Inverse and implicit functions:

1. Solve Exercise 1.4 on page 171 of the Edwards text. (Edwards should have included the assumption \( \partial G/\partial y \neq 0 \) which is needed for the formula to make sense, and as we know is also needed to apply the implicit function theorem. As this Exercise suggests, an implicit function defined by a \( C^2 \) equation is itself \( C^2 \) once the condition \( \partial G/\partial y \neq 0 \) is satisfied. Higher derivatives work in the same way, but — as this Exercise also suggests — you don’t want to derive the general formulas for \( d^3y/dx^3 \) and beyond. Like Edwards (see the brief Section 5), we shall punt on proving the \( C^k \) versions of the implicit and inverse function theorems, whose proofs are somewhat tedious and messy, and introduce no fundamentally new ideas.)

2.–3. Solve Exercise 1.5 on page 171, and Exercise 1.10 on pages 171–172.

4. i) Prove that there exists a differentiable real-valued function \( y(x) \) in some neighborhood of \( x = 0 \) such that

\[
x = e^{-y(x)} y(x)
\]

for all \( x \) in that neighborhood.

ii) Show that \( y(x) \) is a solution of the differential equation \( xy' = y/(1 - y) \), and conclude that \( y \) is a \( C^\infty \) (infinitely differentiable) function of \( x \) near the origin.

iii) Use that differential equation to show that the coefficients \( a_n \) of the Taylor expansion \( y(x) = \sum_{n=1}^{\infty} a_n x^n \) of \( y(x) \) about \( x = 0 \) satisfy the recurrence

\[
a_n = \frac{1}{n-1} \sum_{k=1}^{n-1} ka_k a_{n-k}
\]

for \( n > 1 \); since also \( a_1 = 1 \) (why?) this inductively determines all the \( a_n \). Compute \( a_2 \) through \( a_6 \). (Check: \( a_6 = 54/5 \).) Can you guess a formula for \( a_n \)? Is your guess confirmed by the value of \( a_7 \)?

[I refrain from asking: iv) prove your guess. I know several “elementary” proofs of this, but none that is suitable for a homework problem.]

Inverse and implicit functions in higher dimensions:

5.–8. Solve Exercises 3.1, 3.8, 3.12, and 3.13 of the Edwards text (pages 194 and 195). For 3.13, it will help to recall the formula for the inverse of a \( 2 \times 2 \) matrix. Also check that the formulas of 3.13 hold for the inverse function you found in 3.1 (whose similarity with the formula for the real and imaginary parts of \( 1/(x + iy) \) may be helpful).

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1Definitions of Terms Commonly Used in Higher Math, R. Glover et al.; cf. also Prob. 4.
2Note that while we don’t yet(?) know that convergent power series can be differentiated termwise, we do know that Taylor series can be differentiated termwise, because the \( x^k \) coefficient is defined in terms of the \( k \)-th derivative at zero.
9. Suppose $A_0$ is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 \neq \lambda_j$ for all $j > 1$, so that $\lambda_1$ is an eigenvalue with multiplicity 1 (both geometric and algebraic multiplicity) and the first unit vector $e_1$ is an eigenvector. Prove that there exist $C^1$ functions $\lambda, v$ from a neighborhood $U$ of $A_0$ in the $n^2$-dimensional space of real $n \times n$ matrices to $\mathbb{R}$ and $\mathbb{R}^n$ respectively, such that: $\lambda(A_0) = \lambda_1$, $v(A_0) = e_1$, and for all $A \in U$ we have $Av(A) = \lambda(A)v(A)$ and the first coordinate of $v(A)$ is 1. [Note that the last condition means that $v(A) - e_1$ is in the $n - 1$ dimensional space orthogonal to $e_1$, so in effect we’re looking for a map from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{1+(n-1)} = \mathbb{R}^n$, determined by $n$ conditions which are the coordinates of $Av(A) = \lambda(A)v(A)$]. Give a formula for the directional derivative of $\lambda(A)$ in the direction of an arbitrary $n \times n$ matrix $M$.

[The last problem is the beginning of “perturbation theory” for eigenvalues and eigenvectors, which is an important technique in quantum mechanics and elsewhere.]

This problem set is due Friday, April 11, at 5PM.