Points of low canonical height on elliptic curves and surfaces

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Background; previous work.

Fix a global field, say $\mathbf{Q}$. Let P is a rational point on an elliptic curve E/K. Assume P is a non-torsion point; equivalently, that its canonical height $\hat{h}(P)$ is nonzero, hence positive. How small can it be?

Likewise for $K = \mathbf{C}(t)$, or more generally $K = \mathbf{C}(C)$ for some algebraic curve C over $\mathbf{C}$. Here $\hat{h}(P) \in \mathbf{Q}$, and again we ask: if positive, how small can it be?

[Warning: $P \in E_{\text{tors}} \Rightarrow \hat{h}(P) = 0$, but not $\Leftrightarrow$; the counterexamples are: E a constant curve $C \times E$, and P is a constant section.]

What counts as “small”?...

Conjecture (Lang) Over $\mathbf{Q}$, $\hat{h}(P) \gg \log|\Delta_E|$. For K arbitrary, $\hat{h}(P) \geq C_K \log|\mathrm{N}_{K/\mathbf{Q}}\Delta_{E/K}|$. Over $\mathbf{C}(t)$ [or $\mathbf{C}(C)$ for fixed C], change $\log|\Delta|$ to the discriminant degree.

Theorem (Hindry-Silverman 1988): Lang's conjecture is a consequence of Szpiro's conjecture (known to be $\Leftrightarrow$ the Masser-Oesterlé ABC conjecture). In particular, it is true over $\mathbf{C}(t)$ and other function fields. Moreover,

$$\hat{h}(P) \geq C_K \log|\mathrm{N}_{K/\mathbf{Q}}\Delta_{E/K}| - D_K$$

for constants C_K, D_K depending effectively on the exponent and error terms in Szpiro; in particular, for $K = \mathbf{C}(t)$, Hindry and Silverman obtain this bound with $D_K = 0$ and an explicit C_K.

(In fact, $\log |\mathrm{N}_{K/\mathbf{Q}}\Delta_{E/K}|$ may be replaced by $12h_K(E)$, where h_K is the Faltings height and keeps track of cancellation in Δ.)

So: given K, there should be an absolute minimum for $\hat{h}(P)$; what is it — in particular, what's the smallest height of a nontorsion point on $E/\mathbf{C}(t)$ [a nontorsion section of an elliptic surface over $\mathbf{P}^1$], or on $E/\mathbf{Q}$?

Also: What is the correct value of C_K? Can it really be anywhere as small as the lower bound of $6 \cdot 10^{-11}$ obtained by Hindry and Silverman?

Numerical examples

In “Antwerp” (Tingley, $N \leq 200$): Curve 37A, $\widehat{h}(P) \approx .0511$; then 135A, $\widehat{h}(P) \approx .0295$. In Cremona ($N \leq 10^3$): 280B, $\widehat{h}(P) \approx .0113$, and several others with $\widehat{h}(P) < .02$ (350F, 735F, 430D, 975I). More recently, William Stein searched the extended Cremona tables (see the notes on Matt Baker’s talk at

modular.fas.harvard.edu/edu/mcs/archive/Fall2001/notes/12-10-01/

for the context), which currently go up to $1.2 \cdot 10^4$. Found 32 new cases of $\widehat{h}(P) < .02$, including four with $\widehat{h}(P) < .01$, at $N = 3990$ (giving $\widehat{h}(P) \approx .008914$, the current record), 3630, 1430, 1470, with no clear sign of stopping (e.g. there’s a curve with $N = 11730$ and $\widehat{h}(P) \approx .0154$). For $\widehat{h}(P)/\log|\Delta|$, current record is $1.69 \cdot 10^{-4}$, for a curve of conductor 3476880330 (NDE 2002).

NB for consistency with the $\mathbf{C}(t)$ data, we included a factor of 2 in the values of $\widehat{h}(P)$ for points on elliptic curves over $\mathbf{Q}$; so for instance .008914 is the regulator of the $N = 3990$ curve but twice what's usually called the canonical height of its generator.

