Math 55b: Honors Advanced Calculus and Linear Algebra

Practice Problems (May 3, 2000)

> [This argument] will only carry conviction to those who believe that the infinite sum of small things is always small (except when it is large).[1]

Fourier/Hilbert stuff:

1. (Baire's theorem) Prove that if a complete metric space X is the countable union $\cup_{n=1}^{\infty} S_n$ with each $S_n \subset X$ closed then at least one S_n contains a nonempty open set. [This involves the same argument we used to show that a weakly convergent sequence of operators is bounded.]

2. Give a proof using convolutions of Parseval for functions on $\mathbf{T}$ and/or $\mathbf{Z}/N\mathbf{Z}$.

3. (An instance of Poisson summation) Fix $c > 0$ and define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 1/(x^2 + c^2)$. For $t \in \mathbf{R}$ define $F(t) := \sum_{n \in \mathbf{Z}} f(t + 2\pi n)$. Prove that F is a differentiable function of period 2π, and thus can be regarded as a differentiable function on $\mathbf{T}$. Determine its Fourier series (NB we know $\hat{f}$), and deduce the value of $F(0) = \sum_{n \in \mathbf{Z}} f(2\pi n)$. [Your answer should agree with your result for Problem 5 of the tenth assignment.] Generalize.[2]

The next batch of practice problems develops a proof of the memorable theorem of Müntz which answers the following question: Fix an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of nonnegative real numbers and let V be the $\mathbf{R}$-vector space of functions generated by t^{n_i}, $i = 0, 1, 2, \ldots$ For what $\{n_i\}$ is V dense (A) in $L_2([0,1])$, (B) in $\mathcal{C}[0,1]$? Müntz's Theorem asserts:

A) V is dense in $L_2([0,1])$ iff $\sum_{i=1}^{\infty} 1/n_i$ diverges.
B) V is dense in $\mathcal{C}[0,1]$ iff $n_0 = 0$ and $\sum_{i=1}^{\infty} 1/n_i$ diverges.

Note that the $L_2([0,1])$ and $\mathcal{C}[0,1]$ versions of the Weierstrass Approximation Theorem are the special case $n_i = i$ of Müntz; we assume the Weierstrass theorem in the following proof.

4. For any vectors $x_1, x_2, \ldots, x_m$ in a real Hilbert space $\mathcal{H}$, let $\Delta_m(x_1, \ldots, x_m)$ be the determinant of the $m \times m$ matrix whose ijth entry is the inner

[1] Körner, *Fourier Analysis*, p.296 (in Chapter 60). By now I hope none of our 55b class believes this; keep in mind for the take-home final that I do not believe it either!

[2] This generalizes in many directions; e.g., if $f : \mathbf{Z}/12\mathbf{Z} \to \mathbf{C}$, what is $f(0)+f(3)+f(6)+f(9)$ in terms os $\hat{f}$?

product of x_i with x_j. Recall that if the x_i are linearly independent then this matrix is positive definite, so in particular $\Delta_m(x_1, \ldots, x_m)$ is positive. In this case let V_m be the m-dimensional subspace spanned by the x_i, and show that for any vector $y \in \mathcal{H}$ the distance from y to the nearest point of V_m (i.e. the norm of the projection of y to the orthogonal complement $V_m^\perp$) is the square root of the ratio

$$\Delta_{m+1}(x_1, \ldots, x_m, y) / \Delta_m(x_1, \ldots, x_m).$$

[Note that this problem only uses the finite-dimensional space generated by y and the x_i's; the full Hilbert space $\mathcal{H}$ is only needed for what follows.]

