Math 55b: Honors Advanced Calculus and Linear Algebra

II: Separable inner-product spaces;
orthogonal complements, projections, and duality

The space $l_2$ is much larger than any of the finite-dimensional Hilbert spaces $F^n$ — for instance, it is not locally compact — but it is still small enough to be “separable”; this in fact topologically characterizes $l_2$. This notion is defined as follows:

**Definition.** A metric space is separable if it contains a dense countable subset.

(We consider a finite or even empty set to be countable. The phrase “second countable” is sometimes used for “separable”.)

**Examples.** Any compact set is separable: for each $n = 1, 2, 3, \ldots$ there is a finite $(1/n)$-net, and the union of these nets over $n$ is a countable dense set. The metric space $R$ is separable because $Q \subset R$ is a dense countable subset. The direct sum of two separable spaces is separable, because the Cartesian product of two countable sets is countable. Thus in particular $C$, so also $R^n$ and $C^n$, are separable. Even $l_2$ is separable, because $l_2^{(0)}$ is dense in $l_2$ and $l_2^{(0)}$ is a countable union of separable subsets $F^n$.

If $X$ is a separable metric space, and $S \subset X$ is a subset such that there exists $d_0 > 0$ with $d(s, s') \geq d_0$ for all distinct $s, s' \in S$, then $S$ is countable. Indeed, for each $\epsilon > 0$, we can write $X$ as the countable union of $\epsilon$-neighborhoods (specifically, of $\epsilon$-neighborhoods centered at the points of its dense countable subset), and if $\epsilon < d/2$ each of these neighborhoods contains at most one point of $S$. Thus conversely if $X$ is a metric space containing an uncountable subset $S$ any two of whose points are at least $d_0$ apart, then $X$ is not separable. For example, a discrete space is separable if and only if it is countable. The normed space $l_\infty$ of bounded sequences $(a_1, a_2, \ldots)$ of scalars with the sup norm is not separable, because it contains the uncountable subset $S = \{(a_n)_{n=1}^\infty : a_n = 0$ or $1\}$ with $d_0 = 1$.

**Theorem.** Let $V$ be an inner product space. Then $V$ is separable if and only if it has a countable orthonormal set. In this case every orthogonal set in $V$ is countable.

**Proof:** If $V$ is separable, let $\{v_n\}_{n=1}^\infty$ be a dense sequence in $V$. Clearly $\{v_n\}$ spans $V$ topologically. Discard each $v_n$ that is a linear combination of $v_1, \ldots, v_{n-1}$; the resulting linearly independent sequence may no longer be dense, but its linear span is the same, so is still dense in $V$. Now apply Gram-Schmidt to obtain an orthonormal sequence $w_n$ still with the same linear span, so the $w_n$ constitute a countable orthonormal.
Conversely, if $V$ has a countable ontb it is either isometric with $F^n$ or has a dense subset isometric with $l_2^{(0)}$, and we have shown that each of these spaces is separable.

Finally, if $S$ is any orthonormal subset of an inner product space then any two of its points are at the same distance, namely $\sqrt{2}$. Thus if the space is separable then $S$ must be countable. By normalization the same is true also of orthogonal sets. □

**Corollary.** Every separable Hilbert space is isometric with either $F^n$ (some $n = 0, 1, 2, \ldots$) or $l_2$.

**Orthogonal projections and complements in Hilbert space.** Most uses of the completeness of a Hilbert space go through the following results, which show that orthogonal projections and complements work for a Hilbert space as they do for a finite-dimensional inner product space. In particular, we can identify a Hilbert space $\mathcal{H}$ with its topological dual $\mathcal{B}(\mathcal{H}, F)$ as we did for a finite-dimensional inner-product space.

**Theorem.** Let $V$ be an inner product space, and $W \subset V$ a complete subspace. Then for each $v \in V$ there exists a unique $P(v) \in W$ such that $|v - P(v)| = \min_{w \in W} |v - w|$.

**Proof:** Since $\{|v - w| : w \in W\}$ is a nonempty set bounded below, it has an infimum, call it $d$. Let $\{w_n\}$ be any sequence in $W$ such that $|v - w_n| \to d$. We shall show that $w_n$ is necessarily a Cauchy sequence. Since $W$ is assumed complete, it will follow that $\{w_n\}$ converges to some $w \in W$. Then $|v - w| = d$, so we may set $P(v) = w$.

Suppose that $u_1, u_2 \in W$ with $|v - u_i|^2 \leq d^2 + \epsilon^2$. By the parallelogram law,

$$\left| v - \frac{u_1 + u_2}{2} \right|^2 = \frac{1}{2}(|v - u_1|^2 + |v - u_2|^2) - \frac{1}{4}|u_1 - u_2|^2$$

But the left-hand side is at least $d^2$, and the right hand side at most $d^2 + \epsilon^2 - \frac{1}{4}|u_1 - u_2|^2$. Thus $|u_1 - u_2| \leq 2\epsilon$. In particular, taking $\epsilon = 0$, we see that if $P(v)$ exists then it is unique.

