Math 272y: Rational Lattices and their Theta Functions

October 23: Some applications of the 7-design property for $E_8$

Each shell of the $E_8$ lattice is a spherical 7-design; this follows either from modularity of weighted theta functions (there is no nonzero cusp form of weight less than 12) or from the action of $\text{Aut}(E_8)$ (every invariant homogeneous polynomial of degree < 8 is a polynomial in the norm). We apply this to the first shell, which consists of the 240 roots of $E_8$, obtain refined combinatorial and geometric information about the $E_8$ lattice and its configuration of roots. This can also be used to give several further arguments for the uniqueness of the $E_8$ lattice and the size of $\text{Aut}(E_8)$ — assuming we do use only modular forms, not the invariants of $\theta$ functions (there is no nonzero cusp form of weight less than 8).

Our key tool is a formula for the average of $x^d$ over any sphere about the origin in $\mathbb{R}^n$. For now we need $n = 8$, but we shall later use other $n \geq 2$ as well.

**Proposition.** Let $n, d$ be integers with $d \geq 0$ and $n \geq 2$. For any unit vector $e \in \mathbb{R}^n$, and any real $r > 0$, the average of $\langle v, e \rangle^d$ over the sphere $\langle v, v \rangle = r^2$ is

$$r^d \prod_{j=0}^{d-1} \frac{2j + 1}{n + 2j} = \frac{1}{n} \frac{3}{n + 2} \frac{5}{n + 4} \cdots \frac{d - 1}{n + d - 2} r^d$$

(1)

if $d$ is even, and 0 if $d$ is odd.

The proof is a standard calculation, which we defer to the last page in order to get to our applications more quickly. For now we note that the average must be proportional to $r^d$, and must vanish for odd $d$ by antisymmetry, so we need only confirm the factor $\prod_{j=0}^{d-1} (2j + 1)/(2n + j)$ in (1). For $d = 0$ and $d = 2$ we can do this without integration. Thus for $d = 0$ we are averaging the constant function 1, and indeed (1) gives 1 in that case; for $d = 2$, let $e_1, \ldots, e_n$ be an orthonormal basis, and note that each $\langle v, e_i \rangle^2$ has the same average while $\sum_{i=1}^n \langle v, e_i \rangle^2 = \langle v, v \rangle = r^2$ on our sphere, whence each $\langle v, e_i \rangle^2$ must average to $r^2/n$, again in agreement with (1). For $d \geq 4$, see the Proposition’s proof at the end of this chapter of the notes.

Now fix some nonzero $v_0 \in E_8$, and let $e = v_0/|v_0|$. Since $E_8$ is integral, $\langle v_0, e \rangle = |v_0|^{-1} \langle v_0, v \rangle \in |v_0|^{-1} \mathbb{Z}$ for each $E_8$ root $v$. Let $N_k$ ($k \in \mathbb{Z}$) be the number of roots with $\langle v_0, v \rangle = k$. Then $N_{-k} = N_k$ for all $k$, and $N_k = 0$ if $k^2 > 2 \langle v_0, v_0 \rangle$ by Cauchy-Schwarz. If $\langle v_0, v_0 \rangle < 8$ then this together with the 7-design property gives us enough conditions to determine each $N_k$. For example, if $v_0$ is itself a root (so $|v_0|^2 = 2$), then each $N_{\pm 2} = 1$ because the only $v$ is $\pm v_0$, and we have

$$N_0 + 2N_1 + 2N_2 = \sum_{k=-2}^2 N_k = 240$$

(2)

and

$$2N_1 + 8N_2 = \sum_{k=-2}^2 k^2 N_k = 2 \sum_{v \in E_8, \langle v, v \rangle = 2} \langle v, e \rangle^2 = 2 \cdot 2 \cdot \frac{1}{8} \cdot 240 = 120$$

