Suppose $L$ is a self-dual, positive-definite lattice of rank $n$. Last time we showed that the theta function

$$\theta_L(z) := \Theta_L(e^{2\pi i z}) = 1 + \sum_{k>0} N_k(L)e^{\pi i k z}$$

(with $z \in \mathcal{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$) satisfies functional equations that express $\theta_L(Sz)$ and $\theta_L(T^2z)$ as simple multiples of $\theta_L(z)$, where $S$ and $T$ are the fractional linear transformations

$$S : z \mapsto -1/z, \quad T : z \mapsto z + 1$$

(so $T^2z = z + 2$). Namely, we showed that

$$\theta_L(T^2z) = \theta_L(z), \quad \theta_L(Sz) = (z/i)^{n/2} \theta_L(z),$$

where \((z/i)^{n/2}\) is the $n$th power of the principal square root of $z/i$ (the square root with positive real part). Moreover, if $L$ is even then the first identity in (2) can be replaced by $\theta_L(Tz) = \theta_L(z)$.

We can iterate these functional equations, obtaining a formula for $\theta_L(f(z))$ as a multiple of $\theta_L(z)$ where $f$ is any fractional linear transformation in the group of fractional linear transformations generated by $S, T^2$ (or $S, T$ for $L$ even). We shall see that this makes $\theta_L(f(z))$ a modular form of weight $n/2$, which in turn places very strong constraints on the counts $N_k(L)$. In each case we begin by determining the structure of that group of fractional linear transformations.

Recall that over any field $K$ the group of fractional linear transformations $z \mapsto (az + b)/(cz + d)$ \((ad - bc \neq 0)\) is isomorphic with the group $\text{PGL}_2(K)$ of invertible $2 \times 2$ matrices modulo scalars: the coset \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} K^*\) acts by $z \mapsto (az + b)/(cz + d)$. [This can be seen by direct computation, and explained by considering a fractional linear transformation as a linear change of projective coordinates \(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \).] Our maps $S, T$ correspond to the integer matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of determinant $1$; and $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Thus $S$ and $T$ generate some subgroup $\langle S, T \rangle$ of the “modular group”

$$\Gamma := \text{SL}_2(\mathbb{Z})/\{\pm 1\} = \text{PSL}_2(\mathbb{Z}),$$

acting on $\mathcal{H}$ by projective linear transformations; and $S$ and $T^2$ generate some subgroup of $\langle S, T \rangle$.

Consider first the case of $L$ even, for which $\theta_L$ transforms under all of $\langle S, T \rangle$. It is well-known\footnote{The result that $\langle S, T \rangle = \Gamma$ can be proved in several ways. Perhaps the most elementary approach begins by observing that $ad - bc = 1$ implies $\gcd(a, b) = 1$ and recalling that conversely $\gcd(a, b) = 1$ implies that $ad - bc = 1$ has integer solutions $(c, d)$ that can be computed using the extended Euclidean algorithm. This computation encodes a representation of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a word in $(S, T)$. For example, given one solution $(c, d)$ of $ad - bc = 1$, the complete solution is $\{(c + ma, d + mb) \mid m \in \mathbb{Z}\}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^m \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.} that in fact $\langle S, T \rangle$ is all of $\Gamma$. Serre gives one elegant argument at the start of Chap. VII of A
Serre’s approach is intimately related with lattice basis reduction in the Euclidean plane (identified with \( \mathbb{C} \) in the usual way),\(^2\) though he does not make this connection explicit.

We next review the connection between the structure of the group \( \Gamma = \langle S, T \rangle \) with the hyperbolic geometry of the tiling of \( \mathcal{H} \) by \( \Gamma \)-images of \( \mathcal{F} \). Geometrically, \( \mathcal{F} \) is an ideal isosceles triangle in the hyperbolic plane \( \mathcal{H} \), with angles \( \pi/3, \pi/3, 0 \) at vertices \( e^{\pi i/3}, e^{2\pi i/3}, \infty \). Our group \( \Gamma \) acts on \( \mathcal{H} \) by conformal isometries. The positive imaginary axis bisects \( \mathcal{F} \) into two right ideal triangles, each with angles \( \pi/2, \pi/3, 0 \) and thus congruent. Choose one of these halves, say the one with vertices \( i, e^{2\pi i/3}, \infty \), and call it \( \mathcal{T} \). The other half is then the image of \( \mathcal{T} \) under the (anti-conformal) reflection \( z \mapsto -\bar{z} \) in the imaginary axis. This reflection extends \( \Gamma \) to a group, call it \( \tilde{\Gamma} \), of hyperbolic isometries generated by reflections in the sides of \( \mathcal{T} \). This identifies \( \Gamma \) with the conformal subgroup of \( \tilde{\Gamma} \), which is the hyperbolic triangle group of indices 2, 3, \( \infty \). We shall soon see that the group generated by \( S \) and \( T^2 \) (which acts on theta functions of odd self-dual lattices) is also a hyperbolic triangle group; we shall later encounter several other such groups as transformation groups for theta functions. We thus insert here a description of the general features of such groups.

