Math 55a: Honors Abstract Algebra

Homework Assignment #11 (27 November 2017):

Representations of finite groups

\[ i^2 = j^2 = k^2 = ijk = -1 \]

—W.R. Hamilton, 1843 [cut into a stone on Brougham Bridge, Dublin; see also the final two problems].

We start with some applications of the general theory to permutation representations. Recall that if a finite group \( G \) acts on a finite set \( S \) then \( G^S \) is a representation of \( G \) and the associated character takes any \( g \in G \) to the number of fixed points of \( g \).

1. Let \( V = G^S \). Prove that the dimension of the fixed subspace \( V^G \) is the number of orbits of the action of \( G \) on \( S \), both by identifying \( V^G \) explicitly in terms of the orbit decomposition and by using the formula \( \langle \chi, 1 \rangle \) for that dimension. Deduce that \( G \) is transitive \( \iff \dim V^G = 1 \iff \langle \chi, 1 \rangle = 1 \).

2. Suppose then that \( G \) acts transitively on \( S \). Let \( V_0 \) be the orthogonal complement \( \ominus V \ominus V^G \) of \( V^G \) in \( V \), and let \( \chi_0 \) be its character. Determine \( \langle \chi_0, \chi_0 \rangle \), and deduce that \( V_0 \) is irreducible if and only if \( G \) acts doubly transitively on \( S \). [A group action on \( S \) is said to be “doubly transitive” when it is transitive on ordered pairs \((s_1, s_2)\) with \( s_1, s_2 \in S \) and \( s_1 \neq s_2 \).]

3. i) Show that for \( k = 1, 2, 3, \ldots \) the permutation representation of \( G \) on \( S^k \) is isomorphic with \( V^\otimes k \). Deduce a formula for the number of \( G \)-orbits on \( S^k \). (The action of \( G \) on \( S \) is not required to be transitive, and the action on \( S^k \) is coordinatewise.)

ii) Give a similar formula for the number of \( G \)-orbits on the set \( k^S \) of \( k \)-colorings of \( S^2 \).

iii) Use this formula to show that there are 36 carbon tetrahalides. (A “carbon tetrahalide” is a molecule \( CX_4 \) where each \( X \) is one of the four halogens \( F, Cl, Br, I \), and the four \( X \)'s are vertices of a tetrahedron centered on the \( C \) atom.) Verify this count directly. [Note that there are two kinds of \( CFClBrI \) because only orientation-preserving symmetries are allowed, so an asymmetric molecule is distinct from its mirror image (chemists call such mirror images “enantiomers”); that is, the relevant group \( G \) is what Artin calls the tetrahedral group \( T \) of order 12.]

What about the character of \( \chi_0 \)? Here the formulas are more complicated; the character of \( g \in G \) depends on the full characteristic polynomial of \( \rho(g) \), not just its trace. It’s easier to formulate the result in terms of a generating function, which is a formal power series \( X(g) = \sum_{k=0}^\infty \chi_k(g)T^k \) where \( \chi_k(g) \) is the trace of \( (\otimes^k\rho)(g) \) or \( (\wedge^k\rho)(g) \) respectively.

4. i) For \( \wedge^k \), this generating function is a polynomial of degree \( \dim(V) \), because once \( k > \dim(V) \) the \( k \)-th exterior power is the zero space so the trace is zero as well. Show that this polynomial is the determinant of \( 1 + T\rho(g) \).

ii) For \( \otimes^k \), show that \( X(g) = 1 / \det(1 - T\rho(g)) \).

\(^1\)a.k.a. Broom, which sounds the same in one pronunciation of “brougham”, which is a kind of horse-drawn carriage — and the source of “broughammed” (“traveled by brougham”, or possibly “equipped with a brougham”) which is a candidate for the longest English-language monosyllable. But I digress.

\(^2\)Mathematically a “\( k \)-coloring of \( S^n \)” is just a map from \( S \) to a \( k \)-element set \( C \). The name suggests that we think of \( C \) as a set of “colors” that can be used to paint each element of \( S \). Yet another equivalent description is a partition of \( S \) as the disjoint union of sets \( S_c \) where \( c \) ranges over \( C \) and \( S_c \) is the preimage of \( c \) under the map \( S \to C \).
iii) Now let $G = S_3$ and $V$ be the 3-dimensional permutation representation. Thus $S_3$ acts on the polynomial ring $C[z_1, z_2, z_3] = \bigoplus_{k=0}^\infty \text{Sym}^k V$ by permuting the variables $z_1, z_2, z_3$. Show that $\dim((\text{Sym}^k V)^G)$ is the $X^k$ coefficient of the generating function \(( (1 - X)(1 - X^2)(1 - X^3))^{-1}, \) and explain why this is consistent with the known result that the subring of $C[z_1, z_2, z_3]$ invariant under the action of $S_3$ consists of polynomials in the elementary symmetric functions $z_1 + z_2 + z_3, z_2z_3 + z_3z_1 + z_1z_2,$ and $z_1z_2z_3$ of degrees 1, 2, 3.

