Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #10 (14 April 2000):
Weyl cont'd; differentiation of Fourier series; more Fourier applications

More about Weyl equidistribution:

1. Let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers mod 2π (= elements of $\mathbf{T}$). Assume that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{ist_r} = 0$$

for all nonzero s, except that for $s = \pm 1$ we instead have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{\pm it_r} = 1/2.$$

What can you conclude about the limits

$$\lim_{n\to\infty} \frac{1}{n} \#\{r \in \mathbf{Z} \mid 1 \le r \le n,\ 2\pi a \le t_r \le 2\pi b\}$$

$(0 \le a \le b \le 1)$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(t_r)$$

$(f : \mathbf{T} \to \mathbf{C}$ continuous)? Generalize.

2. Let $\gamma \in \mathbf{R}$ be any irrational number. Prove that the sequence $t_r = r^2\gamma$ is equidistributed mod 1. [Again we need to estimate $\sum_{r=1}^{n} e^{2\pi i s t_r}$ for nonzero integers s. To simplify the notation let z be the complex number $e^{2\pi i s\gamma}$ of absolute value 1, so we need to show that $(\sum_{r=1}^{n} z^{r^2})/n \to 0$ as $n \to \infty$. The trick is to fix some integer h and rewrite our sum for $n > h$ as

$$\sum_{r=1}^{n} z^{r^2} = \frac{1}{h} \sum_{m=1}^{n} \left(\sum_{r=m}^{m+h-1} z^{r^2} \right) + E$$

with the error E bounded by h. By Cauchy-Schwarz this is at most

$$\frac{1}{h} \left(n \sum_{m=1}^{n} \left| \sum_{r=m}^{m+h-1} z^{r^2} \right|^2 \right)^{1/2} + E.$$

Now show that, since z is not a root of unity (why?), the sum over m is at most $hn + Ch^2$, with C depending on γ and h but not on n. Thus $|\sum_{r=1}^{n} z^{r^2}|/n < h^{-1/2} + o(1)$ as $n \to \infty$. Finally let $h \to \infty$.]

With some more courage and perseverance one can repeat this argument to show inductively that for any polynomial $P(x)$ the sequence $\{P(r)\}_{r=1}^{\infty}$ is equidistributed mod 1. provided at least one nonconstant coefficient of P is irrational.

Relating the Fourier expansion of a differentiable function to that of its derivative(s):

3. i) If $f : \mathbf{T} \to \mathbf{C}$ is a $\mathcal{C}^1$ function, express the Fourier coefficients of its derivative in terms of the coefficients $\hat{f}_n$ of f.
ii) Prove that f is $\mathcal{C}^\infty$ if and only if $|n|^k \hat{f}_n \to 0$ as $n \to \pm\infty$ for all k.
iii) Show that the function $F(t) = \sum_{n=1}^{\infty} n^{-5/2} \sin(nt)$ does <u>not</u> have a continuous second derivative on $\mathbf{T}$.

4. (Wirtinger's Inequality) Let $f : [0, \pi] \to \mathbf{R}$ be a $\mathcal{C}^1$ function with right and left derivatives at 0 and π respectively. Assume that $f(0) = f(\pi) = 0$. Prove that $\int_0^\pi f(x)^2\,dx \le \int_0^\pi f'(x)^2\,dx$ with equality if and only if $f(x) = c\sin(x)$ for some $c \in \mathbf{R}$. [Extend f to a $\mathcal{C}^1$ function on $\mathbf{T}$ by $f(-x) = -f(x)$. Challenge: can you prove this inequality without using Fourier analysis? Can you relax the assumption that f is differentiable at the endpoints?]

A few problems on the computation of simple Fourier series, including yet another evaluation of $\zeta(2)$ etc.

5. Determine the Fourier series of the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$ given on $|x| \le \pi$ by $f(x) = e^{c|x|}$ (with c a real constant). Use this to evaluate in closed form $\sum_{n=1}^\infty 1/(n^2 + a^2)$ for $a \in \mathbf{R}$. Check that your answer agrees with the numerical value $\sum_{n=1}^\infty 1/(9n^2 + 1) = .171\ldots$

6. i) Determine the Fourier series of the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$ given on $[0, 2\pi]$ by $f(x) = x(2\pi - x)$.
   ii) For each integer $n > 1$, the Fourier series whose e^{irt} coefficient is r^{-n} converges to a continuous function P_n on $\mathbf{T}$ (by Thm. 9.2 in Körner). Prove that, considered as a function on $[0, 2\pi]$, this P_n is a polynomial of degree n. [Hint: see problem 3.]

7. i) Describe these P_n in terms of the polynomials B_m introduced in the problem 5 of the second 55b problem set.
   ii) Show that the sums $\sum_{r=1}^\infty r^{-n}$ (for n even) and $\sum_{r=0}^\infty (-1)^r (2r+1)^{-n}$ (for n odd) can be computed by evaluting P_n at particular values of t. Deduce again (as in problem 8 of the fifth problem set) that these sums are rational multiples of π^n.

This problem set is due Friday, April 21 in class.