\mathrm{Funct}(\mathbb{S},\mathbb{T}))=\mathrm{Maps}_{2\text{-Cat}}(\mathbb{X}\times\mathbb{S},\mathbb{T}).$$

0.3.7. For example, we have:

$$\mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S})=\mathrm{Funct}([n],\mathbb{T})_{\text{right-lax}}.$$

0.4. **Status of the assertions.**

0.4.1. Unfortunately, the existing literature on $(\infty, 2)$-categories does not contain the proofs of all the statements that we need. We decided to leave some of the statements unproved, and supply the corresponding proofs elsewhere (including the proofs here would have altered the order of the exposition, and would have come at the expense of clarity).

0.4.2. Here is the list of the unproved statements:

Proposition 3.2.6 says that the formation of Gray product commutes with colimits in each variable.

Proposition 3.2.9 asserts the associativity of the Gray product.

Theorems 4.1.3, Theorem 4.3.5, Theorem 4.6.3 and Theorem 5.2.3 are all generalizations of the assertion that the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ (or, rather, its variant $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}$) is fully faithful with specified essential image.

Proposition 4.5.4 gives an explicit description of the Gray product in terms of the functor $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}$.

It is quite possible that references for (some of) the above statements do exist, and we would be grateful if the reader could point them out to us.

1. Recollections: $(\infty, 1)$-categories via complete Segal spaces

As a warm-up to the definition of $(\infty, 2)$-categories, in this section we will recall the description of $(\infty, 1)$-categories as *complete Segal spaces*.

In the case of $(\infty, 2)$-categories, we will follow the same route, but with its inherent complications.

1.1. **The category $\mathbf{\Delta}$.** Let us recall the following from [Chapter I.1]:

1.1.1. For an integer n we let $[n]$ denote the ordinary 1-category symbolically represented as

$$0 \to 1 \to ... \to n.$$

1.1.2. We let $\mathbf{\Delta}$ be the (ordinary) category, whose objects are $[n]$ for $n \in \mathbf{N}$ and whose morphisms are functors $[n_1] \to [n_2]$.

By constrution, $\mathbf{\Delta}$ comes equipped with a fully faithful functor to 1-Cat.

1.1.3. The category $\mathbf{\Delta}$ carries a canonical involution, denoted rev. It acts as identity on objects, and on morphisms it is defined via the commutative diagrams

$$\begin{CD} [n_1] @>{\mathrm{rev}(\alpha)}>> [n_2] \\ @VVV @VVV \\ [n_1]^{\mathrm{op}} @>{\alpha}>> [n_2]^{\mathrm{op}}, \end{CD}$$

where the vertical arrows are the canonical equivalences

$$[n] \to [n]^{\mathrm{op}}, \quad i \mapsto n - i.$$

1.2. **(Complete) Segal spaces.**

1.2.1. Consider the $(\infty,1)$-category of simplicial spaces, i.e., $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$. Let us recall that an object

$$E_\bullet \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is said to be a *Segal* space if the following condition is satisfied:

For any $n = n_1 + n_2$, the natural map

$$E_n \to E_{n_1} \underset{E_0}{\times} E_{n_2},$$

is an isomorphism in Spc.

In the above formula, the maps $E_{n_1} \to E_0 \leftarrow E_{n_2}$ are given by

$$0 \in [0] \mapsto n_1 \in [n_1] \text{ and } 0 \in [0] \mapsto 0 \in [n_2],$$

respectively.

1.2.2. Let $s, t : E_1 \to E_0$ be the "source" and "target" maps. A point $\alpha \in E_1$ is said to be *invertible* if there exists a point

$$\beta \in E_1 \underset{E_0 \times E_0}{\times} (t(\alpha), s(\alpha)),$$

satisfying the following condition:

Note that from the isomorphism $E_2 \simeq E_1 \underset{E_0}{\times} E_1$ and the "composition" map $E_2 \to E_1$, we obtain two points $\alpha \circ \beta$ and $\beta \circ \alpha$ of E_1. Our condition is that both these points be in the essential image of the degeneracy map $E_0 \to E_1$.

It is easy to see that invertibility is a condition on the connected component of E_1 that a given point belongs to. Let $(E_1)^{\mathrm{invert}} \subset E_1$ be the full subspace consisting of invertible arrows.

1.2.3. Recall that a Segal space $E_\bullet$ is said to be *complete* if the above map $E_0 \to (E_1)^{\mathrm{invert}}$ is an isomorphism in Spc.

1.3. **The functor** $\mathrm{Seq}_\bullet$**.** The idea of the functor $\mathrm{Seq}_\bullet$ is very simple: we want to record the datum of an ∞-category by keeping track of *spaces* of n-fold compositions, for every n, along with its simplicial structure as we vary n.

1.3.1. We construct the functor of $(\infty,1)$-categories

$$\mathrm{Seq}_\bullet : 1\text{-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$$

by sending $\mathbf{C} \in 1\text{-Cat}$ to the simplicial space, whose n-simplices is the space

$$\mathrm{Maps}_{1\text{-Cat}}([n], \mathbf{C})$$

of functors $[n] \to \mathbf{C}$.

1.3.2. The functor $\mathrm{Seq}_\bullet$ admits a left adjoint, denoted $\mathfrak{L}$. Tautologically, $\mathfrak{L}$ is the left-Kan extension along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ of the functor tautological functor

$$\mathbf{\Delta} \to 1\text{-Cat},$$

i.e., the functor that sends an ordered finite set to itself, viewed as an ordinary category.

1.3.3. We now quote the following fundamental fact ([Rezk1, JT]):

Theorem 1.3.4. *The above functor* $\mathrm{Seq}_\bullet$ *is fully faithful. Its essential image is the full subcategory of* $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ *that consists of complete Segal spaces.*

1.3.5. The category 1-Cat carries a natural involution, denoted

$$\mathbf{C} \mapsto \mathbf{C}^{\mathrm{op}}.$$

It is uniquely characterized by the property that the the functor $\mathrm{Seq}_\bullet$ intertwines this involution with one on $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$, induced by the functor $\mathrm{rev} : \mathbf{\Delta} \to \mathbf{\Delta}$.

1.4. **Properties of categories and functors in terms of** $\mathrm{Seq}_\bullet$**.** In this subsection we will show how to translate various properties of ∞-categories (such as the property of being ordinary) or functors (such as the property of being fully faithful) into properties of the corresponding simplicial space.

1.4.1. First, let us observe that an $(\infty,1)$-category $\mathbf{C}$ is ordinary if for any $\mathbf{c}_0, \mathbf{c}_1 \in \mathrm{Seq}_0(\mathbf{c})$, the space

$$\mathrm{Seq}_1(\mathbf{C}) \underset{\mathrm{Seq}_0(\mathbf{c})\times \mathrm{Seq}_0(\mathbf{c})}{\times} \{(\mathbf{c}_0, \mathbf{c}_1)\},$$

is *discrete*.

Recall that

$$1\text{-Cat}^{1\text{-ordn}} \subset 1\text{-Cat}$$

denotes the full subcategory that consists of ordinary categories, and that the above inclusion admits a left adjoint, denoted

$$\mathbf{C} \mapsto \mathbf{C}^{1\text{-ordn}}.$$

Sometimes, $\mathbf{C}^{1\text{-ordn}}$ is called the *homotopy category* of $\mathbf{C}$, and is denoted $\mathrm{Ho}(\mathbf{C})$.

1.4.2. We have a fully faithful inclusion

$$\mathrm{Spc} \hookrightarrow 1\text{-Cat}\,.$$

Namely, $\mathbf{C} \in 1\text{-Cat}$ is a space if and only if $\mathrm{Seq}_\bullet(\mathbf{C})$ is degenerate, i.e., the degeneracy map $\mathrm{Seq}_0(\mathbf{C}) \to \mathrm{Seq}_n(\mathbf{C})$ is an isomorphism for every n. Note that given the Segal condition, it is enough to check this for $n = 1$.

The inclusion $\mathrm{Spc} \hookrightarrow 1\text{-Cat}$ admits a right adjoint, given by

$$\mathbf{C} \mapsto \mathbf{C}^{\mathrm{Spc}}.$$

We have

$$\mathrm{Seq}_n(\mathbf{C}^{\mathrm{Spc}}) \simeq \mathrm{Seq}_0(\mathbf{C}), \quad \forall\, n.$$

1.4.3. It follows from the definitions that a functor between $(\infty,1)$-categories $F : \mathbf{C} \to \mathbf{D}$ is *fully faithful* if and only if the corresponding map of spaces

$$\mathrm{Seq}_1(\mathbf{C}) \to \mathrm{Seq}_1(\mathbf{D}) \underset{\mathrm{Seq}_0(\mathbf{D})\times \mathrm{Seq}_0(\mathbf{D})}{\times} (\mathrm{Seq}_0(\mathbf{C}) \times \mathrm{Seq}_0(\mathbf{C})) \tag{1.1}$$

is an isomorphism (in Spc).

Note that if $\mathbf{C}$ and $\mathbf{D}$ are both spaces, then F is fully faithful if and only if it is a *monomorphism*, i.e., the inclusion of a union of connected components. Indeed, the above condition is equivalent to

$$\mathrm{Seq}_0(\mathbf{C}) \to \mathrm{Seq}_0(\mathbf{C}) \underset{\mathrm{Seq}_0(\mathbf{D})}{\times} \mathrm{Seq}_0(\mathbf{C})$$

being an isomorphism.

1.4.4. Recall that notion of a functor being *1-fully faithful*, see [Chapter I.1, Sect. 1.2.4] (for a functor between ordinary categories '1-fully faithful' is what is usually called 'faithful').

It is easy to see that $F : \mathbf{C} \to \mathbf{D}$ is 1-fully faithful if and only if the map (1.1) is a monomorphism.

1.4.5. Recall also the notion of a *1-replete* functor, see [Chapter I.1, Sect. 1.2.5]. It is not difficult to see that this is equivalent to the condition that the functor

$$\mathrm{Seq}_1(\mathbf{C}) \to \mathrm{Seq}_1(\mathbf{D})$$

should be fully faithful.

1.5. **The** $(\infty,1)$**-category** 1-Cat**.** In this subsection we will describe the object

$$\mathrm{Seq}_\bullet(1\text{-Cat}) \in 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

1.5.1. For an $(\infty,1)$-category $\mathbf{I}$, recall that

$$\mathrm{coCart}_{/\mathbf{I}} \subset 1\text{-Cat}_{/\mathbf{I}}$$

denotes the full subcategory consisting of coCartesian fibrations.

Recall that

$$(\mathrm{coCart}_{/\mathbf{I}})_{\mathrm{strict}} \subset \mathrm{coCart}_{/\mathbf{I}}$$

denotes the 1-full subcategory with the same objects, but where 1-morphisms are functors that send arrows that are coCartesian over $\mathbf{I}$ to arrows that are coCartesian over $\mathbf{I}$.

Recall also that we denote by

$$0\text{-coCart}_{/\mathbf{I}} \subset \mathrm{coCart}_{/\mathbf{I}}$$

the full subcategory consisting of coCartesian fibrations in spaces.

The above notation carries over *mutatis mutandis* to the case of *Cartesian* fibrations.

1.5.2. By definition,

$$\mathrm{Seq}_\bullet(1\text{-Cat}) \simeq \mathrm{Maps}_{1\text{-Cat}}([\bullet], 1\text{-Cat}).$$

Applying [Chapter I.1, Sect. 1.4.2], we obtain that

$$\mathrm{Seq}_\bullet(1\text{-Cat}) \simeq (\mathrm{coCart}_{/[\bullet]})^{\mathrm{Spc}}.$$

Under this identification, the full subcategory $\mathrm{Spc} \subset 1\text{-Cat}$ corresponds to

$$(0\text{-coCart}_{/[\bullet]})^{\mathrm{Spc}} \subset (\mathrm{coCart}_{/[\bullet]})^{\mathrm{Spc}}.$$

1.6. **Unstraightening.**

1.6.1. The incarnation of $(\infty,1)$-categories as complete Segal spaces gives an explicit description of the unstraightening functor

$$\mathrm{Maps}_{1\text{-Cat}}(\mathbf{C}, 1\text{-Cat}) \to (\mathrm{coCart}_{/\mathbf{C}})^{\mathrm{Spc}}.$$

Let us be given a functor $F : \mathbf{C} \to 1\text{-Cat}$. Let us describe the complete Segal space of the corresponding coCartesian fibration $\widetilde{\mathbf{C}} \to \mathbf{C}$.

