Venkov’s proof of the Niemeier classification started with the fact that if $N$ is a Niemeier lattice and $P$ is any harmonic polynomial of degree 2 then $\theta_{N,P}$ is a modular cusp form of weight 14, and is therefore zero — whence the roots of $N$ constitute a spherical 2-design. In the special case where $N$ is the Leech lattice, if $P$ is a nonconstant harmonic polynomial of degree $d$ then $\theta_{N,P}$ is a modular form of weight $12 + d$ that vanishes to order at least 2 at the cusp, and is thus zero if $d < 12$, and also if $d = 14$. Hence each shell of the Leech lattice is a spherical 11-design, and indeed an “$11\frac{1}{2}$-design” as Venkov calls it. (In general, for odd $d$ Venkov says a centrally symmetric finite subset $S$ of a sphere is a “$d\frac{1}{2}$-design” if $\sum_{x \in S} P(x) = 0$ for all nonconstant spherical polynomials $P$ with $\deg P \leq d$ or $\deg P = d + 3$.) This property uses only the assumption that $N$ is an even self-dual lattice of rank 24 with no vectors of norm 2, not the fact that $\Lambda_{24}$ is the unique such lattice. As with the $7\frac{1}{2}$-design property for $E_8$, we can thus regard this $11\frac{1}{2}$-design property either as a strong constraint on any such lattice $N$ or as a tool for obtaining detailed information about the configuration of short vectors in $\Lambda_{24}$.

We illustrate this by tabulating some statistics for the distribution of the 196560 vectors in the first shell of $\Lambda_{24}$. We then generalize to self-dual even lattices in $\mathbf{R}^n$ with “extremal” theta series for other $n \equiv 0 \mod 2$, and conclude this chapter with statistics for $n = 32, 48, 72$.

As we did for $E_8$, we fix some nonzero $v_0 \in \Lambda_{24}$, and let $N_k (k \in \mathbb{Z})$ be the number of $v \in \Lambda_{24}$ with $\langle v, v \rangle = 4$ and $\langle v_0, v \rangle = k$. Then $N_{-k} = N_k$ for all $k$, and $N_k = 0$ if $k^2 > 4\langle v_0, v_0 \rangle$ by Cauchy-Schwarz. Here the fact that $\Lambda_{24}$ has no roots gives us a further condition: $N_k = 0$ if $|k| > \frac{1}{2}\langle v_0, v_0 \rangle$, except if $\langle v_0, v_0 \rangle = 4$ when $N_{\pm 4} = 1$. Indeed if $\langle v_0, v \rangle > \frac{1}{2}\langle v_0, v_0 \rangle$ then

$$\langle v - v_0, v - v_0 \rangle = \langle v, v \rangle - 2\langle v_0, v \rangle + \langle v_0, v_0 \rangle < \langle v, v \rangle,$$

but $\langle v, v \rangle = 4$ so $v - v_0 = 0$; likewise if $\langle v_0, v \rangle < -\frac{1}{2}\langle v_0, v_0 \rangle$ then $v + v_0 = 0$. This together with the 11-design property gives us enough conditions to solve for the $N_k$ as long as $\langle v_0, v_0 \rangle \leq 10$. For each of the possible norms $\langle v_0, v_0 \rangle = 4, 6, 8, 10$, we list the number of such $v_0$ (these counts are the $q^2, q^3, q^4, q^5$ coefficients of $\theta_{\Lambda_{24}} = E_4^3 - 720\Delta$), followed by the values of $N_k$ for $|k| \leq 5$.

