Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #9 (7 April 2000):
Life in Hilbert space, and a bit on Fourier and summability methods

Beware the Ides of April! — *the Infernal Revenue Service*[1]

1. (More about separability) Let X be a metric space, and S any subspace. Prove that if X is separable than so is S. Prove that the following are equivalent:
   i) S is separable;
   ii) For each $\epsilon > 0$ there is a countable set of ϵ-neighborhoods in X whose union contains S;
   iii) For each $\epsilon > 0$ there is a countable set of ϵ-neighborhoods in S whose union contains S.

2. i) Show directly (i.e. without Weierstrass approximation or Fourier analysis) that $\mathcal{C}([0,1])$ is separable. (As noted in class, the separability of $L_2([0,1])$ follows.)
   ii) The space $L_2(\mathbf{R})$ is defined as the completion of the space of continuous functions on $\mathbf{R}$ with compact support, relative to the usual inner product $(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$. [Since f,g have comapct support, this is actually an integral over a bounded interval, so it makes sense.] Prove that $L_2(\mathbf{R})$ is separable, and thus isometric with l_2. Can you find an explicit ontb for $L_2(\mathbf{R})$?

3. (Compact subsets of l_2):
   i) Let $r_1, r_2, r_3, \ldots$ be a sequence of positive real numbers. Prove that the "box"
   $$\{x \in l_2 : |x_1| \le r_1, |x_2| \le r_2, \ldots\}$$
   in l_2 is compact if and only if it is bounded, i.e. iff $\sum_{i=1}^{\infty} r_i^2 < \infty$.
   ii) Prove that the "ellipsoid" $\{x \in l_2 : \sum_{i=1}^{\infty} |x_i/r_i|^2 \le 1\}$ is compact iff $r_i \to 0$ as $i \to \infty$.

A linear map $L : V \to W$ between complete normed spaces V, W is said to be **compact** if the closure in W of the image of the closed unit ball $\bar{N}_1(0)$ in V is a compact subset of W. Clearly a compact map must be bounded (why?). Thus the last part of Problem 3 shows that the linear map $l_2 \to l_2$ defined by $(x_1, x_2, x_3, \ldots) \mapsto (r_1x_1, r_2x_2, r_3x_3, \ldots)$ is compact iff $r_i \to 0$.[2] Note that this linear map is self-adjoint, and the unit vectors constitute an orthonormal topological basis of eigenvectors. As was true in the finite dimensional case, there is a **spectral theorem** for compact self-adjoint linear transformations L of an infinite-dimensional separable Hilbert space $\mathcal{H}$, which states that any such L has an ontb of eigenvectors and thus is equivalent to the linear map above under a suitable identification of $\mathcal{H}$ with l_2 (i.e. using those eigenvectors as the unit vectors of l_2). The proof is outlined in the next two problems, under the slightly stronger hypothesis that $L(\bar{N}_1(0))$ [rather than $\overline{L(\bar{N}_1(0))}$] is compact. You may assume that $\mathcal{H}$ is a real Hilbert space; the complex case is essentially the same, but with a few extra wrinkles.

[1] Actually the Ides of April, and of every other month except March, May, July and October, falls on the 13th, not the 15th of the month, to say nothing of the 17th. Too bad.

[2] Here $\{r_i\}$ is any bounded sequence of scalars, which need not be positive reals.

4. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and $L : \mathcal{H} \to \mathcal{H}$ a self-adjoint linear map such that $L(\bar{N}_1(0))$ is compact. Since $K = \{L(x) : x \in \mathcal{H}, |x| \leq 1\}$ is compact, it contains some vector $L(x_1)$ of maximal norm. Prove that (unless L is identically zero, in which case we're done already) $|x_1| = 1$ and x_1 is an eigenvector of L^2. Show also that if x is any vector of $\mathcal{H}$ orthogonal to x_1 then $L^2(x)$ is also orthogonal to x_1, i.e. that L^2 restricts to a linear map from the orthogonal complement of $\langle x_1 \rangle$ to itself.

5. Note that this restriction is itself compact and self-adjoint. Using the result of the previous problem, obtain orthonormal eigenvectors $x_2, x_3, \ldots$ of L^2, and use these to prove the spectral theorem for compact self-adjoint linear operators L on $\mathcal{H}$.

6. Show that for each $x \in [0,1]$ the functional $f \mapsto \int_0^x f(t)\,dt$ on $\mathcal{C}([0,1])$ extends continuously to a functional ϕ_x on $L_2([0,1])$. What is the norm of ϕ_x? Show that for any $f \in L_2([0,1])$ the map $Tf : x \mapsto \phi_{1-x}(f)$ is continuous. Prove that $f \mapsto Tf$ is a continuous linear transformation from $L_2([0,1])$ to $\mathcal{C}([0,1])$, and thus to $L_2([0,1])$. Prove further that T, considered as a linear operator on $L_2([0,1])$, is self-adjoint and compact. Find an ontb of eigenvectors of T.

Finally, a few simple observations on Fourier coefficients, and more about "summability methods".

7. i) If $g : \mathbf{T} \to \mathbf{C}$ is integrable then so is its complex conjugate $\bar{g}$; determine the Fourier coefficients of $\bar{g}$ in terms of those of g, and use this to show that if g is actually real-valued then its n-th coefficient $\hat{g}_n$ is the complex conjugate of $\hat{g}_{-n}$. Find a similar result for even or odd functions g [i.e. functions $g : \mathbf{T} \to \mathbf{C}$ satisfying the identity $g(-t) = g(t)$ or $g(-t) = -g(t)$].
   ii) Prove conversely that if $\overline{\hat{g}_n} = \hat{g}_{-n}$ for all $n \in \mathbf{Z}$ then $g(t) \in \mathbf{R}$ for all $t \in \mathbf{T}$ at which g is continuous. Likewise obtain a converse for the results of (i) for even and odd functions.

8. Let $f : \mathbf{T} \to \mathbf{C}$ be an integrable function, and define for $t \in \mathbf{T}$ and $|z| < 1$

$$\phi_t(z) := \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} z^{|n|}.$$

Prove that if f is continuous at $t \in \mathbf{T}$ then $\phi_t(z)$ approaches $f(t)$ as $z \to 1$, and that if f is continuous at all $t \in \mathbf{T}$ then the limit $\lim_{z\to 1} \phi_t(z) = f(t)$ is uniform in t. [Mimic Fejér's proof, evaluating $\sum_n z^{|n|} e^{ins}$ as a sum of two geometric series.]

9. A series $\sum_{n=-\infty}^{\infty} a_n$ is said to be *Abel summable* if $\phi(z) := \sum_n a_n z^{|n|}$ converges for all $z \in [0,1)$ and $\lim_{z\to 1} \phi(z)$ exists, in which case that limit is called the *Abel sum* of the series. (Thus Problem 8 showed that the Fourier series of f is Abel summable to f wherever f is continuous.)
   i) Prove that if some series $\sum_{n=-\infty}^{\infty} a_n$ is Cesàro summable then it is also Abel summable and the Cesàro and Abel sums are equal. [Of course this result together with Fejér's theorem gives another proof of Problem 8.]
   ii) Find real numbers a_n such that the series $\sum_{n=-\infty}^{\infty} a_n$ is Abel summable but not Cesàro summable. [Thus Abel summability is a strictly weaker condition than Cesàro summability.]

This problem set is due Friday, April 14 in class.