

Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #3 (18 February 2000):
More univariate calculus, and Stone-Weierstrass

Fejér discovered his theorem¹ at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.²

Differential equations, cont'd:

1. Solve Problem 25 on pages 170–1. You should also read the statement of Problem 26 but you need not solve it. We'll give an alternative approach later using contraction mappings.

Some integration techniques. First we show how to integrate an arbitrary rational function:

2. [Partial fractions³] Let k be an algebraically closed field. Let $K = k(x)$, the field of rational functions in one variable x with coefficients in k . Show that the following elements of K constitute a basis for K as a vector space over k : x^n for $n = 0, 1, 2, 3, \dots$, and $1/(x-x_0)^n$ for $x_0 \in K$ and $n = 1, 2, 3, \dots$. (Linear independence is easy. To prove that the span is all of K , consider for any polynomial $Q \in k[x]$ the subspace $V_Q := \{P/Q : P \in k[x], \deg(P) < \deg(Q)\}$ of K , and compare its dimension with the number of basis vectors in V_Q .)
3. Prove that $\tan(x) := \sin(x)/\cos(x)$ is an increasing function on $(-\pi/2, \pi/2)$ mapping this interval bijectively to \mathbf{R} . Prove that the inverse map $\tan^{-1}(x)$ has derivative $1/(x^2 + 1)$. Use this to determine $\int_0^1 (x - x^2)^4 dx / (x^2 + 1)$. what does this tell you about π ?
4. Prove that the integral of any $f \in \mathbf{R}(x)$ is a rational function plus a linear combination of functions of the form $\log|x - x_0|$, $\log((x - x_0)^2 + c)$, and $\tan^{-1}(ax + b)$ ($x_0, a, b, c \in \mathbf{R}, c > 0$).

Next we derive some classical product formulas and integrals. Be careful about justifying all steps!

5. Prove that $\int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x dx$ for all $n \geq 2$. Deduce that

$$\int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \frac{6}{7} \dots \frac{n-1}{n}, & \text{if } n \text{ is odd;} \\ \frac{\pi}{2} \frac{1}{2} \frac{3}{4} \frac{5}{6} \dots \frac{n-1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

¹On Fourier series; see Rudin, pages 199–200 for a sneak preview.

²Körner, *Fourier Analysis*, p.294 (conclusion of Chapter 59: “Weierstrass's proof of Weierstrass's theorem”).

³The decomposition of any $f \in K$ as a linear combination of the basis elements described in this problem is called the “partial fraction decomposition” of f .

6. It follows that

$$\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \cdot \frac{\int_0^{\pi/2} \cos^{2m} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx}.$$

Show that

$$1 < \frac{\int_0^{\pi/2} \cos^{2m} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx} < \frac{\int_0^{\pi/2} \cos^{2m-1} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx} = 1 + \frac{1}{2m},$$

and therefore

$$\frac{\pi}{2} = \lim_{m \rightarrow \infty} \left(\frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \right).$$

[This is usually written as the “infinite product”

$$\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \cdots,$$

attributed to Wallis.]

7. Use the formulas of the previous problem to prove that

$$\lim_{n \rightarrow \infty} \int_0^{\sqrt{n}\pi/2} \cos^n \frac{x}{\sqrt{n}} \, dx = \sqrt{\pi/2}.$$

Now show that $\lim_{n \rightarrow \infty} \cos^n(x/\sqrt{n}) = \exp(-x^2/2)$ for any $x \geq 0$, and use this to prove that⁴

$$\int_0^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi/2}.$$

8. Define $I_n(\lambda)$ for $0 < \lambda < 1$ by

$$I_n(\lambda) = \int_0^{\pi/2} \cos^n x \cos(\lambda x) \, dx \quad (n = 0, 1, 2, \dots).$$

Integrate by parts twice to prove that $(n^2 - \lambda^2)I_n(\lambda) = (n^2 - n)I_{n-2}(\lambda)$ for $n \geq 2$. Then evaluate $I_0(\lambda)$ and $I_1(\lambda)$ to obtain a formula for $I_n(\lambda)$ for all n . Deduce a product formula for $\tan(\pi\lambda/2)$, and verify that Wallis’ product can be recovered from your formula by taking the limit as $\lambda \rightarrow 0$. Can you obtain any further formulas by investigating the behavior of $I_n(\lambda)$ as $n \rightarrow \infty$?

Finally, some (Stone-)Weierstrass stuff:

9. i) Suppose $f : [a, b] \rightarrow \mathbf{R}$ is a continuous function such that $\int_a^b f(x)x^n \, dx = 0$ for each $n = 0, 1, 2, 3, \dots$. Prove that f is the zero function. [This is problem 20 on page 169; it also appeared — without the hint provided there — on a Putnam exam many years ago.]
- ii) Suppose $\alpha, \beta : [0, 1] \rightarrow \mathbf{R}$ are increasing functions such that there exists n_0 with $\int_0^1 x^n d\alpha(x) = \int_0^1 x^n d\beta(x)$ for each integer $n \geq n_0$. Prove that $\alpha_+ - \beta_+$ and $\alpha_- - \beta_-$ are constant functions on $[0, 1)$ and $(0, 1]$ respectively, where $\alpha_{\pm}(x) := \lim_{t \rightarrow x_{\pm}} \alpha(t)$ and β_{\pm} is defined in the same way.
- iii) Solve Problem 21 on page 169.

This problem set due Friday, 25 February, at the beginning of class.

⁴As noted in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral $\int \exp(cx^2) \, dx$ cannot be simplified. We shall give another proof of this result when we come to the change of variable formula for multiple integrals.