<table>
<thead>
<tr>
<th>$\langle v_0, v_0 \rangle$</th>
<th>#</th>
<th>$N_0$</th>
<th>$N_{\pm 1}$</th>
<th>$N_{\pm 2}$</th>
<th>$N_{\pm 3}$</th>
<th>$N_{\pm 4}$</th>
<th>$N_{\pm 5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>196560 = $2^43^25\cdot7\cdot13$</td>
<td>93150</td>
<td>47104</td>
<td>4600</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>16773120 = $2^43^25\cdot7\cdot13$</td>
<td>75900</td>
<td>48600</td>
<td>11178</td>
<td>552</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>398034000 = $2^43^25\cdot7\cdot13$</td>
<td>65780</td>
<td>47104</td>
<td>16192</td>
<td>2048</td>
<td>46</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>4629381120 = $2^43^35\cdot7\cdot13\cdot23$</td>
<td>58806</td>
<td>45100</td>
<td>19450</td>
<td>4050</td>
<td>275</td>
<td>2</td>
</tr>
</tbody>
</table>

As was true for $E_8$, the fact that each of these $N_{\pm k}$ is the same for all $v_0$ of the same norm also reflects the fact that $\text{Aut}(\Lambda_{24})$ (a.k.a. the Conway group $Co_0 = 2.Co_1$) acts transitively on each of the first
four shells of lattice vectors. (See page 181 of the ATLAS [second page of the \(C_{O_1}\) entry] for a list of orbit sizes and stabilizers for the action of \(C_{O_1}\) on nonzero \(v_0\) with \(\langle v_0, v_0 \rangle \leq 32\).) But the common counts \(N_{\pm k} = N_{\pm k}(v_0)\) still carry information about the structure of \(\Lambda_{24}\).

For \(\langle v_0, v_0 \rangle = 4\), the count \(N_0 = 93150\) is the kissing number of the orthogonal complement of \(v_0\) in \(\Lambda_{24}\); this complement is the “laminated lattice” \(\Lambda_{23}\), which is the unique even lattice of rank 23 and discriminant 4 with no roots. This is conjectured to be the maximal kissing number in \(\mathbb{R}^{23}\), even without the lattice condition. The count \(N_2 = 4600\) is what is sometimes called the “necking number” of \(\Lambda_{24}\), that is, the number of lattice vectors at minimal distance from two closest lattice vectors, here 0 and \(v_0\). (This number is well-defined at least if the automorphism group acts transitively on minimal vectors, as is the case here.) It is also the kissing number of the “shorter Leech lattice” \(\Lambda_{0}^{23}\), which is the unique lattice strictly between \(\Lambda_{23}\) and \(\Lambda_{23}^*\), and the unique self-dual lattice of rank at most 23 without vectors of norm 1 or 2 (this too can be deduced from the Niemeier classification via gluing); the minimal norm is 3, attained by the vectors \(v - \frac{1}{2} v_0\) with \(v\) varying over the 4600 “necking” vectors. The stabilizer of \(v_0\) is Conway’s sporadic group \(\tilde{C}_{O_2}\), which is also the determinant +1 subgroup of \(\text{Aut}(\Omega_{23})\), and acts transitively on the 2300 pairs of minimal vectors of \(\Omega_{23}\). As for \(N_{\pm 1} = 47104\), that is the number of minimal vectors in each non-integral coset of \(\Lambda_{23}\) in \(\Lambda_{23}^*\); we shall use this number again when we reach the counts for \(\langle v_0, v_0 \rangle = 10\).

For \(\langle v_0, v_0 \rangle = 6\), the count \(N_0 = 75900\) is the kissing number of the orthogonal complement of \(v_0\) in \(\Lambda_{24}\), which time is unique even lattice of rank 23 and discriminant 6 with no roots. The other \(N_{\pm k}\) again count minimal vectors in cosets of that lattice in its dual. Here the stabilizer of \(v_0\), and the group of determinant +1 automorphisms of the orthogonal complement, is Conway’s sporadic group \(\tilde{C}_{O_2}\). As usual, the minimal vectors \(v\) with \(\langle v_0, v \rangle = \frac{1}{2} \langle v_0, v_0 \rangle\) come in pairs \(\{v, v_0 - v\}\); here there are \(N_{3/2} = 276\) pairs, and it is known that \(\tilde{C}_{O_2}\) acts doubly transitively on them. For the point stabilizer, see the discussion of \(\langle v_0, v_0 \rangle = 10\).

