LOCALIZATION AND THE LONG INTERTWINING OPERATOR FOR REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS

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Introduction

This text is a raft draft, untouched since January 2009. What prevented its completion at the time was the absence of the documented formalism of loop groups acting on categories. Once this formalism is written down (hopefully, soon), this draft will be turned into an actual paper.

Here is an overview of what's done in this paper:

The first author introduced a certain functor, denoted Φ, from the category of Kac-Moody representations at the negative level to that at the positive level. We interpret this functor as a composition of two contravariant functors: one is the usual contragredient duality (from the negative level to itself), and then Verdier duality from the negative level to the positive level (that comes from the fact that the two categories are dual to each other in the sense of Lurie).

According to Kashwara-Tanisaki, the category at the negative level localizes onto the "thin" affine flag space, and the category at the negative level localizes onto the "thick" affine flag space. We show that in terms of the localization functors, Arkhipov's functor corresponds to the "long intertwining operator" from the category of (twisted) D-modules on thin flags to that on thick flags.

In the process we will need to consider the two version of the category of D-modules on Bun_G, recently studied in [DrGa], denoted $\text{D-mod}(\mathrm{Bun}_G)$ and $\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}}$. We show that the category at the positive level naturally localizes on the former, whereas the category at the negative level naturally localizes on the latter.

Date: May 15, 2015.

1. The finite-dimensional case

1.1. The categories.

1.1.1. Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. According to our conventions, we shall denote by $\mathfrak{g}$-mod the canonical object of **DGCat** corresponding to the derived category of $\mathfrak{g}$-modules, and by $\mathbf{H}(\mathfrak{g}\text{-mod})$ its heart, i.e., the abelian category of $\mathfrak{g}$-modules.

By Sect 10.2.3, the category $\mathfrak{g}$-mod identifies with its own dual $(\mathfrak{g}\text{-mod})^\vee$, using the anti-automorphism

$$\tau : U(\mathfrak{g}) \overset{x\mapsto -x,\, x\in\mathfrak{g}}{\longrightarrow} U(\mathfrak{g}).$$

Namely, the pairing $\mathfrak{g}\text{-mod}\otimes\mathfrak{g}\text{-mod}\to\text{Vect}$ is given by

$$M, N \mapsto \langle M, N\rangle_{\mathfrak{g}} := M \underset{U(\mathfrak{g})}{\otimes} N \simeq H_\bullet(\mathfrak{g}, M\otimes N). \tag{1.1}$$

Thus, we have an equivalence

$$\text{Can}_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \to (\mathfrak{g}\text{-mod})^\vee,$$

and in particular,

$$(\mathfrak{g}\text{-mod})^c \to ((\mathfrak{g}\text{-mod})^\vee)^c \simeq (\mathfrak{g}\text{-mod})^{c,o}.$$

We shall denote the resulting contravariant functor $\mathfrak{g}\text{-mod}^c \to \mathfrak{g}\text{-mod}^c$ by $\mathbb{D}_{\mathfrak{g}}$. Explicitly, it is given by

$$\mathbb{D}_{\mathfrak{g}}(M) \simeq \text{Hom}_{\mathfrak{g}\text{-mod}}(M, U(\mathfrak{g})),$$

viewed as a right, and via τ as a left, $\mathfrak{g}$-module.

1.1.2. Let $\mathcal{Y}$ be a smooth scheme. Let $\mathfrak{D}_{\mathcal{Y}}$-mod (resp., Let $\mathfrak{D}_{\mathcal{Y}}^{op}$-mod) be the (canonical object of **DGCat** corresponding to) the derived category of left (resp., right) D-modules on $\mathcal{Y}$, and let $\mathbf{H}(\mathfrak{D}_{\mathcal{Y}}\text{-mod})$ and $\mathbf{H}(\mathfrak{D}_{\mathcal{Y}}^{op}\text{-mod})$ denote their respective hearts, the corresponding abelian categories.

By Sect 10.2.4, the categories $\mathfrak{D}_{\mathcal{Y}}$-mod and $\mathfrak{D}_{\mathcal{Y}}^{op}$-mod are naturally dual to each other with the pairing given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \langle \mathcal{F}_1, \mathcal{F}_2\rangle_{\mathcal{Y}} := \Gamma_{DR}(\mathcal{Y}, \mathcal{F}_1 \underset{\mathcal{O}_{\mathcal{Y}}\text{-mod}}{\otimes} \mathcal{F}_2) \simeq \Gamma(\mathcal{Y}, \mathcal{F}_1 \underset{\mathfrak{D}_{\mathcal{Y}}\text{-mod}}{\otimes} \mathcal{F}_2), \tag{1.2}$$

where $\mathcal{F}_1 \underset{\mathfrak{D}_{\mathcal{Y}}\text{-mod}}{\otimes} \mathcal{F}_2$ is regarded as a sheaf of k-vector spaces in the Zariski topology.

Thus, we obtain an identification

$$\text{Can}_{\mathcal{Y}} : (\mathfrak{D}_{\mathcal{Y}}\text{-mod})^\vee \simeq \mathfrak{D}_{\mathcal{Y}}^{op}\text{-mod},$$

and in particular

$$(\mathfrak{D}_{\mathcal{Y}}\text{-mod})^c \to \left((\mathfrak{D}_{\mathcal{Y}}^{op}\text{-mod})^\vee\right)^c \simeq \left(\mathfrak{D}_{\mathcal{Y}}^{op}\text{-mod}\right)^{c,o}.$$

We shall denote the resulting contravariant functor $(\mathfrak{D}_{\mathcal{Y}}\text{-mod})^c \to \left(\mathfrak{D}_{\mathcal{Y}}^{op}\text{-mod}\right)^c$ by $\mathbb{D}_{\mathcal{Y}}$; this is the usual Verdier duality functor.

1.1.3. Let $\mathcal{Y}$ be as above and suppose that it is acted on by G. In this case we have a natural functor

$$\Gamma(\mathcal{Y},-)^l : \mathfrak{D}_{\mathcal{Y}}\text{-mod} \to \mathfrak{g}\text{-mod},$$

and its left adjoint, denoted $\mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}$ given by $M \mapsto \mathfrak{D}_{\mathcal{Y}} \underset{U(\mathfrak{g})}{\otimes} M$.

In addition, we can consider the functor $\Gamma(\mathcal{Y},-)^r : \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod} \to \mathfrak{g}\text{-mod}$.

Proposition 1.1.4. *Under the identifications*

$$\mathfrak{g}\text{-mod} \simeq (\mathfrak{g}\text{-mod})^\vee \text{ and } \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod} \simeq (\mathfrak{D}_{\mathcal{Y}}\text{-mod})^\vee,$$

the functors $\Gamma(\mathcal{Y},-)^r$ *and* $\mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}$ *are naturally mutually dual.*

Proof. The assertion of the proposition is equivalent to the existence of a canonical isomorphism

$$\langle M, \Gamma(\mathcal{Y},\mathcal{F})\rangle_{\mathfrak{g}} \simeq \langle \mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}(M), \mathcal{F}\rangle_{\mathcal{Y}},$$

which follows from the definitions. □

From Lemma 10.2.1, we obtain:

Corollary 1.1.5. *The left adjoint of the functor* $\Gamma(\mathcal{Y},-)^r$ *is the ind-extension of the functor*

$$\mathbb{D}_{\mathcal{Y}} \circ \mathrm{Loc}_{\mathfrak{g},\mathcal{Y}} \circ \mathbb{D}_{\mathfrak{g}} : (\mathfrak{g}\text{-mod})^c \to (\mathfrak{g}\text{-mod})^c \to (\mathfrak{D}_{\mathcal{Y}}\text{-mod})^c \to \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}\right)^c.$$

1.1.6. Assume for a moment that the structure sheaf $\mathcal{O}_{\mathcal{Y}}$ admits a *right* D-module structure, which is moreover, G-equivariant. This defines an identification $\mathfrak{D}_{\mathcal{Y}}\text{-mod} \to \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}$, which intertwines the functors $\Gamma(\mathcal{Y},-)^l$ and $\Gamma(\mathcal{Y},-)^r$. Under these identifications, the isomorphism of Corollary 1.1.5 can be reformulated as the commutativity of the next diagram

$$\begin{array}{ccc} (\mathfrak{g}\text{-mod})^c & \xrightarrow{\mathbb{D}_{\mathfrak{g}}} & (\mathfrak{g}\text{-mod})^c \\ {\scriptstyle \mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}}\downarrow & & {\scriptstyle \mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}}\downarrow \\ (\mathfrak{D}_{\mathcal{Y}}\text{-mod})^c & \xrightarrow{\mathbb{D}_{\mathcal{Y}}} & (\mathfrak{D}_{\mathcal{Y}}\text{-mod})^c. \end{array} \tag{1.3}$$

1.1.7. *A variant.* Suppose that in the above situation $\mathcal{Y}$ carries an action of another algebraic group, denoted T. In particular, we have a weak action of T on the categories $\mathfrak{D}_{\mathcal{Y}}\text{-mod}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}$. Let $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$ denote the corresponding weakly equivariant categories. From Lemma 10.4.2, we obtain:

Lemma 1.1.8. *The categories* $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ *and* $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$ *are compactly generated.*

Hence, by Sect 10.4.1, we obtain that the duality functor (1.2), which naturally gives rise to a functor

$$\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w} \otimes \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w} \to \mathrm{Vect}^{T,w} = \mathrm{Rep}(T) \overset{\mathrm{Inv}^T}{\to} \mathrm{Vect} \tag{1.4}$$

(here Inv^T is the functor of T-invariants) makes the categories $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$ mutually dual. We shall denote by

$$\mathrm{Can}^{T,w}_{\mathcal{Y}} : \mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w} \to \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$$

the resulting equivalence, and by $\mathbb{D}^{T,w}_{\mathcal{Y}}$ the functor

$$\left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}\right)^c \to \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}\right)^c.$$

These functors commute with $\mathrm{Can}_{\mathcal{Y}}$ and $\mathbb{D}_{\mathcal{Y}}$ via the tautological forgetful functors to $\mathfrak{D}_{\mathcal{Y}}\text{-mod}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}$, respectively.

By Sect 10.5.2, the t-structures on $\mathfrak{D}_{\mathcal{Y}}$-mod and $\mathfrak{D}^{op}_{\mathcal{Y}}$-mod induce t-structures on the categories $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$, whose hearts are the usual categories of weakly T-equivariant D-modules. We have:

Lemma 1.1.9. *The categories $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ and $\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}$ identify with the derived categories of their hearts.*

1.1.10. Assume now that the action of T on $\mathcal{Y}$ commutes with that of G. In this case the functor $\Gamma(\mathcal{Y},-)^l$ naturally lifts to a functor

$$\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w} \overset{\Gamma(\mathcal{Y},-)^l}{\to} \mathfrak{g}\text{-mod} \otimes \mathrm{Rep}(T),$$

and we define the functor $\Gamma^{T,w}(\mathcal{Y},-)^l : \mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w} \to \mathfrak{g}\text{-mod}$ as the composition

$$\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w} \overset{\Gamma(\mathcal{Y},-)^l}{\to} \mathfrak{g}\text{-mod} \otimes \mathrm{Rep}(T) \overset{\mathrm{Inv}^T}{\to} \mathfrak{g}\text{-mod}.$$

The functor $\mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}$ naturally lifts to a functor $\mathfrak{g}\text{-mod} \to \mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ and is the left adjoint of $\Gamma^{T,w}(\mathcal{Y},-)^l$.

1.1.11. Let us return to the setting of Sect 1.1.3. The statement and proof of Proposition 1.1.4 extends to the present situation, i.e., under the identifications

$$(\mathfrak{g}\text{-mod})^\vee \simeq \mathfrak{g}\text{-mod and } \left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}\right)^\vee \simeq \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w},$$

the functors $\mathrm{Loc}_{\mathfrak{g},\mathcal{Y}}$ and $\Gamma^{T,w}(\mathcal{Y},-)^r$ and mutually dual, and the left adjoint to $\Gamma^{T,w}(\mathcal{Y},-)^r$ is the left adjoint of the functor

$$\mathbb{D}_{\mathcal{Y}} \circ \mathrm{Loc}_{\mathfrak{g},\mathcal{Y},T} \circ \mathbb{D}_{\mathfrak{g}} : (\mathfrak{g}\text{-mod})^c \to \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{T,w}\right)^c.$$

1.2. **The equivariant situation.** Let $B \subset G$ be a subgroup. All of the above categories carry a strong action of G, and in particular, of B.

1.2.1. Let $\mathfrak{g}\text{-mod}^B$ denote the corresponding (strongly) B-equivariant category. By Sect 10.5.3, the category $\mathfrak{g}\text{-mod}^B$ inherits a t-structure from $\mathfrak{g}$-mod, whose heart is the abelian category of Harish-Chandra modules with respect to the pair $(\mathfrak{g}, B)$. By Sect 10.5.4, we have:

Lemma 1.2.2.
(a) The category $\mathfrak{g}\text{-mod}^B$ identifies with the derived category of its heart, i.e., $\mathfrak{g}\text{-mod}^B \simeq (\mathfrak{g}, B)$-mod.
(b) The category $\mathfrak{g}\text{-mod}^B$ is compactly generated, and $(\mathfrak{g}\text{-mod}^B)^c$ is the preimage of $\mathfrak{g}$-mod under the forgetful functor.

From Sect 10.4.3, we obtain that the pairing (1.1) gives rise to a functor

$$\mathfrak{g}\text{-mod}^B \otimes \mathfrak{g}\text{-mod}^B \to \mathrm{Vect}^B \overset{H^\bullet_B}{\to} \mathrm{Vect},$$

(here $H^\bullet_B$ denotes the functor of equivariant cohomology), which identifies $\mathfrak{g}\text{-mod}^B$ with its own dual, i.e.,

$$(\mathfrak{g}\text{-mod}^B)^\vee \simeq \mathfrak{g}\text{-mod}^B. \tag{1.5}$$

We shall denote by

$$\mathrm{Can}^B_{\mathfrak{g}} : \mathfrak{g}\text{-mod}^B \to (\mathfrak{g}\text{-mod}^B)^\vee,$$

and by $\mathbb{D}^B_{\mathfrak{g}} : (\mathfrak{g}\text{-mod}^B)^c \to (\mathfrak{g}\text{-mod}^B)^c$ the resulting functors. They commute with $\mathrm{Can}_{\mathfrak{g}}$ and $\mathbb{D}_{\mathfrak{g}}$ via the tautological forgetful functor $\mathfrak{g}\text{-mod}^B \to \mathfrak{g}$-mod.

1.2.3. The pairing (1.5) can be viewed in terms of the equivalence given by Lemma 1.2.2 as follows: By the above lemma, $\mathfrak{g}\text{-mod}^B$ identifies with (canonical object of **DGCat** corresponding to) the derived category $(\mathfrak{g}, B)$-mod of Harish-Chandra modules with respect to the pair $(\mathfrak{g}, B)$. Choose a splitting $B \leftarrow T$. Then the pairing

$$(\mathfrak{g}, B)\text{-mod} \otimes (\mathfrak{g}, B)\text{-mod} \to \text{Vect},$$

corresponding to (1.5) is given by

$$M, N \mapsto \langle M, N\rangle_{(\mathfrak{g},B)} := \left(M \underset{\mathfrak{g};T}{\otimes} N \right) \otimes \det(\mathfrak{t}[1]). \tag{1.6}$$

1.2.4. Let $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^B$ and $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{B;T,w}$ denote the B-equivariant categories corresponding to $\mathfrak{D}_{\mathcal{Y}}\text{-mod}$ and $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$, respectively.

Lemma 1.2.5. *The categories* $\mathfrak{D}_{\mathcal{Y}}\text{-mod}$ *and* $\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{T,w}$ *are compactly generated.*

By Sect 10.4.3, the pairing (1.2) gives rise to a functor

$$\mathfrak{D}_{\mathcal{Y}}\text{-mod}^B \otimes \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^B \to \text{Vect}^B \overset{H^\bullet_B}{\to} \text{Vect},$$

and defines an equivalence

$$\text{Can}^B_{\mathcal{Y}} : \left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^B\right)^\vee \simeq \mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^B;\ \mathbb{D}^B_{\mathcal{Y}} : \left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^B\right)^c \simeq \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^B\right)^c, \tag{1.7}$$

and similarly

$$\text{Can}^{B;T,w}_{\mathcal{Y}} : \left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{B;T,w}\right)^\vee \simeq \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{B;T,w}\right), \tag{1.8}$$

$$\mathbb{D}^{B;T,w}_{\mathcal{Y}} : \left(\mathfrak{D}_{\mathcal{Y}}\text{-mod}^{B;T,w}\right)^c \simeq \left(\mathfrak{D}^{op}_{\mathcal{Y}}\text{-mod}^{B;T,w}\right)^c \tag{1.9}$$

1.2.6. The functors $\Gamma(\mathcal{Y}, -)^l$ and $\text{Loc}_{\mathfrak{g},\mathcal{Y}}$ are naturally compatible with the strong G-actions, and hence with the B-actions. By Sect 10.4, we obtain the mutually adjoint functors

$$\text{Loc}_{\mathfrak{g},\mathcal{Y}} : \mathfrak{D}_{\mathcal{Y}}\text{-mod}^B \leftrightarrows \mathfrak{g}\text{-mod}^B : \Gamma(\mathcal{Y}, -)^l,$$

which are compatible with the dualities in the same sense as in Proposition 1.1.4 and Corollary 1.1.5. If the line bundle $K_{\mathcal{Y}}$ admits a G-equivariant trivialization, then the isomorphism of (1.3) holds for B-equivariant categories as well. Similarly, the assertions of Sect 1.1.11 carry over to the B-equivariant context.

1.3. **The enhanced affine space.** From now on, G will be a reductive group, $B \subset G$ a Borel subgroup, $\mathcal{Y} = G/N$ and T be the Cartan group acting on G/N on the right.

1.3.1. Let χ be a k-point of $\text{Spec}(Z_{\mathfrak{g}})$, where $Z_{\mathfrak{g}} := Z(U(\mathfrak{g}))$. Let $\mathfrak{g}\text{-mod}_\chi$ be the full subcategory of $\mathfrak{g}$-mod consisting of objects, whose localization off χ is 0, i.e.,

$$\mathfrak{g}\text{-mod}_\chi \underset{Z_{\mathfrak{g}}\text{-mod}}{\otimes} Z_{\mathfrak{g}}\text{-mod}_\chi,$$

where $Z_{\mathfrak{g}}\text{-mod}_\chi$ denotes corresponding the full sub-category of $Z_{\mathfrak{g}}$-mod. From Sect 10.1.1, we obtain that $\mathfrak{g}\text{-mod}_\chi$ is compactly generated.

By Sect 10.2.2, we obtain that the duality (1.1) restricts to a pairing $\mathfrak{g}\text{-mod}_\chi \otimes \mathfrak{g}\text{-mod}_{\tau(\chi)}$, which defines an identification

$$\text{Can}_{\mathfrak{g},\chi} : (\mathfrak{g}\text{-mod}_\chi)^\vee \simeq \mathfrak{g}\text{-mod}_{\tau(\chi)}.$$

By Sect 10.5.1, the t-structure on $\mathfrak{g}$-mod induces a t-structure on $\mathfrak{g}\text{-mod}_\chi$. We have:

Lemma 1.3.2. *The natural functor makes* $\mathfrak{g}\text{-mod}_\chi$ *equivalent to the derived category of its heart.*

The above lemma implies that $\mathfrak{g}\text{-mod}_\chi$ identifies with the (canonical object of **DGCat** corresponding to) derived category of $\mathfrak{g}$-modules, on which $Z_\mathfrak{g}$ acts with this generalized central character.

