Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #5 (3 March 2000):
Integration in $\mathbf{R}^k$, and special functions

Similarly, *adv.*: At least one line of the proof of this case is the same as before.[1]

1.–2. Solve Problems 1 and 2 on page 288. Apropos #2, construct a bounded function $f : [0,1] \times [0,1] \to \mathbf{R}$ such that: for each x, the function $y \mapsto f(x,y)$ is Riemann integrable with $\int_0^1 f(x,y)\,dy = 0$ (from which it follows that $\int_0^1 (\int_0^1 f(x,y)\,dy)\,dx = 0$); but there exist y such that the function $y \mapsto f(x,y)$ is not Riemann integrable (whence $\int_0^1 (\int_0^1 f(x,y)\,dx)\,dy$ doesn't even make sense).

3–7. Solve Problems 9 through 13 on pages 290–291. Generalize #13 to the integral of $\prod_{i=1}^k x_i^{r_i}$ over the set of $(x_1, \ldots, x_k)$ with each $x_i \geq 0$ and $\sum_{i=1}^k x_i^{s_i} = 1$. The r_i, s_i can be any real numbers with $r_i > -1$ and $s_i > 0$. (The resulting formula is due to Dirichlet.) In particular, determine the volume of the unit ball in $\mathbf{R}^k$ as a function of k; check that your answer agrees with the known cases $k = 1, 2, 3$. Note what happens to this volume as $k \to \infty$!

8. [Calabi] Prove that

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

(Note that this is an improper integral (why?) so some care will be needed here.) Now let Δ be the triangle $\{(u,v) \in \mathbf{R}^2 | u,v > 0, u+v < \pi/2\}$. Prove that the map $T : \Delta \to \mathbf{R}^2$ defined by $T(u,v) = (\sin u/\cos v, \sin v/\cos u)$ is a $\mathcal{C}^1$ and $\mathcal{C}^1$-invertible map of Δ to the open unit square $0 < x, y < 1$, and compute its Jacobian determinant. Conclude that

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \iint_\Delta 1\,du\,dv = \frac{\pi^2}{8}$$

as desired.

[Can you generalize this to evaluate $\sum_{n=1}^{\infty} 1/n^4$, or show more generally that $\sum_{n=1}^{\infty} 1/n^s$ is a rational multiple of π^s for all positive even s? Can you prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - \cdots = \frac{\pi^3}{32}$$

using this method?]

This problem set due Friday, March 10, at the beginning of class.

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. Note that this does not define an equivalence relation.