Fix positive integers \( e_1, e_2, e_3 \) such that\(^3\) \( 1/e_1 + 1/e_2 + 1/e_3 < 1 \). There is a hyperbolic triangle (unique up to hyperbolic isometry) \( \mathcal{T}(e_1, e_2, e_3) \) with angles \( \pi/e_1, \pi/e_2, \pi/e_3 \). Reflections in the sides of the triangle generate a group of hyperbolic isometries for which \( \mathcal{T}(e_1, e_2, e_3) \) is a fundamental domain. For example, \( (e_1, e_2, e_3) = (2, 4, 6) \) gives the hyperbolic tiling explored by Escher in his “Circle Limit” prints. The orientation-preserving subgroup is called the hyperbolic triangle

\(^2\)…which in turn combines elements of the Euclidean algorithm (see the previous footnote) with Gram-Schmidt orthogonalization.

\(^3\)If \( 1/e_1 + 1/e_2 + 1/e_3 = 1 \) then the triangle is Euclidean, with \( (e_1, e_2, e_3) \) either \( (3, 3, 3) \) (equilateral) or a permutation of \( (2, 4, 4) \) (isosceles right triangle) or \( (2, 3, 6) \) (a 30-60-90 triangle); allowing an ideal vertex we have also permutations of \( (2, 2, \infty) \) (a half-strip). If \( 1/e_1 + 1/e_2 + 1/e_3 > 1 \) then we have a spherical triangle: here \( (e_1, e_2, e_3) \) is up to permutation either \( (2, 2, \infty) \), giving rise to a dihedral group, or one of the exceptional cases \( (2, 3, 3), (2, 3, 4), (2, 3, 5) \), giving rise to the tetrahedral, octahedral, and icosahedral groups respectively.
group of indices \((e_1, e_2, e_3)\); it is generated by rotations through angles \(2\pi/e_1, 2\pi/e_2, 2\pi/e_3\) about the vertices of \(T'(e_1, e_2, e_3)\). Call the rotations \(r_1, r_2, r_3\). Then it is known that the hyperbolic triangle group is generated by \(r_1, r_2, r_3\) subject only to \(r_1^{e_1} = r_2^{e_2} = r_3^{e_3} = 1\) and the one additional relation \(r_1r_2r_3 = 1\); equivalently, the group is

\[
\langle r_1, r_2 \mid r_1^{e_1} = r_2^{e_2} = (r_1r_2)^{e_3} = 1 \rangle.
\]

Ideal vertices (cusps) are allowed: such a vertex has a zero angle and an index of \(\infty\), and the “rotation” about such a vertex is parabolic element of \(\text{Aut}^+(\mathcal{H})\) fixing the cusp. In that case the relation \(r_j^{e_j} = 1\) disappears; in the extreme case of \((e_1, e_2, e_3) = (\infty, \infty, \infty)\) the only relation is \(e_1e_2e_3 = 1\) and we get a free group on two generators. It is also known that the triangle group has no elliptic or parabolic elements other than the conjugates of \(r_1, r_2, r_3\).

In our setting, \(e_1, e_2, e_3 = 2, 3, \infty\) in some order, and \(S\) and \(T\) are the generators that fix \(i\) and the cusp \(i\infty\) respectively. Thus \(ST\) must fix the third vertex, and indeed if \(z = e^{2\pi i/3}\) then

\[
ST(z) = S(z + 1) = -1/(z + 1) = z
\]
because \(z^2 + z + 1 = 0\). We thus expect that \((ST)^3 = 1\), and can verify this directly either by computing \((ST)^3z\) or by checking that \((ST)^3 = (0\ 1\ -1) = -1\). In fact \((S, T \mid S^2 = (ST)^3 = 1)\) is a presentation of \(\Gamma\) (a special case of the two-generator presentation of any triangle group), though we shall have little if any need for this fact.