1.6.2. For each n consider the category

$$([n] \times [n])^{\geq \mathrm{diag}},$$

equipped[1] with the projection on the second coordinate

$$([n] \times [n])^{\geq \mathrm{diag}} \to [n].$$

This is a coCartesian fibration of *ordinary* categories, and consider the corresponding functor

$$\iota_n : [n] \to \text{1-Cat}\,.$$

1.6.3. Now, the space $\mathrm{Seq}_n(\widetilde{\mathbf{C}})$ is described as follows: it is the space of pairs consisting of a functor

$$[n] \to \mathbf{C},$$

and a natural transformation from ι_n to the composite functor

$$[n] \to \mathbf{C} \overset{F}{\to} \text{1-Cat}\,.$$

2. The notion of $(\infty,2)$-category

In this section we give the definition of an $(\infty,2)$-category. In doing so, we will follow C. Barwick in approaching $(\infty,2)$-categories via complete Segal spaces.

Namely, the datum of a $(\infty,2)$-category will consist of an assignment for every n of the $(\infty,1)$-category whose objects are n-fold compositions, and whose morphisms are strings of 2-morphisms.

2.1. **Definition of the $(\infty,1)$-category of $(\infty,2)$-categories.** In this subsection we introduce the $(\infty,1)$-category of $(\infty,2)$-categories, to be denoted 2-Cat.

2.1.1. We define 2-Cat as a full subcategory of $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, defined by the following three conditions:

Condidtion 0: We require that $\mathbf{E}_0 \in \text{1-Cat}$ be a space.

Condidtion 1: We require that for any $n = n_1 + n_2$, the map (i.e., functor between $(\infty,1)$-categories)

$$\mathbf{E}_n \to \mathbf{E}_{n_1} \underset{\mathbf{E}_0}{\times} \mathbf{E}_{n_2}$$

be an isomorphism in 1-Cat (i.e., an equivalence of $(\infty,1)$-categories).

To formulate Condition 2 we note that given Condition 1, the ordinary category $(\mathbf{E}_1)^{\text{1-ordn}}$ contains a 1-full subcategory $((\mathbf{E}_1)^{\text{1-ordn}})^{\mathrm{invert}}$. We let $(\mathbf{E}_1)^{\mathrm{invert}}$ to be the corresponding 1-full subcategory of $\mathbf{E}_1$.

It is easy to see that the degeneracy functor $\mathbf{E}_0 \to \mathbf{E}_1$ (automatically, uniquely) factors as $\mathbf{E}_0 \to (\mathbf{E}_1)^{\mathrm{invert}}$.

Condidtion 2. We require that the above map (i.e., a functor of $(\infty,1)$-categories)

$$\mathbf{E}_0 \to (\mathbf{E}_1)^{\mathrm{invert}}$$

be an isomorphism in 1-Cat (i.e., be an equivalence of $(\infty,1)$-categories).

[1] This is the full subcategory of $[n] \times [n]$ consisting of objects (i,j) with $i \leq j$.

Remark 2.1.2. One can show that, given Conditions 0 and 1, the $(\infty,1)$-category $(\mathbf{E}_1)^{\mathrm{invert}}$ is actually a space. In fact, Conditions 1 and 2 imply that $(\mathbf{E}_\bullet)^{\mathrm{Spc}}$ is a Segal space and the natural map $((\mathbf{E}_1)^{\mathrm{Spc}})^{\mathrm{invert}} \to (\mathbf{E}_1)^{\mathrm{invert}}$ is an isomorphism (of spaces).

From here one deduces that Condition 2 can be replaced by a seemingly weaker condition:

Condition 2': We require that the Segal space $(\mathbf{E}_\bullet)^{\mathrm{Spc}}$ be complete.

2.1.3. We will denote the tautological fully faithful embedding

$$\text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

by $\mathrm{Seq}_\bullet$.

2.1.4. Note that the category 2-Cat has limits, and the functor $\mathrm{Seq}_\bullet$ commutes with limits. Indeed, it suffices to observe that Conditions 0, 1 and 2 above are all stable under taking limits in $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$.

For future reference we record:

Lemma 2.1.5. *The category* 2-Cat *is presentable (in particular, contains colimits).*

2.1.6. The category 2-Cat carries a pair of mutually commuting involutions, denoted

$$\mathbb{S} \mapsto \mathbb{S}^{\text{1-op}} \text{ and } \mathbb{S} \mapsto \mathbb{S}^{\text{2-op}},$$

respectively.

The involution $\mathbb{S} \mapsto \mathbb{S}^{\text{1-op}}$ is uniquely characterized by the property that the functor $\mathrm{Seq}_\bullet$ intertwines it with the involution on $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, induced by the involution $\mathrm{rev} : \mathbf{\Delta} \to \mathbf{\Delta}$.

The involution $\mathbb{S} \mapsto \mathbb{S}^{\text{2-op}}$ is uniquely characterized by the property that the functor $\mathrm{Seq}_\bullet$ intertwines it with the involution on $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, induced by the involution

$$\text{1-Cat} \to \text{1-Cat}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{op}}.$$

In what follows, we shall denote by

$$\mathbb{S} \mapsto \mathbb{S}^{\text{1\&2-op}} \simeq \mathbb{S}^{\text{2\&1-op}}$$

the composition of the above two involutions.

2.2. **Basic properties of $(\infty,2)$-categories.** Since $(\infty,2)$-categories were defined via simplicial $(\infty,1)$-categories, their properties (such as being ordinary) are expressed in such terms.

2.2.1. We have a fully faithful embedding

$$\text{1-Cat} \to \text{2-Cat} \tag{2.1}$$

that makes the diagram

$$\begin{tikzcd} \text{1-Cat} \arrow[r, "\mathrm{Seq}_\bullet"] \arrow[d] & \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d] \\ \text{2-Cat} \arrow[r, "\mathrm{Seq}_\bullet"] & \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{2.2}$$

commute.

2.2.2. The embedding (2.1) admits a right adjoint, to be denoted

$$\mathbb{S} \mapsto \mathbb{S}^{\text{1-Cat}}.$$

This right adjoint can be characterized by the fact that the natural transformation in the diagram

$$\begin{tikzcd} \text{1-Cat} \arrow[r, "\mathrm{Seq}_\bullet"] & \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}} \\ \text{2-Cat} \arrow[r, "\mathrm{Seq}_\bullet"] \arrow[u, "\mathbb{S}\mapsto\mathbb{S}^{\text{1-Cat}}"] & \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}, \arrow[u, "\mathbf{E}_\bullet\mapsto(\mathbf{E}_\bullet)^{\mathrm{Spc}}"'] \end{tikzcd}$$

obtained by passing to right adjoints along the vertical arrows in (2.2), is an isomorphism.

We denote

$$\mathbb{S}^{\mathrm{Spc}} := (\mathbb{S}^{\text{1-Cat}})^{\mathrm{Spc}}.$$

2.2.3. We shall say that an $(\infty,2)$-category $\mathbb{S}$ is *ordinary* if the $(\infty,1)$-category $\mathrm{Seq}_1(\mathbb{S})$ is ordinary.

Let $\text{2-Cat}^{\text{2-ordn}}$ denote the full subcategory of 2-Cat consisting of ordinary 2-categories. It is easy to see that $\text{2-Cat}^{\text{2-ordn}}$ identifies with the category of *ordinary* (a.k.a., usual, classical) 2-categories.

Remark 2.2.4. We always work up to coherent homotopy and what we call an ordinary 2-category is what is often called a 'bicategory' in the literature.

2.2.5. The embedding

$$\text{2-Cat}^{\text{2-ordn}} \hookrightarrow \text{2-Cat}$$

admits a left adjoint, to be denoted

$$\mathbb{S} \mapsto \mathbb{S}^{\text{2-ordn}}.$$

Remark 2.2.6. Note, however, that the diagram

$$\begin{tikzcd} \text{1-Cat} \arrow[r] \arrow[d] & \text{1-Cat}^{\text{1-ordn}} \arrow[d] \\ \text{2-Cat} \arrow[r] & \text{2-Cat}^{\text{2-ordn}} \end{tikzcd}$$

does not commute.

I.e., an $(\infty,2)$-category may be ordinary, only have invertible 2-morphisms, thus being an $(\infty,1)$-category, but not an *ordinary* category.

2.2.7. In what follows we shall refer to points of $\mathrm{Seq}_0(\mathbb{S})$ as *objects* of $\mathbb{S}$. For $s', s'' \in \mathbb{S}$, we consider the category

$$\mathrm{Seq}_1(\mathbb{S}) \underset{\mathrm{Seq}_0(\mathbb{S})\times\mathrm{Seq}_0(\mathbb{S})}{\times} \{s' \times s''\}.$$

We shall refer to it as the *category of morphisms* from s' to s'' and denote it by $\mathbf{Maps}_{\mathbb{S}}(s', s'')$.

We shall use the notation $\mathrm{Maps}_{\mathbb{S}}(s', s'')$ for

$$\mathbf{Maps}_{\mathbb{S}}(s', s'')^{\mathrm{Spc}} \simeq \mathrm{Maps}_{\mathbb{S}^{\text{1-Cat}}}(s', s'').$$

2.2.8. If s', s'' are objects of $\mathbb{S}$, we have

$$\mathbf{Maps}_{\mathbb{S}^{\text{2-ordn}}}(s', s'') \simeq (\mathbf{Maps}_{\mathbb{S}}(s', s''))^{\text{1-ordn}}.$$

2.3. Properties of functors between $(\infty,2)$-categories.

2.3.1. Let $F : \mathbb{S} \to \mathbb{T}$ be a functor between $(\infty, 2)$-categories. We shall say that is *fully faithful* if the resulting functor if $(\infty, 1)$-categories

$$\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}_1(\mathbb{T}) \underset{\mathrm{Seq}_0(\mathbb{T})\times\mathrm{Seq}_0(\mathbb{T})}{\times} (\mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})) \tag{2.3}$$

is an equivalence.

Equivalently, this means that for every $s', s'' \in \mathbb{S}$, the functor

$$\mathbf{Maps}_{\mathbb{S}}(s', s'') \to \mathbf{Maps}_{\mathbb{T}}(F(s'), F(s'')) \tag{2.4}$$

is an equivalence.

2.3.2. We shall say that F is *1-fully faithful* if the functor (2.3) is fully faithful. Equivalently, this means the functor (2.4) is fully faithful for any $s', s'' \in \mathbb{S}$.

2.3.3. We shall say that F is *1-replete* if the functor

$$\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}_1(\mathbb{T})$$

is fully faithful.

A functor which is 1-replete is an equivalence onto what we call a *1-full subcategory*. For an $(\infty, 2)$-category $\mathbb{S}$, its 1-full subcategories are in bijection with those in $\mathbb{S}^{2\text{-ordn}}$, and also with those of $\mathbb{S}^{1\text{-Cat}}$.

2.3.4. We shall say that a functor F is *2-fully faithful* if the functor (2.3) is 1-fully faithful (equivalently, if the functor (2.4) is 1-fully faithful for any $s', s'' \in \mathbb{S}$).

2.3.5. We shall say that F is *2-replete* if the functor (2.3) is 1-replete (equivalently, if the functor (2.4) is 1-replete for any $s', s'' \in \mathbb{S}$).

2.3.6. A functor which is 2-replete is an equivalence onto what we call a *2-full subcategory.*

For an $(\infty, 2)$-category $\mathbb{S}$, its 2-full subcategories are in bijection with those in $\mathbb{S}^{2\text{-ordn}}$. Each such is determined by a subset of $\pi_0(\mathrm{Seq}_1(\mathbb{S}))$, closed under compositions.

2.4. **The $(\infty, 2)$-category $\mathbf{1\text{-}Cat}$.** The main example of an $(\infty, 2)$-category is the $(\infty, 2)$-category of $(\infty, 1)$-categories, denoted $\mathbf{1\text{-}Cat}$. In this subsection we define what it is.