5. Taking $\mathcal{H} = L_2([0,1])$, $x_{i+1} = t^{n_i}$ and $y = t^k$ in problem 7 we find determinants Δ_m, Δ_{m+1} of the form $\det(1/(a_i + b_j))_{i,j=1}^M$. Prove that, for any real numbers $a_1, \ldots, a_M; b_1, \ldots, b_M$ such that none of the $a_i + b_j$ vanishes, the value of this determinant is

$$D_M(a_1, \ldots, a_M) D_M(b_1, \ldots, b_M) \Big/ \prod_{i=1}^M \prod_{j=1}^M (a_i + b_j)$$

where $D_M(r_1, \ldots, r_M) = \prod_{1 \le i < j \le M} (r_i - r_j)$. Use this to compute the L_2 distance from x^k to the space V_m spanned by x^{n_i}, $0 \le i < m$. (Why are these m vectors linearly independent?)

6. Conclude that, provided k is not one of the n_i, the $L_2([0,1])$ closure of $V = \cup_{m=1}^\infty V_m$ contains x^k if and only if $\sum_{i=1}^\infty 1/n_i$ diverges. Use this to deduce part A of Müntz's Theorem.

7. The "only if" half of part B is now easily accessible: prove that if $n_0 > 0$ or $\sum_{i=1}^\infty 1/n_i < \infty$ then $\mathcal{C}[0,1]$ contains functions not in the closure of V. To get the reverse implication we need one more trick: for any $f \in \mathrm{L}_2([0,1])$ define $\int f : [0,1] \to \mathbf{R}$ by $\int f(x) = \int_0^x f(t)\,dt$, i.e. the inner product of f with the characteristic function of $[0,x]$. As part of Problem 6 of the ninth assignment, we showed in effect that $\int$ is a continuous linear map of norm ≤ 1 from $L_2([0,1])$ to $\mathcal{C}[0,1]$. Use this map to finish the proof of Müntz's Theorem.

Some miscellaneous calculus problems follow; there are plenty more in Rudin to choose from:

8. Prove for $r \in \mathbf{Q}$ and $0 < a < b$ that $\int_a^b x^r\,dx = (b^{r+1} - a^{r+1})/(r+1)$ by writing the integral as a limit of Riemann sums. [Use a trick of Fermat: partition $[a,b]$ using a geometric sequence rather than the usual arithmetic one.]

9. (Euler's original evaluation of $\zeta(2)$, etc.) Recall from an earlier problem set the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left[1 - \left(\frac{x}{n\pi} \right)^2 \right]$$

for x in some neighborhood of 0. Taking logarithms of both sides yields

$$\log \frac{\sin x}{x} = \sum_{n=1}^{\infty} \log \left[1 - \left(\frac{x}{n\pi} \right)^2 \right],$$

Now carefully expand the logarithms on the right-hand side in power series about $x = 0$, and compare the leading (x^2) coefficient with that of

$$\log(\sin x/x) = \log(1 - x^2/6 + x^4/120 - + \cdots)$$

to recover Euler's identity. What do the further coefficients of the Taylor expansion tell you?

10. Let $\langle \cdot, \cdot \rangle$ be a (positive-definite) inner product on $\mathbf{R}^n$. Show that the integral of $\exp(-\langle x, x \rangle)$ over $x \in \mathbf{R}^n$ converges absolutely, and calculate this integral in terms of n and the determinant of the positive-definite $n \times n$ matrix whose (i, j) entry is $\langle e_i, e_j \rangle$.

11. Suppose f is a harmonic function on the neighborhood of a closed ball $\bar{B}_r(x)$ in $\mathbf{R}^n$. Prove that $f(x)$ is the average of $f(y)$ over $y \in \bar{B}_r(x)$, and also the average of $f(y)$ over the sphere $|y - x| = r$. Prove that conversely if f is a C^2 function from $\bar{B}_r(x)$ to $\mathbf{R}$ whose average over $B_s(x)$ equals $f(x)$ for each positive $s < r$ then $\Delta f(x) = 0$. (This generalizes the familiar characterization $f(x + r) + f(x - r) = 2f(x)$ for affine-linear functions of one variable $f : [x - r, x + r] \to \mathbf{R}$. An important application is that a gravitational or electrostatic field cannot have points of stable equilibrium!)