Let $R$ be a rectangle with sides parallel to the real and imaginary axes, i.e. of the form $R = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$ for some $a, b, c, d$ with $a < b$ and $c < d$. For a continuous function $f : R \to \mathbb{C}$, we define $\oint_{\partial R} f(z) \, dz$ by

$$\oint_{\partial R} f(z) \, dz = \int_a^b f(x + iy) \, dy - \int_c^d f(x + id) \, dx.$$ 

(Each of the four terms, one for each side $\gamma$ of $R$, is the contour integral $\int_{\gamma} f(z) \, dz$ with $\gamma$ oriented in the counterclockwise direction around the boundary $\partial R$ of $R$; the combination $\oint$ is the contour integral around the boundary $\partial R$ of $R$.)

1. Suppose $F$ is a complex-valued function on a neighborhood of $R$ that is differentiable as a function of a complex variable, with derivative $F'$. Then $\oint_{\partial R} F'(z) \, dz = 0$. The terms corresponding to the integrals around the four sides are $F(b + ic) - F(a + ic)$, $F(b + id) - F(b - ic)$, $F(a + id) - F(a + id)$, and $F(a + ic) - F(a + id)$, and these sum to zero. NB the examples of $f(x) = \text{Re}(z)$ and $f(x) = \text{Im}(z)$ show that $\oint_{\partial R} f(z) \, dz$ isn’t always zero; for those choices, $\oint_{\partial R} f(z) \, dz$ is a nonzero multiple of the area $(b-a)(d-c)$ of $R$.

2. Divide $R$ into rectangles $R_1, R_2$ by choosing some $x_1 \in (a, b)$ and setting

$$R_1 = \{x + iy : a \leq x \leq x_1, c \leq y \leq d\}, \quad R_2 = \{x + iy : x_1 \leq x \leq b, c \leq y \leq d\}.$$ 

Then $\oint_{\partial R} f(z) \, dz = \oint_{\partial R_1} f(z) \, dz + \oint_{\partial R_2} f(z) \, dz$. Likewise for $y_1 \in (c, d)$. It follows by induction that for any partitions $a = x_0 < x_1 < \ldots < x_M = b$ and $c = y_0 < y_1 < \ldots < y_N = d$ we can write $\oint_{\partial R} f(z) \, dz = \sum_{j=1}^M \sum_{k=1}^N \oint_{\partial R_{jk}} f(z) \, dz$ where $R$ is the rectangle of $x + iy$ with $x \in [x_{j-1}, x_j]$ and $y \in [y_{j-1}, y_j]$. (Note that this is consistent with the formulas from the previous problem: certainly $0 = \sum_{j=1}^M \sum_{k=1}^N$ and also the area of $R$ equals the sum of the areas of the $R_{jk}$.

3. [Goursat] Suppose $f$ is a complex-valued function on a neighborhood of $R$ that is differentiable as a function of a complex variable. Then we prove that $\oint_{\partial R} f(z) \, dz = 0$ as follows. Assume the integral is nonzero, and let $C$ be its absolute value, with $C > 0$. Repeatedly applying the result of the previous problem, we obtain a sequence of rectangles $R_n$ (with $R_0 = R$), with each $R_n (n > 0)$ being one quarter of $R_{n-1}$ and satisfying $\left| \oint_{\partial R_n} f(z) \, dz \right| \geq C/4^n$. Then there exists some $z^* \in R$ contained in each $R_n$. Since $f$ is differentiable at $z^*$ we have $f(z) = f(z^*) + f'(z^*)(z-z) + o(|z-z|)$ as $z \to z^*$. But the contour integral over $\partial R$ of the error $o(|z-z|)$ is $o(1/4^n)$, because $|z-z| = O(1/2^n)$ uniformly on $R$ and each of the edges of $R$ has length $O(1/2^n)$. Moreover, $\oint_{\partial R_n} f(z^*) \, dz = \oint_{\partial R_n} f'(z^*)(z-z) \, dz = 0$, because each of $f'(z^*)$ (as a constant function of $z$) and $f'(z^*)(z-z)$ is $f'$ for some $F : C \to \mathbb{C}$ with a complex derivative, namely $F(z) = f(z^*)z$ and $F(z) = f'(z^*)(z-z)^2/2$ respectively. Thus $\oint_{\partial R_n} f(z) \, dz = o(1/4^n)$, contradiction.

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4For any subset $S$ of a metric space (or even a general topological space) $X$, a “neighborhood of $S$” is an open set $N$ of $X$ such that $N \supset S$. See the footnote to problem 4 below.