2.4.1. We introduce the $(\infty, 2)$-category $\mathbf{1\text{-}Cat}$ as follows. We let $\mathrm{Seq}_n(\mathbf{1\text{-}Cat})$ be the 1-full subcategory of $\mathrm{Cart}_{/[n]^{\mathrm{op}}}$, where we restrict 1-morphisms to functors that induce an equivalence over each $i \in [n]$.

The assignment

$$n \mapsto \mathrm{Seq}_n(\mathbf{1\text{-}Cat})$$

clearly defines an object of $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Proposition 2.4.2.

(a) *The above object* $\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat})$ *lies in the essential image of the functor*

$$\mathrm{Seq}_\bullet : \text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

(b) *The resulting object* $\mathbf{1\text{-}Cat} \in \text{2-Cat}$ *is equipped with a canonical identification*

$$(\mathbf{1\text{-}Cat})^{\text{1-Cat}} \simeq \text{1-Cat}.$$

Proof. First, we note that by construction, the $(\infty, 1)$-category $\mathrm{Seq}_0(\mathbf{1\text{-}Cat})$ tautologically identifies with $(1\text{-Cat})^{\mathrm{Spc}}$. Similarly, we have a canonical identification

$$\mathrm{Seq}_\bullet(1\text{-Cat}) := \mathrm{Maps}([\bullet], 1\text{-Cat}) \simeq (\mathrm{Cart}_{/[\bullet]^{\mathrm{op}}})^{\mathrm{Spc}} = (\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat}))^{\mathrm{Spc}},$$

where the second isomorphism is given by [Chapter I.1, Sect. 1.4.5].

It remains to show that the simplicial category $\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat})$ satisfies the Segal condition. Indeed, this follows from the fact that for $n = n_1 + n_2$, the functor

$$\mathrm{Cart}_{/[n]^{\mathrm{op}}} \to \mathrm{Cart}_{/[n_1]^{\mathrm{op}}} \underset{1\text{-Cat}}{\times} \mathrm{Cart}_{/[n_2]^{\mathrm{op}}}$$

is an equivalence.

□

2.4.3. We also have:

Corollary 2.4.4. *For $\mathbf{S}, \mathbf{T} \in \mathbf{1\text{-}Cat}$, there is a canonical equivalence*

$$\mathbf{Maps}_{\mathbf{1\text{-}Cat}}(\mathbf{S}, \mathbf{T}) \simeq \mathrm{Funct}(\mathbf{S}, \mathbf{T}).$$

Proof. Follows from the equivalences:

$$\mathrm{Funct}(\mathbf{S}, \mathbf{T}) \simeq \mathrm{Funct}([1], 1\text{-Cat}) \underset{1\text{-Cat} \times 1\text{-Cat}}{\times} \{(\mathbf{S}, \mathbf{T})\} \simeq \mathrm{Cart}_{/[1]^{\mathrm{op}}} \underset{1\text{-Cat} \times 1\text{-Cat}}{\times} \{(\mathbf{S}, \mathbf{T})\},$$

where the second equivalence is the unstraightening over $[1]$. □

Remark 2.4.5. Suppose in the above definition of $\mathbf{1\text{-}Cat}$, we replace $\mathrm{Cart}_{/[n]^{\mathrm{op}}}$ by $\mathrm{coCart}_{/[n]}$. (Note that the underlying simplicial spaces are both identified with $\mathrm{Maps}([\bullet], 1\text{-Cat})$.)

The latter simplicial category also lies in the essential image of the functor $\mathrm{Seq}_\bullet$, and the resulting $(\infty, 2)$-category identifies with $(\mathbf{1\text{-}Cat})^{2\text{-op}}$.

2.5. **The $(\infty, 2)$-category of functors.** So far, we have defined on the totality of $(\infty, 2)$-categories a structure of $(\infty, 1)$-category. In particular, for $\mathbb{S}, \mathbb{T}$ we have a well-defined *space*

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}).$$

We claim, however, that the above space lifts, in a natural way to an object of 2-Cat.

2.5.1. We have the following basic result:

Theorem 2.5.2 ([Rezk2, BarS])**.** *For $\mathbb{S}, \mathbb{T} \in$ 2-Cat, the functor*

$$\mathbb{X} \mapsto \mathrm{Maps}_{2\text{-Cat}}(\mathbb{X} \times \mathbb{S}, \mathbb{T})$$

is representable.

Note that Theorem 2.5.2 is not at all tautological. By the Adjoint Functor Theorem, it is equivalent to the following one:

Theorem 2.5.3. *The functor*

$$2\text{-Cat} \times 2\text{-Cat} \to 2\text{-Cat}, \quad \mathbb{S}, \mathbb{T} \mapsto \mathbb{S} \times \mathbb{T}$$

commutes with colimits in each variable.

2.5.4. In what follows we shall denote the object representing the functor in Theorem 2.5.2 by

$$\mathrm{Funct}(\mathbb{S}, \mathbb{T}) \in 2\text{-Cat}.$$

Note that by definition

$$(\mathrm{Funct}(\mathbb{S}, \mathbb{T}))^{\mathrm{Spc}} \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}, \mathbb{T}).$$

2.6. $(\infty, 2)$-categories via bi-simplicial spaces.

2.6.1. The category 1-$\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ admits a fully faithful embedding into

$$(\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}} \simeq \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}},$$

given by $(\mathrm{Seq}_\bullet)^{\mathbf{\Delta}^{\mathrm{op}}}$, i.e., apply the functor $\mathrm{Seq}_\bullet : 1\text{-}\mathrm{Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ simplex-wise.

Hence, we obtain a fully faithful embedding

$$\mathrm{Sq}^{\sim}_{\bullet,\bullet} = (\mathrm{Seq}_\bullet)^{\mathbf{\Delta}^{\mathrm{op}}} \circ \mathrm{Seq}_\bullet : 2\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}. \tag{2.5}$$

2.6.2. For a pair of integers $m, n \geq 0$, consider the functor $\mathrm{Sq}^{\sim}_{m,n}$. Since this functor commutes with limits, it is co-representable. We let $[m,n]^{\sim} \in 2\text{-}\mathrm{Cat}$ denote the co-representing object.

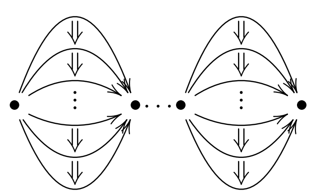
It is easy to see that $[0,n]^{\sim} \simeq [n]$ and $[m,0]^{\sim} \simeq *$. Note that Segal condition implies that the natural maps

$$[m, n_1]^{\sim} \underset{*}{\sqcup} [m, n_2]^{\sim} \to [m, n_1 + n_2]^{\sim} \tag{2.6}$$

and

$$[m_1, n]^{\sim} \underset{[n]}{\sqcup} [m_2, n]^{\sim} \to [m_1 + m_2, n]^{\sim} \tag{2.7}$$

are isomorphisms. Pictorially, one could think of $[m,n]^{\sim}$ as the diagram



2.6.3. The material in the rest of this subsection is included for the sake of completeness.

Note that we have the following explicit description of the *ordinary* 2-category

$$([m,n]^{\sim})^{2\text{-ordn}}.$$

Its objects are integers $0, ..., n$. For $0 \leq i, j \leq n$, the 1-category of morphisms $i \to j$ is described as follows:

(a) It is empty if $i > j$;

(b) It is $\{*\}$ if $i = j$;

(c) For $i < j$ it is the poset of sequences

$$(f_{i,i+1}, ..., f_{j-1,j}) \in \{0, ..., m\}$$

The order relation is

$$(f_{i,i+1}, ..., f_{j-1,j}) \leq (f'_{i,i+1}, ..., f'_{j-1,j}) \Leftrightarrow \forall i \leq k \leq j-1, \ f_{k,k+1} \leq f'_{k,k+1}.$$

2.6.4. We have the following result:

Theorem 2.6.5 ([BarS, Lemma 12.5])**.** *For any* (m,n)*, the tautological functor*

$$[m,n]^{\sim} \to ([m,n]^{\sim})^{2\text{-ordn}}$$

is an equivalence.

In other words, this theorem says that for every m,n the object of 2-Cat, co-representing the functor

$$\mathbb{S} \mapsto \mathrm{Sq}^{\sim}_{m,n}(\mathbb{S}) \simeq \mathrm{Seq}_m(\mathrm{Seq}_n(\mathbb{S})), \quad \text{2-Cat} \to \mathrm{Spc}$$

is an *ordinary* 2-category.

3. Lax functors and the Gray product

One of the key new features of $(\infty,2)$-categories, as compared to $(\infty,1)$-categories, is the notion of *right-lax* (or *left-lax*) functor. We will introduce these notions in the present section.

3.1. **Lax functors.**

The idea of a lax functor is that the composition of 1-morphisms does not have to go to the composition, but rather be connected to it by a (not necessarily invertible) 2-morphism.

3.1.1. Let $\mathbb{S}$ be an $(\infty,2)$-category, thought of as a functor $\mathbf{\Delta}^{\mathrm{op}} \to$ 1-Cat. Let $\mathbb{S}^{\oint}$ be the total space of the corresponding coCartesian fibration over $\mathbf{\Delta}^{\mathrm{op}}$.

By definition, a functor between a pair of $(\infty,2)$-categories $\mathbb{S}$ and $\mathbb{T}$ is a functor

$$\mathbb{S}^{\oint} \to \mathbb{T}^{\oint}$$

that sends *coCartesian arrows to coCartesian* arrows.

3.1.2. We give the following definitions: a map

$$\alpha : [m] \to [n]$$

is said to be:

- *active* if $\alpha(0) = 0$ and $\alpha(m) = n$.
- *idle* if for all $0 \le j \le n$ for which there exist $0 \le i_1, i_2 \le m$ with $\alpha(i_i) \le j \le \alpha(i_2)$, there exists $0 \le i \le m$ such that $\alpha(i) = j$;
- *inert* if for all $0 \le j \le n$ for which there exist $0 \le i_1, i_2 \le m$ with $\alpha(i_i) \le j \le \alpha(i_2)$, there exists a *unique* $0 \le i \le m$ such that $\alpha(i) = j$;

In other words, an inert map sends $i \mapsto i + k$ for some $0 \le k \le n - m$.

3.1.3. We define a *non-unital right-lax functor* from $\mathbb{S}$ to $\mathbb{T}$ to be a functor

$$\mathbb{S}^{\oint} \to \mathbb{T}^{\oint}$$

which takes coCartesian edges over *inert* morphisms of Δ to coCartesian edges.

We define a *right-lax functor* from $\mathbb{S}$ to $\mathbb{T}$ to be a functor

$$\mathbb{S}^{\oint} \to \mathbb{T}^{\oint}$$

which takes coCartesian edges over *idle* morphisms of Δ to coCartesian edges.

We define a *non-unital left-lax functor* (resp., *left-lax functor*) from $\mathbb{S}$ to $\mathbb{T}$ to be a *non-unital right-lax functor* (resp., *right-lax functor*) from $\mathbb{S}^{2\text{-op}}$ to $\mathbb{T}^{2\text{-op}}$.

Remark 3.1.4. At the level of ordinary 2-categories, the notion of 'non-unital right-lax functor' differs from what is called a lax functor in the literature on ordinary 2-categories. In particular, unlike the classical notion, the notion of a non-unital right-lax functor is invariant with respect to equivalence of 2-categories.

On the other hand, the notion of 'right-lax functor', at the level of ordinary 2-categories, agrees with what is usually called a 'normal lax functor' in the literature on ordinary 2-categories.

3.1.5. In what follows, given a pair of $(\infty, 2)$-categories $\mathbb{S}$ and $\mathbb{T}$ we shall symbolically denote right-lax and left-lax functors from $\mathbb{S}$ to $\mathbb{T}$ by

$$\mathbb{S} \dashrightarrow \mathbb{T},$$

to distinguish them from actual (i.e., strict) functors, which we denote by $\mathbb{S} \to \mathbb{T}$.

3.1.6. Let $F : \mathbb{S} \dashrightarrow \mathbb{T}$ be a right-lax functor. Then, for a string of objects $s_0 \overset{\phi}{\to} s_1 \overset{\psi}{\to} s_2$ in $\mathbb{S}$, we are given a (not necessarily invertible) 2-morphism

$$F(\psi) \circ F(\phi) \to F(\psi \circ \phi), \tag{3.1}$$

i.e., a 1-morphism in $\mathbf{Maps}_{\mathbb{T}}(F(s_0), F(s_2))$.

