Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #6 (10 March 2000):
Convexity and Euler's integrals

If you see a good move, don't play it: look for a better one!
Edward Lasker (1868–1941), mathematician and World Chess Champion.

In other words, If you see a complicated way to solve a problem, stop — before carrying it through to completion, check whether a simpler approach is available. In the long run this will save you time and reduce the probability of error.

First, some basic facts about convex functions. Recall that a subset E of a real vector space V is said to be *convex* if $x, y \in E \Rightarrow px + qy \in E$ for all $p, q \in [0,1]$ such that $p + q = 1$. If E is convex, an (upward) *convex function* on E is a function $f : E \to \mathbf{R}$ such that $f(px + qy) \le pf(x) + qf(y)$ for all $x, y \in E$, $p, q \in [0,1]$ with $p + q = 1$; equivalently, f is convex if $\{(x,t) \in V \oplus \mathbf{R} : t > f(x)\}$ is a convex subset of $V \oplus \mathbf{R}$.

1. i) Show that any convex function on a convex open set in $\mathbf{R}^k$ is continuous.
   ii) Let U be a convex open set in $\mathbf{R}^k$, and fix $B \in (0, \infty)$ and a compact subset $K \subset U$. Let $\mathcal{C}$ be the set of all convex functions $f : U \to [-B, B]$. Prove that the restriction of $\mathcal{C}$ to the space of continuous functions on K is equicontinuous.

2. i) Prove the well-known fact that a twice-differentiable function of a single variable is convex if and only if its second derivative is everywhere nonnegative.
   ii) State and prove a generalization of this result for functions of several variables. function

3. [Jensen's inequalities] Let f be a convex function on a convex set E in some real vector space.
   i) If $x_i \in E$, $p_i \ge 0$, and $\sum_{i=1}^n p_i = 1$, prove that $x := \sum_{i=1}^n p_i x_i$ is in E and $f(x) \le \sum_{i=1}^n p_i f(x_i)$. (This contains many classical inequalities as special cases; e.g. the inequality on the arithmetic and geometric means is obtained by taking $E = (0, \infty)$, $f(x) = -\log x$, and $p_i = 1/n$.)
   ii) If $\phi : [a,b] \to E$ is a continuous function and $\alpha : [a,b] \to \mathbf{R}$ is an increasing function such that $\alpha(b) - \alpha(a) = 1$, prove that $x := \int_a^b \phi(t)\, d\alpha(t)$ is in E and $f(x) \le \int_a^b f(\phi(t))\, d\alpha(t)$.

4. We observed that the logarithmic convexity of $\Gamma(x)$, or more generally of any function of the form $f(x) = \int (\alpha(t))^x \beta(t)\, dt$, can be interpreted as the nonnegativity of a certain 2×2 determinant. Generalize this to larger determinants. For instance, prove that for any positive reals $a_1, \ldots, a_n$

the determinant of the $n \times n$ matrix with entires $\Gamma(a_i + a_j)$ is nonnegative, as is the determinant with entries $(a_i + a_j)^{-k}$ for any $k > 0$. [Hint for this last part: what is $\int_0^\infty t^{x-1} e^{-ct}\, dt$?]

Next some problems concerning Euler's Beta and Gamma integrals:

5. The definite integral $\int_{-1}^{1} (1 - t^2)^{x-1}\, dt$ can be evaluated in terms of $B(\cdot, \cdot)$ in two ways: first, by a linear change of variable from $[-1, 1]$ to $[0, 1]$; second, by letting $t^2 = u$. Show that the resulting identity is consistent with formulas 96 (the basic relation between $B(\cdot, \cdot)$ and $\Gamma(\cdot)$) and 102 (the duplication formula).

6. Verify that the product formula 95 is consistent with the Gamma recursion $\Gamma(x + 1) = x\Gamma(x)$, and thus holds to all $x > 0$. Use this formula to obtain an alternative proof of the duplication formula. What is

$$\Gamma\left(\frac{x}{3}\right) \Gamma\left(\frac{x+1}{3}\right) \Gamma\left(\frac{x+2}{3}\right) ?$$

   Generalize.

7. In the third problem set we obtained a formula for $\int_0^{\pi/2} \cos^n x \cos(\lambda x)\, dx$ for any $\lambda \in \mathbf{R}$ and $n = 0, 1, 2, \ldots$. Guess a generalization of this formula that lets n take on suitable non-integral values. Prove your guess.

8. i) Assuming Stirling's formula for $n!$, deduce its validity for $\Gamma(x + 1)$ using the Bohr-Mollerup theorem.
   ii) Assuming that there is an asymptotic formula

$$\Gamma(x + 1) \sim (x/e)^x \sqrt{2\pi x} \left(1 + \sum_{i=1}^{\infty} A_i x^{-i}\right)$$

   (This means that the infinite sum may not converge, but for each n we have

$$\Gamma(x + 1) = (x/e)^x \sqrt{2\pi x} \left(1 + A_1 x^{-1} + A_2 x^{-2} + \cdot + A_n x^{-n} + O(x^{-1-n})\right)$$

   as $x \to \infty$), show that the coefficients A_i may be recursively determined from the recursion $\Gamma(x + 1) = \Gamma(x)$. Use this to compute A_1, A_2.

This problem set due Friday, March 17, at the beginning of class.