Over $\mathbf{C}(t)$, we sort the curves by their discriminant degree $12n$. For $n = 1$ (rational elliptic surface), minimum is 1/30, attained by a one-parameter family (Oguiso and Shioda 1991, using Persson 1990 on possible fiber configurations). For $n = 2$ (K3 surfaces), minimum is 11/420, again attained by a one-parameter family (K.-i. Nishiyama 1996).

For $n = 3$, minimum is 23/840 (NDE 2002, again in one-parameter family; this time with explicit equations, also for the $n = 1, 2$ minima). Note that $11/420 < 23/840 < 1/30$. The minima for $n = 4$, $n = 5$ are probably 41/1540 and 261/10010, each attained by a single explicit curve. The last of these is the current record; note the numerical values:

n	1	2	3	4	5
$\hat{h}(P)$	.03333	.02619	.02738	.02662	.02607

[Even more explicitly: represent all our (E,P) as $(Y^2 + aXY + bY = X^3 + cX^2, (0,0))$; then the record curves have the following (a,b,c) parameters:

$(253, 5320, 1197000)$, for lowest $\widehat{h}(P)$ over $\mathbf{Q}$;

$(15193, -948689280, 13637408400000)$,
for least $\widehat{h}(P)/\log|\Delta|$ over $\mathbf{Q}$; and

$$(t^5 - 4t^3 - t^2 + 4t + 1,\ (t - t^2)(t^2 - t - 1)(t^3 - t - 1),$$
$$(t - t^2)(t^2 - t - 1)^2(t^3 - t - 1)(t^3 + t^2 - 2t - 1)),$$

for least $\widehat{h}(P)$ over $\mathbf{C}(t)$.

Fellow explicit-equation junkies who need yet more can get it at

www.math.harvard.edu/~elkies/low_height.html .]

A better bound on $C := C_{\mathbf{C}(t)}$.

[We concentrate on the function-field case; similar results should be available over number fields after some work at the Archimedean places — but will necessarily hinge on the ABC/Szpiro conjecture, which is still inaccessible in that setting.]

Hindry and Silverman note (p.437) that their lower bound on C "is quite small, but any significant improvement would require new ideas." Using two new ideas, we improved C from $6 \cdot 10^{-11}$ to about $1/25330$ (the exact value is $C_1 := 39086299807/990051106318560$):

Theorem (NDE 2002): *The canonical height of a non-torsion section of a nonconstant elliptic surface of discriminant degree* $12n$ *over* $\mathbf{P}^1(\mathbf{C})$ *is at least* $12C_1 n > n/2111$.

Again if $\mathbf{P}^1(\mathbf{C})$ is replaced by a curve of genus g we prove $\hat{h}(P) > 12C_1 n - D_1(g-1)$.

We also obtain a conjecture for the correct value, namely $C_0 := 3071/10810800 \approx 1/3520$.

Conjecture (NDE 2002): C_0 *is* $\inf(\hat{h}(P)/12n)$ *over nontorsion sections* P *of elliptic surfaces over* $\mathbf{P}^1(\mathbf{C})$*, and* $\liminf(\hat{h}(P)/12h_K(E))$ *over points* P *of positive height on elliptic curves* E *over a given number or function field* K *of characteristic zero.*

We next review the Hindry-Silverman argument and indicate where we improved it and how we surmised the above conjecture.

Ingredients of Hindry-Silverman.

- $\widehat{h}$ is quadratic: $\widehat{h}(mP) = m^2\widehat{h}(P)$.

- $\widehat{h}$ is a sum of local terms: $\widehat{h}(P) = \sum_v \widehat{h}_v(P)$; and $\widehat{h}_v(P) = 0$ unless
  i) $P \equiv 0 \bmod v$, or
  ii) v is a place of singular reduction.

- $\widehat{h}_v(P) \geq 0$ unless v is singular and P does not go through the identity component of the v fiber.

- If the v fiber is multiplicative of degree ν (a "I_ν fiber") then $\widehat{h}_v(mP) \geq \nu B(ma/\nu)$ where $B(x) := \langle x\rangle^2 - \langle x\rangle + 1/6$ (2nd Bernoulli fn.).

- [ABC] There are $\gg n$ singular fibers — in fact, at least $12n/\sigma - O(1)$, with $\sigma = 6$ (σ for Szpiro).

Moreover, if v has discriminant degree d and mP passes through the identity component at v then $\hat{h}_v(mP) \geq d/6$.