2Note that this proof manages to avoid any continuity hypothesis on $f'$, so you cannot obtain the same result by appealing to Green’s theorem even if you already know that approach.
then $y \in R$ has side form an open cover of the compact set $R$. With $n \in S$ more generally: let $z \in S$.

Remarks: i) An important use of contour integration is the evaluation of definite integrals over real intervals. We can already give an example: having shown $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, we can compute $\int_{-\infty}^{\infty} e^{-x^2+2icx} dx$ for any $c > 0$ by applying $\oint f(z) dz = 0$ to $f(z) = e^{-x^2}$ and $R = \{x + iy : -M \leq x \leq M, 0 \leq y \leq c\}$, and letting $M \to \infty$. The vertical contributions approach zero, and we’re left with

$$\int_{-\infty}^{\infty} e^{-(x+iy)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

whence

$$\int_{-\infty}^{\infty} e^{-x^2+2icx} dx = \sqrt{\pi} e^{-c^2}.$$

Averaging $x$ and $-x$ yields the equivalent form $\int_{-\infty}^{\infty} e^{-x^2} \cos cx dx = \sqrt{\pi} e^{-c^2}$, and thus $\int_{0}^{\infty} e^{-x^2} \cos cx dx = \frac{1}{2} \sqrt{\pi} e^{-c^2}$ because the integrand is even.

ii) In general, if $\gamma$ is a line segment from $z_1$ to $z_2$ then $\int_{\gamma} f(z) dz$ can be defined as the definite integral $\int_{0}^{1} f(z_1 + (z_2 - z_1)t) dt$. This lets us define contour integrals over arbitrary polygonal paths, as the sum of the integrals over their component line segments. If $F$ is a function from a neighborhood of $\gamma$ to $\mathbb{C}$ with a complex derivative $F'$, then the formula $\oint f(z) dz = F(z_2) - F(z_1)$ still holds (again by the Fundamental Theorem of Calculus). Thus the integral of $F(z) dz$ over a closed polygonal contour vanishes. If $\Delta$ is a triangle and $f$ is a function from a neighborhood of $\Delta$ to $\mathbb{C}$, then we can tile $\Delta$ by four half-size copies $\Delta_1, \ldots, \Delta_4$ of $\Delta$ (one in the opposite orientation), and check that $\oint_{\Delta} f(z) dz = \sum_{j=1}^{4} \oint_{\partial \Delta_j} f(z) dz$. If $f$ is differentiable as a function of a complex variable, then the Goursat trick applies and we deduce that $\oint_{\Delta} f(z) dz = 0$. We can then do the same with $\Delta$ replaced by any simple polygon, by tiling the polygon with finitely many triangles. 

4. With $f$ as in the previous problem, we can now construct an antiderivative $F : R \to \mathbb{C}$, defined by

$$F(u + iv) = \int_{a}^{u} f(x + ic) dx + i \int_{c}^{v} f(u + iy) dy = \int_{a}^{u} f(x + iv) dx + i \int_{c}^{v} f(a + iy) dy.$$

(These expressions are equal because they differ by $\oint_{R_1} f(z) dz$ for some rectangle $R_1 \subseteq R$, and we already know that such an integral is zero). Better yet, since $f$ is defined and complex-differentiable on some neighborhood $N$ of $R$, we can find a rectangle $R' \subseteq N$ whose interior contains $R$, and make the same definition on $R'$, so that it makes sense to differentiate $F$ also on the boundary of $R$. Now if $a_1 + ib, a_2 + ib \in R'$ with $a_1 < a_2$ then $F(a_2 + ib) - F(a_1 + ib) = \int_{a_1}^{a_2} f(x + ib) dx$, and likewise if $a + ib_1, a + ib_2 \in R'$ with $b_1 < b_2$ then $F(a + ib_2) - F(a + ib_1) = i \int_{b_1}^{b_2} f(a + iy) dy$. Since $f$ is continuous, we can find for all $z \in R$ and $\epsilon > 0$ some $\delta > 0$ such that $N_\delta(z) \subseteq R'$ and $|f(w) - f(z)| < \epsilon$ for all $w \in N_\delta(z)$. It then follows that $|F(w) - F(z) - (w - z)f'(z)| < 2\epsilon |w - z|$ for all $w \in N_\delta(z)$ (note that all the $\delta > 0$ such that $N_\delta(z) \subseteq R'$ and $|f(w) - f(z)| < \epsilon$ for all $w \in N_\delta(z)$. It then follows that $|F(w) - F(z) - (w - z)f'(z)| < 2\epsilon |w - z|$ for all $w \in N_\delta(z)$ (note that all the