We have already encountered \(\langle v_0, v_0 \rangle = 8\). The count \(N_4 = 46\) gives us another entry point into Conway’s characterization of \(\Lambda_{24}\): the orthogonal frame of norm-8 vectors in \(v_0 + 2L\) consists of the 46 vectors \(2v - v_0\), together with \(v_0\) itself and \(-v_0\). Recall that we constructed \(\Lambda_{24}\) as a 2-neighbor of the \(A_1^{24}\) Niemeier lattice \(\{x \in \mathbb{Z}^{24}(1/2) : x \text{ mod } 2 \in \mathcal{G}_{24}\}\). In those coordinates, we may take \(v_0 = 4e_1\); then the 46 vectors \(v \in \Lambda_{24}\) with \(\langle v, v \rangle = \langle v_0, v_0 \rangle = 4\) are \(e_1 \pm \epsilon_j (2 \leq j \leq 24)\), and the 2\(^{11}\) short vectors with \(\langle v_0, v \rangle = 3\) are \(\frac{2}{3} e_1 + \frac{1}{2} \sum_{j=2}^{24} \epsilon_j e_j\) for certain \(\epsilon_j = \pm 1\) determined by the Golay code \(\mathcal{G}_{24}\).

The situation for \(\langle v_0, v_0 \rangle = 10\) recalls our observation for vectors of norm 6 in \(E_8\): in each case, the last nonzero \(N_k\) is 2, and we deduce that \(v\) is uniquely a sum of two minimal vectors. Here we find that every nonzero Leech vector of norm 10 is uniquely the sum of two minimal vectors, necessarily with inner product 1. As a consistency check, the number of such unordered pairs \(\{v_1, v_2\}\) of minimal vectors is \(\frac{1}{2} \cdot 196560 N_1(v_1) = 196560 \cdot 47104/2\), which indeed agrees with the count 4629381120 of norm-10 vectors in \(\Lambda_{24}\). Alternatively we can use the observation that \(\frac{1}{2} N_4(\Lambda_{24}) N_1(v_1) = N_{10}(\Lambda_{24})\) to prove that every \(v_0\) of norm 10 has such a representation \(v_1 + v_2\): if not, then some \(v_0\) would have two different representations, but this is not possible once \(\langle v_0, v_0 \rangle > 8\) because if \(v_0 = v_1 + v_2 = v_1' + v_2'\)
then
\[ 4 \leq \langle v_1 + v'_1 - v_0, v_1 + v'_1 - v_0 \rangle = 8 + 2\langle v_1, v'_1 \rangle - \langle v_0, v_0 \rangle < 2\langle v_1, v'_1 \rangle, \]
whence \( \langle v_1, v'_1 \rangle > 2 \), which we already know is not possible. Thus the stabilizer of \( v_0 \) in \( \text{Aut}(\Lambda_{24}) \) is the stabilizer of the plane spanned by \( v_1, v_2 \). The pointwise stabilizer of \( v_1 \) and \( v_2 \) is yet another sporadic simple group \( \text{McL} \), and the stabilizer of \( \{ v_1, v_2 \} \) is \( \text{Aut}(\text{McL}) = \text{McL} : 2 \). This group was already discovered a few years earlier (1969) by J. McLaughlin, who constructed \( \text{McL} : 2 \) as the automorphism group of a strongly regular graph on 275 vertices. In \( \Lambda_{24} \), these vertices arise as the \( N_4(v_0) = 275 \) short vectors \( v \) with \( \langle v_0, v \rangle = 4 \), and their graph adjacency is determined by their inner products in \( \Lambda_{24} \). Since \( v_1 - v_2 \) has norm 0, this group is also contained in \( CO_3 \), and indeed is the point stabilizer in the action of \( CO_3 \) on the 276 pairs of short vectors that sum to \( v_1 - v_2 \) (which split 1 + 275 + 275 + 1 according to their inner product with \( v_0 \)).

