Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #8 (3 April 2000):
Exterior algebra, differential forms, chains, and integration

\[ so_3 \mathbb{C} \cong so_3 \mathbb{C} \otimes so_3 \mathbb{C} \]
— again see Fulton and Harris, *Representation Theory: a First Course* (Springer, 1991), and elsewhere. This is where problem 5 ultimately leads to: a 4-dimensional inner product space \( V \) is almost equivalent to a pair of 3-dimensional ones \( W_\perp \). This fact from the theory of Lie groups is related to the quaternions, the theory of orientable 4-dimensional manifolds, etc. But for now, back to Chapter 10...

Chains and boundaries in convex sets:

1. Prove that every closed affine 1-chain in a *convex set* \( E \subseteq \mathbb{R}^n \) is the boundary of some affine 2-chain in \( E \).

2. Let \( E \subseteq \mathbb{R}^n \) be a convex set, and \( \gamma : [a,b] \to E \) any \( C^m \) curve. Construct a \( C^m \)-2-chain in \( E \) whose boundary is the difference between \( \gamma \) and the affine 1-simplex \( [\gamma(a), \gamma(b)] \). [Hint: rather than working directly with 2-simplices it will be easier to use a 2-cell and then apply Exercise 17.] Conclude from this and the previous problem that *every closed \( C^m \) 1-chain in a convex set is a boundary of a \( C^m \) 2-chain in the same convex set.*

Since we have shown that \( \partial^2 = 0 \), this means that a 1-chain in a convex set is closed if and only if it is a boundary. The same is true for \( k \)-chains; the proof uses the same basic ideas, but requires rather more bookkeeping. For both your and Travis’ sake I’ll leave the details to a future course in algebraic or differential topology.

The next two problems from Rudin construct and investigate closed but not exact \((n-1)\)-forms on \( \mathbb{R}^n \setminus \{0\} \), generalizing our form \( d\theta \) on the punctured plane. These are a key ingredient in the proof of the Brouwer fixed-point theorem and related results (e.g., the “ham sandwich theorem” and its generalization to \( \mathbb{R}^n \)), at least for sufficiently differentiable functions.


Finally, a sorbet of exterior algebra:

5. i) Let \( V \) be a finite-dimensional real inner product space. Prove that there is a unique inner product on \( \wedge^d V \) such that \( \langle v_1 \wedge \cdots \wedge v_d, (v'_1 \wedge \cdots \wedge v'_d) \rangle = \det((v_i, v'_j))_{i,j=1}^d \) for any \( v_1, \ldots, v_d, v'_1, \ldots, v'_d \in V \).

ii) Now let \( V \) have dimension 4 and \( W = \wedge^2 V \). Fix a generator \( \delta \) of
\wedge^4 V \) such that \( \langle \delta, \delta \rangle = 1 \). (We may take \( \delta = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \) where \( e_1, \ldots, e_4 \) is an orthonormal basis for \( V \).) We then have a bilinear pairing \( \langle \cdot, \cdot \rangle : W \times W \to \mathbb{R} \) defined by \( w \wedge w' = (w, w') \delta \). In the last problem set we showed in effect that this pairing is nondegenerate, and thus identifies \( W \) with \( W^* \). But now that \( V \) has an inner product structure we have another such pairing, \( \langle \cdot, \cdot \rangle \), and thus another identification of \( W \) with its dual. Composing one of these two identifications with the other’s inverse yields a map \( \iota : W \to W \) characterized by \( \langle w, w' \rangle = \langle \iota w, w' \rangle \) for all \( w, w' \in W \). Prove that \( \iota \) is an involution each of whose eigenspaces \( W_\pm := \{ w \in W : \iota w = \pm w \} \) has dimension 3.

iii) For any \( v \in V \), show that \( \{ v \wedge v' | v \in V \} \) and \( \{ v_1 \wedge v_2 | v_i \in V, \langle v, v_i \rangle = 0 \} \) are isotropic subspaces of \( W \) of maximal dimension 3. Show that any 3-dimensional isotropic subspace is of one of these two forms, and that \( \iota \) takes maximal isotropics of one kind to the other.

iv) Recall that any linear transformation \( T \) of \( V \) induces a linear transformation \( \Lambda^2 T \) of \( W \) [by \( \langle \Lambda^2 T \rangle (v \wedge v') = (Tv) \wedge (Tv') \)]. Show that if \( T \) is orthogonal then \( \Lambda^2 T \) takes \( W_+ \) to either \( W_+ \) or \( W_- \). Which?

This abbreviated problem set is due Friday, April 7 in class.