For a left-lax functor, we have a map in the opposite direction: $F(\psi \circ \phi) \to F(\psi) \circ F(\phi)$.

3.1.7. We can introduce 1-full subcategories

$$\text{2-Cat} \hookrightarrow \text{2-Cat}_{\text{right-lax}} \hookrightarrow \text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}$$

of the full subcategory of $\text{1-Cat}^{\mathbf{\Delta}^{\text{op}}}$ formed by coCartesian fibrations, by imposing increasingly weaker conditions on 1-morphisms.

Thus, for a pair of objects $\mathbb{S}, \mathbb{T} \in \text{2-Cat}$ we have the well-defined spaces

$$\text{Maps}_{\text{2-Cat}}(\mathbb{S}, \mathbb{T}) \subset \text{Maps}_{\text{2-Cat}_{\text{right-lax}}}(\mathbb{S}, \mathbb{T}) \subset \text{Maps}_{\text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathbb{S}, \mathbb{T}).$$

All of the above categories contain limits, and the above embeddings commute with limits.

3.2. **The Gray tensor product.** The notion of right-lax functor allows one to introduce the notion of *Gray product* of $(\infty, 2)$-categories. Given $(\infty, 2)$-categories $\mathbb{S}$ and $\mathbb{T}$, the Gray product $\mathbb{S} \circledast \mathbb{T}$ has the same objects as $\mathbb{S} \times \mathbb{T}$. However, for $s_0 \overset{\phi}{\to} s_1$ and $t_0 \overset{\psi}{\to} t_1$, the diagram

$$\begin{tikzcd}
(s_0, t_0) \arrow[r, "{(\mathrm{id}, \psi)}"] \arrow[d, "{(\phi, \mathrm{id})}"'] & (s_0, t_1) \arrow[d, "{(\phi, \mathrm{id})}"] \\
(s_1, t_0) \arrow[r, "{(\mathrm{id}, \psi)}"] & (s_1, t_1)
\end{tikzcd}$$

will *no longer commute*, but will do so up to a *non-invertible* 2-morphism.

The formation of the Gray product will allow us to talk about right-lax natural transformation between functors.

3.2.1. For an n-tuple of $(\infty,2)$-categories $\mathbb{S}_1,\mathbb{S}_2,...,\mathbb{S}_n$ and another $(\infty,2)$-category $\mathbb{T}$, let

$$\mathrm{Maps}_{\text{2-Cat}}(\mathbb{S}_1\circledast...\circledast\mathbb{S}_n,\mathbb{T})\subset\mathrm{Maps}_{\text{2-Cat}^{\mathrm{lax}}}(\mathbb{S}_1\times...\times\mathbb{S}_n,\mathbb{T})$$

to be the subspace given by right-lax functors such that:

(1) For each i and an object $\hat{s}_i\in\prod\limits_{j\neq i}\mathbb{S}_j$, the composite lax functor

$$\mathbb{S}_i\xrightarrow{\mathrm{id}\times\hat{s}_i}\mathbb{S}_1\times...\times\mathbb{S}_n\overset{F}{\dashrightarrow}\mathbb{T}$$

is a strict functor.

(2) For any morphism

$$f=(f_i):(s_1,...,s_n)\to(s_1',...,s_n')$$

in $\mathbb{S}_1\times...\times\mathbb{S}_n$ and $1\leq k\leq n-1$, the 2-morphism in $\mathbb{T}$, corresponding to splitting f as a composition (see (3.1))

$$(s_1,...,s_k,s_{k+1},...,s_n)\xrightarrow{(f_1,...,f_k,\mathrm{id},...,\mathrm{id})}$$
$$\to(s_1',...,s_k',s_{k+1},...,s_n)\xrightarrow{(\mathrm{id},...,\mathrm{id},f_{k+1},...,f_n)}(s_1',...,s_k',s_{k+1}',...,s_n'),$$

is an isomorphism.

3.2.2. For example, if $n=2$ and F is an object of $\mathrm{Maps}_{\text{2-Cat}}(\mathbb{S}_1\circledast\mathbb{S}_2,\mathbb{T})$, for any

$$(s_1,s_2)\overset{(f_1,f_2)}{\to}(s_1',s_2')$$

in $\mathbb{S}_1\times\mathbb{S}_2$, we obtain a 2-morphism in $\mathbb{T}$

$$F(f_1,\mathrm{id}_{s_2'})\circ F(\mathrm{id}_{s_1},f_2)\to F(f_1,f_2)\simeq F(\mathrm{id}_{s_1'},f_2)\circ F(f_1,\mathrm{id}_{s_2}).$$

3.2.3. Since the functor $\mathrm{Maps}_{\text{2-Cat}}(\mathbb{S}_1\circledast\ldots\circledast\mathbb{S}_n,-)$ commutes with limits, it is co-represented by an $(\infty,2)$-category, to be denoted $\mathbb{S}_1\circledast\ldots\circledast\mathbb{S}_n$, and called the *Gray tensor product*.

By definition, we have a tautological projection

$$\mathbb{S}_1\circledast...\circledast\mathbb{S}_n\to\mathbb{S}_1\times...\times\mathbb{S}_n,$$

and a canonically defined lax functor

$$\mathbb{S}_1\times...\times\mathbb{S}_n\dashrightarrow\mathbb{S}_1\circledast...\circledast\mathbb{S}_n,$$

such that the composition

$$\mathbb{S}_1\times...\times\mathbb{S}_n\dashrightarrow\mathbb{S}_1\circledast...\circledast\mathbb{S}_n\to\mathbb{S}_1\times...\times\mathbb{S}_n$$

is the identity functor.

3.2.4. Note also that by construction, we have a canonical identification

$$(\mathbb{S}_n\circledast...\circledast\mathbb{S}_1)^{2\text{-op}}\simeq(\mathbb{S}_1^{2\text{-op}}\circledast...\circledast\mathbb{S}_n^{2\text{-op}}).\tag{3.2}$$

3.2.5. We have the following basic fact[2]

Proposition 3.2.6. *The functor*

$$\text{2-Cat}\times...\times\text{2-Cat}\to\text{2-Cat},\quad\mathbb{S}_1,...,\mathbb{S}_n\mapsto\mathbb{S}_1\circledast\ldots\circledast\mathbb{S}_n$$

commutes with colimits in each variable.

[2]We do not prove it, and we were not able to find a reference.

3.2.7. For a pair of $(\infty,2)$-categories $\mathbb{S}$ and $\mathbb{T}$, recall the $(\infty,2)$-category $\mathrm{Funct}(\mathbb{S},\mathbb{T})$. We introduce its enlargement $\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}$ (which has the same underlying space, but more 1-morphisms) as follows:

We set

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{X},\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}) = \mathrm{Maps}_{2\text{-Cat}_{\text{right-lax}}}(\mathbb{X}\circledast\mathbb{S},\mathbb{T}),$$

where the co-representability is insured by Proposition 3.2.6.

We call 1-morphisms in $\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{right-lax}}$ *right-lax natural transformations.* By definition, given a right-lax natural transformation $\alpha : F_1 \dashrightarrow F_2$, for an object $s\in\mathbb{S}$ we have a 1-morphism

$$\alpha(s) : F_1(s) \to F_2(s)$$

in $\mathbb{T}$, and for a 1-morphism $\phi : s\to s'$, we have a 2-morphism

$$\begin{tikzcd} F_1(s) \arrow[r, "F_1(\phi)"] \arrow[d, "\alpha(s)"'] & F_1(s') \arrow[d, "\alpha(s')"] \arrow[dl, Rightarrow] \\ F_2(s) \arrow[r, "F_2(\phi)"'] & F_2(s'). \end{tikzcd}$$

Similarly, we introduce the $(\infty,2)$-category $\mathrm{Funct}(\mathbb{S},\mathbb{T})_{\text{left-lax}}$.

3.2.8. For $n = n_1+n_2$ and an n-tuple of $(\infty,2)$-categories $\mathbb{S}_1,\dots,\mathbb{S}_n$ consider the right-lax functor

$$\begin{aligned}\mathbb{S}_1\times\dots\times\mathbb{S}_n \simeq (\mathbb{S}_1\times\dots\times\mathbb{S}_{n_1})\times(\mathbb{S}_{n_1+1}\times\dots\times\mathbb{S}_{n_1+n_2}) \dashrightarrow \\ \dashrightarrow (\mathbb{S}_1\circledast\dots\circledast\mathbb{S}_{n_1})\times(\mathbb{S}_{n_1+1}\circledast\dots\circledast\mathbb{S}_{n_1+n_2}) \dashrightarrow \\ \dashrightarrow (\mathbb{S}_1\circledast\dots\circledast\mathbb{S}_{n_1})\circledast(\mathbb{S}_{n_1+1}\circledast\dots\circledast\mathbb{S}_{n_1+n_2})\end{aligned}$$

It follows from the definitions that the above right-lax functor gives rise to a functor

$$\mathbb{S}_1\circledast\dots\circledast\mathbb{S}_{n_1}\circledast\mathbb{S}_{n_1+1}\circledast\dots\circledast\mathbb{S}_{n_1+n_2} \dashrightarrow (\mathbb{S}_1\circledast\dots\circledast\mathbb{S}_{n_1})\circledast(\mathbb{S}_{n_1+1}\circledast\dots\circledast\mathbb{S}_{n_1+n_2}). \tag{3.3}$$

We have the following proposition[3]:

Proposition 3.2.9. *The functor* (3.3) *is an equivalence.*

Remark 3.2.10. It is easy to see that Proposition 3.2.9 implies that the Cartesian monoidal structure on 2-Cat induces a monoidal structure on 2-Cat, given by the Gray product.

3.3. **Cubical 2-categories.**

3.3.1. For an integer k and a k-tuple $n_1,\dots,n_k$ we let

$$[n_1,\dots,n_k]\in 2\text{-Cat}$$

denote the $(\infty,2)$-category

$$[n_1]\circledast\dots\circledast[n_k].$$

3.3.2. From Proposition 3.2.9 we obtain that for $k = k_1+k_2$, the natural functor

$$[n_1,\dots,n_k]\to[n_1,\dots,n_{k_1}]\circledast[n_{k_1+1},\dots,n_{k_1+k_2}]$$

is an equivalence.

[3]We do not prove it, and we were not able to find a reference.

3.3.3. In addition, from Proposition 3.2.6 we obtain that for $1 \leq i \leq k$ and $n_i = n'_i + n''_i$, the natural functor

$$[n_1, ..., n_{i-1}, n'_i, n_{i+1}, ..., n_k] \underset{[n_1,...,n_{i-1},0,n_{i+1},...,n_k]}{\sqcup} [n_1, ..., n_{i-1}, n''_i, n_{i+1}, ..., n_k] \to \to [n_1, ..., n_{i-1}, n_i, n_{i+1}, ..., n_k] \tag{3.4}$$

is an equivalence, where we note that

$$[n_1, ..., n_{i-1}, 0, n_{i+1}, ..., n_k] \simeq [n_1, ..., n_{i-1}, n_{i+1}, ..., n_k].$$

3.3.4. The following proposition is quoted for the sake of completeness[4]:

Proposition 3.3.5. *The $(\infty, 2)$-categories $[n_1, ..., n_k]$ are ordinary.*

3.4. **Squares.** Our primary interest will be the $(\infty, 2)$-categories $[m, n]$.

3.4.1. We consider first the case $m = n = 1$. It follows from the definitions that for $\mathbb{S} \in$ 2-Cat, the space

$$\mathrm{Maps}_{\text{2-Cat}}([1,1], \mathbb{S}) \tag{3.5}$$

identifies canonically with the space of diagrams

$$\begin{tikzcd}
s_{0,0} \arrow[r] \arrow[d] & s_{0,1} \arrow[d] \arrow[dl, Rightarrow] \\
s_{1,0} \arrow[r] & s_{1,1}.
\end{tikzcd}$$

In other words,

$$\mathrm{Maps}_{\text{2-Cat}}([1,1], \mathbb{S}) \simeq (\mathrm{Sq}^{\sim}_{0,2} \times \mathrm{Sq}^{\sim}_{0,2}) \underset{\mathrm{Sq}^{\sim}_{0,1} \times \mathrm{Sq}^{\sim}_{0,1}}{\times} \mathrm{Sq}^{\sim}_{1,1},$$

where both maps $\mathrm{Sq}^{\sim}_{0,2} \to \mathrm{Sq}^{\sim}_{0,1}$ correspond to the unique active map $[1] \to [2]$.

