Math 272y: Rational Lattices and their Theta Functions
25 November 2019: Lattices of level 3 and their theta functions

We briefly describe level-3 lattices and their theta functions, highlighting some differences and new phenomena compared with what we saw for level 2.

Let $L \subset \mathbb{R}^n$ be a lattice of level 3, that is, an even lattice such that $L' := (L^*)^3$ is also even. (As with level 2, we may allow either $L$ or $L'$ to be self-dual, and thus of level 1, though we have dealt with such lattices already.) Since $3$ is odd and squarefree, it is enough to assume that $3L^* \subseteq L$: then if $v \in L^*$ then $3v \in L$ so $\langle v, v \rangle = \frac{1}{3}\langle 3v, v \rangle \in \frac{1}{3}\mathbb{Z}$, and also $\langle v, v \rangle = \frac{1}{9}\langle 3v, 3v \rangle \in \frac{1}{9}\mathbb{Z}$; hence $(L^*)^3$ is an even lattice, as claimed.

Necessarily $n$ is even because disc $L$ is odd. This time any even $n$ may arise; we have seen already the example of $A_2$, which yields a level-3 lattice $A_2^{n/2} \subset \mathbb{R}^n$ for all even $n$.

Next we describe the discriminant forms on the $(\mathbb{Z}/3\mathbb{Z})$-vector spaces

$$D(L) = L^*/L, \quad D(L') = L'^*/L',$$

each of which carries a nondegenerate quadratic form $Q : [v] \mapsto \langle 3(v, v) \rangle \mod 3$.

We recall the structure of quadratic forms over finite fields of odd characteristic. For a finite field $k$ of odd order $q$, it is known that for each $r > 0$ there are two isomorphism classes of nondegenerate quadratic forms on $r$-dimensional vector spaces $V/k$, classified by the discriminant which is well-defined in $k^*/k^{*2}$. For $r$ odd, we can scale a form in one class to the other, so there is only one orthogonal group. For $r$ even, the isomorphism class is scaling invariant, and the two classes are also distinguished by their maximal isotropic spaces: if the discriminant is $(-1)^{r/2}$ times a square then $V$ has isotropic subspaces of dimension $r/2$; otherwise the largest isotropic subspaces have dimension $(r/2) - 1$. (Ultimately the factor $(-1)^{r/2}$ arises because for $r = 2$ the split quadratic form $Q(x_1, x_2) = x_1^2 - x_2^2$ has discriminant $-1$.) We use $Q_+$ for a quadratic form that has an $(r/2)$-dimensional isotropic subspace, and $Q_-$ for a form that does not; as we saw in characteristic 2, the orthogonal sum of two $Q_+$ forms or of two $Q_-$ forms is of type $Q_+$, while the sum of a $Q_+$ and a $Q_-$ is of type $Q_-$. 

Now for $q = 3$ the only nonzero square is 1, so quadratic forms have a well-defined discriminant in $\{ \pm 1 \}$. The quadratic forms on $D(A_2)$ and $D(E_6)$ have discriminants $-1$ and $+1$ respectively, while the form on $D(A_2^2)$ has type $Q_-$. We deduce:

**Proposition.** Let $L \subset \mathbb{R}^n$ be a lattice of level 3 and discriminant $3^r$. Then $r \equiv n/2 \mod 2$. If $n \equiv 2 \mod 4$ then the quadratic form on $D(L)$ has discriminant $(-1)^{(n/2)+r}$. If $4 \mid n$ then the quadratic form on $D(L)$ is of type $Q_+$ or $Q_-$ according as $n$ is $0$ or $4 \mod 8$.

**Proof:** If $r$ is even and the form has type $Q_+$ then the preimage of a maximal isotropic is an even unimodular lattice of rank $n$, so $8 \mid n$. If the type is $Q_-$, the form on the rank $n + 4$ lattice $L \oplus A_2^2$ has type $Q_+$, so $8 \mid n + 4$ and $n \equiv 4 \mod 8$. Conversely, if $r$ is odd then $L \oplus A_2$ has rank $n + 2$ and level 3 with and discriminant $3^{r+1}$, so $4 \mid n + 2$ and $n \equiv 2 \mod 4$. The discriminant is then deduced from the type of the quadratic form on $D(L \oplus A_2)$.

\[\square\]
Now to the modular forms.