We do, however, make good use of the identity \((ST)^3 = 1\). Suppose \(L\) is an even self-dual lattice. Then \(\theta_L(z)\) transforms under \(z \mapsto Tz = z + 1\) and \(z \mapsto Sz = -1/z\). We noted already that iterating the functional equation

\[
\theta_L(Sz) = (z/i)^{n/2}\theta_L(z)
\]
gives \(\theta_L(z) = \theta_L(S^2z)\), which we knew already because \(S^2z = z\) for all \(z\), and thus gives us only a “sanity check” on the functional equation. On the other hand, the identity \((ST)^3z = z\) does give us something new: a proof that \(8|n\). Indeed

\[
\theta_L(ST(z)) = (T(z)/i)^{n/2}\theta_L(T(z)) = (T(z)/i)^{n/2}\theta_L(z)
\]
for all \(z \in \mathcal{H}\), so

\[
\theta_L(z) = \theta_L((ST)^3z) = ((T(z)/i)^{1/2} (STST(z)/i)^{1/2} (TSTST(z)/i)^{1/2})^n \theta_L(z). \tag{3}
\]

Now \(\theta_L(z)\) is not identically zero (because \(\lim_{y \to \infty} \theta_L(iy) = 1\)), so the factor

\[
((T(z)/i)^{1/2} (STST(z)/i)^{1/2} (TSTST(z)/i)^{1/2})^n
\]
in (3) must equal 1. On the other hand, we calculate \(TST(z) = z/(z + 1)\) and \(TSTST(z) = S^{-1}(z) = S(z) = -1/z\), so

\[
\frac{T(z) \ TST(z) \ TSTST(z)}{i} = i^{-3}(z+1) \left(\frac{z}{z+1}\right) \left(\frac{-1}{z}\right) = -1/i^3 = -i.
\]
Hence \((T(z)/i)^{1/2} (TST(z)/i)^{1/2} (TSTST(z)/i)^{1/2}\) is a square root of \(-i\) (which one?), and its \(n\)th power equals 1 if and only if \(8|n\). This completes the proof that if a positive-definite lattice of rank \(n\) is even and self-dual then \(8|n\).

Now the functional equation relating \(\theta_L(z)\) with \(\theta_L(Sz)\) simplifies to \(\theta_L(Sz) = z^{n/2} \theta_L(z)\). Combining this with the translation invariance \(\theta_L(Tz) = \theta_L(z)\), we deduce that for any \((a,b) \in \Gamma = \langle S,T \rangle\) the theta function satisfies

\[
\theta_L \left( \frac{az+b}{cz+d} \right) = (cz+d)^{n/2} \theta_L(z).
\]

(4)

Since \(\theta_L\) is holomorphic on \(H\), and \(\theta_L(z)\) remains bounded as \(\text{Im}(z) \to \infty\), this makes \(\theta_L(z)\) a modular form of weight \(n/2\) for \(\Gamma\).

While the weight of \(\theta_L\) is a multiple of 4, the transformation \((cz+d)^k \phi(z) = \phi(\gamma(z))\) is well-defined for all even \(k\), because \((cz+d)^k\) is the same for either choice of sign in \(\gamma = \pm \frac{a}{c} \frac{b}{d}\).

For any even integer \(k \geq 0\), we say \(\phi\) is a modular form of weight \(k\) for \(\Gamma\) if \(\phi\) is a holomorphic function on \(H\) that satisfies \(\phi(\gamma(z)) = (cz+d)^k \phi(z)\) for all \(\gamma \in \Gamma\) and \(\phi(z)\) remains bounded as \(z \to i\infty\). We shall later need such forms also for \(k \equiv 2 \mod 4\), so we consider them together. For each \(k\), the modular forms of weight \(k\) form a vector space. If \(\phi_1, \phi_2\) are modular forms of weight \(k_1, k_2\) then their product \(\phi_1 \phi_2\) is a modular form of weight \(k_1 + k_2\). Moreover, a constant function is a modular form of weight zero, and there are no nonconstant weight-zero forms by a standard application of the maximum principle.\(^4\) Thus we can package the modular forms of all weights into a graded algebra, which we call

\[
M(\Gamma) := \bigoplus_{k \geq 0} M_k(\Gamma).
\]

(5)

This algebra turns out to be freely generated by the normalized Eisenstein series\(^5\)

\[
E_4 = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1-q^m} = 1 + 240q + 2160q^2 + 6720q^3 + \cdots ,
\]

(6)

\[
E_6 = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1-q^m} = 1 - 504q - 16632q^2 - 122976q^3 - \cdots ,
\]

(7)

of weights 4 and 6 respectively. In general, for each even \(k > 2\) we can construct a modular form \(E_k\) of weight \(k\), with \(\lim_{y \to \infty} E_k(iy) = 1\), as the normalized Eisenstein series given by the

\(^4\)Since \(\phi\) is invariant under \(T\), it descends to a holomorphic function of \(q = e^{2\pi iz}\) on \(0 \leq q \leq 1\); since this function is bounded near \(q = 0\), the singularity at the origin is removable, and \(\phi\) extends to a continuous function on a compact set, so \(|\phi(\cdot)|\) has a maximum somewhere. But then it is constant near that maximum, whether or not it occurs at \(q = 0\). Since \(F\) is connected, this makes \(\phi\) a constant function.

