We have run across several instances of the following problem: given power series
\[ G = \sum_{k \geq -p} g_k q^k, \quad A = \sum_{k \geq k_0} a_k q^k \]
(with \( p > 0 \), this being the order of the pole of \( G \) at \( q = 0 \)), how does the constant coefficient \( C_m \) of \( AG^m \) behave as \( m \to \infty \)? (If we’re interested in some other coefficient \( q^b \) with \( b \) fixed we can recover it from the constant coefficient of \( (q^{-1}A)G^m \), so the \( b = 0 \) question is actually no less general.)

For example, we found a formula for the kissing number of an extremal Type II code of length \( 24m \) in terms of \( C_m \) for \( A = 3q(1+2q)(1+14q+q^2)^2 \) and \( G = (q(1-q)^4)^{-1} \), and will need the asymptotics of this kissing number and of the next coefficient of the extremal weight enumerator. More elementary examples, also with \( p = 1 \), include \( (A, G) = (1, q^{-1} + g_1 q) \), when \( C_m = g_1^{m/2}(m^{m/2}) \) or \( 0 \) according as \( m \) is even or odd (and likewise \( (A, G) = q^{-1} + g_1 q^2 \), giving \( g_1^{m/3}(m^{m/3}) \) if \( 3|m \) and \( 0 \) otherwise), or \( (A, G) = (q^1, G = q^{-1} + 1 + q^{-1}) \), when \( C_m \) is the “king number” counting \( m \)-move paths of a chess king from \((0,0)\) to \((m,i)\).

So far we’ve used only formal manipulations of power series, but for asymptotic estimates we shall need our series to actually converge in some neighborhood of \( q = 0 \) (as indeed they do in all our examples). In each case that we work with, all the coefficients \( a_k \) and \( g_k \) are nonnegative. In that case the same is true for the coefficients of \( AG^m \), so we obtain an upper bound on \( C_m \) by just evaluating at any \( q_0 > 0 \) in the circle of convergence: if the \( q^k \) coefficient of \( AG^m \) is \( C^{(k)}_m \) then
\[
C_m = C^{(0)}_m < \sum_{k = k_0 - mp}^{\infty} C^{(k)}_m q_0^k = A(q_0)G(q_0)^m,
\]
and as \( m \to \infty \) the tightest (i.e., smallest) bound is obtained by choosing for \( q_0 \) the value that minimizes \( G(q_0) \) (which happens at a unique positive \( q_0 \) as long as \( a_k > 0 \) for some \( k > 0 \) [NB we implicitly assume \( a_k > 0 \)], because \( G \) is convex upwards and \( G(q) \to \infty \) as \( q \to 0 \) and also as \( q \) goes to the [possibly infinite] radius of convergence). We must thus assume that the power series defining \( A(q) \) converges at this \( q_0 \). We shall see that, under a very mild further hypothesis, the elementary estimate \( C_m < A(q_0)G(q_0)^m \) is within a factor of order \( \sqrt{m} \) of the correct asymptotic:
\[
(*) \quad C_m \sim \frac{G(q_0)^m}{\sqrt{m}} \left( \alpha_0 + \frac{\alpha_1}{m} + \frac{\alpha_2}{m^2} + \cdots \right),
\]
where
\[
\alpha_0 = \frac{\sqrt{G(q_0)}}{2\pi G''(q_0)} A(q_0).
\]
[Note that (*) starts “\( C_m \sim \cdots \)”, not “\( C_m = \cdots \)”; it’s an asymptotic series, which need not converge to \( C_m \), or indeed to anything: as with Taylor series, an asymptotic series “\( C_m \sim F(m) \sum_{j=0}^{\infty} \alpha_j/m^j \)” means that for each \( j_0 \) the truncated sum \( F(m) \sum_{j=0}^{j_0} \alpha_j/m^j \) is within \( O(F(m)/m^{j_0}) \) of \( C_m \) as \( m \to \infty \).]