(3)
by the 2-design condition and the case \( d = r^2 = 2 \) of Proposition 1. This gives us three independent linear equations in the three unknowns \( N_0, N_1, N_2 \); the solution is

\[
(N_0, N_1, N_2) = (126, 56, 1).
\]  

As a check on this computation we can verify the fourth and sixth moments: for each even \( d \geq 2 \) we have \( \sum_{k=-2}^{2} k^d N_k = 2(56 + 4d) \), which is 144 for \( d = 4 \) and 240 for \( d = 6 \), again in agreement with (1) (multiply the second moment 120 by \( 4 \cdot (3/10) \) to get 144, and multiply this fourth moment by \( 4 \cdot (5/12) \) to get 240).

In particular, the orthogonal complement of \( v_0 \) in \( E_8 \) is an even lattice with 126 roots, so it must be \( E_7 \) because no other root lattice of rank \( \geq 7 \) has Coxeter number \( h \geq 18 \) (and because \( \text{disc} E_7 = 2 \) so an integral lattice cannot properly contain \( E_7 \) with finite index). Thus \( E_8 \) contains \( E_7 \oplus \mathbb{Z} v_0 \) with index 2. We already know that there is a unique choice of gluing of \( E_7 \) to \( A_1 \) to reconstitute \( E_8 \). This proves that \( \text{Aut}(E_8) \) acts transitively on the 240 vectors \( v_0 \) and that \( |\text{Aut}(E_7)| = (2/240) |\text{Aut}(E_8)| \).

Now let \( v_0 \) be one of the 2160 vectors of norm 4, so \( e = v_0/2 \). Then \( |(2|v_0|^2)|^{1/2} = |\sqrt{8}| = 2 \), so \( |\langle v, v_0 \rangle| \) is still bounded by 2. We do not get \( N_{\pm 2} \) for free this time, but still have more than enough linear equations to determine the \( N_k \): again \( N_0 + 2N_1 + 2N_2 = 240 \), and also

\[
2N_1 + 8N_2 = (2 \cdot 4) \cdot \frac{1}{8} \cdot 240 = 240, \quad 2N_1 + 32N_2 = (2 \cdot 4)^2 \cdot \frac{1}{8} \cdot \frac{3}{10} \cdot 240 = 576. \tag{5}
\]

Here the solution is

\[
(N_0, N_1, N_2) = (84, 64, 14) \tag{6}
\]

[and again the sixth moment checks out: \( 2(64 + 4^3 14) = 1920 = 8 \cdot (5/12) 576 \). Here too we identify the orthogonal complement by its number of roots: there are \( N_0 = 84 \), so \( E_7 \) has too many roots, while the next-smaller Coxeter number for rank \( \geq 7 \) is 12 = 84/7 which is attained only by \( E_6 \) and \( D_7 \); so \( v_0 \cap E_8 \cong D_7 \), and again we can reconstruct \( E_8 \) as the unique gluing, this time up to reflection about the \( D_7 \) hyperplane. As a further check, \( N_{\pm 1} = 64 \) is the number of dual vectors of minimal norm 7/4 in each of the non-integral cosets of \( D_7 \) in \( D_7^* \).