1.3.3. The fact that the weak action of T on $\mathfrak{D}_{G/N}\text{-mod}$ canonically extends to a strong one implies, by Sect 10.4.4, that $\mathfrak{D}_{G/N}\text{-mod}^{T,w}$ is a category over the scheme $\mathfrak{t}^*$. Let $\lambda \in \mathfrak{t}^*$ be a weight. Let $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$ be the full subcategory of $\mathfrak{D}_{G/N}\text{-mod}^{T,w}$, whose localization off λ is zero, i.e.,

$$\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda} := \mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda} \underset{\mathrm{Sym}(\mathfrak{t})\text{-mod}}{\otimes} \mathrm{Sym}(\mathfrak{t})\text{-mod}_\chi.$$

From Sect 10.1.1, we obtain that the category $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$ is compactly generated.

Note that the canonical line bundle $K_{G/N}$ admits a canonical trivialization as a G-equivariant line bundle, and the failure of this trivialization to be T-equivariant is given by the character $-2\rho : T \to \mathbb{G}_m$.

By Sect 10.2.2 and Sect 1.1.11, we obtain that the restriction of the pairing (1.4) identifies:

$$\mathrm{Can}^{T,w,\lambda}_{G/N} : \left(\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}\right)^\vee \simeq \mathfrak{D}_{G/N}\text{-mod}^{T,w,-\lambda-2\rho}.$$

We shall denote by $\mathbb{D}^{T,w,\lambda}_{G/N}$ the resulting contravariant functor

$$\left(\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}\right)^c \to \left(\mathfrak{D}_{G/N}\text{-mod}^{T,w,-\lambda-2\rho}\right)^c.$$

By Sect 10.5.1, the t-structure on $\mathfrak{D}_{G/N}\text{-mod}^{T,w}$ induces a t-structure on $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$. We have:

Lemma 1.3.4. *The category* $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$ *identifies with the derived category of its heart.*

1.3.5. Let $\varpi : \mathfrak{t}^* \to \mathrm{Spec}(Z_\mathfrak{g})$ be the Harish-Chandra map. We normalize it so that $\varpi(\lambda) = \varpi(w(\lambda+\rho)-\rho)$.

If $\chi = \varpi(\lambda)$, it is easy to see that the functor $\Gamma^{T,w}(G/N,-)^l$ sends $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$ to $\mathfrak{g}\text{-mod}_\chi$, and that the functor $\mathrm{Loc}_{\mathfrak{g},G/N}$ sends $\mathfrak{g}\text{-mod}_\chi$ to $\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}$.

We have the following theorem of [BB]:

Theorem 1.3.6. *Assume that* λ *is such that* $\lambda+\rho$ *is regular. Then the functors* $\Gamma^{T,w}(G/N,-)^l$ *and* $\mathrm{Loc}_{\mathfrak{g},G/N}$ *define mutually inverse equivalences:*

$$\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda} \leftrightarrows \mathfrak{g}\text{-mod}_\chi.$$

If $\lambda+\rho$ *is, moreover, dominant, then the above functors are exact, i.e., compatible with the t-structures.*

Combining Theorem 1.3.6 with Proposition 1.1.11 we obtain:

Corollary 1.3.7. *The functors* $\Gamma^{T,w}(G/N,-)^l$ *and* $\mathrm{Loc}_{\mathfrak{g},G/N}$ *intertwine the functors*

$$\mathbb{D}^{T,w,\lambda}_{G/N} : \left(\mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda}\right)^c \to \left(\mathfrak{D}_{G/N}\text{-mod}^{T,w,-\lambda-2\rho}\right)^c$$

and

$$\mathbb{D}_{\mathfrak{g},\chi} : (\mathfrak{g}\text{-mod}_\chi)^c \to (\mathfrak{g}\text{-mod}_{\tau(\chi)})^c.$$

1.3.8. Let us now consider the B-equivariant situation. By Sect 10.4.5, the following commutative diagrams of categories are in fact a pull-back squares:

$$\begin{array}{ccc} \mathfrak{g}\text{-mod}^B_\chi & \longrightarrow & \mathfrak{g}\text{-mod}^B \\ \downarrow & & \downarrow \\ \mathfrak{g}\text{-mod}_\chi & \longrightarrow & \mathfrak{g}\text{-mod} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda} & \longrightarrow & \mathfrak{D}_{G/N}\text{-mod}^{B;T,w} \\ \downarrow & & \downarrow \\ \mathfrak{D}_{G/N}\text{-mod}^{T,w,\lambda} & \longrightarrow & \mathfrak{D}_{G/N}\text{-mod}^{T,w}. \end{array}$$

This implies that both $\mathfrak{g}\text{-mod}^B_\chi$ and $\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda}$ are compactly generated, and we have:

$$\mathrm{Can}^B_{\mathfrak{g},\chi} : \left(\mathfrak{g}\text{-mod}^B_\chi\right)^\vee \simeq \mathfrak{g}\text{-mod}^B_{\tau(\chi)} \tag{1.10}$$

and

$$\mathrm{Can}^{B;T,w,\lambda}_{G/N} : \left(\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda}\right)^\vee \simeq \mathfrak{D}_{G/N}\text{-mod}^{B;T,w,-\lambda-2\rho}. \tag{1.11}$$

By Sect 10.5.3, the category $\mathfrak{g}\text{-mod}^B_\chi$ inherits t-structures from $\mathfrak{g}\text{-mod}_\chi$. By Sect 10.5.4, we have:

Lemma 1.3.9. *The category $\mathfrak{g}\text{-mod}^B_\chi$ identifies with the derived category of its heart.*

Similarly, the category $\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda}$ inherits a t-structure from $\mathfrak{D}_{G/N}\text{-mod}^{B;T,w}$. By Sect 10.5.5, we have:

Lemma 1.3.10. *The category $\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda}$ identifies with the derived category of its heart.*

1.3.11. Finally, from Theorem 1.3.6 we obtain that the functors $\Gamma^{T,w}(G/N,-)^l$ and $\mathrm{Loc}_{\mathfrak{g},G/N}$ define mutually inverse equivalences:

$$\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda} \leftrightarrows \mathfrak{g}\text{-mod}^B_\chi,$$

and

$$\mathbb{D}^{B;T,w,\lambda}_{G/N} \circ \mathrm{Loc}_{\mathfrak{g},G/N} \simeq \mathrm{Loc}_{\mathfrak{g},G/N} \circ \mathbb{D}^B_{\mathfrak{g},\chi} \tag{1.12}$$

as functors

$$\left(\mathfrak{g}\text{-mod}^B\right)^c \to \left(\mathfrak{D}_{G/N}\text{-mod}^{B;T,w}\right)^c \text{ and } \left(\mathfrak{g}\text{-mod}^B_\chi\right)^c \to \left(\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,-\lambda-2\rho}\right)^c$$

and

$$\mathbb{D}^B_{\mathfrak{g},\chi} \circ \Gamma^{T,w}(G/N,-)^l \simeq \Gamma^{T,w}(G/N,-)^l \circ \mathbb{D}^{B;T,w,\lambda}_{G/N} \tag{1.13}$$

as functors

$$\left(\mathfrak{D}_{G/N}\text{-mod}^{B;T,w,\lambda}\right)^c \to \left(\mathfrak{g}\text{-mod}^B_{\tau(\chi)}\right)^c.$$

1.4. **Contragredient duality and Arkhipov's functor.** Let G be a reductive group and $B \subset G$ be the Borel subgroup.

1.4.1. Consider the category $\mathfrak{g}\text{-mod}^B$. Let us recall the definition of the contragredient duality functor $\mathrm{Contr}_{\mathfrak{g}} : (\mathfrak{g}\text{-mod}^B) \simeq (\mathfrak{g}\text{-mod}^B)^\vee$. It will correspond to a contravariant functor

$$M \mapsto M^* : (\mathfrak{g}\text{-mod}^B)^c \to (\mathfrak{g}\text{-mod}^B)^c. \tag{1.14}$$

We shall use the description of $\mathfrak{g}\text{-mod}^B$ as $(\mathfrak{g}, B)$-mod given by Lemma 1.2.2. So, it is sufficient to define the functor (1.14) as an exact contravariant self-equivalence on the category of finitely generated $(\mathfrak{g}, B)$-modules.

The latter is defined as follows. Let us choose a splitting $B \hookleftarrow T$. For $M \in \mathbf{H}((\mathfrak{g}, B)\text{-mod})$, let $M \simeq \underset{\nu}{\oplus} M(\nu)$ be the decomposition into weight spaces with respect to T. Since M is assumed finitely generated, all $M(\nu)$ are finite-dimensional. Set $M^* := \underset{\nu}{\oplus} M(\nu)^*$, where $M(\nu)^*$ denotes the linear dual of $M(\nu)$. We define an action of $\mathfrak{g}$ on M^* by conjugating the natural action by (a choice of the representative of) the element $w_0 \in W$. It is clear, however, that the construction of the functor $M \mapsto M^*$ does not depend, up to a canonical isomorphism, on either the choice of T or that of a representative of w_0.

It is clear also that for $M = \Delta_\lambda$–the Verma module with h.w. λ, we have

$$(\Delta_\lambda)^* = \nabla_{-w_0(\lambda)},$$

where for a weight μ, we denote by ∇_μ the dual Verma module with h.w. μ.

This implies that the functor (1.14) sends finitely generated Harish-Chandra modules to finitely generated ones. The fact that $(M^*)^* \simeq M$ implies that an (anti)-self-equivalence.

1.4.2. We define the functor $\Phi : \mathfrak{g}\text{-mod}^B \to \mathfrak{g}\text{-mod}^B$ as the composition

$$\Phi = \mathrm{Can}^B_{\mathfrak{g}} \circ \mathrm{Contr}_{\mathfrak{g}}.$$

We define the functor Ψ as the inverse of Φ, i.e., as the composition in the other order $\mathrm{Contr}_{\mathfrak{g}} \circ \mathrm{Can}^B_{\mathfrak{g}}$. Our goal to describe this functor explicitly.

Recall that for any category $\mathcal{C}$ acted on by G and any $\mathcal{F} \in \mathfrak{D}_G\text{-mod}^{B,B}$ there exists a canonical functor $M \mapsto \mathcal{F} \star M : \mathcal{C}^B \to \mathcal{C}^B$. Let $j_{w_0,!}, j_{w_0,*}$ denote the standard and co-standard objects of $\mathfrak{D}_G\text{-mod}^{B,B}$, corresponding to the element w_0 of the Weyl group.

The main theorem in the finite-dimensional case reads as follows:

Theorem 1.4.3.

$$\Phi \otimes \det(\mathfrak{b}[1]) \simeq j_{w_0,!} \circ - \quad \textit{and} \quad \Psi \otimes \det(\mathfrak{b}^*[-1]) \simeq j_{w_0,*}.$$

Note that since $j_{w_0,!} \circ j_{w_0,*} \simeq \delta_B \simeq j_{w_0,*} \circ j_{w_0,!}$, and hence the functors of convolution with $j_{w_0,!}$ and $j_{w_0,*}$ are mutually inverse, the two assertions in the theorem are equivalent to each other.

1.4.4. Let us give two tautological reformulations of Theorem 1.4.3. We remark that in the infinite-dimensional case, Theorem 1.4.3 will not admit a direct analogue, whereas these reformulations will.

Consider $j_{w_0,!}$ and $j_{w_0,*}$ as $B \times B$-equivariant D-modules on G. We will not distinguish between left and right D-modules on G due to the existence of a $G \times G$-equivariant trivialization of the line bundle K_G. Consider the objects

$$\Gamma(G, j_{w_0,!}), \Gamma(G, j_{w_0,*}) \in (\mathfrak{g} \oplus \mathfrak{g})\text{-mod}^{B\times B} \simeq \mathfrak{g}\text{-mod}^B \otimes \mathfrak{g}\text{-mod}^B \simeq (\mathfrak{g}\text{-mod}^B)^\vee \otimes \mathfrak{g}\text{-mod}^B.$$

Corollary 1.4.5. *The functor* Φ *is given by the kernel*

$$\Gamma(G, j_{w_0,!}) \otimes \det(\mathfrak{b}^*[-1])^{\otimes 2} \in (\mathfrak{g}\text{-mod}^B)^\vee \otimes \mathfrak{g}\text{-mod}^B$$

and the functor Ψ *is given by the kernel*

$$\Gamma(G, j_{w_0,*}) \in (\mathfrak{g}\text{-mod}^B)^\vee \otimes \mathfrak{g}\text{-mod}^B.$$

Corollary 1.4.6. *The diagrams*

$$\begin{array}{ccc} \mathfrak{D}_{G/N}\text{-mod}^{B;T,w} & \xrightarrow{j_{w_0,!}\star -} & \mathfrak{D}_{G/N}\text{-mod}^{B;T,w} \\ {\scriptstyle \Gamma^{T,w}(G/N,-)^l}\downarrow & & \downarrow{\scriptstyle \Gamma^{T,w}(G/N,-)^l} \\ \mathfrak{g}\text{-mod}^B & \xrightarrow{\Phi\otimes\det(\mathfrak{b}[1])} & \mathfrak{g}\text{-mod}^B \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{D}_{G/N}\text{-mod}^{B;T,w} & \xrightarrow{j_{w_0,*}\star -} & \mathfrak{D}_{G/N}\text{-mod}^{B;T,w} \\ {\scriptstyle \Gamma^{T,w}(G/N,-)^l}\downarrow & & \downarrow{\scriptstyle \Gamma^{T,w}(G/N,-)^l} \\ \mathfrak{g}\text{-mod}^B & \xrightarrow{\Psi\otimes\det(\mathfrak{b}^*[-1])} & \mathfrak{g}\text{-mod}^B \end{array}$$

commute.

1.5. **Proof of Theorem 1.4.3.** As a first step, we give the following reformulation of the functor $\mathrm{Contr}_{\mathfrak{g}}$.

Proposition 1.5.1. *For* $M, N \in (\mathfrak{g}\text{-mod}^B)^c$,

$$\mathrm{Hom}_{\mathfrak{g}\text{-mod}^B}(M, N^*) \simeq \mathrm{Hom}_{\mathfrak{D}_G\text{-mod}^{B\times B}}(\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N), j_{w_0,*}).$$

Note that this proposition provides a more comprehensible definition of the contragredient duality functor.

Proof. It is enough it establish the isomorphism at the level of complexes, where both M and N admit Verma flags. We claim that in this case both sides are acyclic off cohomological degree 0, and it would be enough to establish a canonical isomorphism between their 0-th cohomologies. Note that acyclicity for the LHS is evident: there are no higher Exts from a Verma module to a dual Verma module.

Let us make the choices $B \hookleftarrow T$ and w_0 as in the definition of the functor $\mathrm{Contr}_{\mathfrak{g}}$. By definition, the LHS is given by the relative Lie algebra cohomology of $\mathfrak{g}$, relative to $\mathfrak{t}$ with coefficients in $\mathrm{Hom}_k(M \otimes N, k)$, where the action of $\mathfrak{g}$ on N is twisted by w_0 (we shall denote the resulting module by N^{w_0}). Thus, the LHS calculates to

$$\mathrm{Hom}_{(\mathfrak{g},T)\text{-mod}}(M \otimes N^{w_0}, k) \simeq \mathrm{Hom}_k(H_\bullet(\mathfrak{g}; T, M \otimes N^{w_0}), k). \tag{1.15}$$

Since $j_{w_0,*}$ is obtained by averaging with respect to $B \times B$ of the the D-module δ_{w_0}, which is equivariant with respect to T embedded via $t \mapsto (t, w_0(t))$, the RHS is isomorphic to

$$\mathrm{Hom}_{\mathrm{Vect}^T}((\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N))_{w_0}, k), \tag{1.16}$$

where $(\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N))_{w_0}$ is the fiber of $\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N)$ regarded as an object of Vect^T. We have:

$$(\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N))_{w_0} \simeq U(\mathfrak{g}) \underset{U(\mathfrak{g})\otimes U(\mathfrak{g})}{\otimes} (M \otimes N^{w_0}) \simeq M \underset{\mathfrak{g}}{\otimes} N^{w_0}.$$

and we obtain that the RHS in (1.15) is canonically isomorphic to (1.16).

□

As a corollary, we obtain that for $M, N \in (\mathfrak{g}\text{-mod}^B)^c$,

$$\mathrm{Hom}_{\mathfrak{g}\text{-mod}^B}\left(M, j_{w_0,!} \star \mathrm{Contr}_{\mathfrak{g}}(N)\right) \simeq \mathrm{Hom}_{\mathfrak{D}_G\text{-mod}^{B\times B}}\left(\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N), \delta_B\right).$$

I.e., to prove the theorem, it is sufficient to prove the following:

Proposition 1.5.2. *For $M, N \in (\mathfrak{g}\text{-mod}^B)^c$ there exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}\text{-mod}^B}\left(M, \mathbb{D}^B_{\mathfrak{g}}(N)\right) \otimes \det(\mathfrak{b}^*[-1]) \simeq \mathrm{Hom}_{\mathfrak{D}_G\text{-mod}^{B\times B}}\left(\mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M \otimes N), \delta_B\right)$$

Proof. Denote $\mathbb{D}^B_{\mathfrak{g}}(M) =: M_1$ and $\mathbb{D}^B_{\mathfrak{g}}(N) =: N_1$. The LHS is then $\langle M_1, N_1\rangle_{(\mathfrak{g},B)} \otimes \det(\mathfrak{b}^*[-1])$ and the RHS, by (1.3), is isomorphic to

$$\mathrm{Hom}_{\mathfrak{D}_G\text{-mod}^{B\times B}}\left(\delta_B[-\dim(G)], \mathrm{Loc}_{\mathfrak{g}\oplus\mathfrak{g},G}(M_1 \otimes N_1)\right),$$

which can be rewritten as

$$\mathrm{Hom}_{\mathrm{Vect}^B}(k, M_1 \underset{\mathfrak{g}}{\otimes} N_1) \simeq \mathrm{Hom}_{(\mathfrak{g},B)\text{-mod}}(k, M_1 \otimes N_1),$$

and we are done by (1.6). □

2. Representations of the Kac-Moody algebra

2.1. The category of modules.

2.1.1. We fix a level $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ and consider the Kac-Moody extension $\widehat{\mathfrak{g}}_\kappa$. The object of **DGCat**, denoted $\widehat{\mathfrak{g}}_\kappa$-mod, has been introduced in [FG2]. We remind that, by definition, $\widehat{\mathfrak{g}}_\kappa$-mod is generated by compact objects of the form $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathsf{k}}(k)$, where $\mathsf{k} \subset \mathfrak{g}[\![t]\!]$ is a lattice subalgebra.

2.1.2. The category $\widehat{\mathfrak{g}}_\kappa$-mod is acted on strongly by the group ind-scheme $G(\!(t)\!)$ at level κ. In particular, if K is a pro-unipotent open-compact subgroup of $G[\![t]\!]$ we have a full subcategory $\widehat{\mathfrak{g}}_\kappa\text{-mod}^K \subset \widehat{\mathfrak{g}}_\kappa$-mod of K-equivariant objects.