We take this one step further. The action of \( \text{Aut}(\Lambda_{24}) \) on the norm-12 shell cannot be transitive, because \( N_{12}(\Lambda_{24}) = 34417656000 = 2^93^55^313 \cdot 17 \cdot 103 \) but \( \text{Aut}(\Lambda_{24}) \) cannot contain a 103-cycle. (In fact \( |\text{Aut}(\Lambda_{24})| = 8315553613086720000 = 2^{22}3^95^47^211 \cdot 13 \cdot 23 \), as Conway showed in his characterization of the Leech lattice.) There is an easy invariant, though: some but not all vectors \( v_0 \) of norm 12 are the sums of two minimal vectors. Since such a representation, if it exists, is unique, there are
\[ \frac{1}{2} \cdot 196560 \cdot N_2(v_1) = 196560 \cdot 4600/2 = 452088000 \]
choices of \( v_0 \) with \( \langle v_0, v_0 \rangle = 12 \) and \( N_6(v_0) = 2 \), leaving 33965568000 = 2^{12}3^95^313 for which \( \langle v_0, v_0 \rangle = 12 \) and \( N_6(v_0) = 0 \). Each of these subcounts is a factor of \( |\text{Aut}(\Lambda_{24})| \), and indeed it turns out that \( \text{Aut}(\Lambda_{24}) \) acts transitively on each of the subsets \( N_6(v_0) = 2 \) and \( N_6(v_0) = 0 \) of the norm-12 shell. In particular, in each subset the other counts \( N_k(v_0) \) with \( |k| \leq 5 \) must be constant. We can calculate them using the 11-design property, finding
\[
(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 5}) = \begin{cases} (55530, 41472, 22518, 5632, 891, 0), & \text{if } N_6 = 2; \\ (53682, 43056, 21528, 6072, 759, 24), & \text{if } N_6 = 0. \\ \end{cases} \tag{1}
\]
(If we did not know \textit{a priori} that \( N_6(v_0) \leq 2 \), we could deduce this fact from the 11-design calculation: \( N_6(v_0) \) must be even, and if \( N_6(v_0) > 2 \) then \( N_6(v_0) = 24 - 12N_6 < 0 \).)

In the former case, \( v_0 \) is contained in one of the 452088000/6 = 75348000 sublattices of \( \Lambda_{24} \) isomorphic with \( A_2(2) \). Here the stabilizer is not sporadic: \( v_0 \) has stabilizer isomorphic with \( \text{PSU}_6(\mathbb{Z}/2\mathbb{Z}) \), and the stabilizer of \( A_2(2) \) is \( \text{PSU}_6(\mathbb{Z}/2\mathbb{Z}) : S_3 \). The 891 vectors counted by \( N_4(v_0) \) correspond to the maximal isotropic spaces for a unitary form such as \( \sum_{j=1}^6 x_j^3 \) over \( F_4 \); this is reminiscent of the appearance of the configuration of 27 lines on a cubic surface in \( E_6 \), which is the orthogonal complement of \( A_2 \) in \( E_8 \).

In the latter case (when \( N_6 = 0 \) in (1)), we get 24 vectors \( v \) whose projections \( v - \frac{5}{12}v_0 \) have norm \( 4 - (5^2/12) = 23/12 \) and pairwise inner products \(-1/12\). Hence these projections form a regular
Proof. Let $\theta_{N,P} = 0$ for every harmonic polynomial $P$ of degree 2. Here $n = 12\nu$, so $\theta_{N,P}$ is a modular form of weight 6$\nu$ + 2 that vanishes to order at least $\nu/2$ at the cusp, and is thus $\Delta^{\nu/2}$ times a modular form of weight 2. The only such form is zero, so $\theta_{N,P} = 0$. For $\delta > 0$ we are to prove that $\theta_{N,P} = 0$ for every nonconstant harmonic polynomial $P$ of degree at most $d$, and also for $\deg(P) = d + 3$. Here $n = 12\nu - 2\delta$, so $\theta_{N,P}$ is a modular form of weight $6\nu - \delta + \deg(P)$ that again vanishes to order at least $\nu/2$ at the cusp, and is thus $\Delta^{\nu/2}$ times a modular form of weight $\deg(P) - \delta$. If $\deg(P) = \delta + 2$ then this weight is 2, while if $\deg(P) \leq \delta - 1$ then the weight is negative; in either case we deduce that $\theta_{N,P} = 0$. □

