Math 55a: Honors Abstract Algebra

Homework Assignment #5 (1 October 2010):
Linear Algebra V: tensors, more eigenstuff, and a bit on inner products

The terms “proper value”, “characteristic value”, “secular value”, and “latent-value” or “latent root” are sometimes used [for “eigenvalue”] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are “latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf.” We will not adhere to his terminology.


We begin with some basic problems on tensors and tensor products. For the first of these, recall that the “rank” of a linear transformation \( T : U \rightarrow V \) is the dimension of its image \( T(U) \); the rank of a matrix is the rank of the linear transformation it represents.

1. Let \( \{ u_i \}_{i=1}^m \) and \( \{ v_j \}_{j=1}^n \) be bases of the \( F \)-vector spaces \( U \) and \( V \), and consider the general element \( w = \sum_i \sum_j w_{ij} (u_i \otimes v_j) \) of \( U \otimes V \). Prove that \( w \) is the sum of \( r \) pure tensors if and only if the matrix \( (w_{ij}) \) has rank at most \( r \).

2. Let \( V \) be a vector space of finite dimension \( n \) over a field \( F \). We constructed a linear map, the trace, from \( \mathcal{L}(V) \) to \( F \). Hence the map from \( \mathcal{L}(V) \times \mathcal{L}(V) \) to \( F \) taking \( (S, T) \) to the trace of \( ST \) is bilinear. Prove that it is symmetric. For what \( n \) can there exist \( S, T \in \mathcal{L}(V) \) such that \( ST - TS \) is the identity map? (By comparison, observe that the operators \( P \mapsto dP/dz \) and \( P \mapsto zP \) on the infinite-dimensional space \( \mathcal{P} = F[z] \) satisfy \( ST - TS = I \).)

Tensors and eigenstuff:

3. Fix \( a \in \mathbb{C} \), and let \( T : \mathbb{C} \rightarrow \mathbb{C} \) be the map \( z \mapsto az \). This is an \( \mathbb{R} \)-linear operator, so we may consider the linear operator \( T' = T \otimes 1 \) on the complex vector space \( \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \). What are the eigenvalues and eigenvectors of \( T' \)? (Warning: The answer depends on whether \( a \in \mathbb{R} \).)

4. Let \( U, V \) be vector spaces over a field \( F \), equipped with linear operators \( S \in \mathcal{L}(U) \), \( T \in \mathcal{L}(V) \). Consider \( S \otimes T \in \mathcal{L}(U \otimes V) \).
   i) If \( \lambda \in F \) is an eigenvalue of \( S \), and \( \mu \in F \) is an eigenvalue of \( T \), prove that \( \lambda \mu \) is an eigenvalue of \( S \otimes T \).
   ii) If \( U, V \) are finite dimensional and \( F \) is algebraically closed, prove that every eigenvalue of \( S \otimes T \) is the product of an eigenvalue of \( S \) with an eigenvalue of \( T \).
   iii) Show, by constructing a counterexample with finite-dimensional vector spaces \( S, T \) over \( \mathbb{R} \), that (ii) no longer holds when the hypothesis on \( F \) is dropped.

5. Let \( V \) be a finite-dimensional vector space over an algebraically closed field \( F \), and fix \( A, B \in \mathcal{L}(V) \). Consider the linear operator \( T = T_{A,B} : X \mapsto AX + XB \) on \( \mathcal{L}(V) \).
i) Express $T$ in terms of tensor products (via the identification of $\mathcal{L}(V)$ with $V^* \otimes V$).

ii) Describe the eigenvalues of $T$ in terms of the eigenvalues of $A$ and $B$.

iii) Prove that if $F = C$ and all eigenvalues of $A, B$ has positive real part then every $M \in \mathcal{L}(V)$ can be written uniquely as $AX + XB$ for some $X \in \mathcal{L}(V)$.

Apropos eigenstuff... The next result generalizes what we proved in class about involutions (which are the special case $m = 2, \lambda_i = \pm 1$).

6. Suppose $V$ is a vector space over a field $F$ and $T$ is a linear operator on $V$ such that \( \prod_{i=1}^{m}(T - \lambda_i I) = 0 \) for some distinct $\lambda_i \in F$. Prove that $V$ is the direct sum of the $\lambda_i$-eigenspaces of $T$. [NB: $V$ may not be assumed finite-dimensional.]

**Tensor products of $A$-modules.** Like direct sums, quotient spaces, and duals, tensor products can be defined in the same way for modules over rings $A$ that need not be fields. Basic properties such as $M \otimes (N \oplus N') \cong (M \otimes N) \oplus (M \otimes N')$ hold in this more general setting, and for much the same reason; but some new phenomena emerge, as in parts (ii) and (iii) of the next problem:

7. i) Show that if $A$ is a commutative ring with unit, and $I \subseteq A$ is an ideal (an additive subgroup such that $aI \subseteq I$ for all $a \in A$, or equivalently a submodule of the $A$-module $A$), then $(A/I) \otimes_A (A/I)$, the tensor product of the quotient $A$-module $A/I$ with itself, is isomorphic with $A/I$.

ii) On the other hand, show that $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$ is the trivial $\mathbb{Z}$-module $\{0\}$.

iii) For positive integers $m, n$, what is the $\mathbb{Z}$-module $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$?

Finally, a bit about inner products:

8. Solve Exercises 7 and 13 on pages 122, 123 of Axler. For #13, $V$ is either a real or complex inner-product space, which need not be finite dimensional.

9. Is the symmetric bilinear pairing constructed in Problem 2 nondegenerate? When $F = \mathbb{R}$, is it positive definite?

Axler’s exercise #7, as well as the more familiar #6, is often referred to as the “polarization identity”. This shows that a linear transformation preserves the norm if and only if it preserves the inner product [more precisely, it shows the harder, “only if” part of this result]. These are basically also the identities used to prove Propositions 2 and 4 in the next chapter (pages 129, 130).

This problem set is due Friday, 8 October, at the beginning of class.