The space $l_2$ is much larger than any of the finite-dimensional Hilbert spaces $F^n$ — for instance, it is not locally compact — but it is still small enough to be “separable”; this in fact topologically characterizes $l_2$. This notion is defined as follows:

**Definition.** A metric space is **separable** if it contains a dense countable subset.

(We consider a finite or even empty set to be countable. The phrase “second countable” is sometimes used for “separable”.)

**Examples.** Any compact set is separable: for each $n = 1, 2, 3, \ldots$ there is a finite $(1/n)$-net, and the union of these nets over $n$ is a countable dense set. The metric space $\mathbb{R}$ is separable because $\mathbb{Q} \subset \mathbb{R}$ is a dense countable subset. The direct sum of two separable spaces is separable, because the Cartesian product of two countable sets is countable. Thus in particular $\mathbb{C}$, so also $\mathbb{R}^n$ and $\mathbb{C}^n$, are separable. Even $l_2$ is separable, because $l_2^{(0)}$ is dense in $l_2$ and $l_2^{(0)}$ is a countable union of separable subsets $F^n$.

If $X$ is a separable metric space, and $S \subset X$ is a subset such that there exists $d_0 > 0$ with $d(s, s') \geq d_0$ for all distinct $s, s' \in S$, then $S$ is countable. Indeed, for each $\epsilon > 0$, we can write $X$ as the countable union of $\epsilon$-neighborhoods (specifically, of $\epsilon$-neighborhoods centered at the points of its dense countable subset), and once $\epsilon < d/2$ each of these neighborhoods contains at most one point of $S$. Thus conversely if $X$ is a metric space containing an uncountable subset $S$ any two of whose points are at least $d_0$ apart, then $X$ is not separable. For example, a discrete space is separable if and only if it is countable. The normed space $l_\infty$ of bounded sequences $(a_1, a_2, \ldots)$ of scalars with the sup norm is not separable, because it contains the uncountable subset $S = \{\{a_n\}_{n=1}^\infty : \text{each } a_n = 0 \text{ or } 1\}$ with $d_0 = 1$.

**Theorem.** Let $V$ be an inner product space. Then $V$ is separable if and only if it has a countable ontb. In this case every orthogonal set in $V$ is countable.

**Proof:** If $V$ is separable, let $\{v_n\}_{n=1}^\infty$ be a dense sequence in $V$. Clearly $\{v_n\}$ spans $V$ topologically. Discard each $v_n$ that is a linear combination of $v_1, \ldots, v_{n-1}$; the resulting linearly independent sequence may no longer be dense, but its linear span is the same, so is still dense in $V$. Now apply Gram-Schmidt to obtain an orthonormal sequence $w_n$ still with the same linear span, so the $w_n$ constitute a countable ontb.

Conversely, if $V$ has a countable ontb it is either isometric with $F^n$ or has a dense subset isometric with $l_2^{(0)}$, and we have shown that each of these spaces
is separable.

Finally, if $S$ is any orthonormal subset of an inner product space then any two of its points are at the same distance, namely $\sqrt{2}$. Thus if the space is separable then $S$ must be countable. By normalization the same is true also of orthogonal sets. □

**Corollary.** Every separable Hilbert space is isometric with either $\mathbb{F}^n$ (some $n = 0, 1, 2, \ldots$) or $l_2$.

**Orthogonal projections and complements in Hilbert space.** Most uses of the completeness of a Hilbert space go through the following results, which show that orthogonal projections and complements work for a Hilbert space as they do for a finite-dimensional inner product space. In particular, we can identify a Hilbert space $\mathcal{H}$ with its topological dual $\mathcal{B}(\mathcal{H}, \mathbb{F})$ as we did for a finite-dimensional inner-product space.

**Theorem.** Let $V$ be an inner product space, and $W \subset V$ a complete subspace. Then for each $v \in V$ there exists a unique $P(v) \in W$ such that $|v - P(v)| = \min_{w \in W} |v - w|$.

**Proof:** Since $\{ |v - w| : w \in W \}$ is a nonempty set bounded below, it has an infimum, call it $d$. Let $\{w_n\}$ be any sequence in $W$ such that $|v - w_n| \to d$. We shall show that $w_n$ is necessarily a Cauchy sequence. Since $W$ is assumed complete, it will follow that $\{w_n\}$ converges to some $w \in W$. Then $|v - w| = d$, so we may set $P(v) = w$.

Suppose that $u_1, u_2 \in W$ with $|v - u_i|^2 \leq d^2 + \epsilon^2$. By the parallelogram law,

$$|v - \frac{u_1 + u_2}{2}|^2 = \frac{1}{2} (|v - u_1|^2 + |v - u_2|^2) - \frac{1}{4} |u_1 - u_2|^2$$

But the left-hand side is at least $d^2$, and the right hand side at most $d^2 + \epsilon^2 - \frac{1}{4} |u_1 - u_2|^2$. Thus $|u_1 - u_2| \leq 2\epsilon$. In particular, taking $\epsilon = 0$, we see that if $P(v)$ exists then it is unique.