So, how large can $\delta(L)$ get? The dimension of the space of modular forms of weight $n/2$ suggests that $N_{\text{min}}(L)$ might get as large as $2\lfloor n/24 \rfloor + 2$, as it does for $n = 8, 16, 24$, but no larger: $\theta_L$ must be $E_4^{n/8} + \sum_{j=1}^{\lfloor n/24 \rfloor} a_j \Delta^j E_4^{(n/8) - 3j}$ for some $a_j$, and if $N_{\text{min}}(L) > 2\lfloor n/24 \rfloor + 2$ then $a_1, a_2, \ldots, a_{\lfloor n/24 \rfloor}$ are successively determined by the condition that the $q, q^2, \ldots, q^{\lfloor n/24 \rfloor}$ coefficients of $\theta_L$ vanish, at which point $\theta_L$ is completely determined. This is what happened for $n = 8, 16$ (with no $a_j$) and $n = 24$ (with $a_1 = -720$). In general, we call the resulting modular form the extremal theta function of weight $n/2$ (whether or not it is actually the theta function of some lattice); it is the unique modular form $\theta$ of this weight such that $\theta - 1$ vanishes to order greater than $\lfloor n/24 \rfloor$ at the cusp $q = 0$. Conceivably, though, this form’s $q^{\lfloor n/24 \rfloor + 1}$ coefficient might be negative or zero; in the former case, $N_{\text{min}}(L)$ could not be as large as $2\lfloor n/24 \rfloor + 2$ for any even self-dual lattice of rank $n$, and in the latter case, $N_{\text{min}}(L)$ could be yet larger. Siegel\(^1\) showed that in fact the $q^{\lfloor n/24 \rfloor + 1}$ coefficient is always positive, and the result was later generalized to other kinds of extremal theta functions (and extremal weight enumerators for self-dual codes). The general setting is as follows.

Let $f(q)$ and $g(q)$ be power series of the form

$$f(q) = 1 + O(q), \quad g(q) = q + O(q^2),$$

and fix some $r$ with $0 \leq r \leq 1$. (In our setting, $(f, g) = (E_4^3, \Delta)$, and $r = 0, 1/4, 3/4$ according as $n$ is 0, 8, or 16 mod 24.) For a nonnegative integer $k$ and any $a_0, a_1, \ldots, a_k$ there exists a

unique homogeneous polynomial $P(\cdots)$ of degree $k$ such that $f^r P(f, g) = \sum_{j=0}^{k} a_j q^j + O(q^{k+1})$; we construct this polynomial iteratively starting from the leading coefficient $a_0$, as we did when $a_0 = 1$ and $a_1 = \ldots = a_k = 0$. For this choice $(1, 0, 0, \ldots, 0)$ of $a_0, \ldots, a_k$, we find $P$ such that $f^r P(f, g) = 1 + C q^{k+1} + O(q^{k+2})$. We shall show that $C$ is $(k+r)/(k+1)$ times the $1/q$ coefficient of $f'/\left(f^r g^{k+1}\right)$. In particular, $C > 0$ for all $k > 0$ if $f^{1-r}$ and $1/g$ have positive coefficients. In our present setting, $f^{1-r}$ is $E_4$, $E_4^2$, or $E_4^3$, and thus has positive coefficients as desired, while the product expansion for $\Delta$ gives

$$\frac{1}{\Delta} = q^{-1} \left( \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right)^{24} = q^{-1} \left( \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots) \right)^{24}$$

which manifestly has positive coefficients. Note that if $r = 0$ then for any $f, g$ the $1/q$ coefficient of $f'/g^{k+1}$ is well-defined, even though $f$ could have been changed to $f + Ag$ for any constant $A$, because this changes $f'/g^{k+1}$ by $Ag'/g^{k+1}$, which is the derivative of $-Ag^{-k}/(k+1)$ and thus has no $1/q$ term.

