Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #10 (14 April 2000):
Weyl cont’d; differentiation of Fourier series; more Fourier applications

More about Weyl equidistribution:
1. Let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers mod $2\pi$ (= elements of $T$). Assume that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} e^{ist_r} = 0$$

for all nonzero $s$, except that for $s = \pm 1$ we instead have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} e^{\pm it_r} = 1/2.$$ 

What can you conclude about the limits

$$\lim_{n \to \infty} \frac{1}{n} \# \{ r \in \mathbb{Z} | 1 \leq r \leq n, \ 2\pi a \leq t_r \leq 2\pi b \}$$

$(0 \leq a \leq b \leq 1)$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f(t_r)$$

$(f : T \to \mathbb{C}$ continuous)? Generalize.

2. Let $\gamma \in \mathbb{R}$ be any irrational number. Prove that the sequence $t_r = r^2 \gamma$ is equidistributed mod 1. [Again we need to estimate $\sum_{r=1}^{n} e^{2\pi i s t_r}$ for nonzero integers $s$. To simplify the notation let $z$ be the complex number $e^{2\pi i s}$ of absolute value 1, so we need to show that $(\sum_{r=1}^{n} z^r)/n \to 0$ as $n \to \infty$. The trick is to fix some integer $h$ and rewrite our sum for $n > h$ as

$$\sum_{r=1}^{n} z^r = \frac{1}{h} \sum_{m=1}^{n} \left( \sum_{r=m}^{m+h-1} z^r \right) + E$$

with the error $E$ bounded by $h$. By Cauchy-Schwarz this is at most

$$\frac{1}{h} \left( \sum_{m=1}^{n} \left| \sum_{r=m}^{m+h-1} z^r \right|^2 \right)^{1/2} + E.$$ 

Now show that, since $z$ is not a root of unity (why?), the sum over $m$ is at most $hn + Ch^2$, with $C$ depending on $\gamma$ and $h$ but not on $n$. Thus $|\sum_{r=1}^{n} z^r|/n < h^{-1/2} + o(1)$ as $n \to \infty$. Finally let $h \to \infty$.]

With some more courage and perseverance one can repeat this argument to show inductively that for any polynomial $P(x)$ the sequence $\{P(r)\}_{r=1}^{\infty}$ is equidistributed mod 1, provided at least one nonconstant coefficient of $P$ is irrational.

Relating the Fourier expansion of a differentiable function to that of its derivative(s):

3. i) If $f : T \to \mathbb{C}$ is a $C^1$ function, express the Fourier coefficients of its derivative in terms of the coefficients $f_n$ of $f$.
   ii) Prove that $f$ is $C^\infty$ if and only if $|n|^k f_n \to 0$ as $n \to \pm \infty$ for all $k$.
   iii) Show that the function $F(t) = \sum_{n=1}^{\infty} n^{-5/2} \sin(nt)$ does not have a continuous second derivative on $T$. 

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4. (Wirtinger’s Inequality) Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a $C^1$ function with right and left derivatives at 0 and $\pi$ respectively. Assume that $f(0) = f(\pi) = 0$. Prove that $\int_0^\pi f(x)^2 \, dx \leq \int_0^\pi f'(x)^2 \, dx$ with equality if and only if $f(x) = c \sin(x)$ for some $c \in \mathbb{R}$. [Extend $f$ to a $C^1$ function on $\mathbb{T}$ by $f(-x) = -f(x)$. Challenge: can you prove this inequality without using Fourier analysis? Can you relax the assumption that $f$ is differentiable at the endpoints?]

A few problems on the computation of simple Fourier series, including yet another evaluation of $\zeta(2)$ etc.

5. Determine the Fourier series of the $2\pi$-periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ given on $|x| \leq \pi$ by $f(x) = e^{c|x|}$ (with $c$ a real constant). Use this to evaluate in closed form $\sum_{n=1}^\infty 1/(n^2 + a^2)$ for $a \in \mathbb{R}$. Check that your answer agrees with the numerical value $\sum_{n=1}^\infty 1/(9n^2 + 1) = .171 \ldots$

6. i) Determine the Fourier series of the $2\pi$-periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ given on $[0, 2\pi]$ by $f(x) = x(2\pi - x)$.
   ii) For each integer $n > 1$, the Fourier series whose $e^{i\pi t}$ coefficient is $\frac{1}{r^n}$ converges to a continuous function $P_n$ on $\mathbb{T}$ (by Thm. 9.2 in Körner). Prove that, considered as a function on $[0, 2\pi]$, this $P_n$ is a polynomial of degree $n$. [Hint: see problem 3]

7. i) Describe these $P_n$ in terms of the polynomials $B_m$ introduced in the problem 5 of the second 55b problem set.
   ii) Show that the sums $\sum_{r=1}^\infty r^{-n}$ (for $n$ even) and $\sum_{r=0}^\infty (-1)^r(2r + 1)^{-n}$ (for $n$ odd) can be computed by evaluating $P_n$ at particular values of $t$. Deduce again (as in problem 8 of the fifth problem set) that these sums are rational multiples of $\pi^n$. 

This problem set is due Friday, April 21 in class.