Math 55b: Honors Real and Complex Analysis

Homework Assignment #7 (20 March 2017):
Univariate integral calculus

Bitte ve[r]giß alles, was Du auf der Schule gelernt hast;
dennDu hast ist nicht gelernt.  Emil Landau

Admittedly that’s a bit extreme, but it is true that for many of you Stieltjes integrals,
especially of vector-valued functions, are a new path in the familiar territory of integration,
and might require a different kind of thinking. This problem set consists of only
eight problems including four from Rudin, but most are geared towards developing
such “new kinds of thinking”, and a few are somewhat open-ended to suggest further
directions in analysis that we won’t pursue in Math 55.

The first few problems are from Rudin pages 138–139:2

1. [Rudin #3] Define functions \( \beta_j : \mathbb{R} \to \mathbb{R} \) (\( i = 1, 2, 3 \)) as follows: for each \( j \), set \( \beta_j(x) = 0 \) for \( x < 0 \) and \( \beta_j(x) = 1 \) for \( x > 0 \); but \( \beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = 1/2 \). Let \( f : [-1, 1] \to \mathbb{R} \) be any bounded function.
   a) Prove that \( f \in \mathcal{R}(\beta_1) \) iff \( f(0) = \lim_{x \to 0^+} f(x) \), and then \( \int_{-1}^1 f \, d\beta_1 = f(0) \);
   b) State and prove a similar result for \( \mathcal{R}(\beta_2) \);
   c) Prove that \( f \in \mathcal{R}(\beta_3) \) iff \( f \) is continuous at 0, in which case \( \int_{-1}^1 f \, d\beta_j = f(0) \) for each \( j = 1, 2, 3 \).

2. [Rudin #8; “integral test” for convergence of a positive series \( \sum_{n>n_0} f(n) \)] Let \( \alpha : [a, \infty) \to \mathbb{R} \) be an increasing function. Suppose \( f : [a, \infty) \in \mathcal{R}(\alpha) \) on \([a, b] \) for each \( b > a \). The “improper Riemann-Stieltjes integral” \( \int_a^\infty f(x) \, d\alpha(x) \) is then defined as \( \lim_{b \to \infty} \int_a^b f(x) \, d\alpha(x) \) if the limit exists [and is finite]. In that case we say the integral converges; we say it converges absolutely if \( \int_a^\infty |f(x)| \, d\alpha(x) \) also converges. Naturally the “improper Riemann integral” is the special case of this where \( \alpha(x) = x \) for all \( x \). [Likewise for \( \int_a^\infty f \, d\alpha \) converges to \( f_{\alpha}^a - f_{\alpha}^\infty f \, d\alpha \) if both integrals converge.]
   Suppose further that \( f(x) \geq 0 \) and \( f \) is monotone decreasing on \( x \geq 1 \). Prove that \( \int_{-1}^\infty f(x) \, dx \) converges if and only if \( \sum_{n=1}^{\infty} f(n) \) converges.

3. [Integration by parts for improper integrals] Show that in some cases integration
   by parts can be applied to the “improper” integrals defined in the previous problem;
   that is, state appropriate hypotheses, formulate a theorem, and prove it.
   Your hypotheses should be applicable in the following special case: the improper
   integrals \( \int_0^\infty \cos(x) \, dx/(x+1) \) and \( \int_0^\infty \sin(x) \, dx/(x+1) \) converge and are equal.
   Show that one of these two integrals (which one?) converges absolutely, but the
   other does not.

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1Quote taken from Chapter 10 of M. Artin’s Algebra. It roughly translates as “Please forget all
that you have learned in school, for you haven’t [really] learned it.” Don’t complain about the German
transcription, which is presumably of some local dialect — even I recognize that this isn’t the the
German we auf der Schule lernen.