Lemma 2.1.3. *Each of the categories $\widehat{\mathfrak{g}}_\kappa\text{-mod}^K$ is compactly generated;*

$$(\widehat{\mathfrak{g}}_\kappa\text{-mod}^K)^c = \widehat{\mathfrak{g}}_\kappa\text{-mod}^K \cap (\widehat{\mathfrak{g}}_\kappa\text{-mod})^c.$$

2.1.4. For two open compact subgroups $K_1 \subset K_2$, let $\mathbf{e}_{K_2,K_1}$ denote the tautological inclusion functor $\widehat{\mathfrak{g}}_\kappa\text{-mod}^{K_2} \to \widehat{\mathfrak{g}}_\kappa\text{-mod}^{K_1}$, and let Av_{K_2,K_1} be its right adjoint.

By the general formalism of categories acted on by $G(\!(t)\!)$,

$$\widehat{\mathfrak{g}}_\kappa\text{-mod} \simeq \underset{K}{\underset{\longrightarrow}{lim}}\, \widehat{\mathfrak{g}}_\kappa\text{-mod}^K, \tag{2.1}$$

where the colimit is taken with respect to the functors $\mathbf{e}$, and also

$$\widehat{\mathfrak{g}}_\kappa\text{-mod} \simeq \underset{K}{\underset{\longleftarrow}{lim}}\, \widehat{\mathfrak{g}}_\kappa\text{-mod}^K, \tag{2.2}$$

where the limit is taken with respect to the functors Av.

2.2. Duality in the Kac-Moody case.

Let κ' be the opposite level, i.e., $\kappa' = -\kappa - \kappa_{Kil}$. Our current goal is to construct a pairing

$$\langle -, - \rangle_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}}_\kappa\text{-mod} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod} \to \mathrm{Vect}, \tag{2.3}$$

which would identify $(\widehat{\mathfrak{g}}_\kappa\text{-mod})^\vee$ with $\widehat{\mathfrak{g}}_{\kappa'}$-mod.

The construction of the pairing (2.3) depends on an additional choice of trivializing a certain gerbe. This choice can be made by fixing an open-compact subgroup K_0.

2.2.1. The pairing (2.3) is defined as follows. We consider DG pairing

$$\mathbf{C}^+(\widehat{\mathfrak{g}}_\kappa\text{-mod}) \otimes \mathbf{C}^+(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}) \to \mathbf{C}(\mathrm{Vect})$$

given by

$$M, N \mapsto \mathfrak{C}^{\frac{\infty}{2}}(\widehat{\mathfrak{g}}, \mathsf{k}_0, M \otimes N), \tag{2.4}$$

where $\mathfrak{C}^{\frac{\infty}{2}}$ is the semi-infinite Chevalley complex taken with respect to the lattice $\mathsf{k}_0 := \mathrm{Lie}(K_0) \subset \widehat{\mathfrak{g}}$.

It is easy to see that if either M or N is acyclic, then so is $\mathfrak{C}^{\frac{\infty}{2}}(\widehat{\mathfrak{g}}, \mathsf{k}_0, M \otimes N)$.

Since $(\widehat{\mathfrak{g}}_\kappa\text{-mod})^c$ (resp., $(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod})^c$) is a obtained as a quotient category of a subcategory of $\mathbf{C}^+(\widehat{\mathfrak{g}}_\kappa\text{-mod})$ (resp., $\mathbf{C}^+(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod})$) by its intersection with the subcategory of acyclic complexes, we obtain that (2.4) gives rise to a pairing

$$(\widehat{\mathfrak{g}}_\kappa\text{-mod})^c \otimes (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod})^c \to \mathrm{Vect},$$

which we then ind-extend to obtain the pairing (2.3).

Theorem 2.2.2. *The pairing* (2.3) *defines a perfect duality between* $\widehat{\mathfrak{g}}_\kappa$-mod *and* $\widehat{\mathfrak{g}}_{\kappa'}$-mod.

2.2.3. Let $K_1 \subset K_2$ be two open-compact subgroups. We observe:

Lemma 2.2.4. *For* $M_1 \in \widehat{\mathfrak{g}}_\kappa\text{-mod}^{K_1}$, $M_2 \in \widehat{\mathfrak{g}}_\kappa\text{-mod}^{K_2}$,

$$\langle \mathbf{e}_{K_2,K_1}(M_2), M_1\rangle_{\widehat{\mathfrak{g}}} \simeq \langle M_2, \mathrm{Av}_{K_2,K_1}(M_1)\rangle_{\widehat{\mathfrak{g}}}.$$

Therefore, to prove Theorem 2.2.2, it suffices to show that the pairing (2.3) restricts to a perfect pairing

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^K \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^K \to \mathrm{Vect} \tag{2.5}$$

for a fixed open-compact subgroup K.

2.2.5. We are now going to construct an object

$$\mathbb{R}^{K,G((t))}_{\kappa,\kappa'} \in \widehat{\mathfrak{g}}_\kappa\text{-mod}^K \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^K, \tag{2.6}$$

with the property that for $M \in \widehat{\mathfrak{g}}_\kappa\text{-mod}^K$

$$\langle \mathbb{R}^{K,G((t))}_{\kappa,\kappa'}, M\rangle_{\widehat{\mathfrak{g}}} \simeq M \otimes \det.\mathrm{rel}.(\mathsf{k}, \mathsf{k}_0) \in \widehat{\mathfrak{g}}_\kappa\text{-mod}^K, \tag{2.7}$$

and for $N \in \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^K$

$$\langle N, \mathbb{R}^{K,G((t))}_{\kappa,\kappa'}\rangle_{\widehat{\mathfrak{g}}} \simeq N \otimes \det.\mathrm{rel}.(\mathsf{k}, \mathsf{k}_0) \in \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^K. \tag{2.8}$$

The existence of such $\mathbb{R}^{K,G((t))}_{\kappa,\kappa'}$ will imply the duality assertion.

2.2.6. Recall that $\mathfrak{D}^{\kappa,\kappa'}_{G((t))}$-mod denotes the category of twisted D-modules on $G((t))$. It has a natural forgetful functor $\Gamma(G((t)), -)$ to $\widehat{\mathfrak{g}}_\kappa\text{-mod} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}$. The category $\mathfrak{D}^{\kappa,\kappa'}_{G((t))}$-mod is acted on strongly by the group $G((t)) \times G((t))$ at the level (κ, κ') and the above forgetful functor is compatible with the action. In particular, we obtain a functor

$$\Gamma(G((t)), -) : \mathfrak{D}^{\kappa,\kappa'}_{G((t))}\text{-mod}^{K,K} \to \widehat{\mathfrak{g}}_\kappa\text{-mod}^K \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^K.$$

Let $\delta^{\kappa,\kappa'}_{G((t)),K}$ be the canonical object in $\mathfrak{D}^{\kappa,\kappa'}_{G((t))}\text{-mod}^{K,K}$, corresponding to distributions on K inside $G((t))$. We set

$$\mathbb{R}^{K,G((t))}_{\kappa,\kappa'} := \Gamma(G((t)), \delta^{\kappa,\kappa'}_{G((t)),K}).$$

Proposition 2.2.7. *The object* $\mathbb{R}^{K,G((t))}_{\kappa,\kappa'}$ *satisfies* (2.7) *and* (2.8).

Proof. We shall prove (2.7), since (2.8) is similar. It is enough to consider the case $M \in (\widehat{\mathfrak{g}}_\kappa\text{-mod}^K)^c$. By definition, it is enough to construct a functorial quasi-isomorphism of complexes

$$\mathfrak{C}^{\frac{\infty}{2}}(\widehat{\mathfrak{g}}, \mathsf{k}_0, M \otimes \mathbb{R}^{K,G((t))}_{\kappa,\kappa'}) \simeq M \otimes \operatorname{det.rel.}(\mathsf{k}, \mathsf{k}_0).$$

However, the latter has been carried out in [FG1].

□

2.2.8. Let $\mathrm{Can}_{\widehat{\mathfrak{g}}}$ denote the resulting equivalence

$$(\widehat{\mathfrak{g}}_\kappa\text{-mod})^\vee \simeq \widehat{\mathfrak{g}}_{\kappa'}\text{-mod},$$

and let us denote by

$$\mathbb{D}_{\widehat{\mathfrak{g}}} : (\widehat{\mathfrak{g}}_\kappa\text{-mod})^c \to (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod})^c$$

the corresponding contravariant equivalence.

Here is an example of a calculation of this functor. Let $M = \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathsf{k}}(k)$. We claim that $\mathbb{D}(M) \simeq \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa'}}_{\mathsf{k}}(k) \otimes \operatorname{det.rel.}(\mathsf{k}, \mathsf{k}_0)$. Indeed, we have

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}}(\mathbb{D}(M), N) \simeq \langle M, N\rangle_{\widehat{\mathfrak{g}}} \simeq H^\bullet(\mathsf{k}, N) \otimes \operatorname{det.rel.}(\mathsf{k}, \mathsf{k}_0) \simeq$$
$$\simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa'}}_{\mathsf{k}}(k), N) \otimes \operatorname{det.rel.}(\mathsf{k}, \mathsf{k}_0).$$

2.2.9. From now on we shall fix K_0 to be the Iwahori subgroup $\mathbf{I}$.

Consider now the category $\widehat{\mathfrak{g}}_\kappa\text{-mod}^I$, which can be identified with $(\widehat{\mathfrak{g}}_\kappa\text{-mod}^K)^{I/K}$ for any K which is a normal subgroup of I. We have:

Lemma 2.2.10. *The category* $\widehat{\mathfrak{g}}_\kappa\text{-mod}^I$ *is compactly generated. Its compact objects are those which become compact under the forgetful functor to* $\widehat{\mathfrak{g}}_\kappa$-mod.

The pairing (2.3), being $G((t))$-invariant gives rise to a pairing

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^I \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^I \to \mathrm{Vect}^I,$$

which, when composed with $H_I : \mathrm{Vect}^I \to \mathrm{Vect}$ defines a pairing

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^I \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^I \to \mathrm{Vect}\,. \tag{2.9}$$

By Sect 10.4.3, the pairing (2.9) is also perfect. We shall denote by $\mathrm{Can}^I_{\widehat{\mathfrak{g}}}$ the resulting equivalence

$$(\widehat{\mathfrak{g}}_\kappa\text{-mod}^I)^\vee \simeq (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^I),$$

and by $\mathbb{D}^I_{\widehat{\mathfrak{g}}}$ the corresponding contravariant functor

$$(\widehat{\mathfrak{g}}_\kappa\text{-mod}^I)^c \simeq (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^I)^c;$$

these functors are compatible with $\mathrm{Can}_{\widehat{\mathfrak{g}}}$ and $\mathbb{D}_{\widehat{\mathfrak{g}}}$ via the natural forgetful functors to $(\widehat{\mathfrak{g}}_\kappa\text{-mod})^c$ and $(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod})^{c,o}$, respectively.

We remark that the canonical object in $\widehat{\mathfrak{g}}_\kappa\text{-mod}^I \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^I$ that defines $\mathrm{Can}^I_{\widehat{\mathfrak{g}}}$ is

$$\mathbb{R}^{I,G((t))}_{\kappa,\kappa'} := \Gamma(G((t)), \delta_I)^{\kappa,\kappa'}. \tag{2.10}$$

For example, for $M = \Delta_{\kappa,\mu}$–the Verma module with h.w. μ, we have $\mathbb{D}^I_{\widehat{\mathfrak{g}}}(M) \simeq \Delta_{\kappa',-\mu}$.

2.3. **Affine contragredient duality and affine Arkhipov's functor.**

2.3.1. The affine algebra $\widehat{\mathfrak{g}}_\kappa$ we have been dealing with was the extension of the loop algebra $\mathfrak{g}((t))$. In what follows, we shall sometimes add a subscript $\widehat{\mathfrak{g}}_{\kappa,0}$, where $0 \in \mathbb{P}^1$ with t being a global coordinate on $\mathbb{P}^1$.

We shall also consider the affine algebra $\widehat{\mathfrak{g}}_{\kappa,0}$, which is the central extension of the loop algebra $\mathfrak{g}((t^{-1}))$. Of course, one could identify $\widehat{\mathfrak{g}}_{\kappa,0} \simeq \widehat{\mathfrak{g}}_{\kappa,\infty}$, but we prefer not to do that.

2.3.2. From now on, we shall make a distinction positive and negative levels. We shall say that the level κ is negative if (on every simple factor of $\mathfrak{g}$), $\kappa + \frac{\kappa_{Kil}}{2} = c \cdot \kappa_{Kil}$, with $c \notin \mathbb{Q}^{\geq 0}$. We shall say that κ is positive if $\kappa + \frac{\kappa_{Kil}}{2} = c \cdot \kappa_{Kil}$, with $c \notin \mathbb{Q}^{\leq 0}$. Evidently, κ is negative if and only if κ' is positive. We shall say that κ is irrational if $c \notin \mathbb{Q}$ (i.e., irrational=positive $\cap$ negative). We shall say that κ is critical if $c = 0$ (i.e., $\kappa_{crit} = -\frac{\kappa_{Kil}}{2}$).

2.3.3. Let κ be negative. We shall now define the contragredient duality functor

$$\mathrm{Contr}^{0\to\infty}_{\kappa\to\kappa'} : \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \simeq (\widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty})^\vee. \tag{2.11}$$

This will be based on the following:

Lemma 2.3.4. *Let κ be negative.*

(1) *The category $\mathbf{H}(\widehat{\mathfrak{g}}_\kappa\text{-mod}^I)$ is Artinian, i.e., every finitely generated object is of finite length.*

(2) *The triangulated category $(\widehat{\mathfrak{g}}_\kappa\text{-mod}^I)^c$ is equivalent to $D^b(\mathbf{H}(\widehat{\mathfrak{g}}_\kappa\text{-mod}^I)_{f.g.})$.*

As in Sect 1.4.1, it suffices to define the corresponding exact contravariant

$$M \mapsto M^* : \mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0})_{f.g.} \simeq \mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty})_{f.g.}. \tag{2.12}$$

For $M \in \mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0})_{f.g.}$, write $M \simeq \underset{\mu,n}{\oplus} M(\mu,n)$, where μ refers to the grading by means of $T \subset B \subset I$, and $n \in k$ is the degree with respect of the Sugawara L_0 operator. Since M is finitely generated, and κ is negative, each $M(\mu,n)$ is finite-dimensional. We set the vector space underlying M^* to be $\underset{\mu,n}{\oplus} M(\mu,n)^*$. The action of the affine algebra $\widehat{\mathfrak{g}}_{\kappa,\infty}$ is defined as follows: an element $x \otimes t^n$ acts as $\mathrm{Ad}_{w_0}(x) \otimes t^n$; the fact that this action extends continuously to power series follows from the definitions.

It is clear that $(\Delta_{\kappa,\mu,0})^* \simeq \nabla_{\kappa,-w_0(\mu),\infty}$, where the latter denotes the dual affine Verma module, which is known to be finitely generated. This implies that (2.12) constructed above has the required properties, and in particular, gives rise to a functor (2.11), as required.

2.3.5. Let us again assume that κ is negative. We define the functor

$$\Phi^{0\to\infty}_{\kappa\to\kappa'} : \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \simeq \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}$$

to be the composition

$$\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \overset{\mathrm{Contr}^{0\to\infty}_{\kappa\to\kappa'}}{\longrightarrow} (\widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty})^\vee \overset{\mathrm{Can}^I_{\widehat{\mathfrak{g}}}}{\longrightarrow} \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}. \tag{2.13}$$

We define the functor

$$\Psi^{\infty\to 0}_{\kappa'\to\kappa} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{\mathbf{I}_\infty} \simeq \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}$$

to be the inverse of $\Phi^{0\to\infty}_{\kappa\to\kappa'}$.

The goal of the rest of this paper is describe explicitly the kernels

$$\mathbb{S}^{0\to\infty}_{\kappa',\kappa'} \in \widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\mathbf{I}_0} \otimes \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty} \text{ and } \mathbb{S}^{\infty\to 0}_{\kappa,\kappa} \in \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \otimes \widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty}$$

that give rise to the functors $\Phi^{0\to\infty}_{\kappa\to\kappa'}$ and $\Psi^{\infty\to 0}_{\kappa'\to\kappa}$ via the pairing $\langle -,-\rangle_{\widehat{\mathfrak{g}}}$. In addition, we will show how the functors $\Phi^{0\to\infty}_{\kappa\to\kappa'}$ and $\Psi^{\infty\to 0}_{\kappa'\to\kappa}$ are compatible with the localization functors.

2.3.6. Tautologically, we have the following descriptions:

$$\mathbb{S}^{0\to\infty}_{\kappa',\kappa'} = (\mathrm{Id}\otimes\Phi)\,(\mathbb{R}^{I_0,G((t))}_{\kappa',\kappa}) \text{ and } \mathbb{S}^{\infty\to 0}_{\kappa,\kappa} = (\Psi\otimes\mathrm{Id})\,(\mathbb{R}^{I_\infty,G((t^{-1}))}_{\kappa',\kappa}). \tag{2.14}$$

Proposition 2.3.7. *The objects $\mathbb{S}^{0\to\infty}_{\kappa',\kappa'}$ and $\mathbb{S}^{\infty\to 0}_{\kappa,\kappa}$ belong to the hearts of the corresponding t-structures.*

Proof. We have:

$$\Phi^{0\to\infty}_{\kappa\to\kappa'}(\nabla_{\kappa,\mu,0}) = \Delta_{\kappa',w_0(\mu),\infty} \tag{2.15}$$

and, correspondingly,

$$\Psi^{\infty\to 0}_{\kappa'\to\kappa}(\Delta_{\kappa',w_0(\mu),\infty}) \simeq \nabla_{\kappa,\mu,0}, \tag{2.16}$$

to prove the proposition, it suffices to observe that $\mathbb{R}^{I_0,G((t))}_{\kappa',\kappa}$, regarded merely as an object of $\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0})$, has a filtration with subquotients isomorphic to dual Verma modules, and that $\mathbb{R}^{I_\infty,G((t^{-1}))}_{\kappa',\kappa}$, regarded merely as an object of $\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty})$ has a filtration with subquotients isomorphic to Verma modules.

$\square$

3. D-modules on Bun_G

3.1. **The spherical case.**

3.1.1. Let X be a smooth complete curve. Let $\mathrm{Bun}_G(X)$ denote the moduli stack of principal G-bundles on X. In this section we shall establish some basic facts about the category of D-modules on X.

By definition, the category of (left) D-modules, denoted $\mathfrak{D}_{\mathrm{Bun}_G(X)}$-mod on $\mathrm{Bun}_G(X)$ is $\underset{\underset{U}{\longleftarrow}}{lim}\,\mathfrak{D}_U$-mod, where the limit is taken over the partially ordered set of open substacks *of finite type* $U\subset \mathrm{Bun}_G(X)$ and the functors $\mathfrak{D}_{U_2}\text{-mod}\to\mathfrak{D}_{U_1}\text{-mod}$ for $j_{1,2}: U_1\hookrightarrow U_2$ are $j^*_{1,2}\simeq j^!_{1,2}$.

More generally, a choice of a level κ, defines a TDO $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$ on $\mathrm{Bun}_G(X)$, and we set

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_! := \underset{\underset{U}{\longleftarrow}}{lim}\,\mathfrak{D}^\kappa_U\text{-mod}.$$

In the sequel we will establish the following result:

Theorem 3.1.2. *The category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$-mod$_!$ is compactly generated.*

From the definitions (and independent of Theorem 3.1.2) we have the following description of compact objects of $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$-mod$_!$:

Lemma 3.1.3. *An object $\mathcal{F}\in\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$-mod$_!$ is compact if and only if there exists an open sub-stack $U\subset\mathrm{Bun}_G(X)$ of finite type and $\mathcal{F}_U\in(\mathfrak{D}^\kappa_U\text{-mod})^c$, such that $\mathcal{F}|_U=\mathcal{F}_U$ and such that the following equivalent conditions hold:*

- *For any $\mathcal{F}_1\in\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$-mod$_!$, supported on $\mathrm{Bun}_G(X)-U$, we have $\mathrm{Hom}(\mathcal{F},\mathcal{F}_1)=0$.*
- *For any $U\overset{j}{\hookrightarrow}U_1$, we have:*
$$\mathcal{F}|_{U_1}\simeq j_!(\mathcal{F}_U),$$
in particular, the object $j_!(\mathcal{F}_U)$ is defined.

