Math 55a: Honors Abstract Algebra

Homework Assignment #11 (21 November 2016):
Representations of finite groups

\[ i^2 = j^2 = k^2 = ijk = -1 \]

—W.R. Hamilton, 1843 [cut into a stone on Brougham Bridge, Dublin; see also the final two problems].

We start with some applications of the general theory to permutation representations. Recall that if a finite group \( G \) acts on a finite set \( S \) then \( G^S \) is a representation of \( G \) and the associated character takes any \( g \in G \) to the number of fixed points of \( g \).

1. Let \( V = G^S \). Prove that the dimension of the fixed subspace \( V^G \) is the number of orbits of the action of \( G \) on \( S \), both by identifying \( V^G \) explicitly in terms of the orbit decomposition and by using the formula \( \langle \chi, 1 \rangle \) for that dimension. Deduce that \( G \) is transitive iff \( \dim V^G = 1 \) iff \( \langle \chi, 1 \rangle = 1 \).

2. Suppose then that \( G \) acts transitively on \( S \). Let \( V_0 \) be the orthogonal complement \( V \oplus V^G \) of \( V \) in \( V \), and let \( \chi_0 \) be its character. Determine \( \langle \chi_0, \chi_0 \rangle \), and deduce that \( V_0 \) is irreducible if and only if \( G \) acts doubly transitively on \( S \). [A group action on \( S \) is said to be “doubly transitive” when it is transitive on ordered pairs \( (s_1, s_2) \) with \( s_1, s_2 \in S \) and \( s_1 \neq s_2 \).]

3. i) Show that for \( k = 1, 2, 3, \ldots \) the permutation representation of \( G \) on \( S^k \) is isomorphic with \( V^\otimes k \). Deduce a formula for the number of \( G \)-orbits on \( S^k \). (The action of \( G \) on \( S \) is not required to be transitive, and the action on \( S^k \) is coordinatewise.)
   
i) Give a similar formula for the number of \( G \)-orbits on the set \( k^S \) of \( k \)-colorings of \( S \).
   
iii) Use this formula to show that there are 36 carbon tetrahalides. (A “carbon tetrahalide” is a molecule \( CX_4 \) where each \( X \) is one of the four halogens \( F, Cl, Br, I \), and the four \( X \)'s are vertices of a tetrahedron centered on the \( C \) atom.) Verify this count directly. [Note that there are two kinds of \( CFClBrI \) because only orientation-preserving symmetries are allowed, so an asymmetric molecule is distinct from its mirror image (chemists call such mirror images “enantiomers”): that is, the relevant group \( G \) is what Artin calls the tetrahedral group \( T \) of order 12.]

What about the character of \((\text{Sym}^k V, \text{Sym}^k \rho)\) or \((\wedge^k V, \wedge^k \rho)\)? Here the formulas are more complicated; the character of \( g \in G \) depends on the full characteristic polynomial of \( \rho(g) \), not just its trace. It’s easier to formulate the result in terms of a generating function, which is a formal power series \( X(g) = \sum_{k=0}^{\infty} \chi_k(g)T^k \) where \( \chi_k(g) \) is the trace of \((\text{Sym}^k \rho)(g)\) or \((\wedge^k \rho)(g)\) respectively.

4. i) For \( \wedge^k \) this generating function is a polynomial of degree \( \dim(V) \), because once \( k > \dim(V) \) the \( k \)-th exterior power is the zero space so the trace is zero as well. Show that this polynomial is the determinant of \( 1 + T \rho(g) \).
   
i) For \( \text{Sym}^k \), show that \( X(g) = 1 / \det(1 - T \rho(g)) \).
   
iii) Now let \( G = S_3 \) and \( V \) be the 3-dimensional permutation representation. Thus \( S_3 \) acts on the polynomial ring \( \mathbb{C}[z_1, z_2, z_3] = \bigoplus_{k=0}^{\infty} \text{Sym}^k V \) by permuting the variables

\[ \text{a.k.a. Broom, which sounds the same in one pronunciation of “brougham”, which is a kind of horse-drawn carriage — and the source of “broughammed” (“traveled by brougham”, or possibly “equipped with a brougham”) which is a candidate for the longest English-language monosyllable. But I digress.} \]
5. i) Let \((V, \rho)\) be complex representations of finite groups \(G_1, G_2\). Define a representation of \((V, \rho)\) of \(G := G_1 \times G_2\) by \(V = V_1 \otimes V_2\) and \(\rho((g_1, g_2)) = \rho_1(g_1) \otimes \rho_2(g_2)\) for all \(g_1 \in V_1\) and \(g_2 \in V_2\). Find the character of \((V, \rho)\), and deduce that \((V, \rho)\) is irreducible if and only if both \((V_1, \rho_1)\) and \((V_2, \rho_2)\) are irreducible.