Now for each $\epsilon > 0$ there exists $N$ such that $|v - w_n|^2 \leq d^2 + \epsilon^2$ for all $n > N$. Thus $|w_m - w_n| \leq 2\epsilon$ for all $m, n > N$, and we’re done. □

As in the finite-dimensional case, it is then readily seen that $v - P(v) \subset W^\perp$. We thus express any $v \in V$ as a sum of vectors in $W$ and $W^\perp$. This representation is unique because $W \cap W^\perp = \{0\}$. It follows that $P : V \to W$ is a linear transformation. This transformation is called (orthogonal) projection to $W$.

**Corollary.** Under these hypotheses, $V$ is the orthogonal direct sum of $W$ and $W^\perp$, and $W = (W^\perp)^\perp$.

Note that for any $W \subset V$, the orthogonal complement $W^\perp$ is a closed subspace of $V$, and is the same as $\overline{W^\perp}$ where $\overline{W}$ is the closure of $W$ in $V$. Thus $(W^\perp)^\perp$ is a closed subspace of $W$ containing $W$. By the above Corollary, if $\overline{W}$ is complete
then \((W^\perp)^\perp = \overline{W}\). Without this hypothesis on \(\overline{W}\), it could happen that \((W^\perp)^\perp\) is strictly larger than \(\overline{W}\). For instance, let \(V = l_2^{(0)}\) and let \(W\) be the intersection of \(l_2^{(0)}\) with the orthogonal complement of \((1, 1/2, 1/3, 1/4, \ldots)\) in \(l_2\).

Now let \(V\) be a Hilbert space. Then \(W \subset V\) is complete if and only if it is closed. Thus in a Hilbert space \((W^\perp)^\perp = W\) if and only if \(W = \overline{W}\).

We can now conclude that in a separable Hilbert space any orthogonal (orthonormal) set is contained in an orthogonal topological basis (ontb). Using the Axiom of Choice the same can be proved for an arbitrary Hilbert space \(\mathcal{H}\).

We can then define the dimension of \(\mathcal{H}\) as the cardinality of a basis of \(\mathcal{H}\) — once we show that all bases have the same cardinality. We have done this already for a Hilbert space with a countable basis; given the existence of ontb’s, it can be done in general without much further difficulty using the fact that \(c \cdot \aleph_0 = c\)

for any infinite cardinal \(c\).

The topological dual of a normed vector space \(V\) is the vector space \(\mathcal{B}(V, F)\) of continuous linear functionals on \(V\). If \(V\) is an inner product space then any \(v \in V\) may be regarded as the functional \(w \mapsto \langle v, w \rangle\); this is continuous of norm \(|v|\) by Cauchy-Schwarz, so we get an isometric embedding of \(V\) into \(V^*\). In general the image of \(V\) is not all of \(V^*\), since \(V^*\) is necessarily complete (why?). However, this is the only obstruction: a Hilbert space is its own topological dual. More precisely:

**Theorem.** Let \(V\) be a Hilbert space and \(v^* \in V^*\) a continuous linear functional. Then there exists a unique \(v \in V\) such that \(v^*(w) = \langle w, v \rangle\) for all \(w \in V\).

**Proof:** Uniqueness is easy: if \(v_1, v_2\) are two such \(v\)’s then \(v^*(v_1 - v_2) = 0\) yields \(|v_1 - v_2|^2 = 0\) and thus \(v_1 = v_2\). We next prove existence. If \(v^* = 0\) then we of course take \(v = 0\). Else let \(W = \ker v^*\). This is a proper closed subspace of \(V\) and thus has nonzero orthogonal complement \(W^\perp\). We claim that \(W^\perp\) is one-dimensional. Else it would contain two linearly independent vectors \(v_1, v_2\); but we can find scalars \(a_1, a_2\) not both zero such that \(a_1 v^*(v_1) + a_2 v^*(v_2) = 0\), and then \(a_1 v_1 + a_2 v_2\) is a nonzero vector in \(W \cap W^\perp\), which is impossible. For the same reason, if \(v_0\) is a nonzero vector in \(W^\perp\) then \(v^*(v_0) \neq 0\). Choose such \(v_0\), and define

\[
v = \frac{v^*(v_0)}{|v_0|^2} v_0.
\]

Then \(v^*(v_0) = \langle v, v_0 \rangle\). Moreover \(v^*(w) = \langle w, v \rangle\) for all \(w \in V\) since both sides vanish. But we showed that \(v_0\) spans \(W^\perp\), and thus \(v_0\) together with \(W\) span \(V\). Thus \(v^*(w) = \langle w, v \rangle\) for all \(w \in V\) and we are done. \(\Box\)