We have seen that the translation $T : z \mapsto z + 1$ and the involution $w_3 : z \mapsto -1/(3z)$ generate a hyperbolic triangle group, with a cusp at $i\infty$ (fixed by $T$) and elliptic points at $i/\sqrt{3}$ (fixed by $w_3$) and $(i/\sqrt{3} - 1)/2$ (fixed by $w_3 T$, which has order 6). The index-2 subgroup $\Gamma_2(3)$ is also a triangle group, generated by $T$ and $w_3 T w_3 : z \mapsto z/(1 - 3z)$, with cusps at 0 (fixed by $w_3 T w_3$) and $i\infty$, and an elliptic point at $(i/\sqrt{3} - 1)/2$ fixed by $(w_3 T)^2 : z \mapsto -(z + 1)/(3z + 2)$. For $N = 3$ the congruence subgroup $\Gamma_1(N)$ no longer contains $-I_2$, but it still contains either $g$ or $-g$ for every $g \in \Gamma_0(N)$. (This is also the case for $N = 4$ and $N = 6$, and for no larger $N$.) Thus if a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular of some weight $k \in \mathbb{Z}$ for $\Gamma_1(3)$, then $f$ is automatically modular of level 3, with trivial Nebentypus if $k$ is even, and Nebentypus $\chi_3$ (the nontrivial Dirichlet character mod 3) if $k$ is odd.

For $k \in \mathbb{Z}$, let $M_k(\Gamma_1(3))$ be the vector space of modular forms of weight $k$ for $\Gamma_1(3)$, and $S_k(\Gamma_1(3))$ its subspace of cusp forms. The involution $w_3$ acts on these spaces by

$$(w_3 f)(z) = (3^{1/2}z/i)^{-k} f(-1/3z);$$

as we did for $w_2$, we choose the factor $(3^{1/2}z/i)^{-k}$ so that it is positive on the imaginary axis, so in particular $w_3 \theta_L = \theta_L$ if $L$ is a level-3 lattice such as $A_2$ for which $L \cong L'$.

**Theorem.** For any integer $k$,

i) $M_k(\Gamma_1(3))$ has dimension $\max(\lfloor (k + 3)/3 \rfloor, 0)$;

ii) $S_k(\Gamma_1(3))$ has dimension $\dim M_{k-6}(\Gamma_1(3)) = \max(\lfloor (k + 3)/3 \rfloor - 2, 0)$;

iii) If $M_k(\Gamma_1(3))$ or $S_k(\Gamma_1(3))$ has dimension $d$ then the $w_3$-invariant subspace has dimension $d/2$ or $(d + 1)/2$ according as $d$ is even or odd.

That is:

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The proof starts as with the parallel theorems for $\Gamma_1(2)$ (and $\Gamma(1)$). We first show that all nonzero $f \in M_k(\Gamma_1(3))$ have the same number of zeros in a fundamental domain $\mathcal{F}_3$ for $\Gamma_1(3)$, counted with multiplicity. Here the final answer is $k/3$, coming from a contour integral over the third-circle \{$z \in \mathcal{H} : |z| = 1/\sqrt{3}, |\text{Re}(z)| \leq 1/2$\}, and consistent with the count of $k/12$ for a $\Gamma(1)$ form because $\Gamma_1(3)$ has index 4 in $\text{PSL}_2(\mathbb{Z})$. As before (mutatis mutandis), the cusp at $i\infty$ is counted as for $\Gamma_1$ (a nonzero multiple of $q^n + O(q^{n+1})$ has valuation $n$); an $n$-th order zero at an elliptic point in the orbit of $(i/\sqrt{3} - 1)/2$ is counted with multiplicity $n/3$; and at $z = 0$ the valuation of $w_3 f$ is the valuation of $w_3 f$ at $z = i\infty$.