The “very mild hypothesis” is that the coefficients \( g_k \) not be supported on an arithmetic progression \( k \equiv -p \mod d \) for any \( d > 1 \). Note that this hypothesis fails for our examples with \( G = q^{-1} + g_1 q \) and \( G = q^{-1} + g_2 q^2 \) (where \( d = 2 \) and \( d = 3 \)) which led to \( C_m \) depending on \( m \mod d \) (and even when \( C_m \neq 0 \) our asymptotic formula is missing a factor of \( d \)). But it is easy to reduce to the case \( d = 1 \) by a change of variable, replacing \( G(q) \) by \( G(q^{1/d})^{d} \).
To prove (*) we begin by writing the constant coefficient as the average of $AG^m$ over a circle:

$$C_m = \int_{-1/2}^{1/2} A(r e^{2\pi i \theta}) G(r e^{2\pi i \theta})^m d\theta$$

for any $r > 0$ smaller than the radius of convergence of the power series $A(q)$ and $G(q)$. [This is equivalent to the usual formula using a contour integral around the circle $|q| = d$.] We’ll fix $r$ and let $m \to \infty$. The integrand’s absolute value is maximized at $\theta = 0$, and thanks to our additional hypothesis on the $a_k$ this maximum is unique on $\mathbb{R}/\mathbb{Z}$. But for most choices of $r$ we see that increasing $m$ makes the integrand oscillate ever faster at $\theta = 0$, causing massive cancellations that mask the actual asymptotic growth. The exception is the one value $r = q_0$ where $G$, and thus its phase $\Im \log G$, has a critical point (hence the name “stationary phase” for this asymptotic technique). We then have

$$G(r e^{2\pi i \theta})^m = G(q_0) + 2\pi^2 G''(q_0) \theta^2 + O(\theta^3)$$

for small $\theta$. So we expect that for large $m$ and small $\theta$ the integrand behaves like $A(q_0) G(q_0)^m e^{-cm \theta^2}$ where $c = 2\pi^2 G''(q_0)/G(q_0)$, from which the formula $C_m \sim G(q_0)^m a_0/\sqrt{m}$ will follow via the Gaussian integral $\int_{-\infty}^{\infty} e^{-cm \theta^2} d\theta = \sqrt{\pi/cm}$.

To prove this, and obtain further terms of the asymptotic expansion, apply a change of variables to $z = z(\theta)$ such that $G(q_0 e^{2\pi i \theta})$ is actually equal $G(q_0) e^{-z^2/2}$ in a neighborhood $|\theta| < \delta$ of $\theta = 0$. We then write

$$A(r e^{2\pi i \theta}) G(r e^{2\pi i \theta})^m d\theta = \tilde{A}(z) e^{-mz^2/2} dz$$

in that neighborhood, and expand $\tilde{A}(z)$ in a Taylor series with constant term $\tilde{A}(0) = A(q_0)/z'(\theta) = 2\pi (G''(q_0)/G(q_0))^{1/2}$.

For large $m$ we may then estimate $\int_{z(-\delta)}^{z(\delta)} \tilde{A}(z) e^{-mz^2/2} dz$ by the integral of the same integrand form $z = -\infty$ to $+\infty$, and get our asymptotic expansion using the definite integral

$$\int_{-\infty}^{\infty} z^{2k} e^{-mz^2/2} dz = \sqrt{\frac{2\pi}{m}} \frac{(2k - 1)!!}{m^k},$$

where “$(2k - 1)!!$” is $1 \cdot 3 \cdot 5 \cdots (2k - 1) = (2k)!/2^k k!$ (and of course the odd terms $z^{2k-1} e^{-mz^2/2}$ integrate to zero).

**Exercises**

1. Fill in our sketch of the proof of (*). [Warning: at $\theta = \pm \delta$ our new variable $z$ will take complex variables, so the integral $\int_{z(-\delta)}^{z(\delta)}$ must be interpreted as a contour integral to be approximated by $\int_{z = -\infty}^{z = \infty}$.]

2. Adapt this technique to obtain the beginning of the refinement of Stirling’s approximation for

$$n! = \Gamma(n + 1) = \int_0^{\infty} x^n e^{-x} dx$$

to an asymptotic formula:

$$n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + O \left( \frac{1}{n^2} \right) \right)$$

as $n \to \infty$ (approximate the integrand by a Gaussian near its maximum at $x = n$, which is a stationary point, even though no phase is involved here).