To prove our formula, begin by dividing both sides of $f^r P(f, g) = 1 + C q^{k+1} + O(q^{k+2})$ by $f^r g^k$ to obtain $p(f/g) = 1/(f'g^k) + C q + O(q^2)$ where $p$ is the degree-$k$ univariate polynomial $p(X) = P(X, 1)$. Since $g$ and $g/f$ are both of the form $q + O(q^2)$, while $1/f_0 = 1 + O(q^2)$, we can find (unique) power series $G(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ and $F(z) = 1 + O(z)$ such that $g = G(g/f)$ and $f = F(g/f)$. (These power series are holomorphic in a neighborhood of $z = 0$ if $f$ and $g$ have positive radii of convergence, but we need $F, G$ only as formal power series.) Taking $z = g/f$, we find $F(z)^{-r}/G(z)^k = p(1/z) - C z + O(z^2)$, in which the right-hand side is the beginning of the Laurent expansion of $F(z)^{-r}/G(z)^k$ about $z = 0$. Therefore $-C$ is the $z$ coefficient of that expansion. But then $C$ is the residue of $-(F^{-r}/G(z)^k) dz/z^2$ at $z = 0$. Now we use the invariance of the residue under locally invertible transformations. [Again this is a theorem about formal power series, though the easiest proof is to use complex analysis, truncating the series if necessary to ensure convergence.] This yields

$$C = -\text{Res}_{q=0} \frac{1}{f^r g^k} \frac{d(g/f)}{(g/f)^2} = +\text{Res}_{q=0} \frac{g df - f dg}{f^r g^{k+2}}.$$

On the other hand, we have

$$d(f^{1-r}/g^{k+1}) = \frac{(1-r) g df - (k+1) f dg}{f^r g^{k+2}},$$

so the residue at $q = 0$ of $((1-r) g df - (k+1) f dg)/(f^r g^{k+2})$ vanishes. Multiplying (4) by $k+1$ and subtracting the right-hand side of (5), we find that

$$(k+1)C = (k+r) \text{Res}_{q=0} \frac{df}{f^r g(k+1)},$$

whence $C$ is $(k+r)/(k+1)$ times the $1/q$ coefficient of $f'/\left(f^r g^{k+1}\right)$, as claimed.

It follows that a self-dual even lattice $L$ of rank $n > 0$ has $N_{\min}(L) \leq 2[n/24] + 2$. If $L$ attains this bound, it is said to be extremal. Our Lemma then shows that if $L$ is an extremal self-dual even
lattice then every shell of \( L \) is an \( 11\frac{1}{2} \)-design if \( n \equiv 0 \mod 24 \), a \( 7\frac{1}{2} \)-design if \( n \equiv 8 \mod 24 \), and a \( 3\frac{1}{2} \)-design if \( n \equiv 16 \mod 24 \).

We have seen that such \( L \) exist for \( n = 8 \) (\( E_8 \)), \( n = 16 \) (\( E_8^2 \) and \( D_{16}^+ \)), and \( n = 24 \) (\( \Lambda_{24} \)). For \( n = 32 \), King showed\(^2\) that there are more than \( 10^7 \) extremal lattices, so we don’t expect to see this list any time soon, let alone the one for \( n = 40 \). For \( n = 48 \), we again expect such lattices to be rare; three are known (the two described in SPLAG, and a third found by Nebe\(^3\)), but it is anyone’s guess if they are few or plentiful. More recently\(^4\) Nebe constructed an extremal lattice of rank 72. This is the largest \( n \equiv 0 \mod 24 \) for which an extremal \( L \) of rank \( n \) is known, though there are several known examples with \( n > 48 \) in the congruence classes \( 8, 16 \mod 24 \).