In fact, in this case we do not need the classification of root systems: we can show directly that the 14 vectors counted by \( N_2 \), together with \( v_0 \), generate a copy of \( D_8 \) in \( E_8 \); and we have already seen that this determines \( E_8 \) uniquely and proves the transitivity of the action of \( \text{Aut}(E_8) \) on the norm-4 lattice vectors. We begin by observing that the vectors \( v \) with \( \langle v, v \rangle = \langle v, v_0 \rangle = 2 \) pair up under \( v \leftrightarrow v_0 - v \). This can be seen by direct computation, or by writing \( v = v' + e \) (recall that \( e = v_0/2 \)) with \( \langle v', e \rangle = 0 \) and \( \langle v', v' \rangle = 1 \). Now if \( v_1 = v_1' + e \) and \( v_2 = v_2' + e \) are two such vectors then \( \langle v_1', v_2' \rangle = \langle v_1, v_2 \rangle - \langle e, e \rangle = \langle v_1, v_2 \rangle - 1 \in \mathbb{Z} \). By Cauchy-Schwarz, then, either \( v_2' = \pm v_1' \) or \( \langle v_1', v_2' \rangle = 0 \). Hence the 14 vectors \( v' \) constitute an orthonormal frame for \( v_0^\perp \), and those vectors together with \( \pm e \) form an orthonormal frame for \( \mathbb{R}^8 \). In the coordinates of this frame it is easy to see that \( v_0 \) and the vectors \( v_i - v_j \) (for any roots \( v_i, v_j \) with \( \langle v_i, v_0 \rangle = \langle v_j, v_0 \rangle = 2 \)) generate a copy of \( D_8 \) in \( E_8 \), and the rest we have done already.

Using also the sixth moment, we can still give a similar treatment of the next two cases, with \( \langle v_0, v_0 \rangle = 6 \) and \( \langle v_0, v_0 \rangle = 8 \). In the first case, \( |(2|v_0|^2)|^{1/2} = |\sqrt{12}| = 3 \), and the equations are
\[N_0 + 2N_1 + 2N_2 + N_3 = 240 \text{ and } 2 \sum_{k=1}^{3} k^{2d} N_k = 360, 1296, 6480 \text{ for } d = 1, 2, 3, \text{ with solution}
\]
\[(N_0, N_1, N_2, N_3) = (74, 54, 27, 2). \tag{7}\]

This time there are only two vectors attaining the maximal \(\langle v, v_0 \rangle\), and again they must be related by \(v \leftrightarrow v_0 - v\); so we have shown that every \(E_8\) vector of norm 6 is uniquely a sum of two roots, necessarily with inner product 1. (Check: the number of such pairs of roots is \(\frac{2}{4} \cdot 240 \cdot 56 = 240 \cdot 28 = (1^3 + 3^3) \cdot 240\), which is the \(q^3\) coefficient of the normalized Eisenstein series \(\theta_{E_8}\).

Now those two vectors generate a copy of \(A_2\) inside \(E_8\), and their differences \(\pm (2v - v_0)\) are among the 74 roots orthogonal to \(v_0\). The remaining 72 must be orthogonal not just to \(v_0\) but to the entire \(A_2\) copy (the projections to \(A_2 \otimes \mathbb{R}\) land in \(A_2^2\), which has no sufficiently short nonzero vectors orthogonal to \(v_0\)); so the orthogonal complement of this \(A_2\) must be isomorphic with \(E_6\), the only root lattice of rank at most 6 with as many as 72 roots. Again we reconstruct \(E_8\) by gluing \(A_2\) to \(E_6\), and deduce that all rank-6 vectors in \(E_8\) are equivalent under \(\text{Aut}(E_8)\), as are all embeddings \(A_2 \rightarrow E_8\).

For \(\langle v_0, v_0 \rangle = 8\), there are \((1^3 + 2^3 + 4^3) \cdot 240 = 73 \cdot 240\) possible \(v_0\), so they cannot constitute a single orbit under \(\text{Aut}(E_8)\). But there is an easy invariant: 240 of these \(v_0\) are of the form \(2v\) for some root \(v\). We shall see that the remaining 72 \(\times 240\) are equivalent. Here \([\langle 2|v_0|\rangle^2]\)^{1/2} = \(\lfloor \sqrt{16} \rfloor = 4\), but \(\langle v, v_0 \rangle = \pm 4\) can happen only for \(v = \pm v_0/2\), so if \(v\) is not a double root we again have only four unknowns \(N_0, N_1, N_2, N_3\) and just enough linear equations to solve. We find
\[(N_0, N_1, N_2, N_3) = (56, 56, 28, 8). \tag{8}\]