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For each $z$ on the boundary of $R$, find an open square centered on $z$ and contained in $N$. These form an open cover of the compact set $R$, so there is a finite subcover. Add to this subcover the squares centered at the corners of $R$, if they are not in it already. Choose $\epsilon > 0$ such that each of these squares has side $> 2\epsilon$. Then $R'$ can be $x + iy : a - \epsilon \leq x \leq b + \epsilon, c - \epsilon \leq y \leq d + \epsilon$.

More generally: let $S$ be any compact subset of an open set $N$ in any metric space $X$. Then there exists $\epsilon > 0$ such that $N$ contains the “$\epsilon$-neighborhood” $N_\epsilon(S) := \bigcup_{x \in S} B_\epsilon(x)$ of $S$. Proof: if not, find $x_n \in S$ and $y_n \notin N$ such that $d(x_n, y_n) \to 0$; extract a convergent subsequence $\{x_{n_j}\} \to x \in S \subseteq N$, and then $y_{n_j} \to x$, contradiction. (For our application $S = R$ and it is convenient to use the sup metric on $C \cong \mathbb{R}^2$)
horizontal and vertical contours we need are contained in $N_\delta(z)$). Because $\epsilon$ was arbitrary we conclude that $F'(z) = f(z)$ as claimed.

(Once we define $\int_\gamma f(z) \, dz$ for general contours $\gamma$ we can use this result to show that such an integral vanishes over any closed contour contained in $R$.)

5. We next use the complex exponential function to deduce results on integrals over circular contours from our results on contour integrals over rectangular contours.

i) Suppose $0 < r_0 < r$, and let $f$ be a complex-valued function on a neighborhood $N$ of the annulus $\{z \in \mathbb{C} : r_0 \leq |z| \leq r\}$. Assume again that $f$ is differentiable on $N$ as a function of a complex variable. Then

$$\int_{0}^{2\pi} f(r_0 e^{i\theta}) \, d\theta = \int_{0}^{2\pi} f(re^{i\theta}) \, d\theta.$$ 

Indeed let $R$ be the rectangle with $[a, b] = [\log r_0, \log r]$ and $[c, d] = [0, 2\pi]$, and consider the function $g(z) = f(e^{z})$ on a neighborhood of $R$ whose image under $z \mapsto e^{z}$ is contained in $N$. This function is differentiable as a function of a complex variable, because it is the product of $e^{-z}$ and $f(e^{z})$, each of which is differentiable by the complex chain rule. Hence $\oint_{\partial R} g(z) \, dz = 0$. But the horizontal sides of $R$ both map to the interval $[r_0, r] \subset \mathbb{R}$, and their contributions $\pm \int_{r_0}^{r} f(x) \, dx/x$ to $\oint_{\partial R} g(z) \, dz$ cancel out. The vertical sides contribute $i \left( \int_{0}^{2\pi} f(re^{i\theta}) \, d\theta - \int_{0}^{2\pi} f(r_0) e^{i\theta} \, d\theta \right)$. The claimed equality follows.

ii) If furthermore $N$ contains the full disc $\{z \in \mathbb{C} : |z| \leq r\}$, then we can let $r_0 \to 0$ in $\int_{0}^{2\pi} f(r_0 e^{i\theta}) \, d\theta = \int_{0}^{2\pi} f(re^{i\theta}) \, d\theta$ and deduce that

$$f(0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\theta}) \, d\theta.$$ 

(Note that we never use the existence of $f'(z)$ at $z = 0$ itself, only the continuity of $f$ at $z = 0$. We shall soon see that even if $f$ is merely bounded on $0 < |z| < r$ then we can define $f(0)$ by setting $f(0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\theta}) \, d\theta$ to obtain a function that’s not just continuous but even complex-analytic (that is, equal to a convergent power series) at $z = 0$.

