

## Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #11 (21 April 2000):

Fourier variations: Orthogonal polynomials and discrete Fourier analysis

The correct transliteration of Tchebychev's Russian name is a matter of some controversy. Phillip Davis has written a charming book in which this forms the central theme. Without accepting Davis's preferred spelling I do agree with him that only admirers of Čaykovskiy's music are entitled to write Čebysev.<sup>1</sup>

If  $P_n$  are orthogonal polynomials for the inner product  $(f, g) = \int_a^b f(x)g(x)d\alpha(x)$ , and  $\sum_{n=0}^{\infty} a_n P_n$  is the orthogonal expansion of some function  $f$  [so  $a_n = (f, P_n)/(P_n, P_n)$ ] then the  $m$ -th partial sum is  $\int_a^b f(y)K_m(x, y)d\alpha(y)$  where

$$K_m(x, y) = \sum_{n=0}^m \frac{P_n(x)P_n(y)}{(P_n, P_n)}.$$

Remarkably, for any  $\alpha$  we have a formula for  $K_m$  thanks to the three-term recursion ("Theorem 40.9"):

1. [Darboux-Christoffel formula] Find constants  $\kappa_n$  such that

$$K_n(x, y) = \kappa_n \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}$$

provided  $x \neq y$ . Check that your formula works for the Tchebychev polynomials  $T_n$  using the explicit formula  $T_n(\cos \theta) = \cos n\theta$ .

The families of orthogonal polynomials for which an explicit description is known include the *Gegenbauer polynomials*, which are orthogonal with respect to the inner product

$$(f, g)_c := \int_{-1}^1 f(x)g(x)(1-x^2)^c dx$$

(where  $c$  is a real parameter greater than  $-1$ ). Thus they generalize the polynomials of Tchebychev ( $c = -1/2$ ) and Legendre ( $c = 0$ ). There are various formulas and approaches for obtaining the Gegenbauer polynomials. We give here one that works particularly cleanly for the case  $c = 1$ , and is also relevant to this year's Putnam problem B-2:

2. Let  $\mathcal{P} = \mathbf{R}[x]$  be the set of polynomials in  $x$  with real coefficients considered as a real vector space, and let  $A_1 : \mathcal{P} \rightarrow \mathcal{P}$  be the linear operator defined by

$$(A_1 P)(x) = \frac{d^2}{dx^2}[(x^2 - 1)P(x)].$$

Prove that  $A_1$  is self-adjoint with respect to  $(\cdot, \cdot)_1$ . Find, for each  $n = 0, 1, 2, \dots$ , a real  $\lambda_n$  such that some polynomial  $u_n$  of degree  $n$  is a  $\lambda_n$  eigenvalue of  $A_1$ . Show that the  $\lambda_n$  for different  $n$  are distinct. Conclude that  $u_n$  are orthogonal polynomials with respect to  $(\cdot, \cdot)_1$ . Explain the relevance of this to Putnam 1999:B2.<sup>2</sup>

<sup>1</sup>Körner, *Fourier Analysis*, p.200 (conclusion of Chapter 42: "Linkages").

<sup>2</sup>The problem statement was: "Let  $P(x)$  be a polynomial of degree  $n$  such that  $P(x) = Q(x)P''(x)$ , where  $Q(x)$  is a quadratic polynomial and  $P''(x)$  is the second derivative of  $P(x)$ . Show that if  $P(x)$  has at least two distinct roots then it must have  $n$  distinct roots."

3. Generalize  $A_1$  to a differential operator  $A_c$  whose eigen-polynomials are orthogonal polynomials with respect to  $(\cdot, \cdot)_c$ . Verify directly that the  $T_n$  are eigen-polynomials of  $A_{-1/2}$  with the appropriate eigenvalues.
4. If the  $u_n$  in problems 2, 3 are chosen to be monic, what is  $(u_n, u_n)$ , and what is the three-term recurrence they satisfy? Can you generalize Lemma 40.7 in Körner (pages 188–190) to Gegenbauer polynomials for arbitrary  $c$ ?

Discrete Fourier analysis provides a good framework for “Gauss sums” and “Jacobi sums” which are ubiquitous in number theory. For our purposes we shall define them as follows. Fix a prime number  $p$ , and let  $\zeta$  be the primitive  $p$ -th root of unity  $e^{2\pi i/p}$ . A *character* is a homomorphism from the multiplicative group  $(\mathbf{Z}/p)^* = (\mathbf{Z}/p) - \{0\}$  of order  $p - 1$  to the  $(p - 1)$ -st roots of unity. The *Gauss sum* associated to a character  $\psi$  is

$$G(\psi) := \sum_{n=1}^{p-1} \zeta^n \psi(n).$$

The *Jacobi sum* associated to a pair  $\psi_1, \psi_2$  of characters is defined by

$$J(\psi_1, \psi_2) := \sum_{n=2}^{p-1} \psi_1(n) \psi_2(1 - n).$$

For instance, if  $p = 5$  and  $\psi$  takes  $\pm 1$  to 1 and  $\pm 2$  to  $-1$  then  $G(\psi) = \zeta - \zeta^2 - \zeta^3 + \zeta^4 = \sqrt{5}$  and  $J(\psi, \psi) = -1 + 1 - 1 = -1$ .

5. Let  $\psi$  be a nontrivial character (the “trivial character” sends every element of  $(\mathbf{Z}/p)^*$  to 1), and extend it to a complex-valued function on  $\mathbf{Z}/p$ . The Gauss sum  $G(\psi)$  is one value of the discrete Fourier transform of this function. Determine its discrete Fourier transform at all elements of  $\mathbf{Z}/p$ .
6. Prove that

$$J(\psi_1, \psi_2) = G(\psi_1)G(\psi_2)/G(\psi_1\psi_2)$$

provided none of  $\psi_1, \psi_2, \psi_1\psi_2$  is the “trivial character” sending every element of  $(\mathbf{Z}/p)^*$  to 1. What happens if one or more of these characters is trivial? [Hint: remember our formula for  $B(x, y)$  and its proof.]

7. Prove that  $|G(\psi)|^2 = p$  for all nontrivial  $\psi$ . If moreover  $\psi$  is “real”, i.e. sends each element of  $(\mathbf{Z}/p)^*$  to either  $+1$  or  $-1$ , show that  $G(\psi) = \pm\sqrt{p}$  if  $p \equiv 1 \pmod{4}$  and  $G(\psi) = \pm i\sqrt{p}$  if  $p \equiv -1 \pmod{4}$ . Numerically compute  $G(\psi)$  for nontrivial real characters  $\psi \pmod{p}$  for enough values of  $p$  that you detect a pattern in the choices of sign.

I do not ask you to prove this pattern; this sign problem occupied Gauss for years! Our last problem may suggest one way to solve it:

8. Let  $N$  be any positive integer and  $\zeta = e^{2\pi i/N}$ . Let  $M$  be the  $N \times N$  matrix whose  $(a, b)$  entry is  $\zeta^{ab}$ ; i.e.  $M$  is the matrix for the discrete Fourier transform mod  $N$ . Show that  $M^4 = N^2 I_n$ . Deduce that each eigenvalue of  $M$  is  $\pm N^{1/2}$  or  $\pm i N^{1/2}$ . Conclude that there exist integers  $r_N, s_N$  such that  $\sum_{a=1}^N \zeta^{a^2} = (r_N + i s_N) \sqrt{N}$ . Again, compute  $r_N, s_N$  for enough small  $N$  until you can guess a pattern. How much of this pattern can you prove?

This problem set is due Friday, April 28 in class.