This lets us show that for each $k$ the dimensions of $M_k(\Gamma_1(3))$ and $S_k(\Gamma_1(3))$ are no larger than claimed, so it is enough to produce enough modular forms. For $k = 1$, we use $\theta_{A_2}$; this also gives a generator $\theta_{A_2}^3$ of $M_2(\Gamma_1(3))$, and for $M_5(\Gamma_1(3))$ we use $\theta_{A_2}^2$ and $\theta_{E_6}$. We can then construct a weight-6 cusp...
form $\Delta_{(3)} = q + O(q^2)$ as a product of two nonzero linear combinations of $\theta_{A_2}^3$ and $\theta_{E_6}$, one vanishing at $z = i\infty$ and the other at $z = 0$. Now for each $k$ the weight-$k$ monomials in $\theta_{A_2}$ and $\theta_{E_6}$ constitute a basis for $M_k(\Gamma_1(3))$, and their products with $\Delta_{(3)}$ constitute a basis for $S_{k+6}(\Gamma_1(3))$. This also yields the dimensions of the $w_3$-invariant subspaces, once we check that $\theta_{A_2}$ and $\Delta_{(3)}$ are invariant under $w_3$. These two forms generate $M_1(\Gamma_1(3))$ and $S_6(\Gamma_1(3))$ respectively, so must be in either the $+1$ or $-1$ eigenspace for $w_3$, and if either of these forms were in the $-1$ eigenspace it would vanish at $i/\sqrt{3}$, which is not possible by the zero-counting formula.

By comparing zero multiplicities and leading terms, we see that the weight-24 form $\Delta_{(3)}^4$ equals $\Delta(z)\Delta(3z)$, whence

$$\Delta_{(3)}(z) = (\eta(z)\eta(3z))^6 = q \prod_{m=1}^{\infty} ((1 - q^m)(1 - q^{3m}))^6,$$  

which expands to

$$\Delta_{(3)}(z) = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 \cdots.$$  

Likewise the weight-3 forms vanishing at a cusp are

$$\frac{1}{54}(\theta_{E_6} - \theta_{A_2}^3) = \eta(q^3)^9/\eta(q)^3 = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \cdots,$$

$$\frac{1}{2}(3\theta_{E_6} - \theta_{A_2}^3) = \eta(q)^9/\eta(q^3)^3 = 1 - 9q + 27q^2 - 9q^3 - 117q^4 + 216q^5 + 27q^6 \cdots.$$  

If $\text{disc } L = 3^{n/2}$ then $L$ and $L'$ have the same discriminant, and it may happen that $L \cong L'$, as happens for $L = A_2$. For such lattices, $\theta_L$ is invariant under $w_3$, so is a polynomial in $\theta_{A_2}$ and $\Delta_{(3)}$. As before, we deduce a lower bound on $N_{\min}(L)$, which is satisfied if and only if $\theta_L$ is extremal; here this bound is $2[n/12] + 2$, and once more the extremal modular form has positive $q^{[n/12]+1}$ coefficient because the product formula (2) shows that the power series for $1/\Delta_{(3)}$ has positive coefficients. Extremal lattices of level 3 include at least two record sphere packings: $A_2$ for $n = 2$, and the Coxeter-Todd lattice $K_{12}$, which can be constructed as a 3-neighbor of $A_2^6$; write $A_2$ as $\{(x, y, z) \in \mathbb{Z}^3 : x + y + z = 0\}$; form a lattice $(A_2^6)^{+3}$ generated by $A_2^6$ and the dual vector $(v, v, v, v, v, v)$ where $v = (1/3, 1/3, -2/3) \in A_2^6$; then $K_{12}$ is the index-3 sublattice $\{(\bar{x}, \bar{y}, \bar{z}) \in A_2^6 : \sum_{j=1}^6 x_j - y_j \equiv 0 \text{ mod } 3\}$. This lattice has theta series

$$\theta_{K_{12}} = \theta_{A_2}^6 - 36\Delta_{(3)} = 1 + 756q^2 + 4032q^3 + 20412q^4 + 60480q^5 + \cdots.$$  

which can be used to show that $K_{12}$ is the only extremal lattice of rank 12 and level 3.