So: first replace P by $12P$, which always goes through the identity component of each additive fiber. This incurs a factor of 12^2 in the lower bound.

Then use the Fourier expansion of B and positivity of the Fejér kernel(!) to show that $\sum_{|m|<M}(M-|m|)B(mx) \geq 0$ for all $x \in \mathbf{R}$, and $\geq M^2/\nu^2$ if $x \in \nu^{-1}\mathbf{Z}$.

Let $x = 12a/\nu$ and sum over singular v to get

$$2\sum_{m=1}^{M}(M-m)\hat{h}(12mP) \geq \sum_{v}\left(\frac{M^2}{\nu} - \nu M\right)$$

($\nu^{-1}M^2 - \nu M$ becomes $M^2 - dM$ when the fiber at v is additive).

But by ABC, the sum over v of $1/\nu$ or 1 is $\gg \sum_v d = 12n$, so for large enough M we get $\hat{h}(P) \gg n$ as promised.

Unfortunately M must be so large — the optimal value is about $(12\sqrt{2})\sigma$, or 100 for $\sigma = 6$ — that the resulting lower bound on C is tiny.

Fortunately, we can do better!

- No 12^2 penalty

- Better $\sum B$ bounds

First, to reclaim that factor of 12^2. The estimate on $\sum_{m=1}^{M}(M-m)\widehat{h}_v(12mP)$ automatically holds for $\sum_{m=1}^{M}(M-m)\widehat{h}_v(mP)$ too, even if v is additive, since any component of an additive fiber is equivalent (for $\widehat{h}_v$ and d purposes) to a combination of multiplicative ones as follows:

replace ...	... by ...
identity	d [0/1]
III	[1/2] + [0/1]
IV	[1/3] + [0/1]
IV^*	2[1/3] + 2[0/1]
III^*	3[1/2] + 3[0/1]
I_ν^* (near)	2[1/2] + $(\nu+2)$[0/1]
$\mathrm{I}_{2\mu}^*$ (far)	$(\mu+2)$[1/2] + 2[0/1]
$\mathrm{I}_{2\mu+1}^*$ (far)	[1/4] + $(\mu+1)$[1/2] + [0/1]

(Here $[a/\nu]$ means component a of a I_ν fiber.) Replacement also increases the fiber count, so cannot spoil the ABC inequality.

Second, we can greatly improve on

$$2\sum_{m=1}^{M}(M-m)\widehat{h}(mP) \geq \sum_{v}\left(\frac{M^2}{\nu} - \nu M\right).$$

The inequality we got from this must be inefficient beause asymptotic equality is impossible: almost all terms would have to be $[\pm 1/6]$, but then $\widehat{h}(3P)$ would be negative!

As in other contexts (points on curves over $\mathbf{F}_q$, spherical or discrete codes, etc.), we are led to a **linear programming** problem.

Let c_m, s, H be nonnegative reals such that

$$\sum_{m=1}^{\infty} c_m B(ma/\nu) + s(1 - \frac{\sigma}{\nu}) \geq H \qquad (*)$$

holds for all (a, ν). Multiply by ν and sum over the bad fibers to get

$$\left(\sum_{m=1}^{\infty} m^2 c_m\right) \hat{h}(P) \geq \sum_v \nu \left[\sum_m c_m B(ma/\nu)\right]$$

$$\geq (\sum_x \nu H) + s \sum_v (\sigma - \nu) \geq (12n)H - O(1),$$

and hence

$$\hat{h}(P) \geq \frac{H}{\sum_m m^2 c_m} 12n - O(1)$$

(again the $O(1)$ disappears for $\mathbf{C}(t)$ or an elliptic function field). So, we want to minimize $\sum_m m^2 c(m)$ subject to the conditions (*) and $H = 1$.

We took $\sigma = 6$, set up all the inequalities (*) for $m, \nu \leq 60$, sent the resulting linear system on a C program by Michel Berkelaar, and then verified the results with exact arithmetic on GP.