Since the orthogonal complement of \(v_0\) has 56 roots in this case, it must be \(A_7\), and again we can reconstruct \(E_8\) by gluing, this time of the discriminant-8 lattices \(A_7\) and \(\mathbb{Z}v_0\). But here we don’t need the classification of root systems because we can use \(N_3 = 8\): the configuration of eight roots with \(\langle v, v_0 \rangle = 3\) is uniquely determined — any two have inner product 1 — and they generate a lattice \(A_8\), from which we recover the full \(E_8\) lattice (necessarily the same as \(A_8^{\perp 3}\)) by including the translates by \(\pm v_0\). Indeed each projection \(v - \frac{2}{3}v_0\) has norm 7/8, and any two of them have inner product \(\langle v - \frac{2}{3}v_0, v' - \frac{2}{3}v_0 \rangle = -1/8\) (it must be \(\equiv 7/8 \bmod 1\) and of absolute value \(\leq 7/8\); so \(\langle v, v' \rangle = -1/8 + (3/8)^2 \langle v_0, v_0 \rangle = 9/8 - 1/8 = 1\). The 56 differences \(v - v'\) are the roots of the orthogonal complement of \(v_0\). Each of the other \(N_k\), namely \((\frac{8}{8}), (\frac{8}{2}), (\frac{8}{1})\), count vectors of minimal norm 15/8, 12/8, 7/8 in a coset of \(A_7\) in \(A_7^2\).

Somewhat surprisingly, we can even handle \(\langle v_0, v_0 \rangle = 10\) and \(\langle v_0, v_0 \rangle = 12\), even though in these cases there are five variables \(N_0, N_1, N_2, N_3, N_4\) and still only four equations. This is thanks to the additional condition that each \(N_k\) must be a nonnegative integer.

For \(\langle v_0, v_0 \rangle = 10\), the general solution is\footnote{The \(N_k\) coefficients \(70, -56, 28, -8, 1 = (-1)^k \binom{8}{k}\) of \(N_k\) (and also for \(k = -1, -2, -3, -4\)) can be explained by noting that \(\sum_{k=-4}^{4} (-1)^k \binom{8}{k} P(k) = 0\) for all polynomials \(P\) with \(\deg P < 8\), which in turn holds because \(\sum_{k=-4}^{4} (-1)^k \binom{8}{k} f(k)\) is an 8-th finite difference of \(f\).}

\[(N_0, N_1, N_2, N_3, N_4) = (70N_4 - 10, 100 - 56N_4, 28N_4 + 5, 20 - 8N_4, N_4). \tag{9}\]
and nonnegativity of $N_0$ and $N_1$ implies $1/7 \leq N_4 \leq 25/14$; since $N_4 \in \mathbb{Z}$ it follows that $N_4 = 1$ and
\[
(N_0, N_1, N_2, N_3, N_4) = (60, 44, 33, 12, 1). \tag{10}
\]
If $v$ is the unique root such that $\langle v, v_0 \rangle = 4$ then $v' := v_0 - 2v$ has norm $10 - 16 + 8 = 2$ and $\langle v, v' \rangle = 0$; and conversely every decomposition $v_0 = 2v + v'$ with $\langle v, v \rangle = \langle v', v' \rangle = 2$ (and thus $\langle v, v' \rangle = 0$) makes $\langle v_0, v \rangle = 4$. Therefore, every $E_8$ vector of norm $10$ is uniquely $2v + v'$ for some orthogonal roots $v, v'$. The $60 = N_0$ roots orthogonal to $v_0$ are then orthogonal to $v, v'$ as well, and constitute a $D_6$ root system. As we did for $\langle v_0, v_0 \rangle = 6$, we can check that the number of norm-$10$ vectors is the same whether one computes it by counting pairs $(v, v')$ (finding $240 \cdot 126$) or from the theta function (finding $(1^3 + 5^3)240$); and this coincidence gives another proof, once we check that it is not possible for two different orthogonal root pairs $(v, v')$ and $(w, w')$ to give the same $v_0$.