Remark 3.4.2. Using (3.4), for any m, n we can depict $[m, n]$ by the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \cdots \arrow[r] \arrow[dl, Rightarrow] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \bullet \arrow[d] \arrow[dl, Rightarrow] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \cdots \arrow[r] \arrow[dl, Rightarrow] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \bullet \arrow[d] \arrow[dl, Rightarrow] \\
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \cdots \arrow[r] \arrow[dl, Rightarrow] & \cdots \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \cdots \arrow[d] \arrow[dl, Rightarrow] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \cdots \arrow[r] \arrow[dl, Rightarrow] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & \bullet \arrow[d] \arrow[dl, Rightarrow] \\
\bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet.
\end{tikzcd}$$

[4]We do not prove it, and we were not able to find a reference.

3.4.3. Recall the notation $[m,n]^{\sim}$, see Sect. 2.6.2.

Using the description of the space (3.5), we obtain that there is a canonical identification

$$[1,1] \underset{[1]\sqcup[1]}{\sqcup} (* \sqcup *) \simeq [1,1]^{\sim}, \tag{3.6}$$

where the two maps $[1] \to [1,1]$ are

$$[1] \simeq [1,0] \rightrightarrows [1,1],$$

corresponding to the two maps $[0] \rightrightarrows [1]$.

Pictorially, $[1,1]^{\sim}$, which is

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet, \end{tikzcd}$$

is obtained from $[1,1]$, which is

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[dl, Rightarrow] \\ \bullet \arrow[r] & \bullet, \end{tikzcd}$$

by collapsing the vertical edges.

3.4.4. We claim that

$$[m,n] \underset{[m]\sqcup\ldots\sqcup[m]}{\sqcup} (* \sqcup \ldots. \sqcup *) \simeq [m,n]^{\sim}, \tag{3.7}$$

functorial in $[m],[n] \in \mathbf{\Delta}$.

Indeed, unwinding the definitions, to specify a map $\to$ in (3.7), we need to specify a functorial assigment for each $[i],[j] \in \mathbf{\Delta}$ and a pair of maps

$$[i] \to [m] \times [n], [j] \to [m] \times [n] \tag{3.8}$$

a point in $\mathrm{Sq}^{\sim}_{i,j}([m,n]^{\sim})$.

The sought-for point is obtained from the tautological point of $\mathrm{Sq}^{\sim}_{i,j}([i,j]^{\sim})$ by composing with

$$([i],[j]) \to ([m],[n]),$$

where the latter is obtained from (3.8) by projection.

3.4.5. It follows from the construction, that the map (3.7) canonically factors through a map

$$[m,n] \underset{[m]\sqcup\ldots\sqcup[m]}{\sqcup} (* \sqcup \ldots. \sqcup *) \to [m,n]^{\sim}, \tag{3.9}$$

functorial in $[m],[n] \in \mathbf{\Delta}$, where the $n+1$ maps $[m] \to [m,n]$ are given by

$$[m] \simeq [m,0] \to [m,n],$$

corresponding to the $n+1$ maps $[0] \to [n]$.

Proposition 3.4.6. *The map* (3.9) *is an isomorphism.*

Proof. Using (3.4) and (2.6), we obtain that it is sufficient to consider the cases when m and n are equal to 0 or 1.

When m or n are equal to 0, there is nothing to prove. The case of $m = n = 1$ follows from the isomorphism (3.6).

□

3.4.7. Note that by applying Proposition 3.4.6 in the case $n = 1$, we obtain:

Corollary 3.4.8. *For $\mathbb{S} \in$ 2-Cat and $s_0, s_1 \in \mathbb{S}$, there exists a canonical isomorphism*

$$\mathrm{Funct}([1], \mathbb{S})_{\text{right-lax}} \underset{\mathbb{S}\times\mathbb{S}}{\times} \{(s_0, s_1)\} \simeq \mathbf{Maps}_{\mathbb{S}}(s_0, s_1).$$

4. $(\infty, 2)$-categories via squares

Recall that we originally realized 2-Cat as a full subcategory in $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ via the functor $\mathrm{Seq}_\bullet$.

In this subsection, we will discuss a *different* realization of 2-Cat as a full subcategory in $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, this time via the functor that we denote $\mathrm{Seq}^{\mathrm{ext}}_\bullet$.

Recall that for an $(\infty, 2)$-category $\mathbb{S}$, for $n = 0$, the $(\infty, 1)$-category $\mathrm{Seq}_0(\mathbb{S})$ recorded the space $\mathbb{S}^{\mathrm{Spc}}$. For $n = 1$, $(\infty, 1)$-category $\mathrm{Seq}_1(\mathbb{S})$ had as objects 1-morphisms $s_0 \to s_1$ and as morphisms 2-morphisms, i.e., diagrams

$$\begin{tikzcd} s_0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & s_1. \end{tikzcd}$$

The idea of $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ is the following. The $(\infty, 1)$-category $\mathrm{Seq}^{\mathrm{ext}}_0(\mathbb{S})$ will be $\mathbb{S}^{1\text{-Cat}}$. Now, for $n = 1$, the category $\mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$ will have as objects 1-morphisms $s_0 \to s_1$ as before, but as morphisms diagrams

$$\begin{tikzcd} s_0 \arrow[r] \arrow[d] & s_1 \arrow[d] \arrow[dl, Rightarrow] \\ s_0' \arrow[r] & s_1'. \end{tikzcd}$$

I.e., $\mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$ will be the $(\infty, 1)$-category $\mathrm{Funct}([1], \mathbb{S})_{\text{right-lax}}$.

4.1. **The functor** $\mathrm{Sq}_{\bullet,\bullet}$. Before introducing the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$, we introduce the corresponding version, denoted $\mathrm{Sq}_{\bullet,\bullet}$, of the functor $\mathrm{Sq}^{\sim}_{\bullet,\bullet}$. It will have the advantage of respecting more symmetries of the target category, i.e., $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$.

4.1.1. Consider the functor

$$\mathrm{Sq}_{\bullet,\bullet} : 2\text{-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}},$$

defined by

$$\mathbb{S} \mapsto ([m], [n] \mapsto \mathrm{Maps}_{2\text{-Cat}}([m, n], \mathbb{S}))\,.$$

4.1.2. This section is devoted to the discussion of the generalizations of the following fundamental result[5]:

Theorem 4.1.3. *The functor $\mathrm{Sq}_{\bullet,\bullet}$ is fully faithful.*

[5]We do not prove it, and we were not able to find a reference.

4.1.4. Note from (3.9) and Proposition 3.4.6 we obtain a natural transformation

$$\mathrm{Sq}^{\sim}_{\bullet,\bullet} \to \mathrm{Sq}_{\bullet,\bullet},$$

such that for $\mathbb{S} \in 2\text{-Cat}$ and any m, n, the corresponding map

$$\mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{S}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}) \tag{4.1}$$

is a full embedding.

Indeed, if we denote $E_{\bullet,\bullet} = \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S})$, the essential image of (4.1) is the full subspace of $E_{m,n}$ consisting of points, for which for every $[1] \to [m]$ and $[0] \to [n]$, the corresponding point in $E_{1,0}$ is degenerate, i.e., lies in the essential image of $E_{0,0} \to E_{1,0}$.

4.1.5. Let

$$E_{\bullet,\bullet} \mapsto (E_{\bullet,\bullet})^{\mathrm{reflect}}$$

denote the involution on $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$, corresponding to swapping the two factors of $\mathbf{\Delta}^{\mathrm{op}}$.

It follows from (3.2), that there is a canonical identification

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}^{2\text{-op}}) \simeq (\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))^{\mathrm{reflect}}.$$

4.1.6. Let

$$E_{\bullet,\bullet} \mapsto (E_{\bullet,\bullet})^{\text{vert-op}} \text{ and } (E_{\bullet,\bullet})^{\text{horiz-op}}$$

be the involutions on $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$ induced by the involution rev along the first and second copy of $\mathbf{\Delta}^{\mathrm{op}}$, respectively.

Let

$$E_{\bullet,\bullet} \mapsto (E_{\bullet,\bullet})^{\text{vert\&horiz-op}}$$

denote their composition.

It follows that we have a canonical identification

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}^{1\&2\text{-op}}) \simeq (\mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}))^{\text{vert\&horiz-op}}.$$

4.1.7. For $m = 0$, we have a canonical identification

$$\mathrm{Sq}_{0,n}(\mathbb{S}) \simeq \mathrm{Maps}_{2\text{-Cat}}([n], \mathbb{S}) \simeq (\mathrm{Seq}_n(\mathbb{S}))^{\mathrm{Spc}},$$

and similarly for $n = 0$, we have

$$\mathrm{Sq}_{m,0}(\mathbb{S}) \simeq \mathrm{Maps}_{2\text{-Cat}}([m], \mathbb{S}) \simeq (\mathrm{Seq}_m(\mathbb{S}))^{\mathrm{Spc}},$$

Note that (3.4) implies that for $n = n_1 + n_2$, and $\mathbb{S} \in 2\text{-Cat}$, the natural map

$$\mathrm{Sq}_{m,n}(\mathbb{S}) \to \mathrm{Sq}_{m,n_1}(\mathbb{S}) \underset{\mathrm{Sq}_{m,0}(\mathbb{S})}{\times} \mathrm{Sq}_{m,n_2}(\mathbb{S})$$

is an isomorphism and for $m = m_1 + m_2$, the natural map

$$\mathrm{Sq}_{m,n}(\mathbb{S}) \to \mathrm{Sq}_{m_1,n}(\mathbb{S}) \underset{\mathrm{Sq}_{0,n}(\mathbb{S})}{\times} \mathrm{Sq}_{m_2,n}(\mathbb{S}) \tag{4.2}$$

is an isomorphism.

4.2. The functor $\mathrm{Seq}^{\mathrm{ext}}_{\bullet}$.

4.2.1. The isomorphism (4.2) implies that for a fixed n, the functor

$$\text{2-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}, \quad \mathbb{S} \mapsto \mathrm{Sq}_{\bullet,n}(\mathbb{S})$$

lands in the subcategory of Segal spaces. Moreover, it is easy to see that it actually lands in the full subcategory of *complete Segal spaces.*

Hence, we obtain a well-defined functor, to be denoted $\mathrm{Seq}_\bullet^{\mathrm{ext}}$

$$\text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

so that

$$\mathrm{Sq}_{\bullet,\bullet} = (\mathrm{Seq}_\bullet)^{\mathbf{\Delta}^{\mathrm{op}}} \circ \mathrm{Seq}_\bullet^{\mathrm{ext}}.$$

In fact, by definition,

$$\mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}) \simeq \mathrm{Funct}([\bullet], \mathbb{S})_{\text{right-lax}}.$$

4.2.2. From Theorem 4.1.3, combined with Theorem 1.3.4 we obtain:

Corollary 4.2.3. *The functor* $\mathrm{Seq}_\bullet^{\mathrm{ext}}$ *is fully faithful.*

4.2.4. Note that the functor $\mathrm{Seq}_\bullet^{\mathrm{ext}}$ is *different* from the functor $\mathrm{Seq}_\bullet$. For example,

$$\mathrm{Seq}_0^{\mathrm{ext}}(\mathbb{S}) \simeq \mathbb{S}^{\text{1-Cat}},$$

whereas

$$\mathrm{Seq}_0(\mathbb{S}) = \mathbb{S}^{\mathrm{Spc}}.$$

Note that we have a natural transformation between the functors

$$\mathrm{Seq}_\bullet \to \mathrm{Seq}_\bullet^{\mathrm{ext}},$$

and for every $\mathbb{S} \in \text{2-Cat}$ and $n \in \mathbf{\Delta}$, we have a *replete* embedding

$$\mathrm{Seq}_n(\mathbb{S}) \hookrightarrow \mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S}).$$

Indeed,

$$\mathrm{Seq}_n(\mathbb{S}) = \mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S}) \underset{\mathbb{S}^{\text{1-Cat}} \times \ldots \times \mathbb{S}^{\text{1-Cat}}}{\times} (\mathbb{S}^{\mathrm{Spc}} \times \ldots \times \mathbb{S}^{\mathrm{Spc}}).$$

4.3. The category of pairs.

As we have seen above, to $\mathbb{S} \in \text{2-Cat}$, and m, n we can assign two spaces

$$\mathrm{Sq}_{m,n}^{\sim} \subset \mathrm{Sq}_{m,n}.$$

In this subsection, we will see that one can produce an entire array of intermediate spaces, one for each 1-full subcategory $\mathbf{C} \in \mathbb{S}^{\text{1-Cat}}$ with the same space of objects.

