Slogan. Tensor products of vector spaces are to Cartesian products of sets as direct sums of vector spaces are to disjoint unions of sets.

Description. For any two vector spaces $U, V$ over the same field $F$, we will construct a tensor product $U \otimes V$ (occasionally still known also as the “Kronecker product” of $U$ and $V$), which is also an $F$-vector space. If $U, V$ are finite dimensional then so is $U \otimes V$, with $\dim(U \otimes V) = \dim U \cdot \dim V$. If $U$ has basis $\{u_i : i \in I\}$ and $V$ has basis $\{v_j : j \in J\}$, then $U \otimes V$ has basis $\{u_i \otimes v_j : (i, j) \in I \times J\}$.

This notation $u_i \otimes v_j$ is a special case of a map $\otimes : U \times V \rightarrow U \otimes V$, which is bilinear: for each $u_0 \in U$, the map $v \mapsto u_0 \otimes v$ is a linear map from $V$ to $U \otimes V$, and for each $v_0 \in V$, the map $u \mapsto u \otimes v_0$ is a linear map from $U$ to $U \otimes V$. So, for instance,

$$(2u_1 + 3u_2) \otimes (4v_1 - 5v_2) = 8u_1 \otimes v_1 - 10u_1 \otimes v_2 + 12u_2 \otimes v_1 - 15u_2 \otimes v_2.$$

The element $u \otimes v$ of $U \otimes V$ is called the “tensor product of $u$ and $v$”.

Definitions. Such an element $u \otimes v$ is called a “pure tensor” in $U \otimes V$. The general element of $U \otimes V$ is not a pure tensor; for instance you can check that if $\{u_1, u_2\}$ is a basis for $U$ and $\{v_1, v_2\}$ is a basis for $V$ then

$$a_{11}u_1 \otimes v_1 + a_{12}u_1 \otimes v_2 + a_{21}u_2 \otimes v_1 + a_{22}u_2 \otimes v_2$$

is a pure tensor if and only if $a_{11}a_{22} = a_{12}a_{21}$. But any element of $U \otimes V$ is a linear combination of pure tensors. The basis-free construction of $U \otimes V$ is obtained in effect by declaring that $U \otimes V$ consists of linear combinations of pure tensors subject to the condition of bilinearity. More formally, we define $U \otimes V$ as a quotient space:

$$U \otimes V := Z/Z_0,$$

where $Z$ is the (huge) vector space with one basis element $u \otimes v$ for every $u \in U$ and $v \in V$ (that is, $Z$ is the space of formal (finite) linear combinations of the symbols $u \otimes v$), and $Z_0 \subseteq Z$ is the subspace generated by the linear combinations of the form

$$(u + u') \otimes v - u \otimes v - u' \otimes v, \quad u \otimes (v + v') - u \otimes v - u \otimes v',$$

$$(au) \otimes v - a(u \otimes v), \quad u \otimes (av) - a(u \otimes v)$$

for all $u, u' \in U$, $v, v' \in V$, $a \in F$.

Properties. To see this definition in action and verify that it does what we want, let us prove our claim above concerning bases: If $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ are bases for $U$ and $V$ then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ is a basis for $U \otimes V$. Naturally, for any vectors $u \in U$, $v \in V$, we write “$u \otimes v$” for the image of $u \otimes v \in Z$ under the quotient map $Z \rightarrow Z/Z_0 = U \otimes V$.

Let $W$ be a vector space with basis $\{w_{ij}\}$ indexed by $I \times J$. We construct linear maps