\(^5\)Warning: there is an unfortunate but unavoidable notational collision here between the lattice \(E_8\) and the Eisenstein series \(E_k\). Even worse, the usual indexing of Eisenstein series makes \(\theta_{E_8}\) equal \(E_4\), not \(E_8\) which is the theta function of each of the Type II lattices in \(\mathbb{R}^{16}\). We always use a sans serif \(E\) for the Eisenstein series — which also suggests the mnemonic that this \(E\), being thin, has half the weight.
absolutely convergent sums

\[ E_k = \frac{1}{2\zeta(k)} \sum_{c,d \in \mathbb{Z}^2} \frac{1}{(cz + d)^k} = 1 - \frac{2k}{B_k} \sum_{m=1}^{\infty} \frac{m^{k-1} q^m}{1 - q^m} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \quad (8) \]

Here \( B_k \) is the \( k \)-th Bernoulli number, so the coefficients \(-2k/B_k\) for \( k = 4, 6, 8, 10, 12, 14, 16, \ldots\) are 240, -504, 480, -264, 65520/691, -24, 16320/3617, \ldots, and \( \sigma_{k-1}(n) = \sum_{m|n} m^{k-1} \). The sum over \((c, d)\) is readily seen to be modular. We refer to the classic exposition in Serre’s *A Course in Arithmetic* for a proof of the remaining equality in (8) and of the next theorem (his Theorem 4 in Chapter VII):

**Theorem.** The algebra \( M(\Gamma) \) is freely generated over \( \mathbb{C} \) by the modular forms \( E_4 \) and \( E_6 \). In other words, each \( M_n(\Gamma) \) \((n = 0, 2, 4, \ldots)\) has basis

\[ \{E_4^a E_6^b : a, b \geq 0, 4a + 6b = n\}. \quad (9) \]

Hence \( \dim M_n(\Gamma) \) is the number of possible \((a, b)\) in (9), which is \([n/24]\) if \( n \equiv 2 \mod 12 \) and \( 1 + [n/24] \) for other even \( n \geq 0 \):

<table>
<thead>
<tr>
<th>( \frac{n}{8} )</th>
<th>( \frac{n+1}{8} )</th>
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<tr>
<td>( \frac{n}{24} )</td>
<td>( \frac{n+1}{24} )</td>
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(10)

So for example \( E_8 = E_4^2 \), \( E_{10} = E_4 E_6 \), and \( E_{14} = E_4^2 E_6 \); these identities encode surprising convolution identities involving the divisor functions \( \sigma_3, \sigma_5, \sigma_7, \sigma_9, \sigma_{13} \). (“Elementary” but tricky proofs are known.)

The structure of \( M(\Gamma) \) has the following key consequence for theta functions:

**Corollary.** Let \( L \) be a positive-definite self-dual lattice of rank \( n \). If \( L \) is even then

\[ \theta_L = E_4^{n/8} + \sum_{m=1}^{[n/24]} c_m E_4^{n-24m} \Delta^m \quad (11) \]

for some constants \( c_m \) \((m = 1, 2, \ldots, [n/8])\), where

\[ \Delta := \frac{E_3}{12^3} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 \cdots \quad (12) \]
Proof: Because $8|n$, any element of $M_{n/2}(\Gamma)$ is a polynomial in $E_4$ and $E_8^2$. We may thus write

$$\theta_L = \sum_{m=0}^{[n/24]} c_m \theta_{Z_2}^{n-8m} \Delta^m$$

for some $c_m$ ($m = 0, 1, 2, \ldots, [n/24]$), and evaluate at $q = 0$ to obtain $c_0 = N_0(L) = 1$ as desired.

It follows that the coefficients $N_{2k}(L)$ of $\theta_L$ for $k = 1, 2, \ldots, [n/24]$ determine $\theta_L$, because we can use them to calculate the $c_m$ iteratively.

Note that when $n < 24$ there are no undetermined coefficients, so $\theta_L = E_4^{n/8} = 1 + 30nq + O(q^2)$.

Once we classify “root lattices” (integral positive-definite lattices generated by their vectors of norm 2), we will see almost immediately that $L$ is isomorphic with $E_8$ if $n = 8$, and with either $E_8 \oplus E_8$ or $D_{16}^*$ (Serre’s “$E_{16}$”) if $n = 16$; that is, we’ll obtain the classification of even unimodular lattices in $\mathbf{R}^n$ for $n < 24$.