ii) Prove that every irreducible representation of \(G\) arises from the construction in part (i) for some irreducible representations \(V_1 = (V_1, \rho_1)\) and \(V_2 = (V_2, \rho_2)\). [Hint: first show that if \(V_1, V_2\) are irreducible then \(V_1 \otimes V_2\) cannot be isomorphic with \(W_1 \otimes W_2\) unless \(V_1 \cong W_1\) and \(V_2 \cong W_2\).]

The Hamilton quaternions are the skew field \(H\) defined as follows: \(H\) is a 4-dimensional algebra over \(\mathbb{R}\) with basis \(1, i, j, k\) and multiplication characterized by the properties that where 1 is the multiplicative identity while \(i^2 = j^2 = k^2 = ijk = -1\) (so for instance \(ij = k = -ji\)). The quaternion group \(Q_8\) is the subgroup \(\{±1, ±i, ±j, ±k\}\) of \(H^*\). In the last two problems you'll verify that \(H\) is indeed a skew field and construct a representation \(W\) of \(Q_8\) over \(\mathbb{R}\) that is irreducible over \(\mathbb{R}\) but not over \(\mathbb{C}\) and has \(\text{End}_\mathbb{C}(W) \cong H\).

6. i) Let \(A\) be the \(\mathbb{R}\)-vector space of matrices \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) with \(a,b,c,d \in \mathbb{C}\) such that \(a = d\) and \(b = -c\). Prove that \(A\) is closed under matrix multiplication, and every nonzero \(A \in A\) is invertible.

ii) For \(A \in A\) define \(\sigma(A) = \text{tr}(A)I - A\). Prove that \(\sigma\) is an anti-involution of \(A\) (that is, \(\sigma\) is a vector space involution of \(A\) satisfying \(\sigma(\sigma(A)) = \sigma(A)\sigma(A)\) for all \(A, A' \in A\)). Compute \(\text{tr}(A)\) and \(\sigma(A),\sigma(A)A\), and use this to prove that \(\sigma\) contains the inverse of every nonzero \(A \in A\).

iii) Find an isomorphism between \(H\) and \(A\) that identifies \(\sigma\) with the anti-involution taking \(1, i, j, k\) to \(+1, -i, -j, -k\) respectively. (This anti-involution is called “conjugation” in \(H\), and denoted by \(q \leftrightarrow \bar{q}\) as is done for complex conjugation.)

7. i) The identification of \(H\) with \(A\) yields a 2-dimensional complex representation \(V\) of \(H^*\), and thus of \(Q_8\). Prove that the character of any \(q \in H^*\) is the real number \(q + \bar{q}\). Deduce that \(V\) is an irreducible representation of \(Q_8\). [You’ll recognize its character if you went to section last week.]

ii) The action of \(H^*\) on \(H\) by multiplication from the left gives \(H\) the structure of a 4-dimensional real representation \(W\) of \(H^*\), and thus of \(Q_8\). Compute its character for any \(q \in H^*\), and verify that it equals \(2\chi_V(q)\). On the other hand, multiplication from the right by any \(q \in H\) commutes with our action, and this shows that \(\text{End}_{\mathbb{R}}(W)\) contains a copy of \(H\). Prove that in fact \(\text{End}_{\mathbb{R}}(W) \cong H\).

iii) Use this to show that \(W\) is irreducible as a real representation of \(Q_8\).

[Another route to the result of (iii) starts by using the general theory to find that \(W \otimes_{\mathbb{R}} C\) is isomorphic with \(V \oplus V\) as a representation of \(Q_8\); thus if \(W\) were reducible it would be a direct sum of two irreducible real representations of \(Q_8\), with \(-1 \in Q_8\) acting on both by multiplication by \(-1\). But, as with the “unitary trick” for complex representations, any real representation of a finite group has an invariant orthogonal form, so we’d get a homomorphism \(Q_8 \rightarrow O_2(\mathbb{R})\) taking \(-1\) to \(-1\), and this is soon seen to be impossible, e.g. using the facts that \(SO_2(\mathbb{R})\) is commutative and each element of \(O_2(\mathbb{R})\) that is not in \(SO_2(\mathbb{R})\) is an involution.]

This final problem set is due Wednesday, November 30 at 5 P.M.