We remark that the proof of Theorem 3.1.2 will provide an even more explicit description of compact objects in $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}$-mod$_!$.

3.1.4. We define the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_*$ as

$$\underset{U}{\underrightarrow{lim}}\, \mathfrak{D}^\kappa_U\text{-mod},$$

where for $U_1 \subset U_2$, the functor $\mathfrak{D}^\kappa_{U_1}\text{-mod} \to \mathfrak{D}^\kappa_{U_2}\text{-mod}$ is $(j_{1,2})_*$.

Observe that for every open substack $U \subset \mathrm{Bun}_G(X)$ of finite type, we have:

$$(\mathfrak{D}^\kappa_U\text{-mod})^\vee \simeq \mathfrak{D}^{\kappa,op}_U\text{-mod and } \mathfrak{D}^{\kappa,op}_U\text{-mod} \simeq \mathfrak{D}^{\kappa'}_U\text{-mod}. \tag{3.1}$$

Hence, from Lemma 10.1.2 we obtain:

Corollary 3.1.5. *There exists a canonical equivalence:*

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_*\right)^\vee \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X)}\text{-mod}_!.$$

3.1.6. Note that there is a naturally defined functor

$$Q_{*\to !} : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_* \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!. \tag{3.2}$$

Indeed, having such a functor amounts to a compatible collection of functors

$$\mathfrak{D}^\kappa_{U_1}\text{-mod} \to \mathfrak{D}^\kappa_{U_2}\text{-mod}$$

for all pairs U_1, U_2 of open subsets of finite type of $\mathrm{Bun}_G(X)$. The functors in question are defined by

$$\mathfrak{D}^\kappa_{U_1}\text{-mod} \overset{j^*_{12,1}}{\to} \mathfrak{D}^\kappa_{U_1\cap U_2}\text{-mod} \overset{(j_{12,2})_*}{\to} \mathfrak{D}^\kappa_{U_2}\text{-mod},$$

where $j_{i,ij} : U_i \cap U_j \hookrightarrow U_i$.

We shall also prove the following:

Proposition 3.1.7. *If κ is irrational, then the functor $Q_{*\to !}$ of* (3.2) *is an equivalence.*

In fact, we shall prove a stronger assertion:

Proposition 3.1.8. *Let κ be irrational. Then there exists an open substack $U \subset \mathrm{Bun}_G(X)$ of finite type, such that no non-zero object of $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!$ is supported on $\mathrm{Bun}_G(X) - U$.*

3.1.9. Let $\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}\right)^{c,!*}$ denote the small category

$$\underset{U}{\underleftarrow{lim}}\, (\mathfrak{D}_U\text{-mod})^c,$$

where the limit is taken over the poset of open sub-stacks U of finite type.

By construction, it admits a fully faithful functor into $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!$, and receives a fully faithful functor from $\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!\right)^c$. In addition, the Verdier duality functor defines a contravariant equivalence:

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}\right)^{c,!*} \to \left(\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X)}\text{-mod}\right)^{c,!*}.$$

3.1.10. From Theorem 3.1.2 and Corollary 3.1.5, by duality we obtain a fully faithful functor

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_*\right)^c \hookrightarrow \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}\right)^{c,!*}.$$

It is easy to see that the ind-extension of the composition

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_*\right)^c \hookrightarrow \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}\right)^{c,!*} \hookrightarrow \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!,$$

is the functor $Q_{*\to !}$ defined above.

3.2. **Level structure.**

3.2.1. Let $\overline{x} := x_1, ..., x_n$ be a finite collection of points on X. Consider the group-scheme $G(\widehat{\mathcal{O}}_{\overline{x}}) := G(\widehat{\mathcal{O}}_{x_1} \times ... \times \widehat{\mathcal{O}}_{x_n})$. Let K be a group sub-scheme of $G(\widehat{\mathcal{O}}_{\overline{x}})$ of finite codimension.

Let $U \subset \mathrm{Bun}_G(X)$ be an open sub-stack of finite type. Let U^K be the stack (of infinite type) that classifies pairs $(\mathcal{P}_G, \alpha)$, where $\mathcal{P}_G$ is a point of U, and α is a structure of level K on $\mathcal{P}_G$ at $\overline{x}$. Note that for a fixed U and K small enough, U^K is a scheme of finite type. If K is normal in $G(\widehat{\mathcal{O}}_{\overline{x}})$, we have an action of $G(\widehat{\mathcal{O}}_{\overline{x}})/K$ on U^K and $(U^K)/(G(\widehat{\mathcal{O}}_{\overline{x}})/K) \simeq U$.

Set

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_! := \underset{\underset{U}{\longleftarrow}}{lim}\, \mathfrak{D}^\kappa_{U^K}\text{-mod}.$$

Note that unlike the case of $K = G(\widehat{\mathcal{O}}_{\overline{x}})$ (and an appropriate Iwahori variant discussed below), we do not know whether the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!$ is compactly generated or even dualizable. Whether or not or compact objects generate $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!$, they are described by the corresponding version of Lemma 3.1.3.

If K_1 is a normal subgroup of K_2, we obtain that $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_1)}\text{-mod}_!$ is acted on strongly by K_2/K_1 and

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_1)}\text{-mod}_!\right)^{K_2/K_1} \simeq \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_2)}\text{-mod}_!.$$

In particular, we have the tautological forgetful functor

$$\pi^!_{K_1,K_2} : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_2)}\text{-mod}_! \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_1)}\text{-mod}_!,$$

(where π_{K_1,K_2} denotes the map $\mathrm{Bun}_G(X, K_1) \to \mathrm{Bun}_G(X, K_2)$), and its right adjoint

$$(\pi_{K_1,K_2})_* : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_1)}\text{-mod}_! \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K_2)}\text{-mod}_!.$$

3.2.2. We define the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*$ as

$$\underset{\underset{U}{\longrightarrow}}{lim}\, \mathfrak{D}^\kappa_{U}\text{-mod},$$

where for $U_1 \subset U_2$, the functor $\mathfrak{D}^\kappa_{U_1^K}\text{-mod} \to \mathfrak{D}^\kappa_{U_2^K}\text{-mod}$ is $(j^K_{1,2})_*$, where $j^K_{1,2}$ denotes the corresponding open embedding $U_1^K \hookrightarrow U_2^K$.

The equivalence (3.1) continues to hold. In partcular, we obtain a pairing

$$\langle -, -\rangle_{\mathrm{Bun}_G(X,K)} : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_! \otimes \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,K)}\text{-mod}_* \to \mathrm{Vect}\,. \tag{3.3}$$

By Lemma 10.1.2 we obtain:

Corollary 3.2.3. *If one of the categories $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*$ or $\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,K)}\text{-mod}_!$ is dualizable, then so is the other one, and in this case* (3.3) *identies them as each other's duals.*

As in the case of $\mathrm{Bun}_G(X)$ we can introduce the small category, there exists a naturally defined functor

$$Q^K_{*\to !} : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_* \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!.$$

However, for general K, the functor $Q^K_{*\to !}$ is not an equivalence even for irrational κ.

3.2.4. Additionally, as in the case of $\mathrm{Bun}_G(X)$ we can introduce the small category

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*} := \underset{\overleftarrow{U}}{lim}\, (\mathfrak{D}^\kappa_{U^K}\text{-mod})^c,$$

equipped with fully faithful functors

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!\right)^c \hookrightarrow \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*} \hookrightarrow \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!,$$

and Verdier duality defines an contravariant equivalence

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*} \simeq \left(\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*}.$$

However, the conclusion of Sect 3.1.10 holds only conditionally:

Lemma 3.2.5. *If $\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,K)}\text{-mod}_!$ is compactly generated, and hence the functor*

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*\right)^c \to \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*}$$

is well-defined, then the ind-extension of the composition

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*\right)^c \to \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*} \hookrightarrow \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!$$

identifies with the functor $Q^K_{*\to !}$.

3.3. **The Iwahori case.**

3.3.1. Let $\overline{x} := x_1, ..., x_n$ be a finite collection of points on X. Let $\mathrm{Bun}_G(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$) be the moduli stack of principal G-bundles equipped with a reduction of their fibers at $x_1, ..., x_n$ to B (resp., N). The stack $\mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ is acted on by the torus $T^n = T \times ... \times T$.

By the above, we have the categories $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_!$ and $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!$, the latter being acted on strongly by T^n and we have

$$(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!)^{T^n} \simeq \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_!.$$

In addition, we can consider the category

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w} := (\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!)^{T^n,w},$$

which is easily seen to be equivalent to

$$\underset{\overleftarrow{U}}{lim}\, \mathfrak{D}^\kappa_{U'}\text{-mod}^{T^n,w},$$

where U' denotes the preimage of U in $\mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$.

For a collections of weights $\overline{\lambda} = \lambda_1, ..., \lambda_n$, we can consider the corresponding monodromic category $(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!)^{T^n,w,\overline{\lambda}}$, and it is easy to see that we have an equivalence:

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}} := (\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!)^{T^n,w,\overline{\lambda}} \simeq \underset{\underset{U}{\longleftarrow}}{lim}\, \mathfrak{D}^\kappa_{U'}\text{-mod}^{T^n,w,\overline{\lambda}}.$$

Proceeding as above, we define also the corresponding categories as the corresponding co-limits:

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_*,\ \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_*,\ \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_*^{T^n,w},\ \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_*^{T^n,w,\overline{\lambda}}.$$

3.3.2. We shall prove the following:

Theorem 3.3.3. *The category* $(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!)^{T^n,w,\overline{\lambda}}$ *is compactly generated.*

By Corollary 3.2.3, we obtain:

Corollary 3.3.4.

$$\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_*^{T^n,w,\overline{\lambda}}\right)^\vee \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,-\overline{\lambda}}.$$

3.3.5. The construction of Sect 3.1.6 renders to the present situation, i.e., we have a naturally defined functor

$$Q^{T^n,w,\overline{\lambda}}_{*\to !} : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_*^{T^n,w,\overline{\lambda}} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}.$$

We will prove the following generalization of Proposition 3.1.7:

Proposition 3.3.6. *Assume that κ is irrational and $\check{\lambda}_1, ..., \check{\lambda}_n$ are rational. Then the functor* $Q^{T^n,w,\overline{\lambda}}_{*\to !}$ *is an equivalence.*

As in the case of $\mathrm{Bun}_G(X)$, Proposition 3.3.6 follows from the next assertion:

Proposition 3.3.7. *Let κ and $\overline{\lambda}$ be as in Proposition 3.3.6. Then there exists a sub-stack of finite type $U \subset \mathrm{Bun}_G(X)$, such that no non-zero object of* $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}$ *is supported on the complement to* $U' \subset \mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$.

3.3.8. Finally, as in Sect 3.1.9, we can introduce a small category $\left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}\right)^{c,!*}$, equipped with the corresponding functors, and the conclusion of Sect 3.1.10 holds.

3.4. **The full level structure.**

3.4.1. We are now going to define the category

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!$$

of twisted D-modules over "$\mathrm{Bun}_G(X)$ with a full level structure" at $\overline{x}$.

We set

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_! := \underset{\underset{K}{\longrightarrow}}{colim}\, \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!,$$

where the co-limit is taken with respect to the pull-back functors $\pi^!_{K_1,K_2}$.

By Lemma 10.1.2, we have:

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_! \simeq \underset{\underset{K}{\longleftarrow}}{lim}\, \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!,$$

where the limit is taken with respect to the functors $(\pi_{K_1,K_2})_*$.

The latter implies that we also have the equivalence:

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_! \simeq \underset{\underset{U}{\longleftarrow}}{lim}\, \mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod},$$

where

$$\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod} := \underset{\underset{K}{\longrightarrow, \pi^!}}{colim}\, \mathfrak{D}^\kappa_{U^K}\text{-mod} \simeq \underset{\underset{K}{\longleftarrow, \pi_*}}{lim}\, \mathfrak{D}^\kappa_{U^K}\text{-mod}.$$

Although each of the categories $\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}$-mod is compactly generated, we do not know whether $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!$ is.

The category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!$ is naturally acted on by $G(\widehat{\mathcal{O}}_{\overline{x}})$, and for K as above we have:

$$(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!)^K \simeq \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!.$$

3.4.2. We define the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*$ as

$$\underset{\underset{U}{\longrightarrow}}{colim}\, \mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod},$$

with respect to the functors

$$(j^{\widehat{\overline{x}}}_{1,2})_* : \mathfrak{D}^\kappa_{U_1^{\widehat{\overline{x}}}}\text{-mod} \to \mathfrak{D}^\kappa_{U_2^{\widehat{\overline{x}}}}\text{-mod}.$$

By definition, $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*$ is equivalent to

$$\underset{\underset{K}{\longrightarrow}}{colim}\, \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*,$$

with respect to the functors $(\pi_{K_1,K_2})^!$, and also

$$\underset{\underset{K}{\longleftarrow}}{lim}\, \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_*,$$

with respect to the functors $(\pi_{K_1,K_2})_*$.

The assertion of Lemma 3.2.3 and the constructions of Sect 3.2.4 render ditto into the present context.

3.4.3.

Theorem-Construction 3.4.4.
(1) *The action of* $G(\widehat{\mathcal{O}}_{\overline{x}})$ *on the categories*

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_! \text{ and } \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*$$

naturally extends to an action of the group ind-scheme $G(\widehat{\mathcal{K}}_{\overline{x}})$ *at level* κ.
(2) *The pairing*

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_! \otimes \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_* \to \mathrm{Vect}$$

is $G(\widehat{\mathcal{K}}_{\overline{x}})$*-invariant.*

Proof. Fill in. □

4. Localization on Bun_G

4.1. **Definition of localization.**

4.1.1. Let $\widehat{\mathfrak{g}}_{\kappa,\overline{x}}$ be the central extension at level κ of

$$(\mathfrak{g}\otimes\widehat{\mathcal{K}}_{x_1})\oplus ...\oplus(\mathfrak{g}\otimes\widehat{\mathcal{K}}_{x_n}).$$

Let $\widehat{\mathfrak{g}}_{\kappa,\overline{x}}$-mod be the corresponding category of modules. It is naturally acted on by the group ind-scheme $G(\widehat{\mathcal{K}}_{\overline{x}})$ at level κ.

Theorem-Construction 4.1.2.

(1) *There exists a canonical functor*

$$\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}:\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}\to\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!,$$

compatible with the action of $G(\widehat{\mathcal{O}}_{\overline{x}})$.

(2) *The data compatibility of* $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ *with* $G(\widehat{\mathcal{O}}_{\overline{x}})$ *can be naturally lifted to a data of compatibility with the action of* $G(\widehat{\mathcal{K}}_{\overline{x}})$.

4.1.3. *Proof of Theorem 4.1.2(1).* The data of the functor $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ amounts, by definition, to a compatible family of functors

$$\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}:\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}\to\mathfrak{D}^{\kappa}_{U^{\widehat{\overline{x}}}}\text{-mod}.$$

Recall that $(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c$ is a full subcategory in $D^b(\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}))$. Hence, it is sufficient to construct a compatible family of functors

$$L\,\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}:D^b(\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}))\to\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}\to\mathfrak{D}^{\kappa}_{U^{\widehat{\overline{x}}}}\text{-mod}.$$

The latter are obtained as the left derived functor of the naive localization functor

$$\mathbf{H}(\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}):\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})\to\mathbf{H}(\mathfrak{D}^{\kappa}_{U^{\widehat{\overline{x}}}}\text{-mod}).$$

Explicitly, let $K\subset G(\widehat{\mathcal{O}}_{\overline{x}})$ be small enough, so that $\mathsf{k}\cap\mathrm{Stab}_{\mathfrak{g}(\widehat{\mathcal{K}}_{\overline{x}})}(u)=0$ for any $u\in U^{\widehat{\overline{x}}}$. Then

$$L\,\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa,\overline{x}}}_{\mathsf{k}}(k))\simeq\mathbf{H}\,\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa,\overline{x}}}_{\mathsf{k}}(k))\simeq\mathfrak{D}^{\kappa}_{U^K},$$

where $\mathfrak{D}^{\kappa}_{U^K}\in\mathfrak{D}^{\kappa}_{U^K}$-mod is regarded as an object of $\mathfrak{D}^{\kappa}_{U^{\widehat{\overline{x}}}}$-mod via the pull-back functor $\mathfrak{D}^{\kappa}_{U^K}$-mod-mod $\to\mathfrak{D}^{\kappa}_{U^{\widehat{\overline{x}}}}$-mod.

The compatibility of the functors $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}$ among themselves and with the action of $G(\widehat{\mathcal{O}}_{\overline{x}})$ follows from the construction.

4.1.4. In the course of the proof we have shown that for a fixed U, the functor $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}$ sends

$$(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^K)^c \to (\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod}^K)^c,$$

and, hence,

$$(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c \to (\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod})^c.$$

In particular, the restriction of the functor $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ to $(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c$ factors as

$$(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c \overset{\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}}{\longrightarrow} \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}\right)^{c,!*} \hookrightarrow \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!, \tag{4.1}$$

and

$$(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^K)^c \overset{\mathrm{Loc}^K_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}}{\longrightarrow} \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}\right)^{c,!*} \hookrightarrow \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,K)}\text{-mod}_!, \tag{4.2}$$

4.1.5. *Proof of Theorem 4.1.2(2).* Fill in.

□

4.2. **The functor of sections.**

4.2.1. Consider now the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*$.

Theorem-Construction 4.2.2.

(1) *There exists a canonical functor*

$$\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_* : \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_* \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod},$$

compatible with the action of $G(\widehat{\mathcal{O}}_{\overline{x}})$.

(2) *The data compatibility of* $\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*$ *with* $G(\widehat{\mathcal{O}}_{\overline{x}})$ *can be naturally lifted to a data of compatibility with the action of* $G(\widehat{\mathcal{K}}_{\overline{x}})$.

4.2.3. *Proof of Theorem 4.2.2(1).* The data of a functor $\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*$ amounts to a compatible family of functors

$$\Gamma(U^{\widehat{\overline{x}}},-) : \mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod} \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}$$

for open sub-stacks $U \subset \mathrm{Bun}_G(X)$ of finite type.

Each of the latter is determined by the corresponding functor

$$\Gamma(U^{\widehat{\overline{x}}},-) : (\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod})^c \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}.$$

The latter is constructed as the composition of the usual derived functor of global sections

$$R\Gamma(U^{\widehat{\overline{x}}},-) : (\mathfrak{D}^\kappa_{U^{\widehat{\overline{x}}}}\text{-mod})^c \to D^b(\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})) \hookrightarrow D^+(\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})),$$

and the fully faithful embedding

$$D^+(\mathbf{H}(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})) \hookrightarrow \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}$$

of [FG2].

4.2.4. *Proof of Theorem 4.2.2(2).* Fill in.

□

4.3. **Duality.**

4.3.1. The functor $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ does not in general map $(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c$ to the subcategory of compact objects in $\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}$-mod. Therefore, it does not admit a colimit preserving right adjoint.