$$\alpha : W \rightarrow U \otimes V, \quad \beta : U \otimes V \rightarrow W,$$
with \( \alpha(w_{ij}) = u_i \otimes v_j \) and \( \beta(u_i \otimes v_j) = w_{ij} \). We prove that \( \alpha \) and \( \beta \) are each other’s inverse. This will show that \( \alpha, \beta \) are isomorphisms that identify \( w_{ij} \) with \( u_i \otimes v_j \), thus proving our claim. In each case we use the fact that choosing a linear map on a vector space is equivalent to choosing an image of each basis vector. The map \( \alpha \) is easy: we must take \( w_{ij} \) to \( u_i \otimes v_j \). As to \( \beta \), we don’t yet have a basis for \( U \otimes V \), so we first define a map \( \tilde{\beta} : Z \rightarrow W \), and show that \( Z_0 \subseteq \ker \tilde{\beta} \), so \( \tilde{\beta} \) “factors through \( Z_0 \)”, i.e., descends to a well-defined map from \( Z/Z_0 = U \otimes V \). Recall that \( \{ u \otimes v : u \in U, v \in V \} \) is a basis for \( Z \). For all \( u = \sum_i a_i u_i \in U \) and \( v = \sum_j b_j v_j \in V \), we define
\[
\tilde{\beta}(u \otimes v) = \sum_i \sum_j a_i b_j w_{ij}.
\]
Note that this sum is actually finite because the sums for \( u \) and \( v \) are finite, so the sum represents a legitimate element of \( W \). We then readily see that \( \ker \tilde{\beta} \) contains \( Z_0 \), because each generator of \( Z_0 \) maps to zero. We check that \( \beta \circ \alpha \) and \( \alpha \circ \beta \) are the identity maps on our generators of \( W \) and \( U \otimes V \). The former check is immediate: \( \tilde{\beta}(u \otimes v) = w_{ij} \).

The latter takes just a bit more work: it comes down to showing that
\[
u \otimes v - \sum_i \sum_j a_i b_j (u_i \otimes v_j) \in Z_0.
\]

But this is straightforward, since the choice of \( \tilde{\beta}(u \otimes v) \) was forced on us by the requirement of bilinearity. This exercise completes the proof of our claim.

Our initial Slogan, and/or the symbol \( \otimes \) for tensor product, and/or the formula for \( \dim(U \otimes V) \) in the finite-dimensional case, lead us to expect identities such as
\[
V_1 \otimes V_2 \cong V_2 \otimes V_1, \quad (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3),
\]
and
\[
U \otimes (V_1 \otimes V_2) \cong (U \otimes V_1) \otimes (U \otimes V_2).
\]
These are true, and in fact are established by canonical isomorphisms taking \( v_1 \otimes v_2 \) to \( v_2 \otimes v_1 \), \( (v_1 \otimes v_2) \otimes v_3 \) to \( v_1 \otimes (v_2 \otimes v_3) \), and \( u \otimes (v_1, v_2) \) to \( (u \otimes v_1, u \otimes v_2) \). In each case this is demonstrated by first defining the linear maps and their inverses on the level of the \( Z \) spaces and then showing that they descend to the tensor products which are the quotients of those \( Z \)’s. Even more simply we show that
\[
V \otimes F \cong V, \quad V \otimes \{0\} = \{0\}
\]
for any \( F \)-vector space \( V \).

A universal property. Suppose now that we have a linear map
\[
F : U \otimes V \rightarrow X
\]
for some \( F \)-vector space \( X \). Define a function \( f : U \times V \rightarrow X \) by
\[
f(u, v) := F(u \otimes v).
\]
Then this map is bilinear, in the sense described above. Conversely, for any function \( f : U \times V \rightarrow X \) we may define \( \tilde{F} : Z \rightarrow X \) by setting \( \tilde{F}(u \otimes v) = f(u, v) \), and \( \tilde{F} \) descends
to $Z/Z_0 = U \otimes V$ if and only if $f$ is bilinear. Thus a bilinear map on $U \times V$ is tantamount to a linear map on $U \otimes V$; more precisely, there is a canonical isomorphism between the vector space of bilinear maps: $U \times V \rightarrow X$ and the space $\text{Hom}(U \otimes V, X)$ that takes $f$ to the map $u \otimes v \mapsto f(u, v)$. Stated yet another way, every bilinear function on $U \times V$ factors through the bilinear map $(u, v) \mapsto u \otimes v$ from $U \times V$ to $U \otimes V$. This “universal property” of $U \otimes V$ could even be taken as a definition of the tensor product (once one shows that it determines $U \otimes V$ up to canonical isomorphism).

For example, a bilinear form on $V \times V$ is a bilinear map from $V \times V$ to $F$, which is now seen to be a linear map from $V \otimes V$ to $F$, that is, an element of the dual space $(V \otimes V)^*$. We shall come back to this important example later.

