The expression $\delta_{ij}$ is called the Kronecker delta (after the mathematician Leopold Kronecker [1823–1891], who made more substantial contributions to mathematics than this).\textsuperscript{1}

— Corwin and Szczarba, *Calculus in Vector Spaces*, p.124

A bit more about the structure of infinite-dimensional vector spaces:

1. i) Prove that a vector space with a countable\textsuperscript{2} spanning set over a countable field is countable.
   
   ii) Prove that a vector space with a countable spanning set over any field does not have an uncountable linearly independent set.
   
   iii) Prove that if a vector space $V$ has a countable spanning set $S$ then some subset of $S$ is a basis for $V$.

2. Suppose $V$ is a vector space and $U$ a subspace with basis $B_0$. Suppose that for some (finite) $n$ we can extend $B_0$ by $n$ vectors to obtain a basis for $V$. Prove that if $B$ is any basis for $U$, and $B'$ any basis for $V$ that contains $B$, then $\{v \in B': v \notin B\}$ has cardinality $n$. $U$, and thus also $V$, is finite dimensional), once we’ve shown that the dimension is well-defined when finite; but the point is that the result still holds without that assumption.]

Some basics about linear transformations and their matrices:

3.–4. Solve Exercises B-11 (page 68) and D-9, 10, 16 (page 89) from Chapter 3 of the textbook. For B-11, if $S_1 \cdots S_n$ is injective, what if anything can be said of $S_1, S_2, \ldots, S_n$? For the other three exercises, note that “$L(V)$” is Axler’s abbreviation for “$\mathcal{L}(V, V)$” (it is also known as $\text{End}(V) = \text{Hom}(V, V)$).

5. Let $P_n$ be the ($\mathbb{R}$- or $\mathbb{C}$-)vector space of polynomials of degree at most $n$, and $L : P_n \to P_n$ be the linear transformation taking any polynomial $P(x)$ to the polynomial

\[
(L(P))(x) = (x - 3)P''(x)
\]

(here $P''$ is the second derivative $d^2P/dx^2$). Exhibit a matrix for $L$ relative to a suitable basis for $P_n$, and determine the kernel, image, and rank of $L$.

\textsuperscript{1}It is the $(i,j)$ entry of an identity matrix, that is, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise; also $\delta_{ij} = \varphi_j(\nu_i)$ where $(\nu_i)_{i=1}^n$ is a basis for a finite-dimensional vector space, and $(\varphi_j)_{j=1}^m$ its dual basis, see 3.96 on page 102.

\textsuperscript{2}For us “countable” means “finite or countably infinite.”
6. Let \( V, W \) be arbitrary vector spaces over the same field. Show that, for any vector \( v \) in \( V \), the evaluation map \( E_v : \mathcal{L}(V, W) \to W \) defined by \( E_v(L) = L(v) \) for all \( L \in \mathcal{L}(V, W) \) is a linear transformation. If \( V, W \) are finite dimensional, what is the dimension of \( \ker E_v \)?

7. Let \( V, W \) be vector spaces over the rational field \( \mathbb{Q} \). Prove that a map \( T : V \to W \) is linear if and only if \( T(v + v') = Tv + Tv' \) for all \( v, v' \in V \).

(Cf. the italicized note to Exercise 9 on p.58 of the textbook.)

More about duality:

8. We saw that, for any vector spaces \( V, W \), the dual of \( V \oplus W \) is naturally identified with \( V^* \oplus W^* \). What is the dual of \( \oplus_{i \in I} V_i \)? Use this to construct a vector space \( V \) over some field \( F \) such that \( V \) is not isomorphic with \( V^* \).

9. Let \( x_0, \ldots, x_m \) be distinct elements of \( F \). Recall that the \( m + 1 \) vectors \( v_i := (x_0^i, x_1^i, \ldots, x_m^i) \) \((0 \leq i \leq m)\) constitute a basis of \( F^{m+1} \). Describe the dual basis.

10. Finally, suppose \( F \) is a finite field of \( q \) elements, and let \( e \) be a positive integer such that \( 2e < q \). Then we can regard \( \mathcal{P}_{q-2e} \) as a subspace of \( F^F \) by evaluation at the elements of \( F \). Call this subspace \( U \). Show that for any \( v \in V \) there is at most one \( u \in U \) that differs from \( v \) in fewer than \( e \) coordinates; that is, there is at most one polynomial \( P \in \mathcal{P}_{q-2e} \) such that \( P(x) \neq v_x \) holds for fewer than \( e \) elements \( x \in F \). Suppose such \( P \) exists and is nonzero, and let \( d < e \) be the number of \( x \in F \) such that \( P(x) \neq v_x \).

Prove that \( d \) is also the smallest integer \( n \) such that the intersection of \( \mathcal{P}_{q-2e+d} \) with the vector space \( \mathcal{P}_d \) : \( \{ Pv : P \in \mathcal{P}_d \} \) contains some vector \( w \neq 0 \). Explain how this can be used to recover \( u \) in fewer than \( q \) calculations of such \( w \).

(The point of this is that each such calculation can be done “in polynomial time” [i.e. there exist \( C \) and \( k \) such that the calculation requires at most \( Cq^k \) field operations in \( F \), regardless of what \( q, e, v \) might be — one way to do this is “Gaussian elimination”; while trying all \( e \)-element subsets of \( F \) certainly cannot be done in polynomial time. Of course the exceptional case \( u = P = 0 \) can be detected in polynomial time.)

Problem set is due Friday, Sep. 22 in class.