For very large \( n \) (starting around \( 4 \cdot 10^4 \)), the extremal theta function has a negative \( q^{k+2} \) coefficient, so there can be no extremal theta function. This was proved by Mallows, Odlyzko, and Sloane, who furthermore showed\(^5\) that for every \( k_0 \) there is some \( n_0 \) such that once \( n > n_0 \) there is no even self-dual lattice of rank \( n \) with \( N_{\min} > 2 \lfloor n/24 \rfloor - k_0 \), because the theta series would have a negative coefficient.

One can likewise define the notion of an extremal self-dual lattice that need not be even, using \( f = \theta_{Z_2}(q^2) \) and \( g = \Delta_+(q^2) \) (this \( \Delta_+ \), like \( \Delta \), has a product formula that gives positivity of the coefficients of \( 1/g \)). Here we take \( r = \lfloor n/8 \rfloor \), and find that \( N_{\min}(L) \leq \lfloor n/8 \rfloor + 1 \). But in this setting we exhaust the “extremal” lattices (that is, self-dual lattices with \( N_{\min}(L) = \lfloor n/8 \rfloor + 1 \)) much sooner: we have already seen \( Z^n \) (\( n < 8 \)), \( E_8 \), \( D_{12}^+ \), \( (E_7^2)^+ \), \( A_{15}^+ \), and now \( O_{23} \) and \( \Lambda_{24} \); and there are no others.\(^6\)

We conclude with consequences of the spherical-design properties for extremal lattices of ranks 32, 48, and 72. If \( L \) is an extremal lattice of rank 32 then

\[
\theta_L = E_4^1 - 960E_4\Delta = 1 + 146880q^2 + 64757760q^3 + 4844836800q^4 + O(q^5). \tag{7}
\]

The 146880 minimal vectors constitute a spherical 7-design. This lets us determine the counts \( N_k(v_0) \)


only for \( \langle v_0, v_0 \rangle = 4 \) or 6; already for \( \langle v_0, v_0 \rangle = 8 \) there is one degree of freedom:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\langle v_0, v_0 \rangle & \# & N_0 & N_{\pm 1} & N_{\pm 2} & N_{\pm 3} & N_{\pm 4} \\
\hline
4 & 146880 = 2^6 3^4 5 - 17 & 80910 & 31744 & 1240 & 0 & 1 \\
6 & 64757760 = 2^{13} 3 \cdot 5 - 17 \cdot 31 & 66060 & 35640 & 4698 & 72 & 0 \\
8 & 4844836800 = 2^6 3^5 5^2 \cdot 17 \cdot 733 & 70 N_4 + 57040 & 36240 - 56 N_4 & 28 N_4 + 8184 & 496 - 8 N_4 & N_4 \\
\hline
\end{array}
\]

Unfortunately neither the “7\( \frac{1}{2} \)”-design refinement, nor the positivity of the \( N_k \), suffices to determine \( N_4(v_0) \) for \( \langle v_0, v_0 \rangle = 8 \). We already know for geometrical reasons that \( N_4(v_0) \) must be some nonnegative even number and that \( N_4(v_0) \leq 2(32 - 1) = 62 \); and all 32 possible choices yield nonnegative \( N_k \), though \( N_4 = 62 \) makes \( N_3 = 0 \). This does happen, if we apply Leech’s construction to a suitable self-dual doubly even code of length 32; but this is quite unusual: few \( L \) are of this form, and for any extremal \( L \) of rank 32, the average of \( N_4(v_0) \) over all \( v_0 \) of norm 8 is only

\[
146880 \cdot 80910/4844836800 = 1798/733 < 2.453.
\]

If \( L \) is an extremal lattice of rank 48 then

\[
\theta_L = E_6^4 - 1440 E_3^4 \Delta + 125280 \Delta^2 \\
= 1 + 52416000 q^3 + 3900733200 q^4 + 66090221440 q^5 + O(q^6); \tag{8}
\]