In fact for $n = 8$ we can already obtain this result by including one more term of the theta function,\footnote{This proof is modeled after (but quicker and easier than) Conway’s short characterization of the Leech lattice and enumeration of its automorphisms, published in *Inventiones Math.*, in 1969 and reproduced in SPLAG as Chapter 12. We shall encounter several further lattices that allow for similar characterizations and counts. For $E_8$ there are many other proofs, some of which we shall give later in this course.}

$$\theta_L = E_4 = 1 + 240q + 2160q^2 + O(q^3).$$

There are barely enough cosets of $L$ in $2L$ to accommodate 0 and the 240 + 2160 vectors of norm 2 or 4. Indeed suppose $v, v'$ are vectors of norm at most 4 such that $v' \neq \pm v$ but $v \equiv v' \mod 2L$. Then both $v + v'$ and $v - v'$ are in $2L$, and being nonzero vectors must have $\langle v + v', v + v' \rangle \geq 8$ and $\langle v - v', v - v' \rangle \geq 8$. But then

$$16 = 8 + 8 \leq \langle v + v', v + v' \rangle + \langle v - v', v - v' \rangle = 2((v, v) + (v', v')) \leq 2(4 + 4) = 16.$$

Hence equality holds throughout, so $\langle v, v \rangle = \langle v', v' \rangle = 4$ — and then $\langle v, v' \rangle = 0$ because $\langle v + v', v + v' \rangle = 8$. This means that any coset of $L$ in $2L$ that contains a pair $\pm v$ of norm-2 vectors contains no other vectors of norm at most 4, and any coset of minimal norm 4 contains at most 8 pairs $\pm v_i$ of norm-4 vectors. But this requires $240/2 + 2160/16 = 120 + 135 = 255$ nonzero cosets of $L$ in $2L$, which exactly matches the total number $2^8 - 1$ of such cosets. So again equality holds throughout. In particular, for each of the 135 cosets containing a vector of norm 4 we find an orthogonal frame $\pm v_1, \ldots, v_8$ of such vectors. Since these are all congruent mod $2L$, each vector $(v_i - v_j)/2$ is also in $L$; and this gives a sublattice of $L$ congruent to $D_8$, the even sublattice of $Z_8$. But then $D_8 \subset L = L^* \subset D_8^*$. There are only three lattices properly contained between $D_8$ and $D_8^*$: one is $Z_8$ itself, and the others are obtained by augmenting $D_8$ by its translate by $(1, 1, 1, 1, 1, 1, 1, 1)/2$ or $(1, 1, 1, 1, 1, 1, 1, -1)/2$. All three are self-dual, but only the latter two are even, and they are isomorphic to each other by flipping the sign of any coordinate (which preserves the index-2 sublattice $D_8$). This completes the proof, and also lets us count automorphisms of $E_8$: the 135 cosets of norm-4 vectors are all equivalent, and each such coset’s stabilizer is half of $\text{Aut}(D_8)$, so

$$\#(\text{Aut } E_8) = \frac{135}{2} \#(\text{Aut } D_8) = 135 \cdot 2^7 \cdot 8! = 696729600.$$

Moreover, it follows that $\text{Aut } E_8$ acts transitively on the vectors of norm 4, and (with a bit more work) on norm-2 vectors, indeed even on ordered orthogonal pairs of norm-2 vectors. We shall see that $\text{Aut } E_8$ acts transitively on many more such configurations of short vectors.
What then of the subgroup of $\Gamma$ generated by $S$ and $T^2$? Call this subgroup $\Gamma_+$. Since $T^2$ is congruent to the identity mod 2, while $S^2 = 1$, any $\pm (a_b/c_d) \in \Gamma_+$ must be congruent to either the identity or $(0_1/1_0)$ mod 2. We claim that this necessary condition is also sufficient; that is, that $\Gamma_+$ is the preimage in $\Gamma$ of $\{ (0_0/1_1), (1_0/1_1) \}$ under the reduction-mod-2 map $\text{PSL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/2\mathbb{Z})$, and thus that the index $[\Gamma : \Gamma_+]$ is $\#(\text{SL}_2(\mathbb{Z}/2\mathbb{Z}))/2 = 3$. One way to see this is to check that $\Gamma_+$ is itself a hyperbolic triangle group, this time with indices 2, $\infty$, $\infty$. Indeed $S$ is an involution, $T^2$ is parabolic, and $ST^2 : z \mapsto -1/(z + 2)$ is also parabolic, fixing the cusp $z = -1$. Thus $[\Gamma : \Gamma_+]$ is the ratio of the areas of the corresponding hyperbolic triangles, which is $(1 - \frac{1}{2})/(1 - \frac{1}{2} - \frac{1}{3}) = \frac{1}{2}/\frac{1}{6} = 3$; since we have already found an index-3 subgroup of $\Gamma$ that contains $\Gamma_+$, that must be the same as $\Gamma_+$ and we’re done. (Alternatively, prove that $[\Gamma : \Gamma_+] \leq 3$ by checking group-theoretically that each element of $(S, T \mid S^2 = (ST)^3 = 1)$ is either in $(S, T^2)$ or in the $(\langle S, T^2 \rangle)$-coset of $T$ or $TS$, and conclude the proof as before.) As with $\Gamma$, we can construct a fundamental domain $\mathcal{F}_+$ for $\Gamma_+$, consisting of two copies of the associated hyperbolic triangle, which comprise a fully ideal triangle with vertices at $-1, 1, i \infty$; note that $-1$ and 1 are equivalent under $\Gamma_+$ (as the vertices $e^{\pi i/3}$ and $e^{2\pi i/3}$ of $\mathcal{F}$ are equivalent under $\Gamma$). As expected, $\mathcal{F}_+$ can be dissected into hyperbolic triangles that form three $\Gamma$-images of $\mathcal{F}$.