Using $[c, d] = [0, \pi]$ instead of $[0, 2\pi]$, we find instead that

$$\int_{-\pi}^{\pi} f(z) \frac{dz}{z} + \int_{r_0}^{r} f(z) \frac{dz}{z} = i \left( \int_{0}^{\pi} f(r_0 e^{i\theta}) \, d\theta - \int_{0}^{\pi} f(re^{i\theta}) \, d\theta \right).$$ 

Now let $f(z) = e^{icz}$ for some $c > 0$, and let $r_0 \to 0$ and $r \to \infty$. Then the left-hand side is $\int_{r_0}^{r} 2i \sin cz \, \frac{dz}{z} \to \int_{0}^{\infty} 2i \sin cz \, \frac{dz}{z}$. In the right-hand side, the first integral approaches $\pi$, and the second goes to zero (because $e^{icz}$ is small except for $\theta$ near $0$ or $\pi$). Therefore $\int_{0}^{\infty} 2i \sin cz \, \frac{dz}{z} = i\pi$, so we have obtained the famous integral formula

$$\int_{0}^{\infty} \frac{\sin cz}{z} \, dz = \frac{\pi}{2} \quad (c > 0).$$ 

(It is easy to see that the integral does not depend on the choice of $c$, but the fact that it equals $\pi$ is far from obvious.)

6. A nice property of circular discs $D \subset \mathbb{C}$ is that they support non-obvious bijections $w$ that are complex-differentiable on a neighborhood of $D$ and do not preserve the center. We have just proved a formula that, for an arbitrary complex-differentiable function $f$, expresses its value at the center of $D$ as the average of the boundary
values. We next construct \( w \) and use it to generalize our formula from the central value of \( f \) to its value at any interior point of \( D \).

For simplicity we work with the unit disc (so take \( r = 1 \) in the previous problem), and choose real \( z_0 \) with \( |z_0| < 1 \). (Once we have the result for \( z_0 \) real, the general case will follow by applying the formula to the function \( f(cz) \) for suitable \( c \in \mathbb{C}^* \) with \( |c| = 1 \).) Then \( |z| = 1 \) implies \( z \bar{z} = 1 \), and thus also

\[
|z + z_0| = |\bar{z} + z_0| = |z^{-1} + z_0| = |z|^{-1}|1 + z_0 z| = |1 + z_0 z|,
\]

whence \( |(z + z_0)/(1 + z_0 z)| = 1 \). Define

\[
w(z) = \frac{z + z_0}{1 + z_0 \bar{z}}
\]

for \( z \in \mathbb{C} \) such that \( z \neq -z_0^{-1} \). Then we have just shown \( |z| = 1 \implies |w(z)| = 1 \). We claim that also \( |z| < 1 \iff |w(z)| < 1 \). One way to see this is to compute

\[
|1 + z_0 z|^2 = 1 + |z z_0|^2 + 2 \Re(z_0), \quad |z + z_0|^2 = |z|^2 + z_0^2 + 2 \Re(z z_0),
\]

and thus

\[
|1 + z_0 z|^2 - |z + z_0|^2 = (1 + |z z_0|^2) - (|z|^2 + z_0^2) = (1 - z_0^2)(1 - |z|^2),
\]

which has the same sign as \( 1 - |z|^2 \); thus if \( |z| < 1 \) then \( |z + z_0|^2 < |1 + z_0 z|^2 \), so \( |w(z)|^2 < 1 \) and \( |w(z)| < 1 \), while if \( |z| > 1 \) then likewise \( |w(z)|^2 > 1 \).

It follows that \( w \) is a bijection on the unit circle \( |z| = 1 \), and a bijection on the closed unit disc \( D = \{ z \in \mathbb{C} : |z| \leq 1 \} \) that sends 0 to \( z_0 \). Moreover, \( w \) is differentiable as a function of a complex variable; so if \( f \) is a complex-valued function with a complex derivative on a neighborhood of \( D \), then the same is true of \( f \circ w \). But \( (f \circ w)(0) = f(w(z_0)) = f(z_0) \). Since \( f \circ w \) has a complex derivative, we know that \( (f \circ w)(0) \) is the average of the values of \( f \circ w \) on \( |z| = 1 \). It follows that \( f(z_0) \) can also be given by such a formula, though it will be a weighted average:

\[
f(z_0) = (2\pi)^{-1} \int_0^{2\pi} c(\theta)f(e^{i\theta}) \, d\theta,
\]

for some function \( c \) depending on \( z_0 \). We next outline the computation of this function \( c \).