However, we have:

Proposition 4.3.2. *The pairings*

$$\langle -,-\rangle_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}:\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_!\otimes\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*\to\mathrm{Vect}$$

and

$$\langle -,-\rangle_{\widehat{\mathfrak{g}}_{\overline{x}}}:\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}\otimes\widehat{\mathfrak{g}}_{\kappa',\overline{x}}\text{-mod}$$

are compatible with the functors $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ *and* $\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*$.

Proof. The assertion of the proposition reads that for

$$M\in\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod and }\mathcal{F}\in\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}_*,$$

we have a functorial isomorphism

$$\langle\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}(M),\mathcal{F}\rangle_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\simeq\langle M,\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),\mathcal{F})_*\rangle_{\widehat{\mathfrak{g}}_{\overline{x}}}. \tag{4.3}$$

By the construction of both functors, the assertion reduces to showing that for an open sub-stack $U\subset\mathrm{Bun}_G(X)$ of finite type, $M\in(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c$ and and $\mathcal{F}\in(\mathfrak{D}^{\kappa'}_{U^{\widehat{\overline{x}}}}\text{-mod})^c$, we have a functorial isomorphism:

$$\langle\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},U^{\widehat{\overline{x}}}}(M),\mathcal{F}\rangle_{U^{\widehat{\overline{x}}}}\simeq\langle M,\Gamma(U^{\widehat{\overline{x}}},\mathcal{F})\rangle_{\widehat{\mathfrak{g}}_{\overline{x}}}. \tag{4.4}$$

The latter isomorphism follows from the definitions.

□

4.3.3. Recall the functor

$$\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}:(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod})^c\to\left(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}\right)^{c,!*}$$

of (4.1).

Let $\mathbb{D}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}$ denote the contravariant equivalence

$$\left(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}\right)^{c,!*}\to\left(\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}\right)^{c,!*}.$$

As a formal corollary of Proposition 4.3.2, we obtain:

Corollary 4.3.4. *The functors*

$$\mathbb{D}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\circ\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}\circ\mathbb{D}_{\widehat{\mathfrak{g}}_{\overline{x}}}\ \text{and}\ \mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa',\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}$$

mapping

$$(\widehat{\mathfrak{g}}_{\kappa',\overline{x}}\text{-mod})^c\to\left(\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})}\text{-mod}\right)^{c,!*}$$

are isomorphic.

5. Finiteness properties of the category of D-modules on Bun_G

5.1. **The spherical case.**

5.1.1. The goal of this subsection is to prove Theorem 3.1.2. In fact, we shall prove a stronger assertion, explained to us by V. Drinfeld:

Theorem 5.1.2. *The stack* $\mathrm{Bun}_G(X)$ *can be represented as a union of open sub-stacks of finite type* U_α*, such that for every pair* $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$*, the functor* $(j_{\alpha,\beta})_*$ *sends*

$$\left(\mathfrak{D}^\kappa_{U_\alpha}\text{-mod}\right)^c \to \left(\mathfrak{D}^\kappa_{U_\beta}\text{-mod}\right)^c.$$

Let us show how this theorem implies Theorem 3.1.2. Indeed, by Verdier duality we obtain:

Corollary 5.1.3. *For every pair* $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$*, the functor left adjoint to*

$$(j_{\alpha,\beta})_! : \mathfrak{D}^\kappa_{U_\alpha}\text{-mod} \to \mathfrak{D}^\kappa_{U_\beta}\text{-mod},$$

is defined.

From Lemma 10.1.2, we then obtain:

Corollary 5.1.4.

(1) *The category* $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!$ *can be written as* $\underset{\alpha}{\underrightarrow{colim}}\, \mathfrak{D}^\kappa_{U_\alpha}\text{-mod}$*, where for* $U_\alpha \subset U_\beta$*, the functor* $\mathfrak{D}^\kappa_{U_\alpha}\text{-mod} \to \mathfrak{D}^\kappa_{U_\beta}\text{-mod}$ *is* $(j_{\alpha,\beta})_!$*.*

(2) *The category* $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!$ *is compactly generated. Its compact objects are of the form* $(j_\alpha)_!(\mathcal{F})$ *for* $\mathcal{F} \in \left(\mathfrak{D}^\kappa_{U_\alpha}\text{-mod}\right)^c$*, and where* $(j_\alpha)_!$ *denotes the tautological functor*

$$\mathfrak{D}^\kappa_{U_\alpha}\text{-mod} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X)}\text{-mod}_!$$

from point (1).

5.1.5. For the proof of Theorem 5.1.2, we shall need to recall some reduction theory. Let $\check{\Lambda}_G$ denote the coweight lattice of G, and set $\check{\Lambda}^{\mathbb{Q}} := \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \check{\Lambda}$. Let ψ denote the map

$$\check{\Lambda}_G \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_G \simeq \check{\Lambda}^{\mathbb{Q}}_{[G,G]} \oplus \check{\Lambda}^{\mathbb{Q}}_{Z_0(G)} \to \check{\Lambda}^{\mathbb{Q}}_{[G,G]} \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_G.$$

Let $\check{\Lambda}^{pos}_G \subset \check{\Lambda}_G$ denote the positive span of positive simple roots, and $\check{\Lambda}^{pos,\mathbb{Q}}_G \subset \check{\Lambda}^{\mathbb{Q}}_G$ the corresponding rational span. Let $\check{\Lambda}^+_G$ (resp., $\check{\Lambda}^{+,\mathbb{Q}}_G$) denote the dominant cones.

For a parabolic P with Levi factor M consider the quotient $\check{\Lambda}_{G,P} := \check{\Lambda}_G/\check{\Lambda}_{[M,M]_{sc}} \simeq \pi_1(M)$. Let ψ_P denote the map

$$\check{\Lambda}_{G,P} \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_{G,P} \simeq (\check{\Lambda}_{[G,G]}/\check{\Lambda}_{[M,M]})^{\mathbb{Q}} \oplus \check{\Lambda}^{\mathbb{Q}}_{Z_0(G)} \to (\check{\Lambda}_{[G,G]}/\check{\Lambda}_{[M,M]})^{\mathbb{Q}} \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_{G,P}.$$

Let also ϕ_P denote the map

$$\check{\Lambda}_{G,P} \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_{G,P} \simeq \check{\Lambda}^{\mathbb{Q}}_{[G,G]\cap Z_0(M)} \oplus \check{\Lambda}^{\mathbb{Q}}_{Z_0(G)} \to \check{\Lambda}^{\mathbb{Q}}_{[G,G]\cap Z_0(M)} \hookrightarrow \check{\Lambda}^{\mathbb{Q}}_G.$$

Let $\check{\Lambda}^{pos}_{G,P}$ (resp., $\check{\Lambda}^{pos,\mathbb{Q}}_{G,P}$) be the image of $\check{\Lambda}^{pos}_G$ (resp., $\check{\Lambda}^{pos,\mathbb{Q}}_G$) in $\check{\Lambda}_{G,P}$ (resp., $\check{\Lambda}^{\mathbb{Q}}_{G,P}$).

5.1.6. Consider the stack $\mathrm{Bun}_P(X)$ and let $\mathfrak{p}_P$ and $\mathfrak{q}_P$ denote its natural projections to $\mathrm{Bun}_G(X)$ and $\mathrm{Bun}_M(X)$, respectively. Recall that the group $\check{\Lambda}_{G,P}$ enumerates the connected components of the stacks $\mathrm{Bun}_M(X)$ and $\mathrm{Bun}_P(X)$ (the latter are in bijection via the projection $\mathfrak{p}_Q$). For $\check{\lambda}_P \in \check{\Lambda}_{G,P}$ we shall denote by $\mathrm{Bun}_P^{\check{\lambda}_P}(X)$ the corresponding connected component.

Recall that a G-bundle $\mathcal{P}_G \in \mathrm{Bun}_G(X)$ is called "semi-stable" if for every parabolic P, such that $\mathcal{P}_G = \mathfrak{p}_P(\mathcal{P}_P)$ with $\mathcal{P}_P \in \mathrm{Bun}_P^{\check{\lambda}_P}(X)$, then $\psi_P(\check{\lambda}_P) \in -\check{\Lambda}_{G,P}^{pos,\mathbb{Q}}$.

Equivalently, $\mathcal{P}_G$ is semi-stable if for every reduction $\mathcal{P}_B$ of $\mathcal{P}_G$ to the Borel B, we have $\mathcal{P}_B \in \mathrm{Bun}_B^{\check{\lambda}}$ with $\psi(\check{\lambda}) \in -\check{\Lambda}_G^{pos,\mathbb{Q}}$.

It is known that semi-stable bundles form an open sub-stack of finite type, $\mathrm{Bun}_G^{ss}(X)$. Let $\mathrm{Bun}_M^{ss}(X)$ be the corresponding open sub-stack of $\mathrm{Bun}_M(X)$, and let $\mathrm{Bun}_P^{ss}(X)$ be the pre-image of $\mathrm{Bun}_M^{ss}(X)$ in $\mathrm{Bun}_P^{ss}(X)$.

5.1.7. For a parabolic P let $\check{\lambda}_P$ be an element $\check{\Lambda}_{G,P}$. We say that is dominant if $\phi_P(\check{\lambda}_P) \in \check{\Lambda}_G^{+,\mathbb{Q}}$. Let $\check{\Lambda}_{G,P}^+$ denote the cone of dominant elements. We say that $\check{\lambda}_P \in \check{\Lambda}_{G,P}^+$ is regular if it lies off the walls of $\check{\Lambda}_{G,P}^+$, i.e., if $\langle \phi_P(\check{\lambda}_P), \alpha_i \rangle > 0$ for every simple root $\alpha_i \notin [M,M]$.

Theorem 5.1.8.
(1) *Let $\check{\lambda}_P \in \check{\Lambda}_G/\check{\Lambda}_{[M,M]}$ be dominant and regular. Then the map $\mathfrak{p}_P$ defines an isomorphism between $\mathrm{Bun}_P^{\check{\lambda}_P,ss}(X)$ and a locally closed sub-stack of finite type in $\mathrm{Bun}_G(X)$. (We shall denote this sub-stack $\mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$.)*
(2) *The sub-stacks $\mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$ for various pairs $(P,\check{\lambda}_P)$ are pairwise non-intersecting, and their union covers $\mathrm{Bun}_G(X)$, i.e., every geometric point of Bun_G belongs to exactly one $\mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$.*
(3) *If $\mathcal{P}_G \in \mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$ admits a reduction to a parabolic P_1 with parameter $\check{\lambda}_{P_1}$, then the element $\phi_P(\check{\lambda}_P) - \phi_{P_1}(\check{\lambda}_{P_1}) \in \check{\Lambda}_G^{pos,\mathbb{Q}}$.*
(4) *Substacks of the form*

$$\bigcup_{(P,\check{\lambda}_P),\, \phi_P(\check{\lambda}_P)-\check{\lambda}_0 \in \check{\Lambda}_G^{pos,\mathbb{Q}}} \mathrm{Bun}_G^{P,\check{\lambda}_P}(X),$$

for some fixed $\check{\lambda}_0 \in \check{\Lambda}^{+,\mathbb{Q}}$ are closed.

5.1.9. Let P be a parabolic and P^- an opposite parabolic. We shall identify their Levi factors via embedding $M \simeq P \cap P^-$ into both P and P^-.

Lemma 5.1.10. *There exists an element $d \in \mathbb{Q}^{\geq 0}$, which depends on the genus of X, with the following property:*

For every parabolic P and $\check{\lambda}_P \in \check{\Lambda}_{G,P}$ such that $\alpha_i(\varphi_P(\check{\lambda}_P)) \geq d$ for every simple root $\alpha_i \notin [M,M]$, and for every $\mathcal{P}_M \in \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X)$ the following holds:

- *(i) $H^1(X,(N_P)_{\mathcal{P}_M}) = 0$, where N_P is the unipotent radical of P.*
- *(ii) $H^1(X,\mathrm{Lie}((N_P))_{\mathcal{P}_M}) = 0$.*

We call a stratum $\mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$ with $\check{\lambda}$ satisfying the assumption of the lemma "deep enough". The first of these conditions implies that the map $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X) \to \mathrm{Bun}_P^{\check{\lambda}_P,ss}(X)$, given by the embedding is smooth and surjective. The second conditions implies that the map

$$\mathfrak{p}_P^- : \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X) \to \mathrm{Bun}_G(X)$$

is smooth.

5.1.11. Finally, we are ready to prove Theorem 5.1.2. We take the open subsets U_α to be open subsets that are unions of the strata $\mathrm{Bun}_G^{P,\check{\lambda}_P}(X)$, *which include all strata that are not deep enough*. To prove the theorem, it is enough to show that if U is an open subset of finite type of $\mathrm{Bun}_G(X)$ and $V := \mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)$ is a stratum, which is contained and is closed in U, and $\mathcal{F}$ is an object of $\left(\mathfrak{D}^\kappa_{U-V}\text{-mod}\right)^c$, then the $*$-extension of $\mathcal{F}$ to the entire U will belong to $(\mathfrak{D}^\kappa_U\text{-mod})^c$.

Let us denote the morphisms $U - V \hookrightarrow U$ and $V \hookrightarrow U$ by j and i respectively. Since the image of $(\mathfrak{D}^\kappa_U\text{-mod})^c$ under j^* generates $\left(\mathfrak{D}^\kappa_{U-V}\text{-mod}\right)^c$, it sufficient to show that the functor $i^!$ sends $(\mathfrak{D}^\kappa_U\text{-mod})^c$ to $(\mathfrak{D}^\kappa_V\text{-mod})^c$.

Consider the morphism $\mathfrak{p}^-_P : \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X) \to \mathrm{Bun}_G(X)$. By assumption, it is smooth. Let $\widetilde{U}$ and $\widetilde{V}$ be the preimages in $\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X)$ of U and V, respectively. Let $\widetilde{j}$ and $\widetilde{i}$ denote the corresponding morphisms.

Lemma 5.1.12. *For any P and dominant regular $\check{\lambda}_P$, the natural morphism*

$$\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X) \to \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathrm{Bun}_P^{\check{\lambda}_P,ss}(X)$$

is an isomorphism.

By the "deep enough" assumtion, the map $\pi : \widetilde{U} \to U$ is smooth, and its restriction to $\widetilde{V}$, i.e., the map $\widetilde{V} \to V$ is a tower of fibrations into affine spaces. Hence, it is enough to show that

$$\widetilde{i}^!(\widetilde{\mathcal{F}}) \in \left(\mathfrak{D}^\kappa_{\widetilde{V}}\text{-mod}\right)^c,$$

where $\widetilde{\mathcal{F}}$ is the pull-back of $\mathcal{F}$ to $\widetilde{U} - \widetilde{V}$.

5.1.13. However, the latter situation falls into the following paradigm: we have a base stack S (in our case $S = \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X)$), and another stack S' mapping to S (in our case, the fiber of S' over $\mathcal{P}_M \in \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X)$ is $H^1(X, (U_P)_{\mathcal{P}_M})$), in such a way that the morphism $S' \to S$ is affine, and there exists a $\mathbb{G}_m$-action on S' that preserves S and which contracts S' to a section $\widetilde{i} : S \to S'$ (in our case, the $\mathbb{G}_m$ action is given by a regular dominant co-character of $Z_0(M)$). In this case, the functor $\widetilde{i}^!$ sends $\left(\mathfrak{D}_{S'}\text{-mod}^{\mathbb{G}_m}\right)^c$ to $(\mathfrak{D}_S\text{-mod})^c$.

5.1.14. *Proof of Proposition 3.1.8.* We claim that the open sub-stack in question is any open union of the strata $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)$, which contains all those that are not sufficiently deep. In fact, we claim that for any sufficiently deep stratum, the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)}$-mod is zero.

The stratum in question is isomorphic, as a stack, to $\mathrm{Bun}_P^{\check{\lambda}_P,ss}(X)$. The group $Z_0(M)$ acts on it via its adjoint action on P. Moreover, the twisting given by κ is equivariant with respect to $Z_0(M)$ against the character of $\mathrm{Lie}(Z_0(M))$ equal to $-\kappa(\check{\lambda}_P, -)$, which is non-integral. To finish the proof, it suffices to notice that the "sufficiently deep" assumption implies that the above action of $Z_0(M)$ on $\mathrm{Bun}_P^{\check{\lambda}_P,ss}(X)$ is in fact trivial.

5.2. **The Iwahori case.**

5.2.1. As in the spherical case, Theorem 3.3.3 follows from the next result:

Theorem 5.2.2. *For U_α, U_β as in Theorem 5.1.2 with α "deep enough", the functor*

$$(j'_{\alpha,\beta})_* : \mathfrak{D}^\kappa_{U'_\alpha}\text{-mod}^{T^n,w,\overline{\lambda}} \to \mathfrak{D}^\kappa_{U'_\beta}\text{-mod}^{T^n,w,\overline{\lambda}}$$

sends $(\mathfrak{D}^\kappa_{U'_\alpha}\text{-mod}^{T^n,w,\overline{\lambda}})^c$ to $(\mathfrak{D}^\kappa_{U'_\beta}\text{-mod}^{T^n,w,\overline{\lambda}})^c$.

This theorem admits the same corollaries as in the spherical case:

Corollary 5.2.3.

(1) *The functor $(j'_{\alpha,\beta})_!$, left adjoint to $(j'_{\alpha,\beta})^! : \mathfrak{D}^\kappa_{U'_\beta}\text{-mod}^{T^n,w,\overline{\lambda}} \to \mathfrak{D}^\kappa_{U'_\alpha}\text{-mod}^{T^n,w,\overline{\lambda}}$ is defined.*

(2) *The category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}$ is compactly generated and is equivalent to*

$$\underset{\underset{\alpha}{\longrightarrow}}{lim}\, \mathfrak{D}^\kappa_{U'_\alpha}\text{-mod}^{T^n,w,\overline{\lambda}},$$

under the functors $(j'_{\alpha,\beta})_!$.

(3) *The compact objects of $\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}$ are of the form $(j'_\alpha)_!(\mathcal{F})$, where $\mathcal{F}$ is an object of $\left(\mathfrak{D}^\kappa_{U'_\alpha}\text{-mod}^{T^n,w,\overline{\lambda}}\right)^c$, and where $(j'_\alpha)_!$ denotes the tautological functor $\mathfrak{D}^\kappa_{U'_\alpha}\text{-mod} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}$.*

5.2.4. *Proof of Theorem 5.2.2.* Let $(P, \check{\lambda}_P)$ be as in Sect 5.1.7. Let $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$) denote the preimage of the corresponding stratum in $\mathrm{Bun}_G(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$).

In the present situation we modify the definition of pair $(P, \check{\lambda}_P)$ to be deep enough, by making it slightly stronger. Namely, we shall require that

- (i) $H^1(X, (N_P)'_{\mathcal{P}_M}) = 0$, where $(N_P)'$ is the sub-sheaf of $\mathrm{Maps}(X, N_P)$ equal to the kernel of the homomorphism of the latter to N_P^n, corresponding to evaluation at $\overline{x} = x_1, ..., x_n$.
- (ii) $H^1(X, \mathrm{Lie}((N_P))_{\mathcal{P}_M}(-\overline{x})) = 0$.