For $\langle v_0, v_0 \rangle = 12$, the general solution is
\[
(N_0, N_1, N_2, N_3, N_4) = (70N_4 - 164, 216 - 56N_4, 28N_4 - 54, 40 - 8N_4, N_4) \tag{11}
\]
and this time $N_4$ must be $3$, so
\[
(N_0, N_1, N_2, N_3, N_4) = (46, 48, 30, 16, 3). \tag{12}
\]
The root number $N_0 = 46$ of the orthogonal complement forces it to be $A_2 \oplus D_5$, which has discriminant $12$ and again glues uniquely to $\mathbb{Z}v_0$ (up to Aut($A_2 \oplus D_5 \oplus \mathbb{Z}v_0$)) to give $E_8$. For example, the $48 = 3 \cdot 16$ vectors counted by $N_1$ are $(v_0/12) + v_1 + v_2$ where $v_1$ and $v_2$ are vectors of minimal norms $2/3$ and $5/4$ in cosets that generate $A_2^* / A_2$ and $D_5^* / D_5$; the $3$ vectors counted by $N_4$ are $(v_0/3) + v_1$ for the same three vectors $v_1$.

**Appendix: the average of $x^d_1$ over a sphere.** As promised, we conclude with a proof of the Proposition that gives the formula (1) for the average of $x^d_1$ over a radius-$r$ sphere in $\mathbb{R}^n$ when $2|d$.

**Proof of Proposition.** For each $d$, the integral over the sphere $\langle v, v \rangle = r^2$ of $\langle v, e \rangle^d$ is
\[
C_n \int_{-r}^r x^d (r^2 - x^2)^{(n-3)/2} \, dx = C_n r^{d+n-2} \int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} \, dz \tag{13}
\]
for some constant $C_n > 0$ (the surface area of the unit sphere in $\mathbb{R}^{n-1}$). So we need the quotient of the integral (13) by its value at $d = 0$. The factors $C_n$ cancel out, and the ratio $r^{d+n-2}/r^{n-2}$ gives the expected factor of $r^d$; so it remains to evaluate the integral over $|z| < 1$. By symmetry,
\[
\int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} \, dz = (1 + (-1)^d) \int_0^1 z^d (1 - z^2)^{(n-3)/2} \, dz, \tag{14}
\]
and then the change of variable $z^2 = t$ (so $2z \, dz = dt$) gives
\[
\int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} \, dz = \frac{1 + (-1)^d}{2} \int_0^1 t^{(d-1)/2} (1 - t)^{(n-3)/2} \, dt. \tag{15}
\]
We can now evaluate the integral as a special case of the formula

\[
\int_0^1 t^{a-1} (1 - t)^{b-1} \, dt = B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \quad (a, b > 0)
\]  

(16)

for the Beta function. Here this is \(B((d + 1)/2, (n - 1)/2)\), so the quotient we need is

\[
\frac{B((d + 1)/2, (n - 1)/2)}{B(1/2, (n - 1)/2)} = \frac{\Gamma((d + 1)/2)/\Gamma((n + d)/2)}{\Gamma(1/2)/\Gamma(n/2)} = \frac{\Gamma((d + 1)/2)/\Gamma(1/2)}{\Gamma((n + d)/2)/\Gamma(n/2)}.
\]

(17)

By the identity \(\Gamma(s + 1) = s\Gamma(s + 1)\), the numerator \(\Gamma((d + 1)/2)/\Gamma(1/2)\) is \(\prod_{j=0}^{d-1} (j + \frac{1}{2})\), and the numerator \(\Gamma((n + d)/2)/\Gamma(n/2)\) is \(\prod_{j=0}^{d-1} (j + \frac{n}{2})\). Removing the common factor \(2^{d/2}\) recovers the factor \(\prod_{j=0}^{d-1} (2j + 1)/(2n + j)\) in (1), and we are done. \(\square\)