The shells of an extremal lattice of rank $n$ constitute spherical 5-designs if $n \equiv 0$ or $2 \text{ mod } 12$, 3-designs if $n \equiv 4$ or $6 \text{ mod } 12$, and 1-designs in the remaining cases $n \equiv 8$ or $10 \text{ mod } 12$. For example, the regular hexagon is the first shell of $A_2$ and is a circular 5-design, and each shell of $K_{12}$ is a spherical 5-design. For a variation on this theme, the shells of $E_6$ are also 5-designs: since $E_6$ has no roots, any weighted theta function $\theta_{E_6,P}$ with $\deg P > 0$ is a $\Gamma_1(3)$ cusp form with a double zero at $z = 0$, so vanishes as long as $3 + \deg P < 9$. For each of $A_2, E_6, K_{12}$ the 5-design property can also be deduced from the action of the automorphism group, but (as we saw in levels 1 and 2) these
spherical-design properties also persist for lattice in higher dimension whose symmetry groups have more invariants of degree < 6.

Recall that for level-2 lattices we made use of linear combinations of Eisenstein series $E_k(z)$ and $E_k(2z)$ to give exact or approximate formulas for the representation numbers $N_{2n}(L)$. We can give some similar formulas here. For example, $M_2(\Gamma_1(3))$ is generated by

$$E_2(z) = \frac{1}{2}(3E_2(3z) - E_2(z)) = 1 + 12 \sum_{m=1}^{\infty} \left( \frac{mq^m}{1 - q^m} - \frac{3mq^{3m}}{1 - q^{3m}} \right),$$

which must equal $\theta_{A_2}^2$, whence $N_{2n}(A_2^3) = 12\sigma_1(n) - 36\sigma_1(n/3)$ for every positive integer $n$; and $M_4(\Gamma_1(3))$ is generated by $E_4(z)$ and $E_4(3z)$, so

$$\theta_{A_2}(z) = \frac{1}{10}(E_4(z) + 9E_4(3z)), \quad \theta_{A_2E_6}(z) = \frac{1}{40}(13E_4(z) + 27E_4(3z)),$$

and from $\theta_{A_2E_6}$ we also deduce the formula $\frac{1}{40}(E_4(z) + 39E_4(3z))$ for the lattice $(A_2E_6)' = A_2 \oplus E_6'$, and again recover formulas for $N_{2n}(A_2^4)$, $N_{2n}(A_2E_6)$, $N_{2n}(A_2 \oplus E_6')$ as sums over divisors of $n$, e.g.

$$N_{2n}(A_2^4) = 24(\sigma_3(n) + 9\sigma_3(n/3)), \quad N_{2n}(A_2E_6) = 6(13\sigma_3(n) + 27\sigma_3(n/3)).$$

The lattice $A_2E_6$ is characterized by its theta function (and even by its rank 8 and root number of 78); but for $A_2^4$ we find another lattice $(D_4 \oplus (D_4))^{++}$ that has the same theta function, and thus yields our first example of isospectral tori in dimension as low as 8.

But in level 3 we also have lattices of rank $2k$ with $k$ odd, and their theta functions cannot be obtained in the same way from $\Gamma(1)$ Eisenstein series. It is true that we can write formulas such as

$$\theta_{A_2}(z) = E_2(z)^{1/2} = 1 + 6q + 6q^3 + 6q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + \cdots,$$

but this does not give us access to the representation numbers $N_{2n}(A_2)$. Still, there is a known formula for $N_{2n}(A_2)$ as a divisor sum, using the arithmetic of the cyclotomic number field $Q(\sqrt{-3}) = Q(e^{2\pi i/3})$ (because the norms of $A_2$ are the integers $2(x^2 - xy + y^2)$, and $x^2 - xy + y^2$ is also the algebraic norm of a general algebraic integer $x + ye^{2\pi i/3}$ in that cyclotomic field):