As in the $\mathrm{Bun}_G(X)$ case, it is easy to see that all but finitely many strata are "deep enough". The open sub-stacks U_α that appear in Theorem 5.2.2 are open unions of strata that contain all strata that are not deep enough in the new sense. As in the proof of Theorem 5.1.2, for the proof of Theorem 5.2.2, it is sufficient to show that for an open sub-stack $U \subset \mathrm{Bun}_G(X)$ of finite type that contains a stratum $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)$, the !-pullback functor sends

$$(\mathfrak{D}^\kappa_{U'}\text{-mod}^{T^n,w,\overline{\lambda}})^c \to (\mathfrak{D}^\kappa_{\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^c.$$

To prove this, we shall need to refine our stratification. By definition, we have:

$$\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X, \mathbf{I}_{\overline{x}}) \simeq \mathrm{Bun}_P^{\check{\lambda}_P,ss}(X) \underset{(\mathrm{pt}/P)^n}{\times} (B\backslash G/P)^n,$$

where the map

$$\mathrm{Bun}_P \to (\mathrm{pt}/P)^n$$

corresponds to taking the fiber of $\mathcal{P}_P$ at the points $x_1, ..., x_n$.

Let $\overline{w} := w_1, ..., w_n$ be an n-tuple of elements of the set W/W_M. Let $(B\backslash G/P)^{\overline{w}}$ be the corresponding Schubert stratum in $(B\backslash G/P)^n$, and let $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$) be its preimage in $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$).

It is sufficient to show that for an open sub-stack of finite type $U \subset \mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$, which contains a deep enough stratum $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)$, and $\mathcal{F} \in (\mathfrak{D}^\kappa_{U'}\text{-mod}^{T^n,w,\overline{\lambda}})^c$, the !-restriction of $\mathcal{F}$ to the stratum $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ belongs to $(\mathfrak{D}^\kappa_{\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^c$.

5.2.5. Let P^- be the opposite parabolic. Consider the Cartesian products

$$\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathrm{Bun}_G(X, \mathbf{I}_{\overline{x}}),$$

and

$$\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss}(X) \underset{\mathrm{Bun}_G(X)}{\times} \mathrm{Bun}_G(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}),$$

and let $\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \mathbf{I}_{\overline{x}})$ (resp., $\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$) be their locally closed sub-stacks, respectively, corresponding to the condition that the relative position of the reductions of the G-torsor $(\mathcal{P}_G)_{x_i}$ to B and P^- is w_i. (Our conventions are such that given a G-torsor with transversal reductions to P and P^-, there exists exactly one reduction to B, which is in position w with respect to P and P^-.)

By the "deep enough" assumption, the projections

$$\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \mathbf{I}_{\overline{x}}) \to \mathrm{Bun}_{P^-}(X, \mathbf{I}_{\overline{x}}) \text{ and } \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}) \to \mathrm{Bun}_{P^-}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$$

are smooth, and their images contain the strata $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \mathbf{I}_{\overline{x}})$ and $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$, respectively.

Consider the Cartesian product

$$\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}) \underset{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}{\times} \mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}). \tag{5.1}$$

By Lemma 5.1.12, the stack (5.1) identifies with $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ and its projection onto $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ is a tower of affine fibrations.

Hence, we need to show that for

$$\mathcal{F} \in \left(\mathfrak{D}^\kappa_{\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}\right)^c,$$

its !-restriction to the sub-stack (5.1) is a compact object of the corresponding category.

Let i denote the corresponding closed embedding

$$i : \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}) \hookrightarrow \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}).$$

In addition, we have a natural projection

$$\pi : \mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}) \twoheadrightarrow \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}}). \tag{5.2}$$

By the "sufficiently deep" assumption, the morphism π is representable and affine. The two stacks appearing in (5.2) are acted on by $Z_0(M)^n$ (via the embedding $Z_0(M) \hookrightarrow T$ and the action of the latter on G/N), and another copy of $Z_0(M)$ induced by its adjoint action on P^-. Consider the action of $Z_0(M)$ given by

$$z \mapsto (w_1^{-1}(z), ..., w_n^{-1}(z); z^{-1}). \tag{5.3}$$

It acts trivially on $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ and contracts $\mathrm{Bun}_{P^-}^{\check{\lambda}_P,ss,\overline{w}}(X, \overset{\circ}{\mathbf{I}}_{\overline{x}})$ to the image of the section i. Hence, we find ourselves in the paradigm of Sect 5.1.13 (in the monodromic rather than equivariant situation, with the conclusion being the same).

5.2.6. *Proof of Proposition 3.3.7.* We will show that the category $\mathfrak{D}^\kappa_{\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}$ is zero for any $(P, \check{\lambda}_P)$, which is deep enough and any $\overline{w}$.

By the above, it is sufficient to show that the corresponding category

$$\mathfrak{D}^\kappa_{\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}$$

is zero. For that it is sufficient that the corresponding category of twisted D-modules

$$\mathfrak{D}^{\kappa,\check{\lambda}}_{\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})}\text{-mod}$$

on $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \mathbf{I}_{\overline{x}})$ is zero.

Consider again the action of $Z_0(M)$ given by (5.3). The required assertion follows from the fact that the resulting twisting on $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X, \mathbf{I}_{\overline{x}})$ is $Z_0(M)$-equivariant against the character $w_1^{-1}(\lambda_1) + ... + w_n^{-1}(\lambda_n) + \kappa(\check{\lambda}_P, -)$, which is non-integral.

6. Finiteness properties of the localization functor

6.1. Localization in the Iwahori case.

6.1.1. Let us return to the set-up of Sect 4.1. Consider the group-scheme $K = \overset{\circ}{\mathbf{I}}_{\overline{x}} := \overset{\circ}{\mathbf{I}}_{x_1} \times ... \times \overset{\circ}{\mathbf{I}}_{\overline{x}}$. We obtain a functor

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!} : \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!, \tag{6.1}$$

and the corresponding functors

$$\mathrm{Loc}^{\mathbf{I}_{\overline{x}}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!} : \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\mathbf{I}_{\overline{x}}} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_!, \tag{6.2}$$

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!} : \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w} := (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}})^{T^n,w} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_!^{T^n,w}, \tag{6.3}$$

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!} : \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}} := (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}})^{T^n,w,\overline{\lambda}} \to \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_!^{T^n,w,\overline{\lambda}}, \tag{6.4}$$

for a collection $\overline{\lambda} = \check{\lambda}_1, ..., \check{\lambda}_n$ of weights.

By Sect 4.1.4, each of the above functors, when restricted to the corresponding subcategory of compact objects, factors as

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}})^c \to (\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod})^{c,!*},$$

$$\mathrm{Loc}^{\mathbf{I}_{\overline{x}}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\mathbf{I}_{\overline{x}}})^c \to (\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod})^{c,!*},$$

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w})^c \to (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w})^{c,!*},$$

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}})^c \to (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^{c,!*}.$$

6.1.2. In addition, we obtain the functors

$$\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_* : \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}_* \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}}}, \tag{6.5}$$

and the corresponding functors

$$\Gamma(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_* : \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}_* \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\mathbf{I}_{\overline{x}}}, \tag{6.6}$$

$$\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_* : \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w}_* \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w}, \tag{6.7}$$

$$\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_* : \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_* \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}. \tag{6.8}$$

6.1.3. By Proposition 4.3.2 and Theorem 3.3.3, we have:

Corollary 6.1.4. *Under the equivalences*

$$\left(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_!\right)^{\vee} \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}^{T^n,w,-\overline{\lambda}}_!$$

and

$$\left(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}\right)^{\vee} \simeq \widehat{\mathfrak{g}}_{\kappa',\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,-\overline{\lambda}},$$

the functors $\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$ *and* $\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*$ *become the duals of one another.*

Corollary 6.1.5. *The functors*

$$\mathbb{D}_{\mathrm{Bun}_G(X,\widehat{\overline{x}})} \circ \mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} \circ \mathbb{D}_{\widehat{\mathfrak{g}}_{\overline{x}}} \text{ and } \mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa',\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*}$$

mapping

$$(\widehat{\mathfrak{g}}_{\kappa',\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}},T^n,w,-\overline{\lambda}})^c \to \left(\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,-\overline{\lambda}}\right)^{c,!*}$$

are isomorphic.

6.1.6. The main result of the present section is the following:

Theorem 6.1.7. *Assume that κ is positive. Then the functor* $\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}})}$ *of* (6.4) *sends*

$$\left(\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}\right)^c \to \left(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_!\right)^c.$$

By Sect 3.3.8, this theorem can be reformulated as follows:

Theorem 6.1.8. *Assume that κ is positive. Then functor*

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}})^c \to (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^{c,!*}$$

has its image in

$$(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_!)^c \subset (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^{c,!*}.$$

Further, by Corollary 6.1.5, we can reformulate Theorem 6.1.9 as follows:

Theorem 6.1.9. *Assume that κ is negative. Then functor*

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!*} : (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}})^c \to (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^{c,!*}$$

has its image in

$$(\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_{*})^c \subset (\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}})^{c,!*}.$$

I.e., for $M \in (\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}})^c$, the !-stalks of $\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}(M)$ are zero away from an open sub-stack of finite type.

As a formal corollary of Theorems 6.1.7 and 6.1.9, we obtain:

Corollary 6.1.10.
(1) *Let κ be negative. Then there exists a functor*

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),*} : \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}} \to \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_{*},$$

which is the left adjoint to $\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*$. We have:

$$Q^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{*\to !} \circ \mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),*} \simeq \mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}.$$

(2) *Let κ be positive. Then there exists a functor*

$$\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_! : \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}_{*} \to \widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}},$$

right adjoint to $\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}$. We have:

$$\Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_! \circ Q^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{*\to !} \simeq \Gamma^{T^n,w}(\mathrm{Bun}_G(X,\widehat{\overline{x}}),-)_*.$$

6.2. **Proof Theorem 6.1.7.**

6.2.1. By definition, we have to show that for M among a set of compact generators of $\widehat{\mathfrak{g}}_{\kappa,\overline{x}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}$ there exists an open sub-stack of finite type $U_M \subset \mathrm{Bun}_G(X)$ such that for any $\mathcal{F}' \in \mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}$ supported outside of U_M, we have

$$\mathrm{Hom}_{\mathfrak{D}^{\kappa}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}}\left(\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}(M),\mathcal{F}'\right) = 0. \tag{6.9}$$

It is sufficient to take M to be the Verma module, i.e., $\Delta_{\kappa,\overline{\mu},\overline{x}}$, where $\overline{\mu} = \mu_1, ..., \mu_n$ is an n-tuple of weights such that $\mu_i - \lambda_i \in \Lambda_G$. The object

$$\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}(\Delta_{\kappa,\overline{\mu},\overline{x}})$$

is the induced twisted D-module, i.e., for $\mathcal{F}' \in \mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}$, we have:

$$\mathrm{Hom}_{\mathfrak{D}^\kappa_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}}\left(\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_{\overline{x}},T^n,w,\overline{\lambda}}_{\widehat{\mathfrak{g}}_{\kappa,\overline{x}},\mathrm{Bun}_G(X,\widehat{\overline{x}}),!}(\Delta_{\kappa,\overline{\mu},\overline{x}}),\mathcal{F}'\right) \simeq$$

$$\simeq \mathrm{Hom}_{\mathfrak{t}^n}\left(k^{\check{\mu}},\Gamma(\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}}),\mathcal{F}')\right),$$

where $k^{\check{\mu}}$ is the corresponding 1-dimensional representation of $\mathfrak{t}^n$.

6.2.2. For $(P,\check{\lambda}_P,\overline{w})$ let $\iota^{\check{\lambda}_P,\overline{w}}$ denote the locally closed embedding of the corresponding stratum into $\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})$. It is sufficient to take $\mathcal{F}'$ from (6.9) to be of the form $\iota^{\check{\lambda}_P,\overline{w}}_*(\mathcal{F})$ for some

$$\mathcal{F} \in \mathfrak{D}^\kappa_{\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})}\text{-mod}^{T^n,w,\overline{\lambda}}, \tag{6.10}$$

and to show that the vanishing occurs for all but finitely many strata. I.e., we have to show that for all but finitely many strata

$$\mathrm{Hom}_{\mathfrak{t}^n}\left(k^{\check{\mu}},\Gamma(\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_{\overline{x}}),\iota^{\check{\lambda}_P,\overline{w}}_*(\mathcal{F}))\right) = 0. \tag{6.11}$$

We can assume that $\mathcal{F}$ belongs to the heart of the t-structure. Moreover, we can assume that $\mathcal{F}$ is an object of the corresponding category

$$\mathfrak{D}^{\kappa,\overline{\lambda}}_{\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}})}\text{-mod}$$

of twisted D-modules on $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}})$. Multiplying by the line bundle on $\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}})$, corresponding to $\overline{\lambda}-\overline{\mu}$, we are reduced to calculating

$$\Gamma\left(\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}}),\iota^{\check{\lambda}_P,\overline{w}}_*(\mathcal{F})\right) \tag{6.12}$$

for

$$\mathcal{F} \in \mathfrak{D}^{\kappa,\overline{\mu}}_{\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}})}\text{-mod}, \tag{6.13}$$

and showing that it vanishes for all but finitely many strata.

6.2.3. With no restriction of generality we can assume that $(P,\check{\lambda}_P)$ is "deep enough". Recall the projection

$$\mathfrak{q}_P^- : \mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}}) \to \mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}}).$$

It realizes $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}})$ as a quotient of $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})$ with respect to the trivial action of the unipotent group-scheme $\mathfrak{U}$, whose fiber at $\mathcal{P}_M \in \mathrm{Bun}_M(X)$ is

$$\mathrm{Maps}(X,N(P)^{\overline{w}}_{\mathcal{P}_M}),$$

where $N(P)^{\overline{w}} \subset N(P)$ is the sub-sheaf, consisting of sections

$$\{x \mapsto n(x)\,|\,w_i(n(x_i)) \in \mathfrak{n}^+\}.$$

Let $\mathfrak{u}$ denote the sheaf of Lie algebras corresponding to $\mathfrak{U}$.

Consider the pull-back of normal bundle to $\mathrm{Bun}_G^{\check{\lambda}_P,ss}(X)$ inside $\mathrm{Bun}_G(X)$; let $\mathcal{E}_1$ denote its pull-back to $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})$. This is a vector bundle, whose fiber at $\mathcal{P}_M \in \mathrm{Bun}_M(X)$ is the vector space $H^1(X,(\mathfrak{g}/\mathfrak{p})_{\mathcal{P}_M})$.

Consider now the normal bundle to $\mathrm{Bun}_G^{\check{\lambda}_P,ss,\overline{w}}(X,\mathbf{I}_{\overline{x}})$ insider

$$\mathrm{Bun}_G(X,\mathbf{I}_{\overline{x}}) \underset{\mathrm{Bun}_G(X)}{\times} \mathrm{Bun}_G^{\check{\lambda}_P,ss}(X).$$

Let us denote by $\mathcal{E}_2$ its pull-back $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})$. This is a vector bundle, whose fiber at $\mathcal{P}_M \in \mathrm{Bun}_M(X)$ is the vector space

$$\underset{i=1,\dots,n}{\oplus} (\mathfrak{g}/\mathfrak{n}(P))^{w_i}_{(\mathcal{P}_M)_{x_i}},$$

where $(\mathfrak{g}/\mathfrak{n}(P))^w \subset \mathfrak{g}/\mathfrak{n}(P)$ is the subspace , consisting of elements

$$\{g \,|\, w_0 \cdot w_(g) \in \mathfrak{n}^+\}.$$

For $\mathcal{F}$ in Sect 6.13, let $\widetilde{\mathcal{F}}$ denote its pull-back to $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})$. To prove the vanishing of (6.12), it is sufficient to show that for any integers k_1, k_2 and k_3 the space of sections over $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\mathbf{I}_{\overline{x}})$ of the quasi-coherent sheaf

$$\widetilde{\mathcal{F}} \otimes \Lambda^{top}(\mathcal{E}_1) \otimes \Lambda^{top}(\mathcal{E}_2) \otimes \mathrm{Sym}^{k_1}(\mathcal{E}_1) \otimes \mathrm{Sym}^{k_2}(\mathcal{E}_2) \otimes \Lambda^{k_3}(\mathfrak{u}^*) \tag{6.14}$$

vanishes.

We shall again use the torus $Z_0(M)$ that acts trivially on $\mathrm{Bun}_M^{\check{\lambda}_P,ss}(X,\overset{\circ}{\mathbf{I}}_{\overline{x}})$, but with respect to which the quasi-coherent sheaf (6.14) has an equivariant structure. Let us calculate the characters of this action.

6.2.4. The action of $Z_0(M)$ on $\mathcal{E}_2$ is trivial, and hence, so is the action on $\Lambda^{top}(\mathcal{E}_2)\otimes\mathrm{Sym}^{k_1}(\mathcal{E}_2)$.

The action of $Z_0(M)$ on $\widetilde{\mathcal{F}}$ is given by the character

$$w_1^{-1}(\mu_1) + \dots + w_1^{-1}(\mu_n) + \kappa(\check{\lambda}, -),$$

if this character is integral (for otherwise, $\mathcal{F} = 0$).

The action of $Z_0(M)$ on $\mathcal{E}_1$ consists of the characters $\alpha \in \mathfrak{n}(P)$, each coming with multiplicity equal to $(1-g) + \langle \alpha, \check{\lambda}\rangle$. Therefore, the characters η_1 on $\mathrm{Sym}^{k_1}(\mathcal{E}_1)$ are sums of α's from $\mathfrak{n}(P)$.

In particular, the action of $Z_0(M)$ on the line bundle $\Lambda^{top}(\mathcal{E}_1)$ is given by the character

$$\left(\underset{\alpha\in\mathfrak{n}(P)}{\Sigma} \langle \alpha, \check{\lambda}\rangle \cdot \alpha\right) + 2(1-g)\cdot \rho_P,$$

where the first term is easily seen to be equal to $\frac{\kappa_{Kil}}{2}(\check{\lambda}, -)$, and where $2\cdot\rho_P := \underset{\alpha\in\mathfrak{n}(P)}{\Sigma} \alpha$.

The action of $Z_0(M)$ on $\Lambda^{k_3}(\mathfrak{u}^*)$ has characters η_3 equal to sums of α's from $\mathfrak{n}(P)$.

6.2.5. Let $\check{\nu} : \mathbb{G}_m \to Z_0(M)$ be a regular dominant co-character, i.e., $\langle \alpha, \check{\nu}\rangle > 0$ for any $\alpha \in \mathfrak{n}(P)$. The resulting character of $\mathbb{G}_m$ on the space of sections of (6.14) is the sum of three terms:

(1) $(\kappa - \frac{\kappa_{Kil}}{2})(\check{\lambda}, \check{\nu})$
(2) $\langle \eta_1, \check{\nu}\rangle + \langle \eta_2, \check{\nu}\rangle$
(3) $2(1-g)\langle \rho_P, \check{\nu}\rangle + \underset{i=1,\dots,n}{\Sigma} \langle w_i^{-1}(\mu_i), \check{\nu}\rangle$.

We obtain that the integers on the third line are bounded in a way independent of $\check{\lambda}_P$; the integers on the second line are positive, and the integer on the first line is arbitrarily large with $\check{\lambda}_P$ as long as the form $(\kappa - \frac{\kappa_{Kil}}{2})$ is positive definite.

□

7. Arkhipov's functor in the affine case

7.1. The kernel for Arkhipov's functor.

7.1.1. Let κ be a negative level and recall the objects

$$\mathbb{S}^{\infty\to 0}_{\kappa,\kappa} \in \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \otimes \widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty} \simeq \widehat{\mathfrak{g}}_{\kappa,0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty},$$

which defines the functor

$$\Psi^{\infty\to 0}_{\kappa'\to\kappa},$$

and the object

$$\mathbb{S}^{0\to\infty}_{\kappa',\kappa'} \in \widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\mathbf{I}_0} \otimes \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty} \simeq \widehat{\mathfrak{g}}_{\kappa',0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty},$$

which defines the functor

$$\Phi^{\infty\to 0}_{\kappa\to\kappa'},$$

The goal of this section is to describe $\mathbb{S}^{\infty\to 0}_{\kappa,\kappa}$ and $\mathbb{S}^{0\to\infty}_{\kappa',\kappa'}$ explicitly.