Result: the best c_m are supported on $m = 1, \ldots, 18, 20, 21, 22, 24, 28, 30$; equality in (*) holds for $x = a/\nu$ with $\nu \geq 8$ or $x = 1/9$, 2/9, 1/11, 2/11, 3/11, 4/11, 1/13, 3/13, 4/13, 6/13, 1/17, 8/17, 1/23, and their images under the symmetry $x \leftrightarrow 1 - x$.

The resulting $H/(\sum_m m^2 c_m)$ is the new lower bound C_1.

[Explicitly: after clearing denominators, we find $H = 273604098649$, $s = 1620022780846$, and

$$c_m = 1309312950831, 2440728940677, 3164970945729, 2996212783878, 3226831357536, 4050323179854, 2421617838588, 3726398816571, 3156942741837, 3352101048798, 1580761995396, 4044994266168, 796412096352, 2620265310432, 2447062707447, 1545221697888, 385200446373, 1870885387467, 2170280139681, 951160290063, 80415136878, 1230072419973, 508649495349, 165486023508$$

for $m = 1, \ldots, 18, 20, 21, 22, 24, 28, 30.$]

Could this new bound be asymptotically attained?

Probably not. Any configuration of I_ν fibers allowed by ABC can occur over $\mathbf{C}(t)$, but in general the elliptic curve E will have no small section P. A parameter count (which can't be quite right, but does hit ABC exactly) suggests that there must be at least $3n$ bad fibers before a given configuration of I_ν's and $[a/\nu]$'s can be realized. That is, we expect that the Szpiro ratio σ cannot exceed 4. Implementing this yields a new linear program, this time with a simpler solution (actually, as it happens, a line segment of solutions), that leads to our conjectural liminf of $C_0 = 3071/10810800$.

We may take $H = 6142$, $s = 29433$, and $c_1, \ldots, c_{10}, c_{12} =$ 43428, 55878, 65514, 78102, 63474, 88476, 37590, 42480, 25458, 32244, 34002, attaining equality in (*) for $x = a/\nu$ or $1-x$ in {0, 1/2, 1/3, 1/4, 1/5, 2/5, 1/7, 2/7, 1/8, 4/9, 1/11, 1/14}. [The other endpoint also uses $m = 15$ and attains equality also at 3/8.] To attain asymptotic equality, the $[a/\nu]$ components must occur in the proportions

48 : 24 : 33 : 18 : 15 : 17 : 10 : 7 : 6 : 8 : 4 : 5

(this for $n = 65 = 780/12$).

I found no further obstruction; for instance, the resulting configuration satisfies the condition that if $m|m'$ then the naïve height

$$h(mP) = \hat{h}(mP) - \sum_{v} \nu B(ma/\nu)$$

of mP cannot exceed that of $m'P$.

NB This condition does play a key role in computing the minimal $\hat{h}(P)$ for small n; also, points of zero naïve height are the same as integral points.

Apropos integral multiples: for which positive integers m can there be a point P of nonzero height on an elliptic curve over $\mathbf{P}^1(\mathbf{C})$ such that mP is integral? (Over $\mathbf{Q}$, the maximal m known is 31, for a curve of conductor 1830.)

By our Theorem, $m \leq (6C_1)^{-1/2} < 65$. With more linear programming we can show

$$m \in [1, 30] \cup [32, 36] \cup \{39, 40, 42\}.$$

If $\sigma = 4$ then we likewise improve on $m \leq (6C_0)^{-1/2} < 25$, finding $m \in [1, 12] \cup \{14, 15\}$. We have examples showing all $m \leq 12$ actually occur.

The 3071/10810800 conjecture seems plausible but out of reach. For instance, the conjecture that the liminf is at most 3071/10810800 is implied by the following more general conjecture:

Conjecture [NDE 2002]: Let $p_{a/\nu}$ and h be nonnegative numbers such that: $\sum \nu p_{a/\nu} = 1$; $\sum p_{a/\nu} \geq 1/4$ (that is, $\sigma \leq 4$); and the

$$h_m := m^2 h - \sum p_{a/\nu} \nu B(ma/nu)$$

are nonnegative and satisfy $h_m \leq h_{m'}$ whenever $m|m'$. Then there exist (E, P) over $\mathbf{C}(t)$ of large disc. degree d with $(p_{a/\nu} - o(1))d$ fibers of type $[a/\nu]$ and $\widehat{h}(P) = (h + o(1))d$.