![Figure 2: The fundamental domain $\mathcal{F}_+$ dissected into three images of $\mathcal{F}$](image)

So, if $L$ is self-dual then we can iterate our functional equations (2) to find, for each $\gamma = \pm (a_b/c_d) \in \langle S, T^2 \rangle = \Gamma_+$, an identity

$$\theta_L(\gamma(z)) = \epsilon_{c,d}^n(cz + d)^{n/2}\theta_L(z)$$

(13)

for some $\epsilon_{c,d} \in \mathbb{C}^*$ with $\epsilon_{c,d}^8 = 1$. We shall use this as before to prove that $\theta_L$ is in the algebra freely generated by $\text{theta}_Z$ and $E_4$. But there are a few new features here, due to the second cusp and the factors $\epsilon_{c,d}$, that we must address before we can mimic the analysis of $M(\Gamma)$ in Serre. Along the way we shall encounter another important identity relating the theta functions of $L$ and its shadow.

A holomorphic function $\phi : \mathcal{H} \to \mathbb{C}$ satisfying $\phi(\gamma(z)) = \epsilon_{c,d}^n(cz + d)^{n/2}\phi(z)$ for all $\gamma = \pm (a_b/c_d) \in \Gamma_+$ is a weakly modular form of weight $n/2$ for $\Gamma_+$ (more fully, for $\Gamma_+$ and the $\epsilon_{c,d}$; these

---

7We write $\epsilon_{c,d}$ rather than $\epsilon$, because if some $\gamma'$ has the same $(c, d)$ as $\gamma$ then $\gamma' = T^{2k}\gamma$ for some $k \in \mathbb{Z}$, so $\theta_L(\gamma(z)) = \theta_L(\gamma'(z))$, whence $\gamma$ and $\Gamma'$ have the same $c$; also, $\gamma$ determines $(c, d)$ only up to sign, and changing $(c, d)$ to $(-c, -d)$ multiplies $\epsilon_{c,d}$ by $\pm i$. 

---

7
factors are said\(^8\) to form a “multiplier system of weight \(n/2\)” — a multiplier system being a system of factors in an identity such as (13) that is consistent with \(\theta_L(\gamma_1 \gamma_2(z)) = \theta_L((\gamma_1 \gamma_2)(z))\) for all choices of \(\gamma_1, \gamma_2\). To remove the adverb “weakly”, we must check that \(\theta_L(z)\) does not grow too quickly as \(z\) approaches a cusp. All cusps are equivalent under \(\Gamma\), but not under \(\Gamma_+\); so we shall determine the action on \(\theta_L\) of our representatives \(T, TS\) of the nontrivial cosets of \(\Gamma_+\) in \(\Gamma\).

The action of \(T : z \mapsto z + 1\) is easy: this map takes \(e^{\pi iz}\) to \(-e^{\pi iz}\), so for any integral lattice \(L\) we have simply

\[
\theta_L(T(z)) = \theta_L(z + 1) = 1 + \sum_{k=1}^{\infty} (-1)^k N_k(L) e^{\pi ikz}.
\]

The formula for \(\theta_L(TS(z))\) can be expressed in terms of the theta series for the shadow of \(L\). Recall that the “shadow” is the lattice translate consisting of all vectors \(c/\theta\) for \(c\) in the characteristic coset of \(2L\) in \(L\); we denote the shadow by \(s(L)\). Let us define

\[
\Psi_L(q) := \sum_{v \in s(L)} q^{\langle v, v \rangle/2} = 1 + \sum_{k=1}^{\infty} N_{2k}(s(L)) q^k, \tag{14}
\]

\[
\psi_L(z) := \Psi_L(e^{2\pi iz}) = \sum_{k \geq 0} N_k(s(L)) e^{\pi ikz}, \tag{15}
\]

where

\[
N_k(s(L)) = \# \{ v \in s(L) \mid \langle v, v \rangle = k \}. \tag{16}
\]

Then we have:

**Proposition.** For any self-dual lattice \(L\) in \(\mathbb{R}^n\) we have

\[
\theta_L(TS(z)) = (z/i)^{n/2} \psi_L(z) \tag{17}
\]

for all \(z \in \mathcal{H}\).