7.1.2. From now on we shall fix the global curve X to be $\mathbb{P}^1$ with two marked points $x_1 = 0$ and $x_2 = \infty$. We fix a coordinate t on $\mathbb{P}^1$ so that $t(0) = 0$.

Consider the stacks $\mathrm{Bun}_G(\mathbb{P}^1)$, $\mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$. Let $\mathrm{Bun}^{ss}_G(\mathbb{P}^1) \subset \mathrm{Bun}_G(\mathbb{P}^1)$ be the semi-stable locus, which identifies with pt/G. Let $\mathrm{Bun}^{ss}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty) \subset \mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$ be its preimage under the forgetful map $\mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty) \to \mathrm{Bun}_G(\mathbb{P}^1)$. We have an isomorphism:

$$\mathrm{Bun}^{ss}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty) \simeq (G/B) \times (G/B)/G.$$

Let

$$\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty) \subset \mathrm{Bun}^{ss}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$$

be the open sub-stack corresponding to the open G-orbit on $(G/B) \times (G/B)$. We let

$$\jmath : \mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty) \hookrightarrow \mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$$

denote the corresponding morphism.

Note that the TDO on $\mathrm{Bun}_G(\mathbb{P}^1)$, corresponding to any κ, trivializes over $\mathrm{Bun}^{ss}_G(\mathbb{P}^1)$. In particular, the corresponding TDO on $\mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$ trivializes also over $\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)$. Let

$$\mathcal{O}_{\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)} \in \mathfrak{D}^\kappa_{\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)}\text{-mod}$$

denote the corresponding canonical object. Consider the corresponding object

$$\jmath_*(\mathcal{O}_{\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)}) \in (\mathfrak{D}^\kappa_{\mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)}\text{-mod}_*)^c. \tag{7.1}$$

Additionally, we shall consider its Verdier dual, i.e.,

$$\jmath_!(\mathcal{O}_{\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)}) \in (\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)}\text{-mod}_!)^c. \tag{7.2}$$

7.1.3. The following should be regarded as the main result of the present paper:

Theorem 7.1.4. *We have a canonical isomorphism in* $\widehat{\mathfrak{g}}_{\kappa,0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty}$*:*

$$\mathbb{S}^{\infty\to 0}_{\kappa,\kappa} \simeq \Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}), \jmath_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)_*.$$

In addition, we propose the following conjecture:

Conjecture 7.1.5. *We have a canonical isomorphism in* $\widehat{\mathfrak{g}}_{\kappa',0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty}$*:*

$$\mathbb{S}^{0\to\infty}_{\kappa',\kappa'} \simeq \Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}), \jmath_!(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)_!.$$

Theorem 7.1.4 and Conjecture 7.1.5 can be regarded as affine analogs of Theorem 1.4.3 in its incarnation given by Corollary 1.4.5. Later on we shall establish an affine analog of the commutative diagram of Corollary 1.4.6.

7.2. **Proof of Theorem 7.1.4.**

7.2.1. The main step in the proof of the theorem is the following proposition, which is an affine analog of Proposition 1.5.1:

Proposition 7.2.2. *For* $M \in (\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0})^c$, $N \in (\widehat{\mathfrak{g}}_{\kappa,\infty}\text{-mod}^{\mathbf{I}_\infty})^c$ *there exists a canonical isomorphism:*

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}}(M,N^*) \simeq$$
$$\simeq \mathrm{Hom}_{\mathfrak{D}^\kappa_{\mathrm{Bun}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\text{-mod}_*}\left(\mathrm{Loc}^{\mathbf{I}_0,\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),*}(M\otimes N), \jmath_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right).$$

7.2.3. *Proof of Proposition 7.2.2.* As in the finite-dimensional case, it is enough to establish an isomorphism on the level of complexes, which consists of modules with a standard. We will show that if both M and N admit a standard flag, then both sides of the equation in the proposition are acyclic off degree 0 and are isomorphic to one another.

Indeed, since $(\Delta_{\kappa,\infty,\mu})^* \simeq \nabla_{\kappa,0,-w_0(\mu)}$, the acyclicity assertion is true for the LHS. Its 0-th cohomology for any M and N is given by

$$\mathrm{Hom}\left((M\otimes N^{w_0})_{\mathfrak{g}[t,t^{-1}]},k\right).$$

Hence, the full expression is given by

$$\mathrm{Hom}(H_\bullet(\mathfrak{g}[t,t^{-1}];T,M\otimes N^{w_0}),k), \tag{7.3}$$

since the vector space in (7.3) is acyclic off degree 0 if M and N admit standard flags, because in this case the tensor product $M\otimes N$ is free over both Lie sub-algebras

$$t\cdot\mathfrak{g}[\![t]\!]\oplus\mathfrak{n} \text{ and } t^{-1}\cdot\mathfrak{g}[\![t^{-1}]\!]\oplus\mathfrak{n}^-.$$

Remark. The vector space in (7.3) identifies, up to a twist by w_0 with the space of conformal blocks on $\mathbb{P}^1$ with coefficients in M and N. So, the above assertion is the well-known statement that the Hom into the contragredient module can be calculated as conformal blocks.

The RHS of Proposition 7.2.2, by adjunction, is given by

$$\mathrm{Hom}_{\mathfrak{D}^\kappa_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\text{-mod}}\left(\mathrm{Loc}^{\mathbf{I}_0,\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0,\infty},\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\widehat{0},\widehat{\infty}),*}(M\otimes N), \mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\right).$$

However,

$$\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)\simeq \mathrm{pt}/T,$$

and the stabilizer in $\mathfrak{g}((t)) \oplus \mathfrak{g}((t^{-1}))$ of the (unique) point in the corresponding T-torsor is $\mathfrak{g}[t, t^{-1}]$. Hence the RHS of Proposition 7.2.2 is also given by (7.3).

□

7.2.4. *Proposition 7.2.2 ⇒ Theorem 7.1.4.* By definition, we need to show that for $M \in (\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0})^c$ $N \in (\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty})^c$ we have a canonical isomrphism

$$(7.4) \quad \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}}\left(M, (\mathbb{D}_{\widehat{\mathfrak{g}}}(N))^*\right) \simeq$$
$$\simeq \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}}\left(M, \langle \Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1, \widehat{0}, \widehat{\infty}), \jmath_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1, \mathbf{I}_0, \mathbf{I}_\infty)})\right)_*, N\rangle_{\widehat{\mathfrak{g}}_\infty}\right).$$

By Proposition 7.2.2 , the LHS can be rewritten as

$$\mathrm{Hom}_{\mathfrak{D}^\kappa_{\mathrm{Bun}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\text{-mod}_*}\left(\mathrm{Loc}^{\mathbf{I}_0,\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),*}(M \otimes \mathbb{D}_{\widehat{\mathfrak{g}}_\infty}(N)), \jmath_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right),$$

and, by adjunction, further as

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa,0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty}}\left(M \otimes \mathbb{D}_{\widehat{\mathfrak{g}}_\infty}(N), \Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1, \widehat{0}, \widehat{\infty}), \jmath_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)_*\right).$$

The latter is, however, tautologically isomorphic to the RHS of (7.4).

□

8. Localization onto "thick" flags

8.1. **The "thick" affine flag scheme.**

8.1.1. By definition, the thick affine scheme (resp., enhanced thick affine scheme) is $\widetilde{\mathrm{Fl}}^{thick}_G := \mathrm{Bun}_G(\mathbb{P}^1, \overset{\circ}{\mathbf{I}}_0, \widehat{\infty})$ (resp., $\mathrm{Fl}^{thick}_G := \mathrm{Bun}_G(\mathbb{P}^1, \mathbf{I}_0, \widehat{\infty})$). We will be concerned with the categories of equivariant D-modules on it, i.e., one of the categories

$$\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!,\ \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{T,w},\ \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{T,w,\check{\lambda}},$$

defined as in Sect 3.4.

Viewing $\widetilde{\mathrm{Fl}}^{thick}_G$ as endowed with an action of $G(\widehat{\mathcal{K}}_\infty)$, following the recipe of Sect 4.1, we have a functor

$$\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!} : \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod} \to \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{T,w}.$$

Lemma 8.1.2. *Under the equivalence:*

$$\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{T,w} \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty})}\text{-mod}_!^{\overset{\circ}{\mathbf{I}}_0},$$

the functor $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G}$ *identifies with the composition*

$$\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod} \to \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0} \otimes \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod} \overset{\mathrm{Loc}^{\overset{\circ}{\mathbf{I}}_0,T,w}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),!}}{\longrightarrow} \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty})}\text{-mod}_!^{\overset{\circ}{\mathbf{I}}_0,T,w},$$

where the first arrow is the functor $M \mapsto \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa,0}}_{\mathrm{Lie}(\overset{\circ}{\mathbf{I}}_0)}(k) \otimes M$.

8.1.3. We shall consider a particular case of the above set up, restricting ourselves to the $\mathbf{I}_\infty$-equivariant situation. Namely, we will consider the category

$$\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty}.$$

Proposition 8.1.4. *Every object* $\mathcal{F} \in \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty}$ *canonically splits as a direct sum* $\mathcal{F}^0 \oplus \mathcal{F}^{\neq 0}$*, where*

$$\mathcal{F}^0 \in \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty},$$

and

$$\mathcal{F}^{\neq 0} \in \underset{\lambda\neq 0}{\oplus} \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,\lambda),\mathbf{I}_\infty}.$$

Proof. By the definition of $\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty}$, we have to prove the corresponding assertion over each T-stable open sub-stack of finite type

$$U \subset (\widehat{\mathrm{Fl}}^{thick}_G)/\mathbf{I}_\infty \simeq \mathrm{Bun}_G(\mathbb{P}^1, \overset{\circ}{\mathbf{I}}_0, \mathbf{I}_\infty).$$

However, each such sub-stack is a finite union of locally-closed sub-stacks $\mathcal{Y}$, each of the form $\mathrm{pt}/\mathfrak{U}$ for some unipotent group $\mathfrak{U}$. It is sufficient to prove the assertion for each of the categories $\mathfrak{D}^{\kappa'}_{\mathcal{Y}}\text{-mod}^{T,w}$. However, the category $\mathfrak{D}^{\kappa'}_{\mathcal{Y}}$-mod is equivalent to Vect, with the action of T being trivial as a weak action, and the strong action deviating the standard one by a character. Hence, the assertion of the proposition amounts to splitting an object of $\mathrm{Rep}(T)$ into isotypic components.

□

8.2. **The Kashiwara-Tanisaki equivalence.**

8.2.1. Let κ be negative, and hence κ' be positive. Consider the functor

$$\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!} : \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty} \to \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty}$$

equal to the composition of

$$\mathrm{Loc}^{\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!} : \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty} \to \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty},$$

followed by the functor $\mathcal{F} \mapsto \mathcal{F}^0$, given by Proposition 8.1.4.

Proposition 8.2.2. *The above functor* $\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!}$ *sends*

$$(\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty})^c \to (\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty})^c.$$

Proof. By Proposition 8.1.4, for $M \in (\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty})^c$ the restriction of $\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!}(M)$ to every T-stable open sub-stack $U \subset (\widehat{\mathrm{Fl}}^{thick}_G)/\mathbf{I}_\infty$ of finite type, belongs to $(\mathfrak{D}^{\kappa'}_U\text{-mod}_!^{(T,w),\mathbf{I}_\infty})^c$.

We have to show that there exists a large enough open sub-stack of finite type U_M such that

$$\mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty}}\left(\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widetilde{\mathrm{Fl}}^{thick}_G,!}(M), \mathcal{F}\right) = 0$$

for any

$$\mathcal{F} \in \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w),\mathbf{I}_\infty}$$

supported off U. With restriction of generality, we can take $\mathcal{F}$ to be T-equivariant, i.e., an object of

$$\mathfrak{D}^{\kappa'}_{\mathrm{Fl}^{thick}_G}\text{-mod}^{\mathbf{I}_\infty}_! \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}.$$

In this case we have:

$$\mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w),\mathbf{I}_\infty}_!}\left(\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widehat{\mathrm{Fl}}^{thick}_G,!}(M),\mathcal{F}\right)\simeq$$
$$\simeq \mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\text{-mod}_!}\left(\mathrm{Loc}^{\mathbf{I}_0,\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),!}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)\otimes M),\mathcal{F}\right).$$

Since $\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)\otimes M \in (\widehat{\mathfrak{g}}_{\kappa',0,\infty}\text{-mod}^{\mathbf{I}_0,\mathbf{I}_\infty})^c$, our assertion follows from Theorem 6.1.7. □

8.2.3. Consider now the functor

$$\Gamma^{T,w}(\widehat{\mathrm{Fl}}^{thick}_G,-)_!:\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w,0),\mathbf{I}_\infty}_! \to \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty},$$

equal to the composition

$$\Gamma^{T,w}(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),-)_!:\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w,0),\mathbf{I}_\infty}_!\simeq$$
$$\simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty})}\text{-mod}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_\infty}_! \to \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}\otimes\widehat{\mathfrak{g}}^{\mathbf{I}_\infty}_{\kappa',\infty},$$

(see Corollary 6.1.10) followed by the functor

$$\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}\otimes\widehat{\mathfrak{g}}^{\mathbf{I}_\infty}_{\kappa',\infty}\to\widehat{\mathfrak{g}}^{\mathbf{I}_\infty}_{\kappa',\infty}$$

that sends $M\mapsto M^{\overset{\circ}{\mathbf{I}}_0,T}$.

Proposition 8.2.4. *The functors* $\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widehat{\mathrm{Fl}}^{thick}_G,!}$ *and* $\Gamma^{T,w}(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),-)_!$ *are mutually adjoint.*

Proof. By definition, we have to establish a functorial isomorphism for $M\in\in(\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty})^c$ and $\mathcal{F}\in\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w),\mathbf{I}_\infty}_!$

$$(8.1)\quad \mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w),\mathbf{I}_\infty}_!}\left(\mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',\infty},\widehat{\mathrm{Fl}}^{thick}_G,!}(M),\mathcal{F}\right)\simeq$$
$$\simeq\mathrm{Hom}_{\widehat{\mathfrak{g}}^{\mathbf{I}_\infty}_{\kappa',\infty}}\left(M,\Gamma^{T,w}(\widehat{\mathrm{Fl}}^{thick}_G,\mathcal{F})_!\right).$$

The LHS in (8.1) is by definition

$$(8.2)\quad \mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty)}\text{-mod}^{T,w}_!}\left(\mathrm{Loc}^{(\overset{\circ}{\mathbf{I}}_0,T,w),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),!}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)\otimes M),\mathcal{F}\right),$$

which by Proposition 8.2.2 can be rewritten as

$$(8.3)\quad \underset{\overrightarrow{j}}{colim}\ \mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty)}\text{-mod}^{T,w,0}_!}\left(\mathrm{Loc}^{(\overset{\circ}{\mathbf{I}}_0,T,w),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),!}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)\otimes M)/\mathfrak{t}^j,\mathcal{F}\right),$$

where $\mathfrak{t}^j$ denotes the j-th power of the maximal ideal in $\mathrm{Sym}(\mathfrak{t})$.

Consider the modules $\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)/\mathfrak{t}^j \in \widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}$ for $j = 0, 1, ...$ For any $N \in \widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}$ we have:

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w}}\left(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k), N\right) \simeq \underset{\overrightarrow{j}}{colim}\ \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w}}\left(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)/\mathfrak{t}^j, N\right).$$

Hence, the RHS in (8.1) can be rewritten as

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}^{(\overset{\circ}{\mathbf{I}}_0,T,w)\mathbf{I}_\infty}_{\kappa',0,\infty}}\left(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)\otimes M, \Gamma^{T,w}(\widehat{\mathrm{Fl}}^{thick}_G, \mathcal{F})_!\right) \simeq$$

$$\simeq \underset{\overrightarrow{j}}{colim}\ \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty}\text{-mod}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\overset{\circ}{\mathbf{I}}_\infty}}\left(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)/\mathfrak{t}^j\otimes M, \Gamma^{T,w}(\widehat{\mathrm{Fl}}^{thick}_G, \mathcal{F})_!\right),$$

which by adjunction can be further rewritten as

$$\underset{\overrightarrow{j}}{colim}\ \mathrm{Hom}_{\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty)}\text{-mod}^{T,w,0}_!}\left(\mathrm{Loc}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),!}(\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\mathbf{I}_0)}(k)/\mathfrak{t}^j\otimes M), \mathcal{F}\right), \tag{8.4}$$

which is the same as the expression in (8.3).

□

8.2.5. The localization theorem at the positive level, due to Kashiwara-Tanisaki reads as follows:

Theorem 8.2.6. *The functor* $\Gamma^{T,w}(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),-)_!$ *realizes* $\mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w,0),\mathbf{I}_\infty}_!$ *as a direct summand of* $\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}$.

We will show how deduce theorem from another localization result (also due to Kashiwara and Tanisaki), but which takes place at the negative level.

9. Arkhipov's functor and localization

9.1. **From thin to thick flags.**

9.1.1. Let Fl^{thin}_G (resp., $\widetilde{\mathrm{Fl}}^{thin}_G$) be the "thin" affine flag scheme (resp., the "thin" enhanced affine flag scheme), attached to point $0 \in \mathbb{P}^1$, i.e.,

$$\mathrm{Fl}^{thin}_G := G(\widehat{\mathcal{K}}_0)/\mathbf{I}_0,\ G(\widehat{\mathcal{K}}_0)/\overset{\circ}{\mathbf{I}}_0.$$

Note that unlike the corresponding "thick" spaces, the "thin" ones are local, i.e., only depend on the formal neighborhood of $0 \in \mathbb{P}^1$.

For a level κ we can consider the corresponding categories of twisted D-modules

$$\mathfrak{D}^{\kappa}_{\mathrm{Fl}^{thin}_G}\text{-mod and } \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}.$$

The ind-schemes Fl^{thin}_G and $\widetilde{\mathrm{Fl}}^{thin}_G$ are acted on by $G(\widehat{\mathcal{K}}_0)$, and the above categories carry a strong action of this group ind-scheme at level κ.

9.1.2. The rest of this sub-section is devoted to the proof of the following result:

Theorem-Construction 9.1.3. *There exists a canonical equivalence*

$$\Phi^{thin\to thick} : \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} \simeq \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w,0),\mathbf{I}_\infty}_!$$

9.1.4. To construct the functor $\Phi^{think\to thick}$ we recall that the involution $g \mapsto g^{-1}$ defines an equivalence

$$\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Fl}^{thin}_G}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}.$$

The action of $G(\widehat{\mathcal{K}}_0)$ on the category $\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}_!$ defines the convolution functor

$$\begin{aligned}(9.1)\quad \mathfrak{D}^{\kappa'}_{\mathrm{Fl}^{thin}_G}\text{-mod}^{\mathbf{I}_0,T,w,0} \otimes \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}^{\mathbf{I}_0}_! \to \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}_! \simeq \\ \simeq \mathfrak{D}^{\kappa'}_{\widehat{\mathrm{Fl}}^{thick}_G}\text{-mod}^{(T,w,0),\mathbf{I}_\infty}_!.\end{aligned}$$

The sought-for functor $\Phi^{thin\to thick}$ is given by (9.1) by inserting the object

$$\jmath_!(\mathcal{O}_{\mathrm{Bun}^{ss,gen}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}) \in \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)}\text{-mod}_! \simeq \mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}^{\mathbf{I}_0}_!.$$

9.1.5. Note that the functor $\Phi^{thin\to thick}$ admits also the following interpretation. We claim that there exists a canonical map

$$\widetilde{\mathrm{Fl}}^{thin}_G \overset{\mathfrak{r}}{\to} \widetilde{\mathrm{Fl}}^{thick}_G,$$

compatible with the twistings. Here are two (of course, equivalent) ways to view it:

By definiton, $\widetilde{\mathrm{Fl}}^{thin}_G$ classifies the data $(\mathcal{P}_G,\beta,\alpha_0)$, where $\mathcal{P}_G$ is a principal G-bundle on $\mathbb{P}^1$, β is its trivialization on $\mathbb{P}^1-0$, and α is a reduction of $(\mathcal{P}_G)_0$ to N.