**Tensor products of linear maps.** Here is another key example. For any linear maps $S : U \rightarrow U'$ and $T : V \rightarrow V'$ we get a bilinear map $U \times V \rightarrow U' \otimes V'$ taking $(u, v)$ to $S(u) \otimes T(v)$. Thus we have a linear map from $U \otimes V$ to $U' \otimes V'$. We call this map $S \otimes T$. The map $(S, T) \mapsto S \otimes T$ is itself a bilinear map from $\text{Hom}(U, U') \times \text{Hom}(V, V')$ to $\text{Hom}(U \otimes V, U' \otimes V')$, which yields a canonical map $\text{Hom}(U, U') \otimes \text{Hom}(V, V') \rightarrow \text{Hom}(U \otimes V, U' \otimes V')$. At least if $U, U', V, V'$ are all finite dimensional, this map is an isomorphism. This can be seen by choosing bases for $U, V, U', V'$. This yields bases for $U \otimes V$ and $U' \otimes V'$ (the $u_i \otimes v_j$ construction above), for $\text{Hom}(U, U')$ and $\text{Hom}(V, V')$ (the matrix entries), and thus for $\text{Hom}(U \otimes V, U' \otimes V')$ and $\text{Hom}(U, U') \otimes \text{Hom}(V, V')$; and our map takes the $(i, j, i', j')$ element of the first basis to the $(i, j, i', j')$ element of the second. If we represent $S, T, S \otimes T$ by matrices, we get a bilinear map

$$\text{Mat}(m, n) \times \text{Mat}(m', n') \rightarrow \text{Mat}(mm', mn')$$

called the Kronecker product of matrices; the entries of $\mathcal{M}(S \otimes T)$ are the products of each entry of $\mathcal{M}(S)$ with every entry of $\mathcal{M}(T)$.

**Tensor products and duality.** If the above seems hopelessly abstract, consider some special cases. Suppose $U' = V = F$. We then map $U^* \otimes V^*$ to the familiar space $\text{Hom}(U, V^*)$, and the map is an isomorphism if $U, V^*$ are finite dimensional. Thus if $V^*$ are finite dimensional then we have identified $\text{Hom}(V_1, V_2)$ with $V_1^* \otimes V_2$. If instead we take $U' = V' = F$ then we get a map $U^* \otimes V^* \rightarrow (U \otimes V)^*$, which is an isomorphism if $U, V$ are finite dimensional. In particular, if $U = V$ we find that a bilinear form on a finite-dimensional vector space $V$ is tantamount to an element of $V^* \otimes V^*$.

**Changing the ground field.** In another direction, suppose $F'$ is a field containing $F$, and let $V' = V \otimes_F F'$. (When more than one field is present, we’ll use the subscript to indicate the intended ground field for the tensor product. A larger ground field gives more generators for $Z_0$ and thus may yield a smaller tensor product $Z/Z_0$. In most of the applications we’ll have $F = \mathbb{R}$, $F' = \mathbb{C}$.) We claim that $V'$ is in fact a vector space over $F'$. For each $a \in F'$, consider multiplication by $a$ as an $F$-linear map on $F'$. Then $1_V \otimes a$ is a linear map from $V'$ to itself, which we use as the multiplication-by-$a$ map on $V'$. The fact that multiplication by $ab$ is the same as multiplication by $b$ followed by multiplication by $a$ is then a special case of the fact that composition of linear maps is consistent with tensor products:

$$(S_1 \circ S_2) \otimes (T_1 \circ T_2) = (S_1 \otimes T_1) \circ (S_2 \otimes T_2).$$

This in turn is true because it holds on pure tensors $u \otimes v$. We usually think of $V'$ as $V$ with scalars extended from $F$ to $F'$. 

3
If $V$ has dimension $n < \infty$ with basis $\{v_i\}_{i=1}^n$ then $\{v_i \otimes 1\}_{i=1}^n$ is a basis for $V'$. (To see this, begin by using $\{v_i\}$ to identify $V$ with $F \oplus F \oplus \cdots \oplus F$, and tensor the direct sum with $F'$. If $\{v_i\}_i \in I$ is a basis for $V$, it is still true that $\{v_i \otimes 1\}_{i \in I}$ is a basis for $V'$.) If $T : U \rightarrow V'$ is a linear map between $F$-vector spaces then $T \otimes 1$ is an $F'$-linear map from $U \otimes F = U'$ to $V'$; when $U, V'$ are finite dimensional, this map has the same matrix as $T$ as long as we use the bases $\{u_i \otimes 1\}, \{v_j \otimes 1\}$ for $U', V'$. We usually think of $T \otimes 1$ as $T$ with scalars extended from $F$ to $F'$.