4.3.1. Let $\text{2-Cat}^{\mathrm{Pair}}$ be the following $(\infty,1)$-category. Its objects are pairs $(\mathbb{S}, \mathbf{C})$, where $\mathbb{S} \in$ 2-Cat, and $\mathbf{C}$ is a 1-full subcategory in $\mathbb{S}^{\text{1-Cat}}$ such that $\mathbf{C}^{\mathrm{Spc}} = \mathbb{S}^{\mathrm{Spc}}$.

For a pair of objects $(\mathbb{S}_1, \mathbf{C}_1)$ and $(\mathbb{S}_2, \mathbf{C}_2)$, the space of morphisms between them consists of functors $F : \mathbb{S}_1 \to \mathbb{S}_2$, such that the induced functor $\mathbf{C}_1 \to \mathbb{S}_2$ factors (automatically uniquely) via $\mathbf{C}_2$.

The ∞-categorical structure on $\text{2-Cat}^{\mathrm{Pair}}$ is uniquely determined by the requirement that the forgetful functor

$$\mathrm{OblvSubcat} : \text{2-Cat}^{\mathrm{Pair}} \to \text{2-Cat}, \quad (\mathbb{S}, \mathbf{C}) \mapsto \mathbb{S}$$

should be 1-fully faithful.

4.3.2. The above functor OblvSubcat admits a left and a right adjoints, given by

$$\mathbb{S} \mapsto (\mathbb{S}, \mathbb{S}^{\mathrm{Spc}}) \text{ and } \text{ and } \mathbb{S} \mapsto (\mathbb{S}, \mathbb{S}^{\text{1-Cat}}),$$

respectively.

4.3.3. We define the functor

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet} : \text{2-Cat}^{\mathrm{Pair}} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$$

as follows.

For $(\mathbb{S}, \mathbf{C}) \in \mathrm{Pair}$ we let $\mathrm{Sq}^{\mathrm{Pair}}_{m,n}(\mathbb{S}, \mathbf{C})$ be the full subspace of $\mathrm{Sq}_{m,n}(\mathbb{S})$, consisting of points such that for every $[1] \to [m]$ and $[0] \to [n]$, the resulting point of

$$\mathrm{Sq}_{1,0}(\mathbb{S}) \simeq (\mathrm{Seq}_1(\mathbb{S}))^{\mathrm{Spc}}$$

belongs to

$$\mathrm{Seq}_1(\mathbf{C}) \subset (\mathrm{Seq}_1(\mathbb{S}))^{\mathrm{Spc}}.$$

The sought-for functor

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S}, \mathbf{C}) : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Spc}$$

is uniquely determined by the requirement that the embeddings

$$\mathrm{Sq}^{\mathrm{Pair}}_{m,n}(\mathbb{S}, \mathbf{C}) \hookrightarrow \mathrm{Sq}_{m,n}(\mathbb{S})$$

upgrade to a natural transformation

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S}, \mathbf{C}) \to \mathrm{Sq}_{\bullet,\bullet} \circ \mathrm{OblvSubcat}.$$

Note that we have

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S}, \mathbb{S}^{\text{1-Cat}}) \simeq \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S}) \text{ and } \mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}(\mathbb{S}, \mathbb{S}^{\mathrm{Spc}}) \simeq \mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{S}).$$

4.3.4. We have the following generalization of Theorem 4.1.3[6]:

Theorem 4.3.5. *The functor* $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}$ *is fully faithful.*

4.3.6. Note that for a given $(\mathbb{S}, \mathbf{C}) \in \text{2-Cat}^{\mathrm{Pair}}$, we have

$$\mathrm{Sq}^{\mathrm{Pair}}_{0,\bullet}(\mathbb{S}, \mathbf{C}) \simeq \mathrm{Sq}_{0,\bullet}(\mathbb{S}) \simeq \mathrm{Seq}_\bullet(\mathbb{S}^{\text{1-Cat}}),$$

while

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,0}(\mathbb{S}, \mathbf{C}) \simeq \mathrm{Seq}_\bullet(\mathbf{C}).$$

In addition, for $n = n_1 + n_2$, the natural map

$$\mathrm{Sq}^{\mathrm{Pair}}_{m,n}(\mathbb{S}, \mathbf{C}) \to \mathrm{Sq}^{\mathrm{Pair}}_{m,n_1}(\mathbb{S}, \mathbf{C}) \underset{\mathrm{Sq}^{\mathrm{Pair}}_{m,0}(\mathbb{S}, \mathbf{C})}{\times} \mathrm{Sq}^{\mathrm{Pair}}_{m,n_2}(\mathbb{S}, \mathbf{C})$$

is an isomorphism and for $m = m_1 + m_2$, the natural map

$$\mathrm{Sq}^{\mathrm{Pair}}_{m,n}(\mathbb{S}, \mathbf{C}) \to \mathrm{Sq}^{\mathrm{Pair}}_{m_1,n}(\mathbb{S}, \mathbf{C}) \underset{\mathrm{Sq}^{\mathrm{Pair}}_{0,n}(\mathbb{S}, \mathbf{C})}{\times} \mathrm{Sq}^{\mathrm{Pair}}_{m_2,n}(\mathbb{S}, \mathbf{C}) \tag{4.3}$$

is an isomorphism.

[6] We do not prove it, and we were not able to find a reference.

4.3.7. As in Sect. 4.2.1, we obtain that there exists a well-defined functor

$$\mathrm{Seq}_\bullet^{\mathrm{Pair}} : 2\text{-}\mathrm{Cat}^{\mathrm{Pair}} \to 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

so that

$$\mathrm{Sq}_{\bullet,\bullet}^{\mathrm{Pair}} = (\mathrm{Seq}_\bullet)^{\mathbf{\Delta}^{\mathrm{op}}} \circ \mathrm{Seq}_\bullet^{\mathrm{Pair}}.$$

From Theorem 4.3.5, combined with Theorem 1.3.4, we obtain:

Corollary 4.3.8. *The functor* $\mathrm{Seq}_\bullet^{\mathrm{Pair}}$ *is fully faithful.*

4.4. **Left adjoint functors.**

4.4.1. By construction, the functor $\mathrm{Seq}_\bullet^{\mathrm{ext}}$ commutes with limits. Hence, by the Adjoint Functor Theorem, it admits a left adjoint, to be denoted $\mathfrak{L}^{\mathrm{ext}}$.

Similarly, the functor

$$\mathrm{Seq}_\bullet : 2\text{-}\mathrm{Cat} \to 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

admits a left adjoint, to be denoted $\mathfrak{L}$.

It is clear that when we restrict $\mathfrak{L}$ to the full subcategory $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}} \subset 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, the resulting functor lands in $1\text{-}\mathrm{Cat} \subset 2\text{-}\mathrm{Cat}$, thereby providing the left adjoint to the functor

$$\mathrm{Seq}_\bullet : 1\text{-}\mathrm{Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

4.4.2. We have the natural transformations of functors $2\text{-}\mathrm{Cat}^{\mathrm{Pair}} \to 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$

$$\mathrm{Seq}_\bullet \circ \mathrm{OblvSubcat} \to \mathrm{Seq}_\bullet^{\mathrm{Pair}} \to \mathrm{Seq}_\bullet^{\mathrm{ext}} \circ \mathrm{OblvSubcat}.$$

Composing with $\mathfrak{L}^{\mathrm{ext}}$ and the co-unit of the adjunction, we obtain the natural transformations

$$\mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet \circ \mathrm{OblvSubcat} \to \mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet^{\mathrm{Pair}} \to \mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet^{\mathrm{ext}} \circ \mathrm{OblvSubcat} \to \mathrm{OblvSubcat},$$

where the last arrow is an isomorphism by Corollary 4.2.3.

We claim:

Proposition 4.4.3. *The natural transformations*

$$\mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet \circ \mathrm{OblvSubcat} \to \mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet^{\mathrm{Pair}} \to \mathfrak{L}^{\mathrm{ext}} \circ \mathrm{Seq}_\bullet^{\mathrm{ext}} \circ \mathrm{OblvSubcat} \to \mathrm{OblvSubcat}$$

are isomorphisms.

Proof. By adjunction, we need to show that for $(\mathbb{S}, \mathbf{C}) \in 2\text{-}\mathrm{Cat}^{\mathrm{Pair}}$ and $\mathbb{T} \in 2\text{-}\mathrm{Cat}$, the natural map

$$\mathrm{Maps}_{2\text{-}\mathrm{Cat}}(\mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}), \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{T})) \to \mathrm{Maps}_{2\text{-}\mathrm{Cat}}(\mathrm{Seq}_\bullet^{\mathrm{Pair}}(\mathbb{S}, \mathbf{C}), \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{T})),$$

given by the inclusion $\mathrm{Seq}_\bullet^{\mathrm{Pair}}(\mathbb{S}, \mathbf{C}) \hookrightarrow \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S})$, is an isomorphism.

Note, however, that the above map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Maps}_{2\text{-}\mathrm{Cat}}(\mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}), \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{T})) \arrow[r] & \mathrm{Maps}_{2\text{-}\mathrm{Cat}}(\mathrm{Seq}_\bullet^{\mathrm{Pair}}(\mathbb{S}, \mathbf{C}), \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{T})) \\
\mathrm{Maps}_{2\text{-}\mathrm{Cat}^{\mathrm{Pair}}}((\mathbb{S}, \mathbb{S}^{1\text{-}\mathrm{Cat}}), (\mathbb{T}, \mathbb{T}^{1\text{-}\mathrm{Cat}})) \arrow[r] \arrow[u] & \mathrm{Maps}_{2\text{-}\mathrm{Cat}^{\mathrm{Pair}}}((\mathbb{S}, \mathbf{C}), (\mathbb{T}, \mathbb{T}^{1\text{-}\mathrm{Cat}})), \arrow[u]
\end{tikzcd}$$

where the vertical arrows are isomorphisms by Corollary 4.3.8. Now, the bottom horizontal arrow is an isomorphism, since the functor

$$\mathbb{T} \mapsto (\mathbb{T}, \mathbb{T}^{1\text{-}\mathrm{Cat}}), \quad 2\text{-}\mathrm{Cat} \to 2\text{-}\mathrm{Cat}^{\mathrm{Pair}}$$

is the right adjoint to OblvSubcat.

□

4.4.4. Let

$$\mathfrak{L}^{\mathrm{Sq}} : \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \to 2\text{-Cat},$$

denote the left adjoint of the functor $\mathrm{Sq}_{\bullet,\bullet}$. Tautologically, we have:

$$\mathfrak{L}^{\mathrm{Sq}} = \mathfrak{L}^{\mathrm{ext}} \circ \mathfrak{L}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

From Proposition 4.4.3 we obtain:

Corollary 4.4.5. *The natural transformations*

$$\mathfrak{L}^{\mathrm{Sq}} \circ \mathrm{Sq}^{\sim}_{\bullet,\bullet} \circ \mathrm{OblvSubcat} \to \mathfrak{L}^{\mathrm{Sq}} \circ \mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet} \to \mathfrak{L}^{\mathrm{Sq}} \circ \mathrm{Sq}_{\bullet,\bullet} \circ \mathrm{OblvSubcat} \to \mathrm{OblvSubcat}$$

are isomorphisms.

4.5. **The Gray product via Squares.**

4.5.1. Let $\mathbb{S}$ and $\mathbb{T}$ be a pair of $(\infty,2)$-categories. Consider the following object of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$

$$(\mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{S}^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{T}). \tag{4.4}$$

I.e., its space of m,n-simplices is

$$\mathrm{Sq}^{\sim}_{n,m}(\mathbb{S}^{2\text{-op}}) \times \mathrm{Sq}^{\sim}_{m,n}(\mathbb{T}).$$

We claim that we have a canonically defined map

$$\mathfrak{L}^{\mathrm{Sq}}\left((\mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{S}^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{T})\right) \to \mathbb{S} \circledast \mathbb{T}, \tag{4.5}$$

functorial in $\mathbb{S}$ and $\mathbb{T}$.

