

Math 55b: Honors Advanced Calculus and Linear Algebra

The Riemann-Stieltjes integral of a vector-valued function

At the end of Chapter 6 Rudin defines the integrals of functions from $[a, b]$ to \mathbf{R}^k one coordinate at a time. This approach is problematic in infinite-dimensional vector spaces... Fortunately it is not too hard to adapt our definitions and results concerning Riemann-Stieltjes integration to functions from $[a, b]$ to an arbitrary normed vector space, as long as that space is complete — a condition we must impose so that the limiting process implicit in \int will converge under reasonable hypothesis. We shall need to pay more attention to “Riemann sums”, which Rudin is able to relegate to a fraction of Thm. 6.7 when treating integrals of real-valued functions.

Let V , then, be a *complete* normed metric space. Fix an interval $[a, b] \subset \mathbf{R}$, an increasing function $\alpha : [a, b] \rightarrow \mathbf{R}$, and a bounded function $f : [a, b] \rightarrow V$. For each partition $P : a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, and any choice of $t_i \in [x_{i-1}, x_i]$, we call

$$R(P, \vec{t}) := \sum_{i=1}^n (\alpha(x_i) - \alpha(x_{i-1})) f(t_i)$$

a *Riemann sum* for $\int_a^b f(x) d\alpha(x)$. (We suppress f, α from the notation $R(P, \vec{t})$ because f, α are fixed for this discussion.) When $V = \mathbf{R}$ we can estimate all the Riemann sums above and below by $U(P)$ and $L(P)$, and try to make the difference between these upper and lower bounds arbitrarily small by choosing a sufficiently fine partition P . For an arbitrary V there is no “above” and “below”, and thus no $U(P)$ and $L(P)$; but we can still formulate a generalization of $U(P) - L(P)$ that bounds how much ambiguity the choice of \vec{t} entails. For any c, d with $a \leq c < d \leq b$, let

$$E(c, d) := \sup_{t, t' \in [c, d]} |f(t) - f(t')|;$$

note that the sup exists because f is bounded. Then, for any partition P define

$$\Delta(P) := \sum_{i=1}^n (\alpha(x_i) - \alpha(x_{i-1})) E(x_{i-1}, x_i).$$

Then

$$\left| R(P, \vec{t}') - R(P, \vec{t}) \right| \leq \Delta(P)$$

for any choices of $t_i, t'_i \in [x_{i-1}, x_i]$.

When $V = \mathbf{R}$ this $\Delta(P)$ coincides with $U(P) - L(P)$ (why?). For this to work in the general setting, we must verify that if P^* refines P then $\Delta(P^*) \leq \Delta(P)$. By induction we need only check that if $a \leq x < y < z \leq b$ then

$$(\alpha(y) - \alpha(x))E(x, y) + (\alpha(z) - \alpha(y))E(y, z) \leq (\alpha(z) - \alpha(x))E(x, z).$$

But this is clear from $E(x, y) \leq E(x, z)$ and $E(y, z) \leq E(x, z)$. Furthermore, we need the following: if P^* refines P , any Riemann sums for P^* and P differ by at most $\Delta(P)$. Again it is enough to prove this for a one-point refinement of $[x, z]$ to $[x, y] \cup [y, z]$, when we claim

$$\begin{aligned} & |(\alpha(y) - \alpha(x))f(t_1) + (\alpha(z) - \alpha(y))f(t_2) - (\alpha(z) - \alpha(x))f(t)| \\ & \leq (\alpha(z) - \alpha(x))E(x, z). \end{aligned}$$

To prove this, we write the LHS as

$$|(\alpha(y) - \alpha(x))(f(t_1) - f(t)) + (\alpha(z) - \alpha(y))(f(t_2) - f(t))|$$

and use $|f(t_i) - f(t)| \leq E(x, z)$.

We can now recast Thm. 6.6 as a definition for Riemann-Stieltjes integrability of f : we say that f is *integrable* with respect to $d\alpha$ if for each ϵ there is a partition P such that $\Delta(P) < \epsilon$. [By Thm. 6.6, this is equivalent to Rudin's definition in the case $V = \mathbf{R}$.] Theorems 6.8 through 6.10 then generalize immediately to sufficient conditions for integrability of vector-valued functions.

But you'll notice that we have managed to define integrability without defining the integral! The definition as follows: if f is integrable with respect to $d\alpha$, its *integral* $\int_a^b f(x) d\alpha(x)$ is the unique $I \in V$ such that $|I - R(P, \vec{t})| \leq \Delta(P)$ for all Riemann sums $R(P, \vec{t})$. Of course this requires proof of existence and uniqueness. Uniqueness is easy: if two distinct I, I' worked, we'd get a contradiction by choosing P such that $\Delta(P) < \frac{1}{2}|I' - I|$. We next prove existence. First we construct I . Let $\epsilon_m \rightarrow 0$ and choose P_m such that $\Delta(P_m) < \epsilon_m$. Without loss of generality we may assume P_m refines $P_{m'}$ for each $m' < m$, by replacing P_m by a common refinement of P_1, \dots, P_m (this cannot increase $\Delta(P_m)$). Choose for each m an arbitrary Riemann sum $R_m := R(P_m, \vec{t}(m))$. These constitute a Cauchy sequence in V : if $m' < m$ then $|R_{m'} - R_m| \leq \Delta(P_{m'}) < \epsilon_{m'}$. We now at last use the completeness of V to conclude that $\{R_m\}$ converges. Let I be its limit. This of course is essentially the way we obtain the integral of a real-valued function. We claim that I satisfies our requirements for an integral even in our vector-valued setting. Consider any Riemann sum $R = R(P, \vec{t})$. For each m , let R_m^* be a Riemann sum for a common refinement of P_m and P . Then $|R_m^* - R| \leq \Delta(P)$ and $|R_m^* - R_m| \leq \Delta(P_m) < \epsilon_m$. Thus $|R_m - R| < \Delta(P) + \epsilon_m$. Letting $m \rightarrow \infty$ we obtain $|I - R| \leq \Delta(P)$ as desired, Q.E.D.