Recall that we’ve fixed power series $F = F_0 = 1 + O(q)$ and $\Delta = q + O(q^2)$ and an integer $a \in [0, t)$, and for every positive integer $m$ defined $W_m$ to be the unique power series of the form $W_m = F_0 F(F, \Delta) = 1 + O(q^{m+1})$ with $F_m$ some homogeneous polynomial of degree $m$. We found that the $q^{m+1}$ coefficient $N_{m+1}$ of $W_m$ is $(tm + a)/(tm + t)$ times the $q^{-1}$ coefficient of $F_0^{-a} F^{m}/\Delta^{m+1} \equiv dF_0 F_0^{-1-a}/\Delta^{m+1}$, where $F^0 = dF/dq$. We deduced that if $F_0$ has nonnegative coefficients, and $1/\Delta$ has positive $q^k$ coefficient for all $k \geq -1$, then $N_{m+1} > 0$. In our setting these power series also satisfy the conditions we gave for $N_{m+1}$ to have an asymptotic expansion of the form

$$N_{m+1} \sim \frac{1}{\sqrt{m}} \Delta(q_0)^{-m} \left( \alpha_0 + \frac{\alpha_1}{m} + \frac{\alpha_2}{m^2} + \cdots \right),$$

where $q_0$ is the positive number at which $\Delta$ is minimal (e.g. $q_0 = 1/5$ for the $F, \Delta$ that arise for Type II codes), and $\alpha_0 > 0$. We next give a formula for $N_{m+2}$ that shows that under the same hypotheses $N_{m+2}/N_{m+1} = O(1) - c_0 m$ for some $c_0 > 0$, namely the constant coefficient of $1/\Delta$ (which is minus the $q^2$ coefficient of $\Delta$). In particular $N_{m+2} < 0$ for $m$ sufficiently large, so $W_m$ cannot be a weight enumerator or theta function.

We start as before by setting $z = \Delta/F = q + O(q^2)$ and expanding $1/(F_0^a F^m)$ in powers of $z$ as $\sum_{k=0}^{\infty} b_k z^k$ to find that

$$\frac{1 - W_m}{F_0^a F^m} = \sum_{k=m+1}^{\infty} b_k z^k.$$

We noted already that $N_{m+1} = -b_{m+1}$. To find $N_{m+2}$ we must determine $b_{m+2}$. We again write it as a residue at $z = 0$, this time of $(1/(F_0^a F^m)) \frac{dz}{z}$ (with an extra factor of $z$ in the denominator), and use invariance of the residue and integration by parts to find that $-b_{m+2}$ is $(tm + a)/(tm + 2t)$ coefficient of $F_0^{-a} F'(q)/\Delta^{m+2}$. Using again the example of $G_{24}$, we compute

$$\frac{F_0^{-a} F'(q)}{\Delta^{m+1}} = \frac{(42 + 6q) F_0^0}{\Delta^{3}} = 42q^{-3} + 3450q^{-2} + 121578q^{-1} + 2416506 + 30193194q + \cdots$$

and so $-b_3 = \frac{4}{3} 121578 = 40526$, which is confirmed by direct computation. In particular, $-b_{m+2}$ is asymptotic to a constant positive multiple of $-b_{m+1}$ (the multiplier being the value of $F$ at $q_0$).

But $N_{m+2}$ is not simply $-b_{m+2}$, because $-b_{m+1}$ contributes too, multiplied by the $q^{m+2}$ coefficient of $F_0^a F^m z^{m+1}$. (In our example that’s

$$F^2 = \Delta^2 / F = \frac{(1 - 4q + O(q^2))^2}{1 + 42a + O(q^2)} = 1 - 50q + O(q^2),$$

and $40526 - 50 \cdot 759$ does come to 2576, the number of weight-12 codewords (“[umbral] dodecads”) of $G_{24}$.) In general, the $q^{m+2}$ coefficient of $F_0^a F^m z^{m+1}$ is the $q$ coefficient of

$$q^{-(m+1)} F_0^a F^m z^{m+1} \sim \frac{F_0^a}{q} (\Delta/q)^m,$$

which is a constant plus $m$ times the $q^2$ coefficient of $\Delta$, which as we already observed is $-c_0 < 0$. So we’ve proven that $N_{m+2} = (-c_0 m + O(1)) N_{m+1} < 0$ for large $m$.

Now suppose we fix some $j > 0$ and relax the extremality condition by allowing power series of the form $W = F_0^a P(F, \Delta) = 1 + O(q^{m+1-\delta})$, i.e. adding to $W_m$ an arbitrary linear combination of the $j$ monomials $F^h \Delta^{m-h}$ with $0 \leq h < j$. I claim that even then one of the coefficients $N_{m+1-j}, N_{m+2-j}, \ldots, N_{m+1}, N_{m+2}$ is bound to be negative for $m$ large enough. We do this by finding the linear relation $\sum_{i=0}^{j} \gamma_i \Delta^{m+2-i} = 0$ that any such power series must satisfy, and showing that for large enough $m$ the coefficients $\gamma_i$ are all positive by calculating $\gamma_i = (c_0m)^{i+j} + O(m^{i-j})$. This in turn follows from our estimates for $N_{m+1}$ and $N_{m+2}$ together with the observation that in each monomial $F^h \Delta^{m-h}$ the $q^{m+2}$ coefficient is $(-c_0 m)^{h+2-i} / (h + 2 - i) + O(m^{h+1-i})$ for each $i = 0, 1, \ldots, h + 2$ (and the fact that for each row of Pascal’s triangle other than the zero row the alternating sum vanishes). This completes the proof of the Mallows-Odlyzko-Sloane theorem of 1975.