4.5.2. Indeed, the datum of a map (4.5) is equivalent to that of a map in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$

$$(\mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{S}^{2\text{-op}}))^{\mathrm{reflect}} \times \mathrm{Sq}^{\sim}_{\bullet,\bullet}(\mathbb{T}) \to \mathrm{Sq}_{\bullet,\bullet}(\mathbb{S} \circledast \mathbb{T}). \tag{4.6}$$

The datum of the map (4.6) (when we require functoriality in $\mathbb{S}$ and $\mathbb{T}$) is equivalent to that of a map of bi-cosimplicial objects in 2-Cat

$$[m,n] \to ([n,m]^{\sim})^{2\text{-op}} \circledast [m,n]^{\sim}. \tag{4.7}$$

The functors (4.7) are constructed as follows. Consider the composition of (right-lax) functors

$$\begin{aligned}[m] \times [n] \xrightarrow{\mathrm{diag}} [m] \times [n] \times [m] \times [n] \dashrightarrow [m] \circledast [n] \circledast [m] \circledast [n] \to \\ \to ([m] \circledast [n]) \circledast ([m] \circledast [n]) \simeq ([n]^{2\text{-op}} \circledast [m]^{2\text{-op}})^{2\text{-op}} \circledast ([m] \circledast [n]) \simeq \\ \simeq ([n] \circledast [m])^{2\text{-op}} \circledast ([m] \circledast [n]) = ([n,m])^{2\text{-op}} \circledast [m,n] \to ([n,m]^{\sim})^{2\text{-op}} \circledast [m,n]^{\sim}\end{aligned} \tag{4.8}$$

Now, unwinding the construction, one checks that the composite map in (4.8) indeed gives rise to a map (4.7) (the corresponding 2-morphisms are isomorphisms).

4.5.3. We have the following result[7]:

Proposition 4.5.4. *The map* (4.5) *is an equivalence.*

4.6. **Cubes.**

[7] We do not prove it, and we were not able to find a reference.

4.6.1. Let k be an integer ≥ 1. The assignment

$$(n_1, ..., n_k) \mapsto [n_1, ..., n_k]$$

gives a functor

$$\mathbf{\Delta}^{\times k} \to 2\text{-Cat}\,.$$

Hence, we obtain a well-defined functor

$$\mathbf{Cu} : 2\text{-Cat} \to \mathrm{Spc}^{(\mathbf{\Delta}^{\mathrm{op}})^{\times k}}$$

that sends $\mathbb{S}$ to

$$(n_1, ..., n_k) \mapsto \mathrm{Maps}_{2\text{-Cat}}([n_1, ..., n_k], \mathbb{S}).$$

4.6.2. We have the following generalization of Theorem 4.1.3[8]:

Theorem 4.6.3. *For any $k \geq 2$, the corresponding functor* **Cu** *is fully faithful.*

5. Essential image of the functor $\mathrm{Sq}_{\bullet,\bullet}$

The goal of this section is to describe the essential image of the functors $\mathrm{Sq}_{\bullet,\bullet}$ and $\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet}$.

5.1. **Invertible angles.**

5.1.1. Let $E_{\bullet,\bullet}$ be an object of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$. We shall say that $E_{\bullet,\bullet}$ is a double category if for every n, the objects $E_{\bullet,n}$ and $E_{n,\bullet}$ of $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ are complete Segal spaces.

5.1.2. Let $E_{\bullet,\bullet}$ be a double category. Let $(E_{1,1})^{\ulcorner} \subset E_{1,1}$ be the full subspace consisting of squares in which the right vertical side and the bottom horizontal side are degenerate. I.e., these are diagrams

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "\beta"'] & y \arrow[d, "\mathrm{id}"] \\ y \arrow[r, "\mathrm{id}"] & y. \end{tikzcd} \tag{5.1}$$

Let $(E_{1,1})^{\lrcorner} \subset E_{1,1}$ be the full subspace consisting of squares in which the left vertical side and the top horizontal side are degenerate. I.e., these are diagrams

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & x \arrow[d, "\beta"] \\ x \arrow[r, "\alpha"] & y. \end{tikzcd}$$

[8] We do not prove it, and we were not able to find a reference.

5.1.3. We define the full subspace

$$(E_{1,1})^{\ulcorner,\mathrm{invert}} \subset (E_{1,1})^{\ulcorner}$$

of *invertible angles* as follows. A point (5.1) is invertible if the following two conditions hold:

(I) There exists a point in $E_{2,1}$

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & x \arrow[d, "\beta'"] \\ x \arrow[r, "\alpha"] \arrow[d, "\beta"'] & y \arrow[d, "\mathrm{id}"] \\ y \arrow[r, "\mathrm{id}"] & y, \end{tikzcd} \tag{5.2}$$

in which the lower square is the original (5.1), the top square is in $(E_{1,1})^{\lrcorner}$, and the outer square is degenerate (i.e., pulled back via $[1] \times [1] \to [1] \times [0]$).

(II) There exists a point in $E_{1,2}$

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & x \arrow[r, "\alpha"] \arrow[d, "\beta"'] & y \arrow[d, "\mathrm{id}"] \\ x \arrow[r, "\alpha'"] & y \arrow[r, "\mathrm{id}"] & y, \end{tikzcd} \tag{5.3}$$

in which the right square is the original (5.1), the left square is in $(E_{1,1})^{\lrcorner}$, and the outer square is degenerate (i.e., pulled back via $[1] \times [1] \to [0] \times [1]$).

5.1.4. The following is an elementary check:

Lemma 5.1.5. *The restriction maps*

$$(E_{1,1})^{\ulcorner,\mathrm{invert}} \to E_{1,0} \text{ and } (E_{1,1})^{\ulcorner,\mathrm{invert}} \to E_{0,1},$$

given by taking (5.1) *to its left vertical side and its top horizontal side, are* monomorphisms.

5.2. **Description of the essential image.** Let $(\mathbb{S}, \mathbf{C})$ be an object of 2-$\mathrm{Cat}^{\mathrm{Pair}}$, and consider a point $\beta \in \mathrm{Sq}^{\mathrm{Pair}}_{1,0}(\mathbb{S}, \mathbf{C})$. This point represents an 1-morphism in $\mathbf{C}$, and the same point can be represented by an element in $\alpha \in \mathrm{Sq}^{\mathrm{Pair}}_{0,1}(\mathbb{S}, \mathbf{C})$, completing β to a point (5.1) in $(\mathrm{Sq}^{\mathrm{Pair}}_{1,1}(\mathbb{S}, \mathbf{C}))^{\ulcorner,\mathrm{invert}}$.

It turns out that the property that one can complete a point in $\mathrm{Sq}^{\mathrm{Pair}}_{1,0}(\mathbb{S}, \mathbf{C})$ to a point in $(\mathrm{Sq}^{\mathrm{Pair}}_{1,1}(\mathbb{S}, \mathbf{C}))^{\ulcorner,\mathrm{invert}}$ characterizes the essential image of 2-$\mathrm{Cat}^{\mathrm{Pair}}$ in $\mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$.

5.2.1. Let $E_{\bullet,\bullet}$ be again a double category. We shall say that $E_{\bullet,\bullet}$ is *anti-clockwise reversible* if the map

$$(E_{1,1})^{\ulcorner,\mathrm{invert}} \to E_{1,0}$$

from Lemma 5.1.5 is an isomorphism.

We shall say that $E_{\bullet,\bullet}$ is *reversible* if both maps in Lemma 5.1.5 are isomorphisms.

5.2.2. We now give the following sharpening of Theorems 4.1.3 and 4.3.5[9]:

Theorem 5.2.3.
(a) *The essential image of the functor*

$$\mathrm{Sq}_{\bullet,\bullet} : 2\text{-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$$

is the full subcategory consisting of reversible double categories.
(b) *The essential image of the functor*

$$\mathrm{Sq}^{\mathrm{Pair}}_{\bullet,\bullet} : 2\text{-Cat}^{\mathrm{Pair}} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$$

is the full subcategory consisting of anti-clockwise reversible double categories.

5.3. **The $(\infty, 2)$-category $\mathbf{1\text{-}Cat}$ via squares.** In Sect. 2.4 we introduced the $(\infty, 2)$-category $\mathbf{1\text{-}Cat}$. In this subsection we will describe its essential image under the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$.

5.3.1. Consider the following object, denoted $\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat})$ of $1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$. Namely,

$$\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{1\text{-}Cat}) := \mathrm{Cart}_{/[n]^{\mathrm{op}}}.$$

Note the difference between $\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{1\text{-}Cat})$ and $\mathrm{Seq}_n(\mathbf{1\text{-}Cat})$: the two have the same objects, while the latter has fewer morphisms.

5.3.2. We claim:

Proposition 5.3.3.
(a) *The object* $\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat})$ *lies in the essential image of the functor*

$$\mathrm{Seq}^{\mathrm{ext}} : 2\text{-Cat} \to 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

(b) *The resulting object of* 2-Cat *identifies canonically with* $\mathbf{1\text{-}Cat}$.

Proof. First, it is clear that simplicial category $\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat})$ is obtained from $\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat})$ by the procedure of Sect. 4.2.4.

The fact that the simplicial category $\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat})$ satisfies the Segal condition follows in the same way as in the case of $\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat})$.

Consider the bi-simplicial space

$$\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1\text{-}Cat}) := \mathrm{Seq}_\bullet(\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat})).$$

It is easy to see that it is a *complete* Segal space along each row and column, so it is a double category.

By Theorem 5.2.3(a), in order to prove the proposition, we need to show that $\mathrm{Sq}_{\bullet,\bullet}(\mathbf{1\text{-}Cat})$ is reversible.

For every n, consider the 1-full subcategory

$$\mathrm{Funct}([n], 1\text{-Cat}) \simeq (\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{\mathrm{strict}} \subset \mathrm{Cart}_{/[n]^{\mathrm{op}}},$$

and the corresponding full bi-simplicial subspace

$$\mathrm{Seq}_\bullet((\mathrm{Cart}_{/[\bullet]^{\mathrm{op}}})_{\mathrm{strict}}) =: \mathrm{Sq}'_{\bullet,\bullet}(\mathbf{1\text{-}Cat}) \subset \mathrm{Sq}_{\bullet,\bullet}(\mathbf{1\text{-}Cat}).$$

Note that

$$\mathrm{Sq}'_{m,n}(\mathbf{1\text{-}Cat}) \subset \mathrm{Sq}_{m,n}(1\text{-Cat})$$

[9] We do not prove it, and we were not able to find a reference.

is an isomorphism when either m or n equals 0. Hence, its is enough to show that $\mathrm{Sq}'_{\bullet,\bullet}(\mathbf{1\text{-}Cat})$ is reversible.

However, by construction, $\mathrm{Sq}'_{\bullet,\bullet}(\mathbf{1\text{-}Cat})$ identifies with $\mathrm{Sq}_{\bullet,\bullet}(\text{1-Cat})$, where $\text{1-Cat} \in \text{1-Cat}$ is regarded as an $(\infty,2)$-category. In particular, it is reversible.

□

6. The $(\infty,2)$-category of $(\infty,2)$-categories

In this section we upgrade the structure of $(\infty,1)$-category on the totality of $(\infty,2)$-categories to a structure of $(\infty,2)$-category. I.e., we will define an $(\infty,2)$-category $\mathbf{2\text{-}Cat}$ equipped with an identification

$$(\mathbf{2\text{-}Cat})^{\text{1-Cat}} \simeq \text{2-Cat}\,.$$

We will show that for $\mathbb{S},\mathbb{T} \in \mathbf{2\text{-}Cat}$, the $(\infty,1)$-category $\mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S},\mathbb{T})$ is canonically equivalent to $(\mathrm{Funct}(\mathbb{S},\mathbb{T}))^{\text{1-Cat}}$, where $\mathrm{Funct}(\mathbb{S},\mathbb{T})$ is the $(\infty,2)$-category of functors defined in Sect. 2.5.4.

We will also show that the $(\infty,2)$-category $\mathbf{1\text{-}Cat}$ sits inside $\mathbf{2\text{-}Cat}$ as a full subcategory.