By definition, $\widetilde{\mathrm{Fl}}^{thick}_G$ classifies the data $(\mathcal{P}_G,\gamma,\alpha_0)$, where $\mathcal{P}_G$ is a principal G-bundle on $\mathbb{P}^1$, β is its trivialization on the formal neighborhood $\mathcal{D}_\infty$ of $\infty\in\mathbb{P}^1$, and α is a reduction of $(\mathcal{P}_G)_0$ to N.

The map $\mathfrak{r}$ acts identically on the data of $(\mathcal{P}_G,\alpha_0)$, and attaches to a datum of β its restriction to $\mathcal{D}_\infty\subset\mathbb{P}^1-0$, twisted by w_0.

The group-theoretic interpretation (although non-rigorous) is as follows:

$$\widetilde{\mathrm{Fl}}^{thin}_G \simeq G(\widehat{\mathcal{K}}_0)/\overset{\circ}{\mathbf{I}}_0 \overset{\mathrm{id}\times w_0}{\hookrightarrow} G[t,t^{-1}]\backslash\left(G(\widehat{\mathcal{K}}_0)/\overset{\circ}{\mathbf{I}}_0 \times G(\widehat{\mathcal{K}}_\infty)\right) \simeq G'_{out\,\infty}\backslash G(\widehat{\mathcal{K}}_\infty) \simeq \widetilde{\mathrm{Fl}}^{thick}_G,$$

where $G_{out\,\infty}=G[t]$, and $G'_{out\,\infty}\subset G_{out\,\infty}$ is the sub-group of maps whose value at 0 belongs to N.

Let p denote the canonical projection

$$\widetilde{\mathrm{Fl}}^{thick}_G \to \widetilde{\mathrm{Fl}}^{thick}_G/\mathbf{I}_\infty \simeq \mathrm{Bun}_G(\mathbb{P}^1,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty).$$

Let q denote the composition

$$\mathsf{p}\circ\mathfrak{r}:\widetilde{\mathrm{Fl}}^{thin}_G \to \mathrm{Bun}_G(\mathbb{P}^1,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty).$$

Since Fl^{thin}_G is ind-proper, it is easy to see that the functor $\Phi^{thin\to thick}$ is canonically isomorphic to $\mathsf{q}_!$.

9.1.6. Let us show that $\Phi^{thin\to thick}$ is an equivalence.

First, it is easy to see that $\Phi^{thin\to thick}$ sends

$$(\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)})^c \to (\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty})^c.$$

Secondly, from the affine Bruhat decomposition, it is easy to see that the compact generators of $(\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty}$ are in the essential image of $(\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)})^c$.

Hence, it remains to show that $\Phi^{thin\to thick}$ is fully faithful on $(\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)})^c$. For that end, it is enough to show that the corresponding functor

$$\Phi^{thin\to thick} : \mathfrak{D}^{\kappa}_{\mathrm{Fl}^{thin}_G}\text{-mod}^{\mathbf{I}_0} \to \mathfrak{D}^{\kappa'}_{\mathrm{Fl}^{thick}_G}\text{-mod}_!^{\mathbf{I}_\infty}$$

is fully faithful.

For $\mathcal{F}_1, \mathcal{F}_2 \in (\mathfrak{D}^{\kappa}_{\mathrm{Fl}^{thin}_G}\text{-mod}^{\mathbf{I}_0})^c$, by [FG1], we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\Phi^{thin\to thick}(\mathcal{F}_1), \Phi^{thin\to thick}(\mathcal{F}_2)) \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathrm{Hom}(\delta_{1_{\mathrm{Fl}^{thin}_G}}, \mathcal{F}_1^* \star \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\Phi^{thin\to thick}(\delta_{1_{\mathrm{Fl}^{thin}_G}}), \Phi^{thin\to thick}(\mathcal{F}_1^* \star \mathcal{F}_2)), \end{array}$$

so we can assume that $\mathcal{F}_1 = \delta_{1_{\mathrm{Fl}^{thin}_G}}$. In the latter case, the required assertion reduces to the statement that for $\mathcal{F} \in \mathfrak{D}^{\kappa}_{\mathrm{Fl}^{thin}_G}\text{-mod}^{\mathbf{I}_0}$, the map

$$H_{\delta_{1_{\mathrm{Fl}^{thin}_G}}}(\mathrm{Fl}^{thin}_G, \mathcal{F}) \to H_c(\mathrm{Fl}^{thin,gen}_G, \mathcal{F}) \tag{9.2}$$

is an isomorphism, where $\mathrm{Fl}^{thin,gen}_G \subset \mathrm{Fl}^{thin}_G$ is the big affine Bruhat cell. The fact that (9.2) is another instance of Sect 5.1.13.

9.2. **The basic commutative diagram.**

9.2.1. Assume again that κ is negative. Recall the functor

$$\Gamma^{T,w} : \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{(T,w,0)} \to \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}.$$

It admits a right adjoint, denoted $\mathrm{Loc}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G}$.

The following theorem is due to Kashiwara-Tanisaki (see, e.g., [FG3] for a proof):

Theorem 9.2.2. *The functor*

$$\Gamma^{T,w} : \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} \to \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}$$

realizes the former category as a direct summand of the latter.

The right adjoint to the functor in Theorem 9.2.2 is denoted $\mathrm{Loc}^{\mathbf{I}_0,(T,w,0)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G}$. It is isomorphic to the composition of

$$\mathrm{Loc}^{\mathbf{I}_0}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G} : \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} \to \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0},$$

followed by the functor

$$\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0} \to \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)}$$

defined as the projection on the corresponding direct summand, in complete analogy with Sect 8.2.1.

9.2.3. The goal of this section is to prove the following theorem:

Theorem 9.2.4. *We have a commutative diagram of functors:*

$$\begin{array}{ccc} \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} & \xrightarrow{\Phi^{thin\to thick}} & \mathfrak{D}^{\kappa'}_{\widehat{\widetilde{\mathrm{Fl}}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \\ {\scriptstyle \mathrm{Loc}^{\mathbf{I}_0,(T,w,0)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thick}_G,!}} \\ \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xrightarrow{\Phi^{0\to\infty}_{\kappa\to\kappa'}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}. \end{array}$$

Of course, as the horizontal arrows appearing in the theorem are equivalences, its statement is equivalent to the following one:

Theorem 9.2.5. *We have a commutative diagram of functors:*

$$\begin{array}{ccc} \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} & \xleftarrow{\Psi^{thick\to thin}} & \mathfrak{D}^{\kappa'}_{\widehat{\widetilde{\mathrm{Fl}}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \\ {\scriptstyle \mathrm{Loc}^{\mathbf{I}_0,(T,w,0)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thick}_G,!}} \\ \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xleftarrow{\Psi^{\infty\to 0}_{\kappa'\to\kappa}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}, \end{array}$$

where $\Psi^{thick\to thin}$ is the inverse functor to $\Phi^{thin\to thick}$.

9.2.6. By Theorem 9.2.2, the functor $\mathrm{Loc}^{\mathbf{I}_0,(T,w,0)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}^{thin}_G}$ is also the left adjoint to

$$\Gamma^{T,w}:\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)}\to\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0}.$$

Combining this with Theorems 9.2.4 and 9.2.5, we obtain that the following:

Corollary 9.2.7. *We have the following commutative diagrams of functors:*

$$\begin{array}{ccc} \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} & \xrightarrow{\Phi^{thin\to thick}} & \mathfrak{D}^{\kappa'}_{\widehat{\widetilde{\mathrm{Fl}}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \\ {\scriptstyle \Gamma^{T,w}(\widetilde{\mathrm{Fl}}^{thin}_G,-)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma^{T,w}(\widetilde{\mathrm{Fl}}^{thick}_G,-)} \\ \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xrightarrow{\Phi^{0\to\infty}_{\kappa\to\kappa'}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}. \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} & \xleftarrow{\Psi^{thick\to thin}} & \mathfrak{D}^{\kappa'}_{\widehat{\widetilde{\mathrm{Fl}}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \\ {\scriptstyle \Gamma^{T,w}(\widetilde{\mathrm{Fl}}^{thin}_G,-)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma^{T,w}(\widetilde{\mathrm{Fl}}^{thick}_G,-)} \\ \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xleftarrow{\Psi^{\infty\to 0}_{\kappa'\to\kappa}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}. \end{array}$$

The above corollary theorem should be regarded as an affine analog of Corollary 1.4.6.

Remark. Note that the first commutative diagram of Corollary 9.2.7 follows tautologically from Conjecture 7.1.5. In its turn, the commutativity of this diagram would imply Theorems 9.2.4 and 9.2.5 by Theorem 9.2.2.

9.2.8. *Proof of Theorem 8.2.6.* This follows as a combination of Corollary 9.2.7 and Theorem 9.2.2.

9.3. **Proof of Theorem 9.2.5.**

9.3.1. Let us describe the functor $\Psi^{thick\to thin}$ explicitly. As $\Phi^{thin\to thick}$ is an equivalence, $\Psi^{thick\to thin}$ equals its right adjoint.

Here is a geometric description of $\Psi^{thick\to thin}$ in terms of Sect 9.1.5. It equals the composition of

$$\mathsf{q}^! : \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}^{thick}_G}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \to \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{T,w,0},$$

followed by the averaging functor

$$\mathrm{Av}_{\mathbf{I}_0} : \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{T,w,0} \to \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)},$$

right adjoint to the forgetful functor

$$\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{\mathbf{I}_0,(T,w,0)} \to \mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}^{thin}_G}\text{-mod}^{T,w,0}.$$

However, for the purpose of proof of Theorem 9.2.5, we shall need a description of the functor $\Psi^{thick\to thin}$ in terms of the action of $G(\widehat{\mathcal{K}}_0)$ on the category $\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}_!$ as in Sect 9.1.4.

9.3.2. Consider the following general paradigm. Let $\mathcal{C}$ be a category acted on by $G(\widehat{\mathcal{K}}_0)$ at level κ'. For two open-compact subgroups $K', K'' \subset G(\widehat{\mathcal{K}}_0)$ we have the functors

$$\mathrm{Conv}^{K',K''} : \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)}\text{-mod}^{K',K''} \otimes \mathcal{C}^{K''} \to \mathcal{C}^{K'}$$

and

$$\text{co-Conv}^{K',K''} : \mathcal{C}^{K'} \to \mathcal{C}^{K''} \otimes \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)}\text{-mod}^{K',K''} \simeq \mathcal{C}^{K''} \otimes \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}.$$

Let $\mathcal{C}^\vee$ denote the category of all (co-limit preserving) functors $\mathcal{C} \to \mathrm{Vect}$, whether or not $\mathcal{C}$ is dualizable. For $\mathcal{F}^\vee \in (\mathcal{C}^{K''})^\vee$ consider the functor

$$\text{co-Conv}^{K',K''}_{\mathcal{F}^\vee} : \mathcal{C}^{K'} \to \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}$$

equal to the composition of $\text{co-Conv}^{K',K''}$, followed by the functor

$$\mathcal{C}^{K''} \otimes \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'} \overset{\mathcal{F}^\vee(-)\otimes \mathrm{Id}}{\longrightarrow} \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}.$$

In what follows we will need the following properties of these functors. First, let $A : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between categories acted on by $G(\widehat{\mathcal{K}}_0)$, and let $A^\vee : \mathcal{C}_2^\vee \to \mathcal{C}_1^\vee$ be its dual.

Lemma 9.3.3. *For $\mathcal{F}^\vee \in (\mathcal{C}_2^{K''})^\vee$ the composition*

$$\mathcal{C}_1^{K''} \overset{A}{\to} \mathcal{C}_2^{K''} \overset{\text{co-Conv}^{K',K''}_{\mathcal{F}^\vee}}{\longrightarrow} \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}$$

is canonically isomorphic to

$$\text{co-Conv}^{K',K''}_{A^\vee(\mathcal{F}^\vee)} : \mathcal{C}_1^{K''} \to \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}.$$

Secondly, let us assume that $G(\widehat{\mathcal{K}}_0)/K''$ and ind-proper, and that $\mathcal{F}$ is a compact object of $\mathcal{C}^{K''}$. Let $\mathcal{F}^\vee$ be the corresponding object of $\mathcal{C}^\vee$, i.e., $\mathrm{Hom}(\mathcal{F}, -)$.

Lemma 9.3.4. *Under the above circumstances, the functor $\mathrm{Conv}^{K',K''}(-,\mathcal{F})$ sends compact objects to compact ones, and its right adjoint is given by $\text{co-Conv}^{K',K''}_{\mathcal{F}^\vee}$.*

9.3.5. Returning to our situation, we obtain that the functor $\Psi^{thick\to thin}$ is given by

$$\text{co-Conv}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_0}_{j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})},$$

which maps

$$\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}_G^{thick}}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty}\simeq\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\overset{\circ}{\mathbf{I}}_0,\mathbf{I}_\infty)}\text{-mod}_!^{T,w,0}\simeq\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}_!^{\overset{\circ}{\mathbf{I}}_0,T,w,0}\to$$
$$\to\mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/\mathbf{I}_0}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}\simeq\mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\overset{\circ}{\mathbf{I}}_0,T,w,0}\simeq\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\mathbf{I}_0,(T,w,0)}.$$

By Theorem 7.1.4, the assertion of Theorem 9.2.5 reduces to the commutativity of the following diagram of functors:

$$\begin{array}{ccc}
\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\mathbf{I}_0,(T,w,0)} & \xleftarrow{\text{co-Conv}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_0}_{j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})}} & \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}_G^{thick}}\text{-mod}_!^{(T,w,0),\mathbf{I}_\infty} \\
\uparrow \mathrm{Loc}^{\mathbf{I}_0,(T,w,0)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}_G^{thin}} & & \mathrm{Loc}^{(T,w,0),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}_G^{thick},!}\uparrow \\
\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xleftarrow{\langle\Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)_*,-\rangle_{\widehat{\mathfrak{g}}_\infty}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}.
\end{array}$$

However, it is easy to see that it is sufficient to check the commutativity of the following diagram instead:

$$\begin{array}{ccc}
\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\mathbf{I}_0,(T,w)} & \xleftarrow{\text{co-Conv}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_0}_{j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})}} & \mathfrak{D}^{\kappa'}_{\widetilde{\mathrm{Fl}}_G^{thick}}\text{-mod}_!^{(T,w),\mathbf{I}_\infty} \\
\uparrow \mathrm{Loc}^{\mathbf{I}_0,(T,w)}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}_G^{thin}} & & \mathrm{Loc}^{(T,w),\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa,0},\widetilde{\mathrm{Fl}}_G^{thick},!}\uparrow \\
\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{\mathbf{I}_0} & \xleftarrow{\langle\Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)_*,-\rangle_{\widehat{\mathfrak{g}}_\infty}} & \widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}.
\end{array} \tag{9.3}$$

9.3.6. Note that the counter-clockwise circuit in (9.3) is the composition

$$\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}\overset{\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\overset{\circ}{\mathbf{I}}_0)}(k)\otimes-}{\longrightarrow}\widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\mathbf{I}_0}\otimes\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}\overset{\mathrm{Loc}^{\mathbf{I}_0,\mathbf{I}_\infty}_{\widehat{\mathfrak{g}}_{\kappa',0,\infty},\mathrm{Bun}_G(X,\widehat{0},\widehat{\infty}),!}}{\longrightarrow}$$
$$\longrightarrow\mathfrak{D}^{\kappa'}_{\mathrm{Bun}_G(X,\widehat{0},\mathbf{I}_\infty)}\text{-mod}_!^{\overset{\circ}{\mathbf{I}}_0,T,w}\overset{\text{co-Conv}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_0}_{j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})}}{\longrightarrow}\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\mathbf{I}_0,(T,w)}.$$

By Lemma 9.3.3, this can be rewritten as

$$\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}\overset{\mathrm{Ind}^{\widehat{\mathfrak{g}}_{\kappa',0}}_{\mathrm{Lie}(\overset{\circ}{\mathbf{I}}_0)}(k)\otimes-}{\longrightarrow}\widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{\mathbf{I}_0}\otimes\widehat{\mathfrak{g}}_{\kappa',\infty}\text{-mod}^{\mathbf{I}_\infty}\longrightarrow$$
$$\overset{\text{co-Conv}^{(\overset{\circ}{\mathbf{I}}_0,T,w,0),\mathbf{I}_0}_{\Gamma\left(\mathrm{Bun}_G(\mathbb{P}^1,\widehat{0},\widehat{\infty}),j_*(\mathcal{O}_{\mathrm{Bun}_G^{ss,gen}(\mathbb{P}^1,\mathbf{I}_0,\mathbf{I}_\infty)})\right)}}{\longrightarrow}\mathfrak{D}^{\kappa}_{\widetilde{\mathrm{Fl}}_G^{thin}}\text{-mod}^{\mathbf{I}_0,(T,w)}.$$

9.3.7. The commutativity of (9.3) follows now from the following observation. Let us return to the paradigm of Sect 9.3.2. Let $\mathcal{C} := \widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}$. Let $K', K'' \subset G(\widehat{\mathcal{K}}_0)$ be open-compact subgroups (in our situation, we take $K' = \mathbf{I}_0$, $K'' = \overset{\circ}{\mathbf{I}}_0$).

Consider the functor

$$\widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{K''} \simeq \left(\widehat{\mathfrak{g}}_{\kappa',0}\text{-mod}^{K''}\right)^{\vee} \to \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}$$

given by

$$M \mapsto \text{co-Conv}_M^{K',K''}(\text{Ind}_{\mathfrak{k}''}^{\widehat{\mathfrak{g}}_{\kappa',0}}(k)).$$

Lemma 9.3.8. *The above functor is canonically isomorphic to the localization functor*

$$\text{Loc}^{K''}_{\widehat{\mathfrak{g}}_{\kappa,0},G(\widehat{\mathcal{K}}_0)/K'} : \widehat{\mathfrak{g}}_{\kappa,0}\text{-mod}^{K''} \to \mathfrak{D}^{\kappa}_{G(\widehat{\mathcal{K}}_0)/K'}\text{-mod}^{K''} \simeq \mathfrak{D}^{\kappa'}_{G(\widehat{\mathcal{K}}_0)/K''}\text{-mod}^{K'}.$$

10. Appendix: DG categories

10.1.

10.1.1.

Lemma 10.1.2.

10.2.

Lemma 10.2.1.

10.2.2.

10.2.3.

10.2.4.

10.3.

10.4.

10.4.1.

Lemma 10.4.2.

10.4.3.

10.4.4.

10.4.5.

10.5.

10.5.1.

10.5.2.

10.5.3.

10.5.4.

10.5.5.

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