Now for each $\epsilon > 0$ there exists $N$ such that $|v - w_n|^2 \leq d^2 + \epsilon^2$ for all $n > N$. Thus $|w_m - w_n| \leq 2\epsilon$ for all $m, n > N$, and we’re done. □

As in the finite-dimensional case, it is then readily seen that $v - P(v) \subset W^\perp$. We thus express any $v \in V$ as a sum of vectors in $W$ and $W^\perp$. This representation is unique because $W \cap W^\perp = \{0\}$. It follows that $P : V \to W$ is a linear transformation. This transformation is called (orthogonal) projection to $W$.

**Corollary.** Under these hypotheses, $V$ is the orthogonal direct sum of $W$ and $W^\perp$, and $W = (W^\perp)^\perp$. 

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Note that for any $W \subset V$, the orthogonal complement $W^\perp$ is a closed subspace of $V$, and is the same as $\overline{W}^\perp$ where $\overline{W}$ is the closure of $W$ in $V$. Thus $(W^\perp)^\perp$ is a closed subspace of $W$ containing $W$. By the above Corollary, if $\overline{W}$ is complete then $(W^\perp)^\perp = \overline{W}$. Without this hypothesis on $\overline{W}$, it could happen that $(W^\perp)^\perp$ is strictly larger than $\overline{W}$. For instance, let $V = l_2^{(0)}$ and let $W$ be the intersection of $l_2^{(0)}$ with the orthogonal complement of $(1, 1/2, 1/3, 1/4, \ldots)$ in $l_2$.

Now let $V$ be a Hilbert space. Then $W \subset V$ is complete if and only if it is closed. Thus in a Hilbert space $(W^\perp)^\perp = W$ if and only if $W = \overline{W}$.

We can now conclude that in a separable Hilbert space any orthogonal (orthonormal) set is contained in an orthogonal topological basis (ontb). Using the Axiom of Choice the same can be proved for an arbitrary Hilbert space $\mathcal{H}$.

We can then define the dimension of $\mathcal{H}$ as the cardinality of a basis of $\mathcal{H}$ — once we show that all bases have the same cardinality. We have done this already for a Hilbert space with a countable basis; given the existence of ontb’s, it can be done in general without much further difficulty using the fact that $c \cdot \aleph_0 = c$ for any infinite cardinal $c$.

The topological dual of a normed vector space $V$ is the vector space $\mathcal{B}(V, \mathbb{F})$ of continuous linear functionals on $V$. If $V$ is an inner product space then any $v \in V$ may be regarded as the functional $w \mapsto \langle v, w \rangle$; this is continuous of norm $|v|$ by Cauchy-Schwarz, so we get an isometric embedding of $V$ into $V^*$. In general the image of $V$ is not all of $V^*$, since $V^*$ is necessarily complete (why?). However, this is the only obstruction: a Hilbert space is its own topological dual. More precisely:

**Theorem.** Let $V$ be a Hilbert space and $\nu^* \in V^*$ a continuous linear functional. Then there exists a unique $v \in V$ such that $\nu^*(w) = \langle v, w \rangle$ for all $w \in V$.

**Proof:** Uniqueness is easy: if $v_1, v_2$ are two such $v$'s then $\nu^*(v_1 - v_2) = 0$ yields $|v_1 - v_2|^2 = 0$ and thus $v_1 = v_2$. We next prove existence. If $\nu^* = 0$ then we of course take $v = 0$. Else let $W = \ker \nu^*$. This is a proper closed subspace of $V$ and thus has nonzero orthogonal complement $W^\perp$. We claim that $W^\perp$ is one-dimensional. Else it would contain two linearly independent vectors $v_1, v_2$; but we can find scalars $a_1, a_2$ not both zero such that $a_1 \nu^*(v_1) + a_2 \nu^*(v_2) = 0$, and then $a_1 v_1 + a_2 v_2$ is a nonzero vector in $W \cap W^\perp$, which is impossible. For the same reason, if $v_0$ is a nonzero vector in $W^\perp$ then $\nu^*(v_0) \neq 0$. Choose such $v_0$, and define

$$v = \frac{\nu^*(v_0)}{|v_0|^2} v_0.$$ 

Then $\nu^*(v_0) = \langle v, v_0 \rangle$. Moreover $\nu^*(w) = \langle v, w \rangle$ for all $w \in W$ since both sides vanish. But we showed that $v_0$ spans $W^\perp$, and thus $v_0$ together with $W$ span $V$. Thus $\nu^*(w) = \langle v, w \rangle$ for all $w \in V$ and we are done. \(\square\)