7. Before setting out on the calculation, note that we have a very strong hint and sanity check on the result: it must work for power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) that converge on \( |z| < R \) for some \( R > 1 \). Indeed we know that

\[
(\ast) \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) \, d\theta
\]

for such \( f \), and that the integral vanishes for \( n < 0 \). (Note that the case \( n = 0 \) recovers the formula for \( f(0) \) that we've already proved.) Thus

\[
(\ast\ast) \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} e^{-in\theta} z_0^n \right) f(e^{i\theta}) \, d\theta
\]

(the exchange of sum and integral is easily justified here), and the sum over \( n \) is a geometric series, equal to \( 1/(1 - z_0 e^{-i\theta}) \). This can’t quite be right, not even for \( z_0 = 0 \), because it’s not real-valued; but we can fix it by adding \( \sum_{n=1}^{\infty} e^{+in\theta} z_0^n \) (which is the complex conjugate minus 1), because \( f(z) \sum_{n=1}^{\infty} z^n z_0^n \) is a differentiable
function on $|z| < R$ that vanishes at $z = 0$, so its average over $|z| = 1$ vanishes. This gives
\[ \frac{1}{1 - z_0 e^{-i \theta}} + \frac{1}{1 - z_0 e^{i \theta}} - 1 = \frac{1 - z_0^2}{1 + z_0^2 - 2z_0 \cos \theta}, \]
which is indeed the formula we shall obtain for the multiplier of $(2\pi)^{-1} f(e^{i \theta}) \, d\theta$. Now we have
\[ f(z_0) = f(w(0)) = (f \circ w)(0) = \frac{1}{2\pi} \int_0^{2\pi} (f \circ w)(e^{i \theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(w(e^{i \theta})) \, d\theta, \]
and we saw that we can write $w(e^{i \theta}) = e^{i \psi}$ for some real $\psi$, so
\[ f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i \psi}) \frac{d\psi}{\psi'(\theta)} \]
if we regard $\psi$ as a function of $\theta$. By the chain rule (which we know applies also to complex analytic functions),
\[ ie^{i \psi(\theta)} \psi'(\theta) = ie^{i \theta} w'(e^{i \theta}), \]
and $w'(z) = (1 - z_0^2)/(1 + z_0 z)^2$ (in general the derivative of $(az + b)/(cz + d)$ is $(ad - bc)/(cz + d)^2$). We write everything in terms of $w = e^{i \psi}$, including $z = e^{i \theta} = (w - z_0)/(1 - z_0 w)$. This gives
\[ \psi'(\theta) = \frac{w/z}{w'(z)} = \frac{w(1 + z_0 z)^2}{(1 - z_0^2) z}, \]
which can be written as $(1 - z_0^2)/((1 - w z_0)(1 - w^{-1} z_0))$. The denominator expands to $1 + z_0^2 - (w + w^{-1} z_0) = 1 + z_0^2 - 2z_0 \cos \psi$, which matches our prediction once we change the variable name from $\psi$ to $\theta$:
\[ f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i \theta}) \frac{1 - z_0^2}{1 + z_0^2 - 2z_0 \cos \theta} \, d\theta. \]
To get from this “Poisson integral formula” to (**), we subtract
\[ 0 = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i \theta}) \sum_{n=1}^{\infty} e^{i n \theta} z_0^n \, d\theta \]
and finally obtain the power series $f(z_0) = \sum_{n=0}^{\infty} a_n z_0^n$ with $a_n$ given by the integral (**).

8. Some additional properties of analytic functions soon follow. For example, an analytic function $f$ on some disc $B_r(z_0)$ cannot have a sequence of zeros $z_k$ (solutions of $f(z) = 0$) such that $z_k \to z_0$, unless $f$ is the zero function. Indeed if $f$ is not the zero function then it has a power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n = (z - z_0)^{\alpha_0} f_1(z)$ with $c_{\alpha_0} \neq 0$ and $f_1(z) = \sum_{n=0}^{\infty} c_{\alpha_0 + n} (z - z_0)^n$; since $f_1$ is itself an analytic function on $B_r(z_0)$, it is a fortiori continuous, and then $f_1(z_0) = c_{\alpha_0} \neq 0$ implies that $f_1$ is nonzero in some neighborhood of $z_0$, whence $f(z) \neq 0$ for all $z \neq z_0$ in that neighborhood. This means that if $f, g$ are analytic functions on $B_r(z_0)$, and $\{z_k\}$ is a sequence in $B_r(z_0)$ such that $z_k \to z_0$, then $f(z_k) = g(z_k)$ for all $k$ (or even for infinitely many $k$) implies $f(z) = g(z)$ for all $z \in B_r(z_0)$. (Proof: consider the analytic function $f - g$.) This soon gives the “reflection principle(s)” for analytic functions and the process of “analytic continuation”.