$$N_{2n}(A_2) = 6 \sum_{d|n} \chi_3(d),$$

where $\chi_3$ is again the nontrivial Dirichlet character mod 3. The coefficients of $\theta_{E_6}$ suggest similar patterns; for example, when $p$ is prime, $N_{2p}(E_6) = 72(p^2 + 1)$ for $p \equiv 1 \mod 3$ and $N_{2p}(E_6) = 90(p^2 - 1)$ for $p \equiv -1 \mod 3$ (and also for $p = 3$). This suggests that the construction of Eisenstein series might be modified to produce modular forms of odd weight for $\Gamma(1)$ whose coefficients are even-power divisor sums weighted by $\chi_3$, and more generally modular forms of level $N$ whose coefficients are divisor sums weighted by the Nebentypus character. We next carry out such a construction.

Recall that for even $k$ the Eisenstein series of weight $k$ for $\Gamma(1)$ is proportional to $\sum_{m,n} (mz + n)^{-k}$, the sum ranging over all $(m, n) \neq (0, 0)$. We generalize as follows: fix a prime $^1N > 1$, a nontrivial

\footnote{The construction generalizes to non-prime levels $N$ and characters of conductor $N$, but in that setting there are more cusps to check, especially when the level is not squarefree; for now we introduce the idea for prime $N$ only.}
Dirichlet character $\chi \mod N$, and an integer $k > 2$ such that $(-1)^k = \chi(-1)$, define

$$G_k(z, \chi) = \sum_{m,n \in \mathbb{Z}} \overline{\chi}(n)(Nmz + n)^{-k}. \quad (11)$$

The sum converges absolutely because $k > 2$, and the parity condition $(-1)^k = \chi(-1)$ is needed to prevent the sum from simplifying to zero due to cancellation between the $(m, n)$ and $(-m, -n)$ terms. For $g = \frac{a}{c} \frac{b}{d} \in \Gamma_1(n)$ we have

$$G_k(gz, \chi) = (cz + d)^k \sum_{m,n \in \mathbb{Z}} \overline{\chi}(n)(((Nam + cn)z + (Nbm + dn))^{-k}. \quad (12)$$

Since $N|c$, the coefficient $Nam + cn$ of $z$ is a multiple of $N$; combining this with $\det(g) = 1$, we see that as $(m, n)$ ranges over all nonzero integer vectors in $\mathbb{Z}^2$, so does

$$(m', n') := (Nam + cn, Nbm + dn).$$

But the constant coefficient $n' = Nbm + dn$ is congruent to $dn$ mod $N$, so $\chi(n') = \chi(d)\chi(n)$. Therefore

$$G_k(gz, \chi) = \chi(d)(cz + d)^k \sum_{m',n' \in \mathbb{Z}} \overline{\chi}(n')(mz + n)^{-k} = \chi(d)(cz + d)^k G_k(z, \chi). \quad (13)$$

Thus $G_k(z, \chi)$ is at least weakly modular of level $N$ with Nebentypus $\chi$. In particular, $G_k(z, \chi) = G_k(z + 1, \chi)$ so $G_k(z, \chi)$ has an expansion in powers of $q = e^{2\pi i z}$. The constant coefficient is

$$G_k(i\infty, \chi) = 2 \sum_{n=1}^{\infty} \overline{\chi}(n)n^{-k} = 2L(k, \chi), \quad (14)$$

which is known to be an algebraic multiple of $\pi^k$ under our hypothesis that $(-1)^k = \chi(-1)$; for example,

$$L(3, \chi_3) = 4\pi^3/3^{9/2}, \quad L(5, \chi_3) = 4\pi^5/3^{13/2}, \quad L(7, \chi_3) = 56\pi^7/3^{19/2}, \quad (15)$$

and in general $L(k, \chi)$ is a rational multiple of $N^{1/2}\pi^k$ if $\chi$ is a real character. By (14) we have

$$G_k(z, \chi) = 2L(k, \chi) + 2 \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \overline{\chi}(n)(Nmz + n)^{-k} = 2L(k, \chi) + 2 \sum_{m=1}^{\infty} \sum_{n_0 \mod N} \overline{\chi}(n_0) \sum_{n \equiv n_0 \mod N} (Nmz + n)^{-k}. \quad (16)$$