**Symmetric and alternating tensor squares.** The tensor square $V \otimes^2$ of $V$ is defined by

$$V \otimes^2 := V \otimes V.$$  
Likewise we can define tensor cubes and higher tensor powers. (Of course $V \otimes^1$ is $V$ itself; what should $V \otimes^0$ be?) Our isomorphism $V_1 \otimes V_2 \cong V_2 \otimes V_1$ then becomes an isomorphism $s$ from $V \otimes V$ to itself. This map is not the identity (unless $V$ has dimension 0 or 1), but it is always an involution; that is, $s^2$ is the identity. The subspace of $V \otimes V$ fixed under $s$ is the symmetric square of $V$, denoted $\text{Sym}^2 V$. It can also be defined as a quotient $Z/Z_1$, with $Z$ as in the definition of $V \otimes V$, and $Z_1$ generated by $Z_0$ and combinations of the form $v_1 \otimes v_2 - v_2 \otimes v_1$. Likewise we may define the symmetric cube and higher symmetric powers of $V$ by declaring $\text{Sym}^2 V$ to be the subspace of $V \otimes^k$ invariant under arbitrary permutations of the $k$ indices. If $V$ has finite dimension $n$ then $\text{Sym}^2 V$ has dimension $(n^2 + n)/2$; do you see why? What does this correspond to in terms of our motivating Slogan? Can you determine the dimension of $\text{Sym}^k V$ for $k = 3, 4, \ldots$?

We can also regard $\text{Sym}^2 V$ as the +1-eigenspace of $s$. Since $s^2 = 1$, we know that the only possible eigenvalues are $\pm 1$. What then of the $-1$ eigenspace? Usually this is called the alternating square of $V$, denoted by $\wedge^2 V$, and can be obtained as the quotient of $Z$ by the subspace generated by $Z_0$ and combinations of the form $v_1 \otimes v_2 + v_2 \otimes v_1$; the image of $v_1 \otimes v_2$ in $\wedge^2 V$ is denoted by $v_1 \wedge v_2$. The caveat “usually” is necessary because in characteristic 2 one cannot distinguish between $-1$ and $+1$! Note however that if $2 \neq 0$ then the identity $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$ entails $v \wedge v = 0$ for all $v$. Conversely, in any characteristic the identity $v \wedge v = 0$ entails $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$ for all $v_1, v_2 \in V$. In other words, the subspace $Z_2$ of $Z$ generated by $Z_0$ and all elements of the form $v \wedge v$ contains all combinations $v_1 \otimes v_2 + v_2 \otimes v_1$. Proof:

\[v_1 \otimes v_2 + v_2 \otimes v_1 = (v_1 + v_2) \otimes (v_1 + v_2) - (v_1 \otimes v_1) - (v_2 \otimes v_2) - B,\]

where $B \in Z_0$ (why?). So, we actually define $\wedge^2 V$ to be $Z/Z_2$; this is identical with the $-1$ eigenspace of $s$ when $2 \neq 0$, and does what we want it to even when $2 = 0$. If $V$ has finite dimension $n$ then $\wedge^2 V$ has dimension $(n^2 - n)/2$, and if $V$ has basis $\{v_i\}_{i \in I}$ for some totally ordered index set $I$ then $\wedge^2 V$ has basis $\{v_i \wedge v_j : i < j\}$. We will later define higher alternating powers $\wedge^k V$ of dimension $\binom{n}{k}$ (so $\wedge^k V$ will correspond to unordered $k$-tuples under our Slogan). The key ingredient is the existence of the sign homomorphism from the group of permutations of $\{1, 2, \ldots, k\}$ to the two-element group $\{\pm 1\}$.

If we apply the $\text{Sym}^k$ construction to the space $V^* \otimes V$ of bilinear forms on $V$, we obtain the space of homogeneous polynomial functions of degree $k$ from $V \rightarrow F$. For instance, a symmetric bilinear form on $V$ is an element of $\text{Sym}^2 V^*$. Likewise $\wedge^2 V^*$ consists of the alternating (a.k.a. antisymmetric) bilinear forms on $V$. 