It will follow that \(|z|^{-n/2} \theta_L(TS(z))\) remains bounded as \(z \to i \infty\), which is what we need to check at the cusp \(1 = TS(\infty)\) to verify that \(\theta_L\) is modular (not just weakly modular) for \(\Gamma_+\).

**Proof** of (17): Let \(c\) be a characteristic vector. Then

\[
\theta_L(T(z)) = \theta_L(z + 1) = \sum_{v \in L} (-1)^{\langle v, c \rangle} e^{\pi i \langle v, v \rangle z} = \sum_{v \in L} e^{\pi i \langle (v, c) \rangle}
\]

because \((-1)^c = e^{\pi ic}\) for any integer \(c\). We now apply Poisson summation to the sum. The Fourier transform of \(\exp(\pi i \langle x, x \rangle z)\) is the integral over \(x \in \mathbb{R}^n\) of \(\exp(\pi i \langle x, x \rangle z + \langle x, c + 2y \rangle)\), which is to say the value at \(y + \frac{c}{2}\) of the Fourier transform of \(\exp(\pi i \langle x, x \rangle z)\), which we already know is \((z/i)^{-n/2}\exp(-\pi i \langle x, x \rangle / z)\). Poisson summation then gives

\[
\theta_L(T(z)) = (z/i)^{-n/2} \sum_{v \in L} e^{-\pi i \langle v + \frac{c}{2}, v + \frac{c}{2} \rangle / z} = (z/i)^{-n/2} \sum_{v \in s(L)} e^{-\pi i \langle v, v \rangle / z} \tag{19}
\]

\(^8\)See e.g. Iwaniec’s *Topics in Classical Automorphic Forms*, 2.6.
which is \((z/i)^{-n/2} \psi_L(S(z))\); replacing \(z\) by \(Sz\) we recover (17), Q.E.D.

**Corollary.** If \(L\) is a positive-definite self-dual lattice of rank \(n\) then every characteristic vector \(c\) has \(\langle c, c \rangle \equiv n \mod 8\).

**Proof:** We have seen that all characteristic vectors have the same norm \(\mod 8\); denote this common residue \(\mod 8\) by \(s\). Then \(\psi_L(t + 1) = e^{\pi is(L)/4} \psi_L(t)\). We claim that \(s = n\), or equivalently that

\[
\psi_L(t + 1) = e^{\pi in/4} \psi_L(t).
\]

Using (17), together with \(S^2 = (ST)^3 = 1\) and \(\theta_L(T^2z) = \theta_L(z)\), we calculate

\[
\left(\frac{t + 1}{i}\right)^{n/2} \psi_L(t + 1) = \theta_L(TST(t)) = \theta_L(ST^{-1}S(t)) = (T^{-1}S(t)/i)^{n/2} \theta_L(T^{-1}S(t)) = \left(\frac{i(t + 1)}{t}\right)^{n/2} \theta_L(TS(t))
\]

(in which we used \(S^2 = (ST)^3 = 1\) and the invariance of \(\theta_L\) under \(T^2\), and again wrote \(n/2\) power to mean \(n\)th power of principal square root). Comparing with (17) yields the desired identity (20), Q.E.D.

Now to describe the \(\tfrac{1}{2} \mathbb{Z}\)-graded algebra

\[
\mathbf{M}(\Gamma_+) := \bigoplus_{\substack{k \geq 0 \atop 2k \in \mathbb{Z}}} M_k(\Gamma_+).
\]

of modular forms for \(\Gamma_+\). Recall (e.g. Theorem 3 in Serre, Chap. VII) that any nonzero modular form of weight \(k\) for \(\Gamma\) has \(k/12\) zeros in \(\mathcal{F}\), counted with appropriate multiplicity (including zeros at the cusp, and zeros at \(i\) and \(e^{2\pi i/3}\) counted with half the usual multiplicity). A similar result and proof works for \(\Gamma_+\) and \(\mathcal{F}_+\), but with a total of \(k/4\) — as it must be because if we start with a form in \(M_k(\Gamma)\) then we must multiply \(k/12\) by the index \([\Gamma : \Gamma_+]\) to get the total multiplicity in \(\mathcal{F}_+\). A new feature is that due to the “multiplier system” the multiplicity of a zero at the cusp \(\pm 1\) is \(\equiv n/8 \mod 1\); we have in effect seen this in our proof of \(\langle c, c \rangle \equiv n \mod 8\). At \(i\infty\), we assign multiplicity \(m\) to the zero of \(q^{m/2} + O(q^{(m+1)/2})\). In the determination of \(\mathbf{M}(\Gamma)\) it was crucial that the weight-12 form \(\Delta\) had a zero at \(i\infty\) and nowhere else in \(\mathcal{F}\). For \(\Gamma_+\), we can use instead the weight-4 form we call

\[
\Delta_+ = \frac{1}{16}(\theta_2^2 - E_4) = q^{1/2} - 8q + 28q^{3/2} - 64q^2 + 126q^{5/2} - 224q^3 + 344q^{7/2} - 512q^4 \cdots.
\]