Note, however, that the structure of $(\infty,2)$-category on the totality of $(\infty,2)$-categories is not the end of the story: the latter must in fact form an $(\infty,3)$-category. However, we will not pursue this here.

6.1. The $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ model for $\mathbf{2\text{-}Cat}$.

6.1.1. We introduce the $(\infty,2)$-category $\mathbf{2\text{-}Cat}$ to be the full subcategory in

$$\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\mathbf{1\text{-}Cat}),$$

whose objects are functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{1\text{-}Cat}$ such that, when regarded as functor $\mathbf{\Delta}^{\mathrm{op}} \to \text{1-Cat}$, they belong to

$$\text{2-Cat} \overset{\mathrm{Seq}^{\mathrm{ext}}_\bullet}{\hookrightarrow} \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\text{1-Cat}).$$

I.e., we take the $(\infty,1)$-category 2-Cat realized as a full subcategory of $\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\text{1-Cat})$ *via the functor* $\mathrm{Seq}^{\mathrm{ext}}_\bullet$ and extend it to an $(\infty,2)$-category by addinng non-invertible 2-morphisms to be those given by extending the target $(\infty,1)$-category 1-Cat to the $(\infty,2)$-category $\mathbf{1\text{-}Cat}$.

By construction, we have

$$(\mathbf{2\text{-}Cat})^{\text{1-Cat}} \simeq \text{2-Cat}\,.$$

Remark 6.1.2. Note that in giving the above definition, it is important that we are dealing with the functor $\mathrm{Seq}^{\mathrm{ext}}_\bullet$, rather than $\mathrm{Seq}_\bullet$.

6.1.3. By unwinding the definition, we obtain the following description of the functor

$$\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{2\text{-}Cat}) : \mathbf{\Delta}^{\mathrm{op}} \to \text{1-Cat}\,.$$

Namely, $\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{2\text{-}Cat})$ is the full subcategory in

$$\Big(\mathrm{Funct}\big([n],\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\mathbf{1\text{-}Cat})\big)_{\text{right-lax}}\Big)^{\text{1-Cat}} \subset$$

$$\subset \Big(\mathrm{Funct}\big(\mathbf{\Delta}^{\mathrm{op}},\mathrm{Funct}([n],\mathbf{1\text{-}Cat})_{\text{right-lax}}\big)\Big)^{\text{1-Cat}} \simeq$$

$$\simeq \mathrm{Funct}\Big(\mathbf{\Delta}^{\mathrm{op}},\big(\mathrm{Funct}([n],\mathbf{1\text{-}Cat})_{\text{right-lax}}\big)^{\text{1-Cat}}\Big) = \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{Cart}_{/[n]^{\mathrm{op}}}),$$

consisting of objects $\mathbf{E}_\bullet$ that satisfy the following:

- As an object of $(\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}))^{\mathrm{Spc}}$, we require that $\mathbf{E}_\bullet$ belong to

$$\begin{aligned}\mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{\Delta}^{\mathrm{op}} \times [n], 1\text{-}\mathrm{Cat}) \simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Funct}([n], 1\text{-}\mathrm{Cat})) \simeq \\ \simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{\Delta}^{\mathrm{op}}, (\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{\mathrm{strict}}) \subset \\ \subset \mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}) = (\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}))^{\mathrm{Spc}};\end{aligned}$$

- For every $i \in [n]$, we require that the resulting object $\mathbf{E}_{\bullet,i} \in \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-}\mathrm{Cat})$ lie in the essential image of the functor

$$\mathrm{Seq}^{\mathrm{ext}}_\bullet : 2\text{-}\mathrm{Cat} \to \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-}\mathrm{Cat}) = 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

6.2. **Identifying the categories of maps.** Let $\mathbb{S}$ and $\mathbb{T}$ be two objects of $\mathbf{2\text{-}Cat}$. The first test on whether the above definition of $\mathbf{2\text{-}Cat}$ is reasonable, is whether or not the $(\infty,1)$-category $\mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S},\mathbb{T})$ indeed recovers the $(\infty,1)$-category $(\mathrm{Funct}(\mathbb{S},\mathbb{T}))^{1\text{-}\mathrm{Cat}}$ of functors from $\mathbb{S}$ to $\mathbb{T}$, defined in Sect. 2.5.4.

6.2.1. We claim:

Proposition-Construction 6.2.2. *For* $\mathbb{S},\mathbb{T} \in \mathbf{2\text{-}Cat}$, *the* $(\infty,1)$-*category* $\mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S},\mathbb{T})$ *identifies canonically with*

$$(\mathrm{Funct}(\mathbb{S},\mathbb{T}))^{1\text{-}\mathrm{Cat}}.$$

The rest of this subsection is devoted to the proof of this proposition.

6.2.3. Unwinding the definition of $\mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S},\mathbb{T})$, and using [Chapter I.1, Sect. 1.4.5], we obtain that for $\mathbf{I} \in 1\text{-}\mathrm{Cat}$, we have a canonical isomorphism

$$\mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{I}, \mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S},\mathbb{T})) \simeq \mathrm{Maps}_{\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-}\mathrm{Cat})}(\mathbf{I} \times \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}), \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{T})),$$

functorial in $\mathbf{I}$.

Thus, in order to prove the proposition, we need to construct an identification

$$\mathrm{Maps}_{2\text{-}\mathrm{Cat}}(\mathbf{I} \times \mathbb{S},\mathbb{T}) \simeq \mathrm{Maps}_{\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-}\mathrm{Cat})}(\mathbf{I} \times \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}), \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{T})). \tag{6.1}$$

functorial in $\mathbf{I}$.

6.2.4. Recall the $(\infty,2)$-category $\mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}}$, defined so that

$$\mathrm{Maps}(\mathbb{S}', \mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}}) := \mathrm{Maps}(\mathbf{I} \circledast \mathbb{S}', \mathbb{T}).$$

Note that for every n we have a canonical fully faithful embedding

$$\mathbf{Maps}(\mathbf{I}, \mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{T})) \hookrightarrow \mathrm{Seq}^{\mathrm{ext}}_n(\mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}}).$$

Indeed, for every m we have

$$\mathrm{Seq}_m(\mathbf{Maps}(\mathbf{I}, \mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{T}))) = \mathrm{Maps}(\mathbf{I} \times [m], \mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{T})) \simeq \mathrm{Maps}((\mathbf{I} \times [m]) \circledast [n], \mathbb{T}),$$

while

$$\begin{aligned}\mathrm{Seq}_m(\mathrm{Seq}^{\mathrm{ext}}_n(\mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}})) = \\ = \mathrm{Maps}([m] \circledast [n], \mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}}) = \mathrm{Maps}(\mathbf{I} \circledast ([m] \circledast [n]), \mathbb{T}),\end{aligned}$$

and the embedding in question comes from the projection

$$\mathbf{I} \circledast ([m] \circledast [n]) \simeq \mathbf{I} \circledast [m] \circledast [n] \simeq (\mathbf{I} \circledast [m]) \circledast [n] \to (\mathbf{I} \times [m]) \circledast [n].$$

6.2.5. Thus, we obtain that the right-hand side in (6.1), interpreted as

$$\mathrm{Maps}_{\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\text{1-Cat})}(\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}), \mathbf{Maps}(\mathbf{I}, \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{T}))),$$

admits a fully faithful embedding into

$$\mathrm{Maps}_{\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}},\text{1-Cat})}(\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}), \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}})) \simeq \\ \simeq \mathrm{Maps}(\mathbb{S}, \mathrm{Funct}(\mathbf{I},\mathbb{T})_{\text{left-lax}}) = \mathrm{Maps}(\mathbf{I} \circledast \mathbb{S}, \mathbb{T}).$$

Furthermore, it is easy to see that the essential image of the right-hand side in (6.1) in $\mathrm{Maps}(\mathbf{I} \circledast \mathbb{S}, \mathbb{T})$ equals that of the fully faithful embedding

$$\mathrm{Maps}(\mathbf{I} \times \mathbb{S}, \mathbb{T}) \hookrightarrow \mathrm{Maps}(\mathbf{I} \circledast \mathbb{S}, \mathbb{T}), \tag{6.2}$$

thereby giving rise to the sought-for isomorphism (6.1).

6.3. **Another interpretation for 1-Cat.** We will now show that **1-Cat**, as defined in Sect. 2.4, embeds fully faithfully into **2-Cat**.

6.3.1. Let us (temporarily) denote by

$$\mathbf{1\text{-}Cat}' \subset \mathbf{2\text{-}Cat}$$

the full subcategory, defined so that $(\mathbf{1\text{-}Cat}')^{\mathrm{Spc}}$, viewed as a subspace of $(\mathbf{2\text{-}Cat})^{\mathrm{Spc}}$, equals

$$(\text{1-Cat})^{\mathrm{Spc}} \subset (\text{2-Cat})^{\mathrm{Spc}} \simeq (\mathbf{2\text{-}Cat})^{\mathrm{Spc}}.$$

We are going to prove:

Proposition-Construction 6.3.2. *There is a canonical equivalence of $(\infty,2)$-categories*

$$\mathbf{1\text{-}Cat} \simeq \mathbf{1\text{-}Cat}',$$

extending the identification

$$(\mathbf{1\text{-}Cat})^{\text{1-Cat}} \simeq \text{1-Cat} \simeq (\mathbf{1\text{-}Cat}')^{\text{1-Cat}}.$$

The rest of this subsection is devoted to the proof of this proposition.

6.3.3. We construct the functor $\mathbf{1\text{-}Cat} \to \mathbf{1\text{-}Cat}'$ in the guise of a map of simplicial categories:

$$\mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat}) \to \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbf{1\text{-}Cat}'). \tag{6.3}$$

We recall that

$$\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{1\text{-}Cat}) = \mathrm{Cart}_{/[n]^{\mathrm{op}}}$$

and

$$\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{1\text{-}Cat}') \subset \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}}) \subset \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \text{1-Cat}_{/[n]^{\mathrm{op}}}).$$

The sought-for functor in (6.3) is given by

$$\mathrm{Cart}_{/[n]^{\mathrm{op}}} \to \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \text{1-Cat}_{/[n]^{\mathrm{op}}}), \quad (\mathbf{E} \to [n]) \mapsto \\ \left([m] \mapsto \mathrm{Funct}([m], \mathbf{E}) \underset{\mathrm{Funct}([m],[n]^{\mathrm{op}})}{\times} [n]^{\mathrm{op}}\right),$$

where $[n]^{\mathrm{op}} \to \mathrm{Funct}([m],[n]^{\mathrm{op}})$ is given by

$$[n]^{\mathrm{op}} = \mathrm{Funct}(\{*\}, [n]^{\mathrm{op}}) \to \mathrm{Funct}([m],[n]^{\mathrm{op}}).$$

It is easy to see that the image of the above map indeed lands in $\mathrm{Seq}^{\mathrm{ext}}_n(\mathbf{1\text{-}Cat}')$.

6.3.4. It follows from the construction that the resulting functor $\mathbf{1\text{-}Cat} \to \mathbf{1\text{-}Cat}'$ makes the diagram

$$\begin{tikzcd} (\mathbf{1\text{-}Cat})^{\text{1-Cat}} \arrow[r] & (\mathbf{1\text{-}Cat}')^{\text{1-Cat}} \\ \text{1-Cat} \arrow[u, "\sim"] \arrow[r, "\mathrm{Id}"] & \text{1-Cat} \arrow[u, "\sim"'] \end{tikzcd}$$

commute.

Hence, it remains to show that it is fully faithful. However, it follows from the construction that for $\mathbf{S}, \mathbf{T} \in \text{1-Cat}$, the diagram

$$\begin{tikzcd} \mathbf{Maps}_{\mathbf{1\text{-}Cat}}(\mathbf{S}, \mathbf{T}) \arrow[r] & \mathbf{Maps}_{\mathbf{1\text{-}Cat}'}(\mathbf{S}, \mathbf{T}) \\ \mathrm{Funct}(\mathbf{S}, \mathbf{T}) \arrow[u, "\text{Corollary 2.4.4}", "\sim"'] \arrow[r, "\mathrm{Id}"] & \mathrm{Funct}(\mathbf{S}, \mathbf{T}) \arrow[u, "\sim", "\text{Proposition 6.2.2}"'] \end{tikzcd}$$

commutes, establishing the required fully-faithfulness.