The irreducible representations of the direct product of two finite groups:

5. i) Let \( (V_1, \rho_1) \) and \((V_2, \rho_2)\) be complex representations of finite groups $G_1, G_2$. Define a representation of \((V, \rho)\) of $G := G_1 \times G_2$ by $V = V_1 \otimes V_2$ and $\rho(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$ for all $g_1 \in G_1$ and $g_2 \in G_2$. Find the character of $(V, \rho)$, and deduce that $(V, \rho)$ is irreducible if and only if both $(V_1, \rho_1)$ and $(V_2, \rho_2)$ are irreducible.

ii) Prove that every irreducible representation of $G$ arises from the construction in part (i) for some irreducible representations $V_1 = (\rho_1, g_1)$ and $V_2 = (\rho_2, g_2)$. [Hint: first show that if $V_1, V_2$ are irreducible then $V_1 \otimes V_2$ cannot be isomorphic with $W_1 \otimes W_2$ unless $V_1 \cong W_1$ and $V_2 \cong W_2$.]

The Hamilton quaternions are the skew field $\mathbb{H}$ defined as follows: $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication characterized by the properties that where 1 is the multiplicative identity while $i^2 = j^2 = k^2 = \bar{ij} = k = -ij). The quaternion group $Q_8$ is the subgroup \{±1, ±i, ±j, ±k\} of $\mathbb{H}^\ast$. In the last two problems you’ll verify that $\mathbb{H}$ is indeed a skew field and construct a representation $W$ of $Q_8$ over $\mathbb{R}$ that is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$ and has $\text{End}_{Q_8}(W) \cong \mathbb{H}$.

6. i) Let $A$ be the $\mathbb{R}$-vector space of matrices \(
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\) with $a, b, c, d \in \mathbb{C}$ such that \(\bar{a} = d \) and $\bar{b} = -c$. Prove that $A$ is closed under matrix multiplication, and every nonzero $A \in A$ is invertible.

ii) For $A \in A$ define $\sigma(A) = \text{tr}(A)I - A$. Prove that $\sigma$ is an anti-involution of $A$ (that is, $\sigma$ is a vector space involution of $A$ satisfying $\sigma(AA') = \sigma(A')\sigma(A)$ for all $A, A' \in A$). Compute $A\sigma(A)$ and $\sigma(A)A$, and use this to prove that $A$ contains the inverse of every nonzero $A \in A$.

iii) Find an isomorphism between $\mathbb{H}$ and $A$ that identifies $\sigma$ with the anti-involution taking $1, i, j, k$ to $+1, -i, -j, -k$ respectively. (This anti-involution is called “conjugation” in $\mathbb{H}$, and denoted by $q \leftrightarrow \bar{q}$ as is done for complex conjugation.)

7. i) The identification of $\mathbb{H}$ with $A$ yields a 2-dimensional complex representation $V$ of $\mathbb{H}^\ast$, and thus of $Q_8$. Prove that the character of any $q \in \mathbb{H}^\ast$ is the real number $q + \bar{q}$. Deduce that $V$ is an irreducible representation of $Q_8$. [You’ll recognize its character if you went to section last week.]

ii) The action of $\mathbb{H}^\ast$ on $\mathbb{H}$ by multiplication from the left gives $\mathbb{H}$ the structure of a 4-dimensional real representation $W$ of $\mathbb{H}^\ast$, and thus of $Q_8$. Compute its character for any $q \in \mathbb{H}^\ast$, and verify that it equals $2\chi_V(q)$. On the other hand, multiplication from the right by any $q \in \mathbb{H}$ commutes with our action, and this shows that $\text{End}_{Q_8}(W)$ contains a copy of $\mathbb{H}$. Prove that in fact $\text{End}_{Q_8}(W) \cong \mathbb{H}$.

iii) Use this to show that $W$ is irreducible as a real representation of $Q_8$.

[Another route to the result of (iii) starts by using the general theory to find that $W \otimes_R \mathbb{C}$ is isomorphic with $V \oplus V$ as a representation of $Q_8$; thus if $W$ were reducible it would be a direct sum of two irreducible real representations of $Q_8$, with $-1 \in Q_8$ acting on both by multiplication by $-1$. But, as with the “unitary trick” for complex representations, any real representation of a finite group has an invariant orthogonal form, so we’d get a homomorphism $Q_8 \to \text{O}_2(\mathbb{R})$ taking $-1$ to $-1$, and this is soon seen to be impossible, e.g. using the facts that $\text{SO}_2(\mathbb{R})$ is commutative and each element of $\text{O}_2(\mathbb{R})$ that is not in $\text{SO}_2(\mathbb{R})$ is an involution.]

This final problem set is due Wednesday, December 6 at 5 P.M.