Since \(\Delta_+\) vanishes at \(i\infty\), it has no other zeros; thus for each half-integer \(k\) we see that multiplication by \(\Delta_+\) is an isomorphism from \(M_{k-4}(\Gamma_+)\) to the subspace of \(M_k(\Gamma_+)\) consisting of forms vanishing at \(i\infty\). For \(k \geq 0\) this subspace has codimension 1 because it is the kernel of a linear function and does not contain \(\theta_2^{2k}\). We now have all the ingredients to prove, exactly as Serre does for \(\mathbf{M}(\Gamma)\):
Theorem. The algebra $M(\Gamma_+)$ is freely generated over $\mathbb{C}$ by the modular forms $\theta_{\mathbb{Z}}$ and $E_4$. In other words, each $M_{n/2}(\Gamma)$ $(n = 0, 1, 2, 3, \ldots)$ has basis

$$\{\theta_{\mathbb{Z}}^a E_4^b : a, b \geq 0, a + 8b = n\}.$$  (22)

An easy consequence is the classification of positive-definite self-dual lattices of rank up to 7:

Proposition. If $n < 8$ then every self-dual lattice in $\mathbb{R}^n$ is isomorphic with $\mathbb{Z}^n$.

Proof: Here $|n/8| = 0$, so $\theta_L = \theta_{\mathbb{Z}}$. Comparing coefficients, we deduce $N_k(L) = N_k(\mathbb{Z}^n)$ for all $k$. In particular, $N_1(L) = 2n$, because $N_1(\mathbb{Z}^n) = 2n$, as may be seen either directly or by expanding $\theta_{\mathbb{Z}}^2$ in powers of $q^{1/2}$. Thus $L$ contains $n$ pairs $\pm v_i$ $(1 \leq i \leq n)$ of vectors with $\langle v_i, v_i \rangle = 1$. For $i \neq j$ we then have $|\langle v_i, v_j \rangle| < 1$ by Cauchy-Schwarz; since $L$ is integral, it follows that $\langle v_i, v_j \rangle = 0$. That is, the $v_i$ are orthonormal. Therefore $L$ contains their $\mathbb{Z}$-span, call it $L_0$, which is isomorphic with $\mathbb{Z}^n$. But then $L_0 \subseteq L = L^* \subseteq L_0^* = L_0$, so $L = L_0 \cong \mathbb{Z}^n$, Q.E.D.

For any $n$, the characteristic vectors in $\mathbb{Z}^n$ all have norm at least $n$ (with equality for the $2^n$ vectors $\pm 1, \ldots, \pm 1$), so $\psi_{\mathbb{Z}^n}(z) = O(|q|^{n/4})$ as $z \to i\infty$. Thus $\theta_{\mathbb{Z}}$, has a zero at the cusp $z = \pm 1$ that contributes $n/8$ to the total count of zeros with multiplicity. Again we deduce, as with $\Delta$ and $\Delta_+$, that there are no other zeros. We conclude this chapter of the notes by using that observation about $\theta_{\mathbb{Z}}$ to prove that $\mathbb{Z}^n$ is the unique self-dual lattice in $\mathbb{R}^n$ whose characteristic vectors all have norm at least $n$. If $L$ is any such lattice then $\psi_L(z) = O(|q|^{n/4})$ as $z \to i\infty$, so $\theta_L$ vanishes at the cusp $z = \pm 1$ to at least the same order as $\theta_{\mathbb{Z}}$. Hence $\theta_L/\theta_{\mathbb{Z}}$ is a modular form of weight 0, and is thus constant; as usual, it follows from $\Theta_L(0) = 1 = \Theta_{\mathbb{Z}}(0)$ that $\theta_L = \theta_{\mathbb{Z}}$. In particular, $L$ has $n$ pairs of vectors of norm 1. This implies $L \cong \mathbb{Z}^n$, using the same argument as in the previous paragraph. This answers a question that arose in the geometry of 4-manifolds. With some further work (including the use of root lattices), this technique even lets us classify all self-dual lattices in $\mathbb{R}^n$ with no characteristic vectors of norm less than $n - 8$.  

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