The counts of vectors of norm 6, 8, 10 factor as

\[
52416000 = 2^9 3^2 5^3 7 \cdot 13, \\
3900733200 = 2^5 3^7 5^3 7^3 \cdot 13, \\
66090221440 = 2^{11} 3^8 5 \cdot 7 \cdot 13 \cdot 23 \cdot 47, \tag{9}
\]

and the counts \( N_k(v_0) \) for these three cases are

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\langle v_0, v_0 \rangle & N_0 & N_{\pm 1} & N_{\pm 2} & N_{\pm 3} & N_{\pm 4} & N_{\pm 5} & N_{\pm 6} \\
\hline
6 & 23766960 & 12608784 & 1678887 & 36848 & 0 & 0 & 1 \\
8 & 20582240 & 12816384 & 2905728 & 192512 & 2256 & 0 & 0 \\
10 & 18409300 & 12612600 & 3898200 & 475300 & 17150 & 100 & 0 \\
\hline
\end{array}
\]

For \( \langle v_0, v_0 \rangle = 12 \), the average \( N_6 = N_6(v_0) \) is

\[
52416000 \cdot 23766960 \\
437824977480000 = 990290 \cdot 348037 = 2.845+,
\]

and \( N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 5} \) are

\[
16802688 + 924 N_6, \ 12270384 - 792 N_6, \ 4642848 + 495 N_6, \ 833592 - 220 N_6, \ 58656 + 66 N_6, \ 1176 - 12 N_6,
\]

with \( N_{\pm 5} = 12(98 - N_6) \). Using also the \( 11\frac{1}{2} \)-design condition, we likewise analyze \( \langle v_0, v_0 \rangle = 14 \): here \( N_7(v_0) \) is either 0 or 2, with the latter occurring for

\[
\frac{1}{2} 52416000 \cdot 12608784 = 330451011072000
\]

7
choices of \( v_0 \), which is \( 12/551 < 2.2\% \) of the number \( N_{14}(L) = 15173208925056000 \) of lattice vectors of norm 14; we then find that the possible counts \((N_0, N_{\pm 1}, N_{\pm 2}, \cdots, N_{\pm 7})\) are

\[
(15558368, 11883840, 5190387, 1215048, 133992, 5496, 53, 0),
\]
\[
(15573680, 11870838, 5198263, 1211770, 134848, 5390, 49, 2).
\] (10)

For extremal \( L \) of rank 72,

\[
\theta_L = E_4^0 - 2160 E_4^6 \Delta + 965520 E_4^3 \Delta^2 - 27302400 \Delta^3
= 1 + 6218175600 q^4 + 15281788354560 q^5 + 9026867482214400 q^6 + O(q^7),
\] (11)

with factorizations

\[
6218175600 = 2^4 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 37,
\]
\[
15281788354560 = 2^{16} \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37, \ldots;
\]

the nonzero counts \( N_k(v_0) \) are

\[
(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 8}) = (2603658750, 1512243200, 280928256, 13959168, 127800, 1)
\]

for \( \langle v_0, v_0 \rangle = 8 \), and

\[
(N_0, N_{\pm 1}, N_{\pm 2}, N_{\pm 3}, N_{\pm 4}, N_{\pm 5}) = (2328777990, 1508892000, 396819000, 37926000, 1056125, 5680)
\]

for \( \langle v_0, v_0 \rangle = 10 \).

For the numbers 565866362880, 45792819072000, \ldots of minimal vectors in putative extremal lattices of dimensions \( n = 96, 120, \ldots \), see OEIS Sequence 34597 (http://oeis.org/A034597). By the time we reach \( n = 96 \) the 11-design property gives the counts \( N_k(v_0) \) only for minimal vectors \( v_0 \): they are

\[
219453729516, 137524268000, 32881785375, 2733804000, 66118250, 341056
\]

for \( k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \) respectively, and of course 1 for \( k = \pm 10 \) and zero otherwise.