The innermost sum is

$$N^{-k} \sum_{n_1 \in \mathbb{Z}} (nz + (n_0/N) + n_1)^{-k} = \frac{(-2\pi i)^k}{(k - 1)!N^k} \sum_{d=1}^{\infty} d^{k-1}e^{2\pi idn_0/N}q^{md} \quad (17)$$
where \( q = e^{2\pi i z} \) (for the last step, see formula (32) on page 92 of Serre’s *A Course in Arithmetic*). Multiplying this by \( \overline{\chi}(n_0) \) and summing over \( n_0 \) yields

\[
\frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{d=1}^{\infty} d^{k-1} \tau_d(\overline{\chi}) q^{md},
\]

where \( \tau_d(\overline{\chi}) \) is the *Gauss sum*

\[
\tau_d(\overline{\chi}) := \sum_{n_0 \mod N} \overline{\chi}(n_0) e^{2\pi i d n_0 / N} = \chi(d) \tau_1(\overline{\chi}).
\]

It is known that \( \tau_1(\overline{\chi}) \) is a complex number of absolute value \( N^{1/2} \); for example,

\[
\tau_1(\chi_3) = e^{2\pi i / 3} - e^{-2\pi i / 3} = \sqrt{3} i.
\]

Putting everything together, we find that

\[
G_k(z, \chi) = 2L(k, \overline{\chi}) + 2 \frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \chi(d) d^{k-1} q^{md}.
\]

For example,

\[
G_3(z, \chi_3) = \frac{8\pi^3}{3^{9/2}} \left( 1 - 9 \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \chi_3(d) d^{k-1} q^{md} \right) = \frac{8\pi^3}{3^{9/2}} [1 - 9 (q + 3q^2 - q^3 - 13q^4 \cdots)].
\]

The bracketed series agrees with the linear combination \((3\theta^3_{A_2} - \theta_{E_6})/2\) of our generators of \( M_3(\Gamma_1(3)) \).

We check the growth condition at the cusp \( z = 0 \), and obtain an independent modular form of weight \( k \), by applying \( w_N \), which takes \( G_k(z, \chi) \) to a multiple of

\[
z^{-k} G_k(1/(Nz), \chi) = \sum_{m \in \mathbb{Z}, (m,n) \neq (0,0)} \overline{\chi}(n)(nz - m)^{-k}.
\]

This is simpler: the constant term vanishes because \( \overline{\chi}(n) = 0 \) for \( n = 0 \), and the \( n \neq 0 \) terms are

\[
2 \sum_{n=1}^{\infty} \overline{\chi}(n) \sum_{m \in \mathbb{Z}} (nz - m)^k = 2 \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \overline{\chi}(n) \sum_{d=1}^{\infty} d^{k-1} q^{md}.
\]

Removing the factor \( 2(-2\pi i)^k/(k-1)! \) leaves the series whose \( q^n \) coefficient is \( \sum_{d|n} \overline{\chi}(n/d) d^{k-1} \).

For example, \( w_3 G_3(z, \chi) \) is a multiple of

\[
q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \cdots.
\]

Multiplying (25) by 9, 27, or 81 and adding to the series \( 1 - 9(q + 3q^2 - q^3 - 13q^4 \cdots) \) of (22) recovers the theta functions of \( E_6 \), \( A_2^3 \), \( E_6 \) respectively. For example, \( N_{2n}(E_6') = 9 \sum_{d|n} (\chi_3(n/d) - \chi_3(d)) \), which includes the formula for \( N_{2n}(E_6) \) (as \( N_{6n}(E_6') \)) and vanishes as it should for \( n \equiv 1 \mod 3 \).

\(^2\)In general if \( \chi \) is a real character to prime modulus \( N \) then \( \tau_1(N) = N^{1/2} \) or \( N^{1/2} \) according to whether \( N \equiv 1 \) or \( -1 \mod 4 \).