2For the first of these, cf. also Rudin #1: Suppose \( \alpha : [a, b] \to \mathbb{R} \) is increasing, and continuous
at \( x_0 \). Define \( f : [a, b] \to \mathbb{R} \) by \( f(x) = 0 \) if \( x \neq x_0 \) and \( f(x_0) = 1 \). Then \( f \in \mathcal{R}(\alpha) \) [i.e. \( f \) is integrable with respect to \( \alpha \)], and \( \int_a^b f(x) \, d\alpha = 0 \).
4. [Bernoulli polynomials] Prove that for each positive integer $m$ there exists a polynomial $B_m$ such that $\sum_{i=1}^n i^{m-1} = B_m(n)$ for all positive integers $n$. [Hint: What must the polynomial $B_m(x) - B_m(x-1)$ be? The map taking any polynomial $P(x)$ to the polynomial $Q(x) := P(x) - P(x-1)$ is linear.] Determine the leading coefficient of $B_m$, and deduce the value of $\int_0^b x^{m-1} \, dx$ for any $b > 0$ (and thus also of $\int_a^b x^{m-1} \, dx$) without using the Fundamental Theorem of Calculus. Beyond the leading term, what further patterns can you detect in the coefficients of $B_m$? Can you prove any of these patterns? (You may need to go at least to $m = 6$ or $m = 7$ to see what’s going on; a computer algebra system could help to handle the linear algebra manipulations.)

5. [Fermat] Prove using the Riemann-sum definition of the integral that $\int_a^b x^{r-1} \, dx = (b^r - a^r)/r$ for every nonzero rational number $r$ and all real $a, b$ such that $0 < a < b$. [Note: since Fermat predated Newton, the solution cannot use the Fundamental Theorem of Calculus. Besides the special case that $r$ is a positive integer, addressed in the previous problem, you might also find a solution for the special case that $1/(r-1)$ is a positive integer — but this will not directly lead you to a solution of the general case.]

6. In the vint handout on integration of vector-valued functions, you might have expected a theorem to the effect that such a function is integrable (as defined there) with integral $I$ if and only if for each $\epsilon > 0$ there exists a partition $P$ of whose Riemann sums differ from $I$ by vectors of norm at most $\epsilon$. Certainly the existence of such $P$ is a consequence of integrability, but in fact the converse implication does not hold! Prove this by finding a normed vector space $V$ and a function $f : [0, 1] \rightarrow V$ such that $\Delta(P) = 1$ for any partition $P$ (and thus $f \notin \mathscr{A}$), but nevertheless for each $\epsilon$ there exist partitions $P$ such that every Riemann sum $R(P, \bar{f})$ for $\int_0^1 f(x) \, dx$ has norm at most $\epsilon$. [Hints: $f$ cannot be continuous or even nearly (e.g. piecewise) continuous, because then our vector version of Thm. 6.8 would yield integrability; in fact the function I have in mind is discontinuous everywhere. Moreover, $V$ cannot be finite dimensional. Thus the example is rather pathological — but it is also simple enough that it can be described and proved in a short paragraph.]

Finally, (indefinite) integration of arbitrary rational functions:

7. [Partial fractions] Let $k$ be an algebraically closed field. Let $K = k(x)$, the field of rational functions in one variable $x$ with coefficients in $k$. Show that the following elements of $K$ constitute a basis for $K$ as a vector space over $k$: $x^n$ for $n = 0, 1, 2, 3, \ldots$, and $1/(x - x_0)^n$ for $x_0 \in k$ and $n = 1, 2, 3, \ldots$. (Linear independence is easy. To prove that the span is all of $K$, consider for any polynomial $Q \in k[x]$ the subspace $V_Q := \{ P/Q : P \in k[x], \deg(P) < \deg(Q) \}$ of $K$, and compare its dimension with the number of basis vectors in $V_Q$.)

8. Prove that the integral of any $f \in \mathbb{R}(x)$ is a rational function plus a linear combination of functions of the form $\log |x - x_0|$, $\log((x - x_0)^2 + c)$, and $\tan^{-1}(ax + b)$ ($x_0, a, b, c \in \mathbb{R}$, $c > 0$).

This problem set due Monday, 27 March, at the beginning of class.

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3 Not that it matters for our purposes, but the “Bernoulli polynomials” usually seen in the literature differ from our $B_m$ by an additive constant.

4 Along the way we again encounter a natural example of a vector space with an uncountable algebraic basis (assuming $k$ is uncountable, e.g. $k = \mathbb{C}$).